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# How Effective are Smooth Compositions in Predictive Control of TS Fuzzy Models?

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**Abstract** In this article, we study the structural properties that smooth compositions bring to predictive control of TS fuzzy models and examine how they affect the uncertainties, parameter variations of the system and environmental noises to die out. We have employed the smoothness structure of compositions to convert the MPC cost function of TS fuzzy model of the nonlinear systems to an incremental iterative algorithm. Hence, the proposed algorithm does not linearize the nonlinear dynamics, neither requires solving an NP optimization problem in MPC and, therefore, is very fast and simple. The connectivist identification—MPC approach—can be employed for the systems with the long-range horizons. Therefore, the technique is beneficial to the dead-time and non-minimum phase systems. The stability analysis of the algorithm has been carried out, and the performance of the smooth TS fuzzy identification–controller scheme to the classical ones has been compared on a non-min phase test problem with different uncertainties and working environments, including (a) the normal working conditions, (b) with the additive noises, (c) with the parametric changes, (d) with the additive time-varying disturbances to demonstrate the robust behavior of the smooth compositions.

**Keywords** Fuzzy control · Fuzzy IF–THEN systems (TSK) · Smooth compositions · System identification · Model predictive control (MPC) · Unstable systems

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## 1 Introduction

Soft computing methods are being used for identification and control of nonlinear and complex systems based on the input–output data collected from the original system [1, 2]. There are many applications of artificial neural network and fuzzy modeling framework for the identification and model-based control purpose in the industry and academia [3, 4]. Such methods have quite interesting ability in presenting the industrial processes with different types of data. The advantage of fuzzy models is that they can also include the operator information for dealing with the concept of uncertainty and handling the probabilistic logics [5, 6]. The inclusion of information about the process in the generation of the mathematical model makes the control task also capable of coping with the various nonlinear behaviors such as limit cycles, or where large changes in the operating conditions can be anticipated during the routine operation, such as the systems with the time-varying parameters, in batch processes or during the start-up and the shutdown of the continuous processes [7, 8].

Some researchers have demonstrated the universal approximation properties of the fuzzy logic-based models [3]. It is widely recognized that the fuzzy models can approximate any nonlinear function to any degree of accuracy in a convex compact region. However, in many applications it is desired to go beyond and have a model to approximate the nonlinear function on a smooth surface to get better performance and stability properties [9, 10]. The first contribution of this manuscript is to make TS fuzzy model of the nonlinear systems using smooth fuzzy compositions. It facilitates to avoid abrupt changes and discontinuities in the input–output mapping, especially in the region around the steady states, when both error and change in error are approaching zero. The continuity of the function and its derivative is defined as the smoothness property [1, 9, 11].

### 1.1 Smooth Fuzzy Compositions

After introduction of topological structures [12], different researchers have studied the concept of smooth fuzzy topological spaces [13] and their properties and characterizations in different compact, disconnected and bi-topological spaces [14–17].

Recently, some new smooth compositions have been presented and have been employed for modeling static input–output mapping of the dynamical systems in [2, 18, 19]. All the contributions in application of smooth compositions for advanced control of dynamic systems have employed the relational modeling framework [20].

The identification process then will be consisting of the estimation of the unknown relational matrix from the input–output data [20, 21]. Even though the fuzzy relational matrix can be quite easily developed online, this advantage must be viewed in the context of their known limitations. Firstly, their use is normally limited to the processes with a small number of variables in view of the potential large size of the matrix and the computational requirements. The relational fuzzy modeling approaches generally require significant computational effort, especially if a high number of variables and reference fuzzy sets are involved. A first-order relational model of a system consisting of 2 inputs and 1 output, where 7 reference sets are used by each variable, will generate the matrix of 2401 elements [22]. Another problem posed by the fuzzy relational modeling framework is that there exists no simple approach for deriving the controller output analytically, which makes it necessary that for any change in the system one needs to resort to the numerical approaches, for doing the modeling process again. It increases the already mentioned large computational requirements of the model. Also, the controller design problems arise as a result of incomplete rule bases. It is to say, the fuzzy relational approach does not provide rules that can be expressed linguistically. Hence, this technique would be difficult to use interactively with the human in loop, making it difficult to update and modify the matrix using the heuristic knowledge [5, 6]. Therefore, the first motivation of the present work has been to overcome such barriers by presenting an algorithm for identification of IF–THEN fuzzy models, rather than relational fuzzy models for more effective employment of the smooth compositions.

It is worth reminding the slight alteration of the definition of a smooth fuzzy topology built from the employment of the smooth fuzzy norms by fuzzy relations which is associated with the concept of composition of binary numbers and relations in the earlier works [20, 23, 24], rather than the topology built from the employment of the same norms in the IF–THEN model, which more relates to the concept of fuzzy numbers as introduced by Zadeh [25]. This is to say, the main difference of two approaches of the relational smooth fuzzy models and IF–THEN smooth fuzzy models is that whether or not it is more practical that the functions be presented through fuzzy numbers of the fuzzy topology or one should restrain to only the constant zero and one fuzzy sets 0 and 1 of the smooth fuzzy relations; We think the first one is preferable and will contribute on development of the IF–THEN smooth fuzzy connectivist modeling control scheme in this contribution.

Alongside, the other difficulty in smooth fuzzy relational models is that they suffer from the lack of analyzability. Hence, our other motivation has been not only to develop a new TS fuzzy modeling framework using the smooth compositions, but also to construct models which could be used more efficiently for study on the numerical behavior, speed of convergence and the stability of the algorithm in the model predictive controller (MPC) design phase.

MPC is one of the methods that have been considered largely for the purpose of fuzzy logic model-based control of nonlinear processes. It can run the complex nonlinear dynamics toward the desired point employing the system data combined with prior knowledge [3, 8]. This control strategy is based on online optimization algorithms and can employ the long-range predictive horizons to secure the stability and optimality of the unstable processes. The fuzzy model structure of this paper is obtained through a harmonious selection of components, which employs the fuzzy smooth structure to simplify its subsequent controller design.

MPC has been employed for the relational fuzzy model systems with smooth compositions in [20], where the authors have attempted to make the one-step-ahead model predictive control of the nonlinear process. Contrarily, we have developed MPC for TS fuzzy model with long-range horizon. Using the long-range horizon, one can predict the impact of the current process input to the future process output, to handle the uncertainty in the system and the models' mismatches during the closed-loop control performance. Therefore, our proposed algorithm is beneficial for the multivariable systems, to run the system back to the feasibility region in the cases of failure in the actuators, to stabilize the non-minimum phase and dead-time systems.

The originality of the paper in computational facilitation of the algorithm is that using the smoothness property of the fuzzy models in the predictive controller design, we propose a systemic iterative algorithm without need of solving an NP-hard optimization problem in every step, which widens the area of application of the algorithm for the industrial applications without the need to a high computational power. In the earlier works of AmirAskari and Menhaj [20], they have identified the smooth fuzzy model through relational matrix, and hence, they again resulted in the NP-hard optimization problem using the non-derivative-based methods, without any effective employment of the smoothness structure of the model. Besides, their formulation is based on the triangular membership function which is not widely being used; however, we have put no restriction on the shape of the membership function in the formulation development. Although we have used normal membership function to simplify the expression of the algorithm, the algorithm can be extended for any kind of the membership function readily.

Therefore, we believe that the present work paper improves the general form of the fuzzy smooth models both theoretically (for the theoretical algorithm development and comparative analysis of the simulation results) and practically (for facilitation in the numerical implementation of the algorithm to propone to the industry).

We illustrate application of the combined identification–control scheme on different simulations. It has been applied to a non-minimum phase system, taken from the literature. We have considered the variations of the prediction horizon as an important MPC tuning parameter and applied different working conditions to study the effect of smooth compositions in the performance of the predictive controller to the TS fuzzy models. According to [7], smooth fuzzy continuity is equivalent to fuzzy continuity on all the cuts that together form the decomposition of the smooth fuzzy topology. Therefore, it is expected that smooth fuzzy models show more robustness to the parametric changes and uncertainties in the predictive control rather than the classical fuzzy models, by structure.

The paper is organized as follows. First, we present the structure of fuzzy models for dynamic systems which comprises a review on the different fuzzy compositions. Then, the identification problem is addressed and we present an identification scheme employing the smooth fuzzy compositions. In the next section, we employ the model constructed through the proposed identification algorithm for the purpose of model predictive control of the systems in the long horizons. We have provided several simulations of the proposed uniform identification–control design procedure to study the effectiveness of the smooth compositions in predictive control of TS fuzzy models. We will end the paper with the conclusions.

## 2 Problem Definitions

Consider a MIMO system with m inputs $u \in U \subset R^m$ and $p$ outputs $y \in Y \subset R^p$.

$$y(k+1) = f(\xi(k), u(k)). \tag{1}$$

The input vector $u(k) \in R^m$ contains the input variables, and the regression vector $\xi(k)$ includes the current and lagged inputs and outputs:

$$\begin{aligned}\xi(k) &= \left[\acute{y}_1, \acute{y}_2, \ldots, \acute{y}_p, \acute{u}_1, \ldots, \acute{u}_m\right]^{\mathrm{T}} \\ \acute{y}_1 &= \left[y_i(k), y_i(k-1), \ldots, y_i\left(k - n_{yi}\right)\right], i = 1, \ldots, p \\ \acute{u}_j &= \left[u_j(k), u_j(k-1), \ldots, u_j\left(k - n_{uj}\right)\right], j = 1, \ldots, m\end{aligned}$$

where $n_{yi}$ and $n_{uj}$ specify the number of lagged values for the $i$th output and $j$th input, respectively. We can define a fuzzy inference for this system as

$$\begin{aligned}&R_{li}: \text{if } \xi_1 \in \Omega^{li,1} \text{ and } \cdots \text{ and } \xi_p \in \Omega^{li,p} \text{ and} \\ &\xi_{p+1} \in \Omega^{li,p+1} \text{ and } \cdots \text{ and } \xi_{p+m} \in \Omega^{li,p+m} \\ &\text{then } Y_{li}(k+1) = \theta_l(\xi(k), u(k)), l = 1, \ldots, r,\end{aligned}$$

where $\Omega^{li}$ are the associated interval of existence of the fuzzy set, $\xi_1$ is the first element of the vector $\xi$, and $\theta_l$ is the linguistic consequent parameters of the $l$th fuzzy rule, $\theta = [0,1]^{rp}$ and $r$ is the number of the rules for the $i$th output. The output value is calculated from the predicted output corresponding to each rule via the center of gravity method:

$$y_i(k+1) = \frac{\sum_{l=1}^{r} \beta_{li}\theta_{li}}{\sum_{l=1}^{r} \beta_{li}} \tag{2}$$

Based on the definition, $\beta$ is the degree of membership function for the antecedent (states + input) variables as follows:

$$\beta_i : \underbrace{U \times \cdots \times U}_{m \text{ times}} \times \underbrace{Y \times \cdots \times Y}_{p \text{ times}} \rightarrow [0,1]^l \tag{3}$$

where the symbol $\times$ represents the Cartesian product in the fuzzy sets. It can be calculated through the $s$–$t$ composition where $s$ and $t$ are some $t$-conorm and $t$-norm, respectively.

The mostly used fuzzy composition (sometimes called $s$–$t$ composition) is max–min. However, other fuzzy compositions also have been introduced in the literature [18, 23] and some has been collected in Table 1. We would refer the smooth composition II as "atan" composition and the smooth composition III as "acos" composition, according to the mathematical definition, in the rest of the paper.

Employing different $t$-norm and $s$-norm from Table 1 can give rise to the different levels of accuracy in modeling of the dynamical systems upon the context, which has been studied in the literature [3]. From them, the smooth fuzzy compositions can make the fuzzy model such that the output is a deferential function of the input variables. Hence, the different schemes of gradient-based methods can be used later for the adaptive tuning of the fuzzy model

**Table 1** Fuzzy compositions

| | s-Norm | t-Norm |
|---|---|---|
| Classical compositions | $T(a,b) = \max(a,b)$ | $T(a,b) = \min(a,b)$ |
| | $S(a,b) = a + b - ab$ | $T(a,b) = ab$ |
| Smooth compositions | $I : S_S(a,b) = \frac{r \cdot d \cdot \beta^{-\log_\beta(d) - \log_\beta(r)}}{(\beta-1)}$, $r = (\beta - 1)a + 1,\ s = (\beta - 1)b + 1,\ \beta \in (1, \infty)$ | $I : T_S(a,b) = 1 - \cos\left(\frac{2}{\pi}\cos^{-1}(1-a)\cos^{-1}(1-b)\right)$ |
| | $II : S_S(a,b) = 1 - \frac{4}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{4}(1-a)\right)\tan\left(\frac{\pi}{4}(1-b)\right)\right)$ | $II : T_S(a,b) = \frac{4}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{4}a\right)\tan\left(\frac{\pi}{4}b\right)\right)$ |
| | $III : S_S(a,b) = \frac{2}{\pi}\cos^{-1}\left(\cos\left(\frac{\pi}{2}a\right)\cos\left(\frac{\pi}{2}b\right)\right)$ | $III : T_S(a,b) = 1 - \frac{2}{\pi}\cos^{-1}\left(\sin\left(\frac{\pi}{2}a\right)\sin\left(\frac{\pi}{2}b\right)\right)$ |
| | $IV : S_S(a,b) = \cos\left(\frac{2}{\pi}\cos^{-1}a\cos^{-1}b\right)$ | $IV : T_S(a,b) = \cos\left(\cos^{-1}a + \cos^{-1}b - \frac{2}{\pi}\cos^{-1}a\cos^{-1}b\right)$ |
| Yager compositions | $S_Y(a,b) = \min\left\{1, \sqrt[p]{a^p + b^p}\right\}\ p > 0$ | $T_Y(a,b) = 1 - \min\left\{1, \sqrt[p]{(1-a)^p + (1-b)^p}\right\} p > 0$ |
| Lukasiewicz compositions | $S(a,b) = \max(a + b, 1)$ | $T(a,b) = \max(a + b - 1, 0)$ |
| Hamacher composition | $S_H(a,b) = \frac{a+b-(2-\gamma)ab}{1-(1-\gamma)ab}\ \gamma \geq 0$ | $T_H(a,b) = \frac{ab}{\gamma+(1-\gamma)\Vert a+b-ab \Vert}\ \gamma \geq 0$ |

parameters upon the time-varying plant parameters and the uncertainties of the plant. We would employ this idea for TS fuzzy model identification and long-range horizon predictive control, to be described in the sequent.

## 3 Generation of the Fuzzy Smooth Model

In the process of system identification, we train the fuzzy model to capture the functioning of the real plants. We can view this process as an application of an optimization method to the fuzzy models, very similar to the process of training neural networks, where the least square optimization problem is solved. At every sampling time, we consider a target value $t_i(k)$ for the system output $y_i(k)$ and, correspondingly, define the overall performance index $\Xi$ of the model as

$$J = \frac{1}{2}\Xi(t - y)^2 \tag{4}$$

The parameters of the fuzzy model can be tuned through solving the minimization problem of the performance index. It leads us to have a general method of modifying the fuzzy model at every sampling time $k$. The goal is to use the performance index to find the optimal shape of the membership functions. Therefore, the variables to find will be the center and the width of the fuzzy membership functions. To simplify the procedure, we consider the normal membership functions with the gradient-based variables' update algorithm:

$$\rho_{ld}(k+1) = \rho_{ld}(k) - \alpha_\rho \frac{\partial J(k)}{\partial \rho_{ld}} \tag{5}$$

$$\theta_{li}(k+1) = \theta_{li}(k) - \alpha_b \frac{\partial J(k)}{\partial \theta_{li}} \tag{6}$$

where $\rho = [c_{ld}, \delta_{ld}]$ are parameters of the normal membership functions, $\alpha_\rho$ and $\alpha_b$ are the step lengths in the gradient-based optimization and $l = 1 \cdots, r$, $d = 1, \cdots, m + p$ are the number of the system rules and the system inputs, respectively. The error's derivatives are straightforward, and the interested readers are referred to "Appendix 1" for more details.

## 4 Fuzzy Model-Based Control

In this section, we intend to employ the smooth fuzzy model developed in the last section to construct a uniform online identification-predictive control framework for the nonlinear processes. In order to facilitate the explanation of the algorithm development, we consider a single-input single-output dynamics; however, we emphasize that the results are readily extendable to the multi-input multi-output processes.

We consider the following cost function for the model predictive control purpose:

$$J = \frac{1}{2T}\sum_{t=1}^{T}\left[e^2(k+t) + \lambda u^2(k+t-1)\right] \tag{7}$$

where the tracking error is defined as

$$e(k+t) := r(k+t) - y(k+t) \tag{8}$$

$r(k+t)$ is the reference and $y(t+k)$ is the output of the plant both at $(k+t)$th sampling time instant. We choose $\lambda \geq 0$ as the penalty factor and $T$ as the control horizon. Based on the minimization of the cost function $J$, we derive a sequence of the optimal increase in input signal $\Delta u(k), \ldots, \Delta u(k+T-1)$; however, just the first increase signal is applied to the system. At the next time instant $k+1$, the whole process will be repeated. To derive the control law, we consider the simple case, where the input signal of the process is comprised of two membership functions as

$$u(k) = \chi_1 u_1(k) + \chi_2 u_2(k) \tag{9}$$

where $\chi_1(k) = \frac{\mu_1}{\mu_1+\mu_2}$, $\chi_2(k) = \frac{\mu_2}{\mu_1+\mu_2}$. The input signal at the next time step will be

$$u(k+1) = u(k) + \alpha \Delta u(k) \tag{10}$$

Or, in other formulation,

$$u(k) = [\chi_1, \chi_2]\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \alpha[\chi_1, \chi_2]\begin{bmatrix} \Delta u_1(k) \\ \Delta u_2(k) \end{bmatrix}$$

The incremental input signals $\Delta u_1(k)$ and $\Delta u_2(k)$ are given as

$$\Delta u_\varrho(k) = \frac{-\partial J}{\partial u_\varrho(k)}, \varrho = 1, 2. \tag{11}$$

with the length step $\alpha (0 < \alpha \leq 1)$.

Based on the definition, we have

$$\frac{\partial J}{\partial u_\varrho(k)} = \frac{1}{T}\sum_{t=1}^{T}\left[-(r(k+t) - y(k+t))\frac{\partial y(k+t)}{\partial u_\varrho(k)} + \lambda u(k+t-1)\frac{\partial u(k+t-1)}{\partial u_\varrho(k)}\right] \tag{12}$$

We assume free $u(k)$ and $(k+t) = u(k)$, $t = 1, 2, \ldots, N-1$. When one considers $i, j = 1, y = \frac{\sum_{l=1}^{r}\beta_l\theta_l}{\sum_{l=1}^{r}\beta_l}$ and $Y_l(k+1) = \theta_l(\xi(k), u(k))$, $l = 1, \ldots, r$, the state vector and inputs become

$$\xi(k) = [\acute{y}, \acute{u}]^T$$
$$\acute{y} = \left[y(k), y(k-1), \ldots, y\left(k-n_y\right)\right]$$
$$\acute{u} = [u(k), u(k-1), \ldots, u(k-n_u)]$$

The increment of the input signal can be obtained by taking the derivatives

$$\frac{\partial y(k+t)}{\partial u_\varrho(k)} = \frac{\sum_{l=1}^{r}\theta_l\left(\frac{\partial\beta_l(k+t)}{\partial u_\varrho(k)}\right) - y(k+t)\sum_{l=1}^{r}\frac{\partial\beta_l(k+t)}{\partial u_\varrho(k)}}{\sum_{l=1}^{r}\beta_l(k+t)}, \quad \varrho = 1, 2 \tag{13}$$

In Eq. (13), the value of the derivative $\frac{\partial\beta_l(k+t)}{\partial u_\varrho(k)}$ can be computed after the model and inference structure selection. For the sake of illustration, we consider the following model structure defined by the smooth fuzzy composition:

$$\beta_l(k+t) = S(T(R, U(k+t-1), Y(k+t-1))) \tag{14}$$

where $U$ and $Y$ are fuzzy values in $[0, 1]$. For input prediction horizon with $t = 1$, (i.e., $\frac{\partial y(k+1)}{\partial u_\varrho(k)}$), the only term depending on $u(k)$ is $U(k)$. Therefore,

$$\frac{\partial\beta_l(k+1)}{\partial u_\varrho(k)} = \frac{\partial S}{\partial T(\cdot,\cdot)}\frac{\partial T(\cdot,\cdot)}{\partial U(k)}\frac{\partial U(k)}{\partial u_\varrho(k)} \tag{15}$$

where

$$\frac{\partial U(k)}{\partial u_\varrho(k)} = \frac{\partial f(u'(k), \varrho_u)}{\partial u_\varrho(k)}$$

and $f(\cdot,\cdot)$ is the membership function with the parameters $\varrho_u = [c_u, \delta_u]$, $c_u$ is the membership function center and $\delta_u$ is the membership function width, obtained in the identification phase.

For the input prediction horizon with $t = 2$, the terms depending on $u(k)$ are $\beta(k+1)$ and

$$\beta_l(k+2) = S(T(R, U(k+1), Y(k+1))). \tag{16}$$

Therefore,

$$\frac{\partial\beta_l(k+2)}{\partial u_\varrho(k)} = \frac{\partial S}{\partial T(\cdot,\cdot)} \times \left[\frac{\partial T(\cdot,\cdot)}{\partial U(k+1)}\frac{\partial U(k+1)}{\partial u_\varrho(k)} + \frac{\partial T(\cdot,\cdot)}{\partial Y(k+1)}\frac{\partial Y(k+1)}{\partial u_\varrho(k)}\right], \quad \varrho = 1, 2$$

$$\frac{\partial Y(k+1)}{\partial u_\varrho(k)} = \frac{\partial Y(k+1)}{\partial y(k+1)}\frac{\partial y(k+1)}{\partial u_\varrho(k)}, \quad \varrho = 1, 2$$

where

$$\frac{\partial Y(k+1)}{\partial y(k+1)} = \frac{\partial f_l\left(\acute{y}(k+1), \varrho_y\right)}{\partial y(k+1)},$$

and $\frac{\partial y(k+1)}{\partial u_\varrho(k)}$ is calculated above in (13).

For $i = 3$, the only term depending on $u(k)$ is $\beta(k+2)$; hence,

**Table 2** Method for obtaining the increment of the input signal. The derivatives are obtained along the corresponding sequences for being used in (13) per iteration

| Iteration | $k+1$ | $k+2$ | $k+3$ | $\cdots$ | $k+T$ |
|---|---|---|---|---|---|
| Term available for computations from the previous instant | $u(k)$ | $\frac{\partial y(k+1)}{\partial u_\varrho(k)}$ | $\frac{\partial y(k+2)}{\partial u_\varrho(k)}$ | $\cdots$ | $\frac{\partial \beta_l(k+T-1)}{\partial u_\varrho(k)}$ |
| Model derivative computation | $\frac{\partial \beta_l(k+1)}{\partial u_\varrho(k)}$ | $\frac{\partial \beta_l(k+2)}{\partial u_\varrho(k)}$ | $\frac{\partial \beta_l(k+3)}{\partial u_\varrho(k)}$ | $\cdots$ | $\frac{\partial \beta_l(k+T)}{\partial u_\varrho(k)}$ |
| State derivative computation | $\frac{\partial y(k+1)}{\partial u_\varrho(k)}$ | $\frac{\partial y(k+2)}{\partial u_\varrho(k)}$ | $\frac{\partial y(k+3)}{\partial u_\varrho(k)}$ | $\cdots$ | $\frac{\partial y(k+T)}{\partial u_\varrho(k)}$ |

$$\frac{\partial \beta_l(k+3)}{\partial u_\varrho(k)} = \frac{\partial S}{\partial T(\cdot,\cdot)}\left[\frac{\partial T(\cdot,\cdot)}{\partial \beta_l(k+2)}\frac{\partial \beta_l(k+2)}{\partial u_\varrho(k)}\right],\ \varrho = 1,2 \tag{17}$$

which is calculated above. For $i > 3$, $\beta_l(k+i-1)$ is the only term depending on $u(k)$ which can be calculated recursively (Table 2).

*Remark 1* We can extend the control design procedure and the identification process readily for other definitions of the membership functions involved or to the systems with multi-inputs and multi-outputs.

**Lemma 1** *Assume that there exists a feasible solution for the control problem in (7). Then, the system dynamics will converge to track the reference signal as k* $\to \infty$.

*Proof* Assume that we are at the time k and implement the optimal input $u(k) = u_0^*$ that runs the system to the state $x(k+1)$.

$$J_{x(k)} = \min_{u(k)} \frac{1}{2T}\sum_{t=1}^{T}\left[e^2(k+t) + \lambda u^2(k+t-1)\right] \tag{18}$$

$$J_{x(k)} = \min_{u(k)} \frac{1}{2T}\left[e^2(k+1) + \lambda u^2(k)\right] + J_{x(k)}(k+1) \tag{19}$$

□

At this time, we can determine the associated optimal control input to the system over the horizon 1 to $T+1$:

$$J_{x(k+1)} = \min_{u(k+1)} \frac{1}{2T}\sum_{t=1}^{T}\left[e^2(k+t+1) + \lambda u^2(k+t)\right] \tag{20}$$

However, we can employ the previous sequence of optimal moves followed by zero as well: $u(k+1) = u_0^*$. As this sequence of input is not optimal,

$$J_{x(k+1)} \le J_{x(k)} - \min_{u(k)} \frac{1}{2T}\left[e^2(k+t) + \lambda u^2(k+t-1)\right] \tag{21}$$

As the value of minimization is positive for $(e,u) \ne (0,0)$, the sequence of the optimal costs is strictly decreasing for all $(e,u) \ne (0,0)$., i.e., $J(k+2) \le J(k+1) \le J(k)$. From the other hand, by the definition in Eq. (7), we have $0 \le J$. It means that the sequence of the cost functions $J(k), J(k+1), J(k+2)$ is converging to zero and $e \to 0$ as well.

**Corollary 1** *The feasibility of the control input and state variables implies that the MPC controller will run the state trajectory to zero.*

*Proof* It can be proved by change of parameters from Lemma 1, considering that

$J(\cdot) > 0, (e,u) \ne (0,0), J(e=0, u=0) = 0.$

□

For the unstable systems, the question will be how to determine the interval $T$ or at least an upper bound such that the system enters the positive invariant set. Several algorithms for the proper selection of the control horizon $T$ have been introduced in the literature.

**Lemma 2** *The obtained control law is continuous and smooth.*

*Proof* Since the control law u is obtained by the derivation and linear combination of some smooth and continuous functions, the control input is continuous and smooth. □

**Lemma 3** *The cost function* $J(\cdot)$ *is convex, continuous and smooth.*

*Proof* Since the cost function is obtained by the derivation and linear combination of some cosine smooth and continuous functions, it is continuous and smooth. The convexity of the cost function can be proved easily from Eq. (7). □

**Corollary 2** *The control function is the optimal control sequence, and the system trajectory is the corresponding optimal trajectory.*

*Proof* The corollary can be concluded from the convexity of the cost function $J(\cdot)$ in Lemma 3. □

The overall procedure of the connectivist smooth fuzzy identification and MPC control scheme is portrayed in Fig. 1.

*Remark 2* As it is shown in Fig. 1, we could join the learning capacities of the adaptive smooth TS fuzzy modeling scheme to the iterative method of MPC controller design to reach a uniform framework with the parallel processing features.

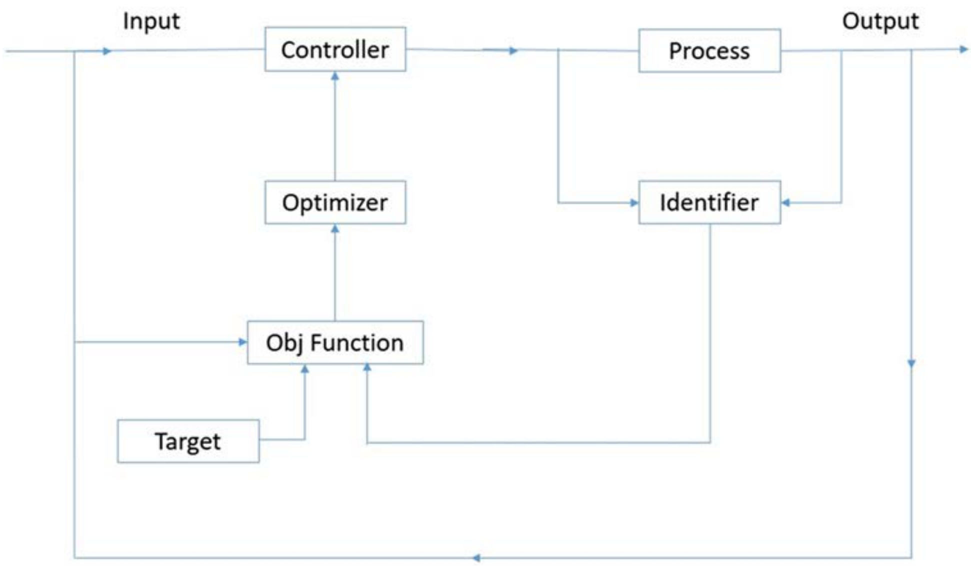


**Fig. 1** Overall scheme of the presented connectivist smooth TS fuzzy identification–control approach

*Remark 3* During the fuzzy adaptation process, the membership functions represent linguistic terms of smooth TS fuzzy model interference, which are comprehensible to a human. This aspect, which has been forgotten in the earlier works using relational fuzzy models [18, 20, 23], is one of the strengths of fuzzy modeling. Actually, the blind performance index used at the relational smooth fuzzy modeling-based tuning of the membership functions of the earlier works causes semantically meaningless linguistic terms at the model interfaces.

*Remark 4* In the present work, we have developed a systematic incremental controller using the smoothness and continuity properties of the model structure, to employ the online membership function calibration of the model with the least online computational burdens. This is while the MPC design for TS fuzzy models is typically based on the minimization of a NP-hard problem [20] or employs some kind of linearization algorithm [7].

**Lemma 4** *The rate of convergence of the control function is quadratic.*

*Proof* Based on Lemma 3, derivative of the control function is smooth almost everywhere, and its second derivate is continuous. Hence, when the initial point of the control signal and the system states are sufficiently close to the optimal points and the derivative function is not zero, the optimization algorithm will converge quadratically. □

**Corollary 3** *The smooth fuzzy MPC control function will converge faster than the classical fuzzy MPC and to a more stable solution.*

*Proof* Considering the quadratic rate of convergence for the control function in the smooth TS fuzzy models and the linear rate of convergence of the classical fuzzy model, the corollary can be concluded straight from Lemma 4. □

*Remark 5* The algorithm in Table 3 will converge only if the assumptions in the proof of Lemma 1 are satisfied. The most common difficulty is to choose a proper initial point of search in the basin of convergence of the algorithm. The suggested remedy is to run the algorithm from the several random initial points.

We show the effectiveness of the proposed uniform smooth TS fuzzy connectivist identification–control approach on different simulations of a non-minimum phase example below.

## 5 Illustrative Example

### 5.1 Smooth Fuzzy IF–THEN Model for Identification: MPC Control of a Non-minimum Phase System

In this section, we intend to illustrate the effectiveness of the proposed approach through an example based on [17]. We also study the role of extending control horizon on the overall performance of the controlled system. Consider the following discrete-time nonlinear system:

$$y(k+1) = -u(k) + 1.2u(k-1) + 1.4\exp\left(-y^2(k)\right) - 0.6y(k-1) \tag{22}$$

The open-loop response shown in Fig. 2 indicates that the process is indeed highly nonlinear.

Initially we have modeled the system through the proposed smooth fuzzy modeling scheme. Then, we controlled

**Table 3** Model predictive algorithm for the smooth TS fuzzy model

Concept: Assume that the smooth TS fuzzy model is available and we want to implement the predictive control strategy to track the reference signal.

Initialization:

Choose the maximum iteration and the simulation horizon.

Main Steps:

1. While NOT max iterations
2. Update $\frac{\partial y(k+t)}{\partial u_e(k)}$ based on Eq. (13) and instructions of Table 2
3. Update the incremental input signal $\Delta u(k)$ based on Eq. (11)
4. Update the solution $u(k+1) = u(k) + \alpha \Delta u(k)$
5. Apply the input to the system

End loop

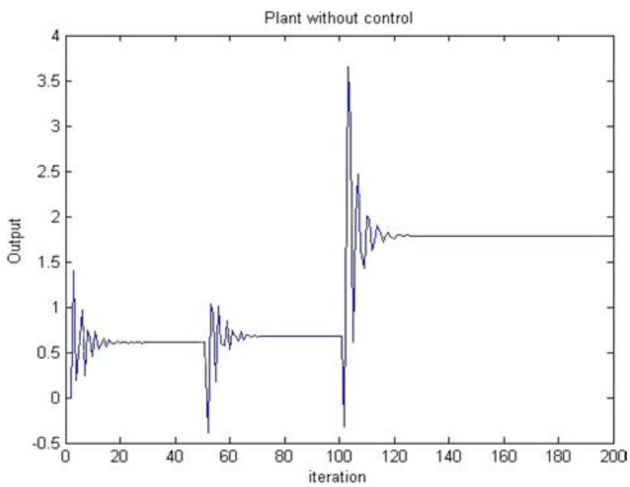


**Fig. 2** Open loop of the plant

the system in different control horizons $T$. We have taken $\alpha = 0.5$ and $\lambda = 0$ for this purpose.

System simulation is conducted to study how change of the control horizon and the selection of fuzzy composition affect the controller's performance. We have tested the controller performance in different time horizons.

### *5.1.1 Fuzzy Controller Design*

A series of simulations are conducted to examine the effect of fuzzy composition for predictive control of TS fuzzy models. In testing the set point tracking capability, the set point has been changed in a train of pulses. The dynamic response of the system and systems' input is depicted in the same figure. Figure 3 demonstrates the closed-loop dynamic responses in three control horizons using different fuzzy compositions: two "atan" and "acos" TS smooth fuzzy controllers and the classical product–sum TS fuzzy controller. Apparently, the control dynamics with all three compositions are satisfying and good.

### *5.1.2 Disturbance Rejection Performance*

To make the control problem more realistic, values of the parameters have been randomly varied and different disturbances have been added to the system.

We have employed both the classical fuzzy structure and smooth fuzzy structures in the comparative scheme to examine the effectiveness of the smooth compositions.

To give a quantitative measure of the controller accuracy, the performance function accounts for the error as $F(t) = e(t) \times e(t)$ has been employed. The comparison of the best performance of different compositions is shown in Tables 4, 5, 6 and 7.

From Figs. 3, 4, 5 and 6, it is observed that the smooth fuzzy compositions perform better than the classical compositions in overall.

### *5.1.3 Additive Noise*

To show how different compositions influence the control performance, we have considered noise in the environment which is added to the system (usually not measurable variable). Obviously, this leads to degradation in the normal performance of the controllers. The performance of the smooth fuzzy MPC scheme in the presence of the additive noise is shown in Fig. 4. We have considered the followings as the additive noise to the system:

$$y(k+1) = -u(k) + 1.2u(k-1) + 1.4\exp\left(-y^2(k)\right) - 0.6y(k-1) + 0.05 * R \tag{23}$$

where $R$ is a random noise signal. It can be seen from Fig. 4 and Table 5 that smooth fuzzy model and MPC controller are more robust to the additive noise and arrives at a better solution, and faster rather than the classical product–sum fuzzy structure.

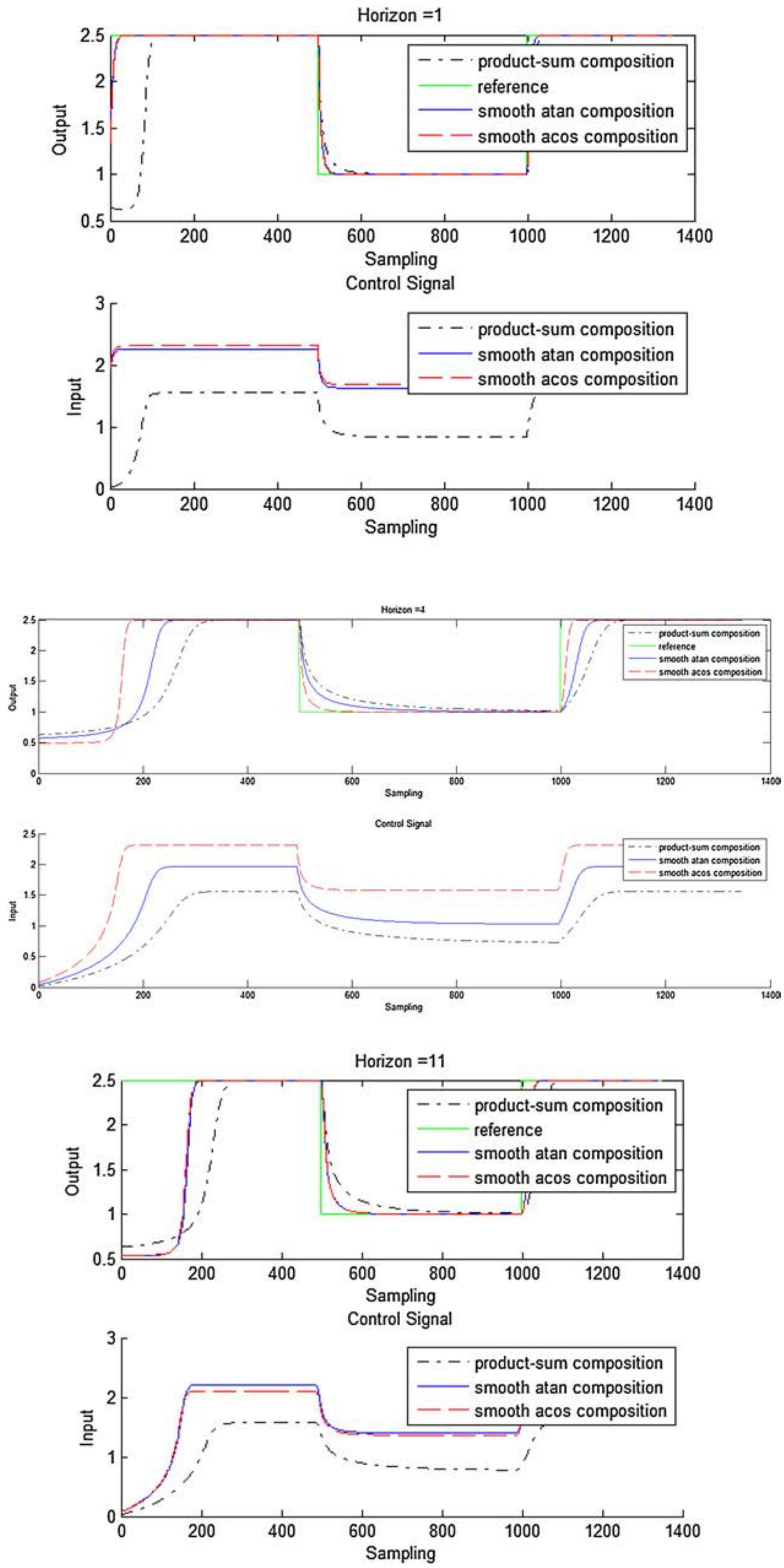


**Fig. 3** Comparison of three compositions under the normal working conditions: (top) short-term control horizon $T = 1$, (middle) medium-term horizon $T = 4$, (bottom) long-term horizon $T = 11$

**Table 4** Performance comparison of the system with different control horizons under the normal conditions

| | RMS error | | | Rise time | | |
|---|---|---|---|---|---|---|
| | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) |
| Product–sum composition | 0.9011 | 1.0352 | 1.0472 | 393 | 456 | 547 |
| Smooth atan composition | 0.4567 | 0.5412 | 0.7745 | 310 | 306 | 476 |
| Smooth acos composition | 0.4710 | 0.5412 | 0.7868 | 309 | 306 | 473 |

**Table 5** Performance comparison of the system with different control horizons in the presence of the additive noises

| | RMS error | | | Rise time | | |
|---|---|---|---|---|---|---|
| | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) |
| Product–sum composition | 3.22 | 0.8587 | 1.0602 | 418 | 388 | 571 |
| Smooth atan composition | 1.02 | 0.7291 | 0.8289 | 353 | 365 | 459 |
| Smooth acos composition | 1.23 | 0.5937 | 0.9283 | 300 | 368 | 439 |

**Table 6** Performance comparison of the system with different control horizons in the presence of the parametric changes

| | RMS error | | | Rise time | | |
|---|---|---|---|---|---|---|
| | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) |
| Product–sum composition | 0.8701 | 0.8940 | 0.9175 | 400 | 418 | 496 |
| Smooth atan composition | 0.7140 | 0.7230 | 0.7399 | 302 | 317 | 445 |
| Smooth acos composition | 0. 5061 | 0. 6433 | 0.7737 | 302 | 309 | 459 |

**Table 7** Performance comparison of the system with different control horizons in the presence of the additive time-varying disturbances

| | RMS error | | | Rise time | | |
|---|---|---|---|---|---|---|
| | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) | Short term ($T = 1$) | Medium term ($T = 4$) | Long term ($T = 11$) |
| Product–sum composition | 0.8448 | 0.8554 | 1.0347 | 372 | 390 | 472 |
| Smooth atan composition | 0.5302 | 0.5573 | 0.9512 | 303 | 306 | 461 |
| Smooth acos composition | 0. 4678 | 0.5370 | 0.6578 | 304 | 307 | 344 |

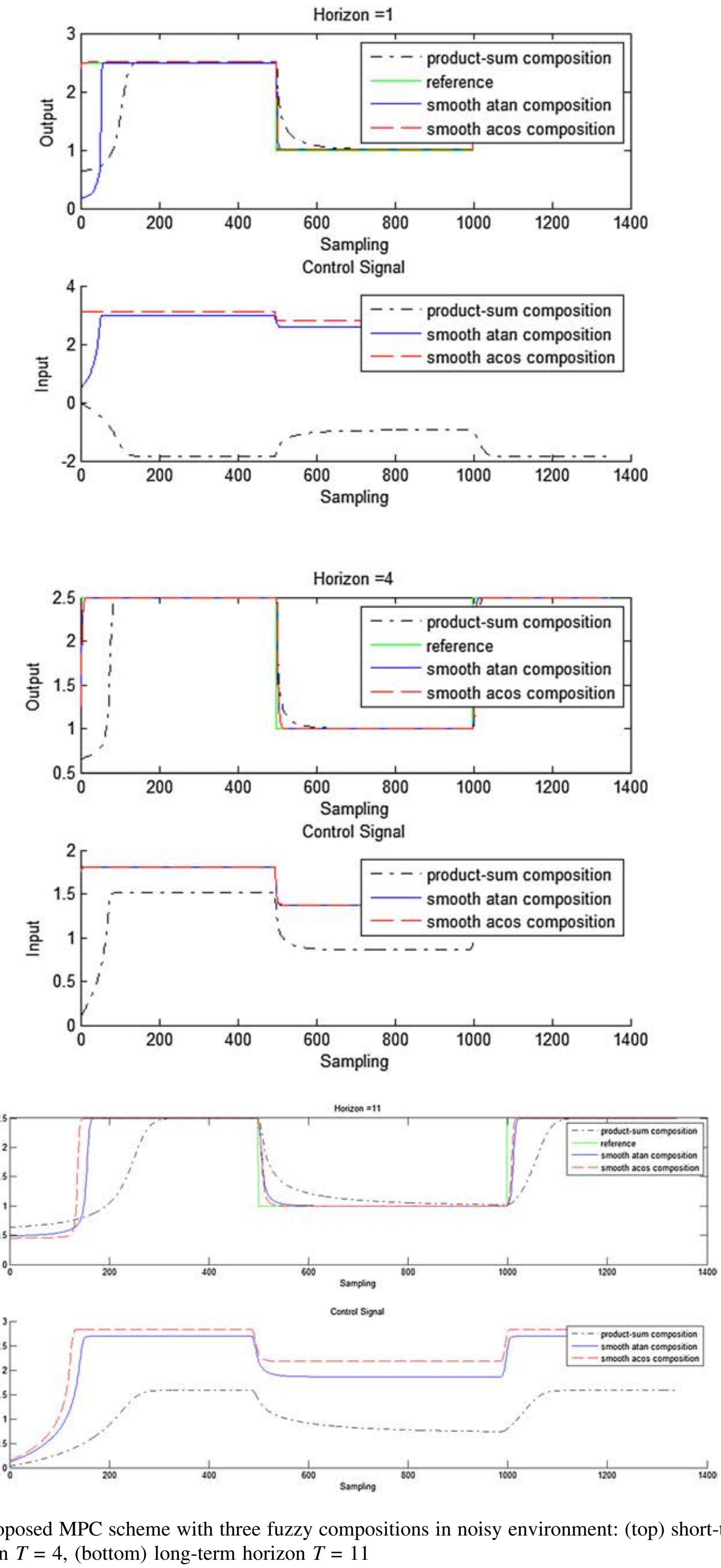


**Fig. 4** Performance of the proposed MPC scheme with three fuzzy compositions in noisy environment: (top) short-term control horizon $T = 1$, (middle) medium-term horizon $T = 4$, (bottom) long-term horizon $T = 11$

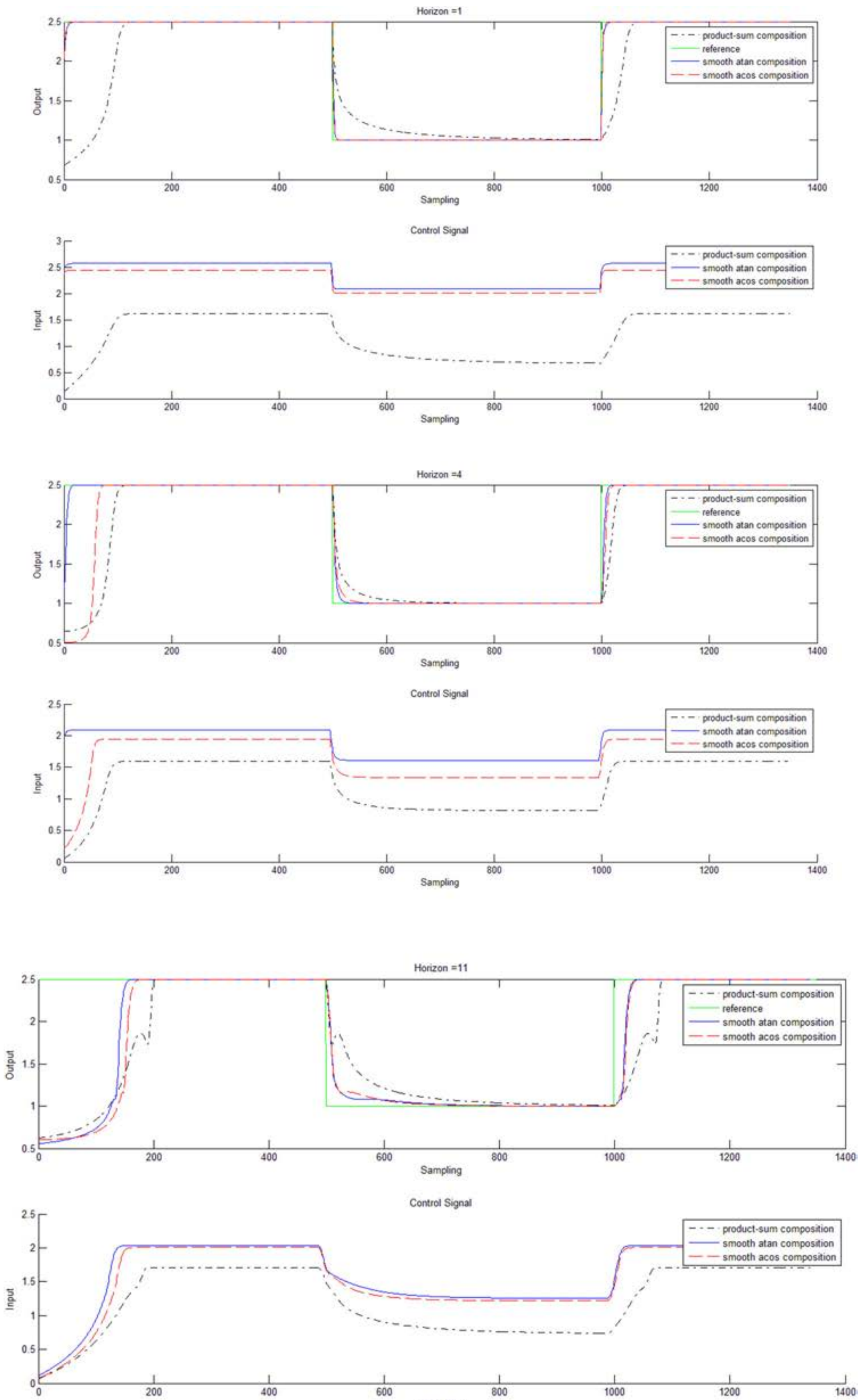


**Fig. 5** Comparison of three compositions with change in the parameters of the plant: (top) short-term control horizon $T$ = 1, (middle) medium-term horizon $T$ = 4, (bottom) long-term horizon $T$ = 11

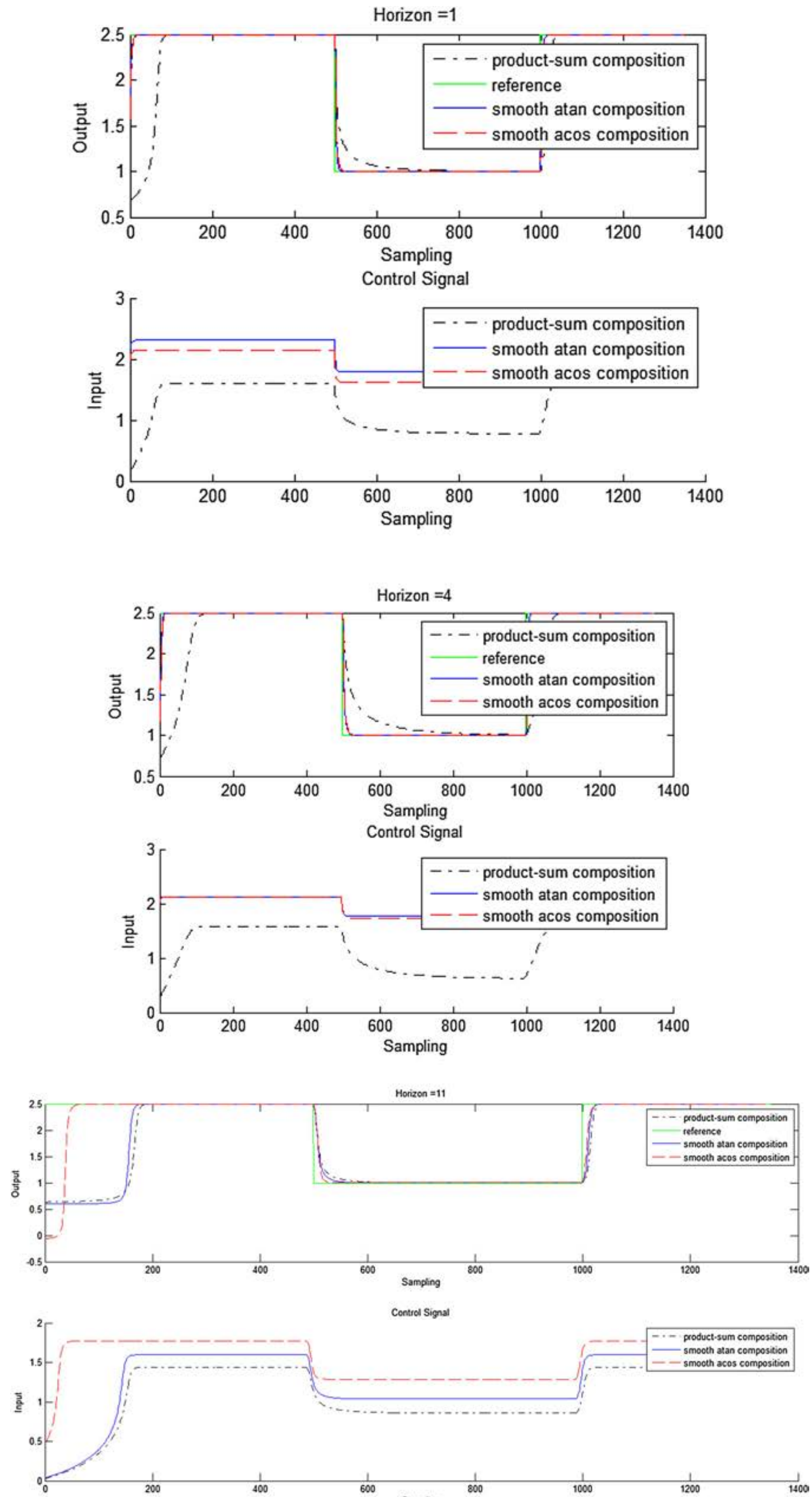


**Fig. 6** Comparison of three compositions with the additive time-varying disturbances: (top) short-term control horizon $T = 1$, (middle) medium-term horizon $T = 4$, (bottom) long-term horizon $T = 11$

### 5.1.4 Parametric Changes

We also have studied how the change in parameters (usually not measurable) can impact the controller performances. Obviously, this leads to degradation in the normal performance of the controllers. The disturbance rejection performance of the proposed smooth scheme is observed to be excellent as is shown in Fig. 5. We have considered the parametric changes to the system as

$$\begin{aligned} y(k+1) = & -u(k) + 1.2u(k-1) \\ & + (1.4 + 0.08 * R)\exp\left(-y^2(k)\right) - 0.6y(k-1) \end{aligned} \tag{24}$$

with $R$ as defined above. It can be seen from Fig. 5 and Table 6 that the smooth fuzzy model and MPC controller are more robust to the parametric changes and arrive at a better solution, and faster rather than the classical product–sum fuzzy structure.

### 5.1.5 Additive Time-Varying Disturbances

We also have studied how time-varying disturbances (usually not measurable) can impact the controller performance. This leads to degradation in the normal performance of the controllers, too. The controller managed to achieve the desired reference trajectory under the constant disturbance of

$$\begin{aligned} y(k+1) = & -u(k) + 1.2u(k-1) + 1.4\exp\left(-y^2(k)\right) \\ & - 0.6y(k-1) * 0.05 * \sin(k) \end{aligned} \tag{25}$$

It can be seen from Fig. 6 and Table 7 that the smooth fuzzy model and MPC controller are more robust to the additive time-varying disturbances and arrive at a better solution, rather than the classical product–sum fuzzy structure.

The study of dynamic responses shows that the smooth TS fuzzy identification-predictive controllers have strong disturbance rejection capabilities, in all the figures, in comparison with the classical fuzzy one.

Also, as it can be seen, employment of the classical compositions in the fuzzy implementation of the non-minimum phase system with the short horizon leads to smooth controller operation with the small system overshoot/undershot, which is compensated with the smooth fuzzy compositions. Moreover, the performed comparative study on the equal initial conditions for the different fuzzy compositions shows that the classical fuzzy representation suffers from the instability in the system performance or shows higher level of performance delay, depending on the control horizon.

Taking into account that most of the real-time processes under control have a smooth nature and the possibility of parametric changes and noise in the environment of the plant is relatively high, it can be concluded that the proposed smooth TS fuzzy model may be a promising solution in the fuzzy MPC scheme.

In all the figures, it can be distinguished that both the employed smooth "atan" and "acos" compositions provide more satisfactory and stable performance with superiority to the classical fuzzy controllers. More analytical study on the robustness and stability of the control system is beyond the scope of the current paper. The interested reader can consult Ref. [1].

The key features and main results of developing the presented smooth fuzzy TS modeling–control scheme through the simulations can be briefly summarized as follows:

(a) The accuracy of modeling and control with the smooth fuzzy compositions is highly better than the classical fuzzy models, which is clear from the comparison of the errors in the simulations. In the longer horizons, the superiority of the smooth compositions is more obvious. Hence, smooth compositions are even more propone for the unstable system, non-minimum phases and/or delayed systems.

(b) The smooth compositions bring about higher speed of convergence in the controller as shown in Figs. 4, 5 and 6.

(c) Employment of the smooth compositions brings about faster transient responses and lesser rise time to the outputs.

(d) The classical fuzzy composition shows lower level of stability in the presence of the additive time-varying disturbances and parametric variation rather than the smooth fuzzy compositions and the signal variation is higher.

Beside the mentioned points, based on Eqs. (11)–(13), the controller input can be computed incrementally, while the common practice requires solving a MIP optimization problem, for every iteration from the scratch, or linearization of the nonlinear dynamics. Hence, the presented approach lowers very much the computational burdens, although this has not demonstrated explicitly here.

Bearing the points in mind, we will work for the implementation of the smooth TS fuzzy identification–control algorithm in the processes that it is required to make up a fast simultaneous measurement and control scheme. The connectivist approach for the measurement-based modeling and model-based control will lower the down-time production and provide a feasible solution to the challenge of precise and high level of accuracy in the validation and calibration phases, with the minimal level of being underscored by the parameter variations, perturbations and noises. This potentially would give the dynamical systems, possibility of working at higher speeds up to video rate and also utilization for the examination of live processes.

## 6 Conclusions

Several interesting properties of smooth fuzzy compositions have been cited and proved in [1], and robustness advantage of smooth fuzzy models has been reported in almost all the contributions in the field [2, 18, 20, 23]. However, they cannot be employed for the practical cases and industrial systems until an easy and industrial implementable algorithm appears. This manuscript is a response to this requirement, and we formulated a scheme for identification and long-horizon MPC control of the smooth TS fuzzy models in the general form. We also have studied the effectiveness of the proposed connectivist smooth TS fuzzy identification and control framework.

Hence, the overall achievement of the paper is twofold. One seeks to contribute to the state of smooth fuzzy compositions by proposing a general and systematic expert free modeling–controller design methodology which comprises lower computational complexities, and then to extend the operation range of smooth IF–THEN fuzzy models for the practical applications. A gradient-based optimization approach has been developed to convert the MPC cost function of the smooth TS fuzzy model into an incremental controller design problem which results in a very fast and simple controller, in comparison with the other fuzzy MPC approaches that solve the problem through the Hessian and gradient approximation or by solving MIP optimization problem.

Different simulations on a non-minimum phase unstable system were presented to illustrate the superiority of the smooth compositions in the presence of disturbances and noises with different control horizons. Three fuzzy compositions for extracting fuzzy MPC controllers along the predicted trajectory based upon the TS fuzzy models have been compared. According to the test results, we can say that the overall smooth TS fuzzy modeling–control scheme is very much suitable for the adaptive control of time-changing dynamics and noisy systems. It is demonstrated that the smooth fuzzy compositions are the best choice to respond in time to the disturbances, with the high level of stability in case of changing reference.

## 7 Future Works

Using the results of the present study on smooth TS fuzzy model identification-predictive controller design, the long-range prediction horizons for the dead-time systems can be addressed in the future work. Also, it will be possible to deal with the systems' constraints on the manipulated variables by handling them through proper application of the merit functions and penalty parameters. Moreover, it would be possible to consider the multi-objective optimization criteria for the nonlinear processes. We believe that the future works also can dedicate to study the analytical robustness conditions in the controller design phase. The application of the developed algorithm to an industrial dynamic is undergone in the group.

**Compliance with Ethical Standards**

**Conflict of interest** Authors declare that there exists no conflict of interest.

**Ethical Approval** This article does not contain any studies with human participants or animals performed by any of the authors.

## Appendix 1

In order to drive error derivatives, we study the identification process in more detail. To begin with, we write the gradient descent method formula and define the vectors as follows:

$$\frac{\partial J}{\partial \rho_{ld}} = \frac{\partial J}{\partial y}\frac{\partial y}{\partial y_i}\frac{\partial y_i}{\partial \acute{y}_{li}}\frac{\partial \acute{y}_{li}}{\partial \xi_{ld}}\frac{\partial \xi_{ld}}{\partial \beta_{ld}}\frac{\partial \beta_{ld}}{\partial \rho_{ld}}$$

But to complete the formulation we need to take partial derivative of each variable separately.

First, we define the fuzzy variables $\left\{\acute{\xi}_1, \acute{\xi}_2, \ldots, \acute{\xi}_r\right\}$ at every time instant as

$$\begin{aligned}\acute{\xi}_l &= \left[\xi_{l1}, \xi_{l2}\ldots, \xi_{l,m+p}\right] \\ &= \left[\beta_{l1}(\xi_1), \beta_{l2}(\xi_2), \ldots, \beta_{l,m+p}(\xi_{m+p})\right],\ l = 1\ldots, r\end{aligned}$$

and $\acute{\xi} = \left[\acute{\xi}_l\right]_{l=1}^{r}$, where $\beta(\cdot)$, as stated above, is value of the membership function for the fuzzy set. In general, this function can be written as

$$\beta_{ld}(\cdot) = \exp\left(\frac{-1}{2}\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}}\right)^2\right).$$

Therefore, for making up the gradient descent method formula, $\frac{\partial \xi_{ld}}{\partial \rho_{ld}}$ can be written as

$$\frac{\partial \xi_{ld}}{\partial c_{ld}} = \exp\left(\frac{-1}{2}\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}}\right)^2\right)\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ij}^2}\right) \tag{26}$$

$$\frac{\partial \xi_{ld}}{\partial \delta_{ld}} = \exp\left(\frac{-1}{2}\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}}\right)^2\right)\left(\frac{(\xi_{ld} - c_{ld})^2}{\delta_{ld}^3}\right) \tag{27}$$

Based on the compositional rule inference, we can say that estimation of the output, according to our notation, is

$$\acute{y}_{li} = s - \text{norm}\left(t - \text{norm}\left(\acute{\xi}_l, R_l\left(\acute{\xi}, y_i\right)\right)\right)$$

for all $l = 1, \ldots, r$. Let us abbreviate $S : s - \text{norm}$ and $T : t - \text{norm}$ in the following.

To facilitate the explanation of the procedure of taking the derivation of $\frac{\partial \acute{y}_{li}}{\partial \xi_{ld}}$, we assume a simple system and put $\acute{\xi}_l = [\xi_{l1}, \xi_{l2}]$ and $c = R\left(\acute{\xi}, y_i\right)$. Then, based on the properties of $t$ norms, we have

$$\acute{y}_{li} = S(T(T(\xi_{l1}, \xi_{l2}), c)) = S(T(\xi_{l1}, c), T(\xi_{l2}, c))$$

We define: $\Lambda_1 = T(\xi_{l1}, c)$ and $\Lambda_2 = T(\xi_{l2}, c)$, then

$$\begin{aligned} \acute{y}_{li} &= S(\Lambda_1, \Lambda_2) \\ \frac{\partial \acute{y}_{li}}{\partial \xi_{l1}} &= \frac{\partial S}{\partial \Lambda_d} \frac{\partial \Lambda_d}{\partial \xi_{l1}} = \acute{S}^1 \acute{T}^1,\ d = 1, 2, \end{aligned} \tag{28}$$

where $\acute{S}^1$ and $\acute{T}^1$ are the first-order derivatives of the compositions, which will be calculated below. If there exist more state variables in the augmented state vector, $\xi_l' = [\xi_{l1}, \xi_{l2} \cdots, \xi_{l,m+p}]$ we could continue in the same manner and write as

$$\frac{\partial \acute{y}_{li}}{\partial \xi_{ld}} = \acute{S}^{m+p-1} \acute{T}^{m+p-1} \ldots \acute{S}^1 \acute{T}^1. \tag{29}$$

Hence, to derive the gradient descent method formulation, the general formula for the error derivation will be

$$\begin{aligned} \frac{\partial J}{\partial c_{ld}} &= \frac{\partial J}{\partial y} \frac{\partial y}{\partial y_i} \frac{\partial y_i}{\partial \xi_{ld}} \frac{\partial \xi_{ld}}{\partial \beta_{ld}} \frac{\partial \beta_{ld}}{\partial c_{ld}} \\ &= e(k) \cdot \left(\frac{\theta_{li} - y_i}{\sum_{i=1}^r \beta_{li}}\right) \cdot \left(\acute{S}^{m+p-1} \acute{T}^{m+p-1} \ldots \acute{S}^1 \acute{T}^1\right) \\ &\quad \cdot \exp\left(\frac{-1}{2}\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}}\right)^2\right)\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}^2}\right) \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial J}{\partial \delta_{ld}} &= \frac{\partial J}{\partial y} \frac{\partial y}{\partial y_i} \frac{\partial y_i}{\partial \xi_{ld}} \frac{\partial \xi_{ld}}{\partial \beta_{ld}} \frac{\partial \beta_{ld}}{\partial \delta_{ld}} \\ &= e(k) \cdot \left(\frac{\theta_{li} - y_i}{\sum_{i=1}^r \beta_{li}}\right) \cdot \left(\acute{S}^{m+p-1} \acute{T}^{m+p-1} \cdots \acute{S}^1 \acute{T}^1\right) \\ &\quad \cdot \exp\left(\frac{-1}{2}\left(\frac{\xi_{ld} - c_{ld}}{\delta_{ld}}\right)^2\right)\left(\frac{(\xi_{ld} - c_{ld})^2}{\delta_{ld}^3}\right) \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial J}{\partial \theta l_i} &= \frac{\partial J}{\partial y} \frac{\partial y}{\partial y_i} \frac{\partial y_i}{\partial \theta_{li}} \\ &= e(k) \cdot \left(\frac{\theta_{li} - \beta_{li}}{\sum_{i=1}^r \beta_{li}}\right) \end{aligned} \tag{32}$$

## Appendix 2

Let us take $B = S_B(T_B(\xi_{l1}, c), T_B(\xi_{l2}, c))$, and, $\Lambda_1 = T_B(\xi_{l1}, c), \Lambda_2 = T_B(\xi_{l2}, c)$, where,

$$\begin{aligned} T_B(\xi_{l1}, c) &= \frac{4}{\pi} \tan^{-1}\left(\tan\left(\frac{\pi}{4} \xi_{l1}\right) \tan\left(\frac{\pi}{4} c\right)\right) \\ S_B(\Lambda_1, \Lambda_2) &= 1 - \frac{4}{\pi} \tan^{-1}\left(\tan\left(\frac{\pi}{4}(1 - \Lambda_1)\right) \tan\left(\frac{\pi}{4}(1 - \Lambda_2)\right)\right) \end{aligned}$$

Now, the first-order derivative will be

$$\begin{aligned} \acute{B} &= \acute{S}_B^1 \acute{T}_B^1 \\ &= \frac{4}{\pi} \frac{1}{1 + \left(\tan\left(\frac{\pi}{4} \xi_{l1}\right) \tan\left(\frac{\pi}{4} c\right)\right)^2} \times \frac{\pi}{4} \sec^2\left(\frac{\pi}{4} \xi_{l1}\right) \tan\left(\frac{\pi}{4} c\right) \\ &\quad \times \frac{-4}{\pi} \times \frac{1}{1 + \left(\tan\left(\frac{\pi}{4}(1 - \Lambda_1)\right) \times \tan\left(\frac{\pi}{4}(1 - \Lambda_2)\right)\right)^2} \\ &\quad \times \sec^2\left(\frac{\pi}{4}(1 - \Lambda_1)\right) \times \frac{-\pi}{4} \times \tan\left(\frac{\pi}{4}(1 - \Lambda_2)\right) \end{aligned} \tag{33}$$

Now, similarly, take $C = S_C(T_C(\xi_{l1}, c), T_C(\xi_{l2}, c))$ with $\Lambda_1 = T_C(\xi_{l1}, c), \Lambda_2 = T_C(\xi_{l2}, \xi_{l1})$, and,

$$\begin{aligned} T_C(\mu_a(\cdot), \mu_b(\cdot)) &= 1 - \frac{2}{\pi} \cos^{-1}\left(\sin(\xi_{l1}) \sin\left(\frac{\pi}{2} c\right)\right) \\ S_C(\Lambda_1, \Lambda_2) &= \frac{2}{\pi} \cos^{-1}\left(\cos\left(\frac{\pi}{2} \Lambda_1\right) \cos\left(\frac{\pi}{2} \Lambda_2\right)\right) \end{aligned}$$

Then, the first-order derivative is

$$\begin{aligned} \acute{C} &= \acute{S}_C^1 \acute{T}_C^1 \\ &= \frac{2}{\pi} \times \frac{-1}{\sqrt{1 - \left(\cos\frac{\pi}{2} \Lambda_1 \cos\frac{\pi}{2} \Lambda_2\right)^2}} \\ &\quad \times \frac{\pi}{2} \times \left(-\sin\frac{\pi}{2} c\right) \times \cos\left(\frac{\pi}{2} \xi_{l1}\right) \\ &\quad \times \frac{2}{\pi} \frac{-1}{\sqrt{1 - \left(\sin\frac{\pi}{2} \xi_{l1} \sin\frac{\pi}{2} c\right)^2}} \times \frac{\pi}{2} \times \left(-\sin\frac{\pi}{2} \Lambda_1\right) \\ &\quad \times \cos\left(\frac{\pi}{2} \Lambda_2\right) \end{aligned} \tag{34}$$

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