Efficiency assessment and target setting using a fully fuzzy DEA approach ☆

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Abstract

Data Envelopment Analysis (DEA) is a non-parametric methodology for efficiency assessment. This paper proposes a new radial, input-oriented and fully fuzzy DEA approach, based on an LU-fuzzy partial order (L for lower, U for upper), for assessing the relative efficiency of a set of Decision Making Units (DMUs). The proposed approach involves a radial input contraction, Phase I, and an additive slacks maximization, Phase II. Each phase is first formulated as a fully fuzzy linear programming (FFLP), and then it is transformed into a multiobjective optimization problem. The latter is solved using the lexicographic weighted Tchebycheff method. The proposed fully fuzzy DEA approach provides, for each unit, a fuzzy efficiency measure and a fuzzy target operating point. A classification of the efficiency status of the units is also presented. Computational experiences and comparison with other fuzzy DEA approaches are reported.

Keywords: `efficiency assessment, fuzzy data, fully fuzzy linear programming, multiobjective optimization`

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1. Introduction

In order to assess the relative efficiency of a set of homogeneous Decision-Making Units (DMUs), a non-parametric methodology, namely, Data envelopment analysis (DEA), is generally used. DEA only requires data about the inputs consumed and the outputs produced by the DMUs. A DMU is said to be inefficient if it can be shown that it can produce its current outputs with fewer inputs or if it can produce more outputs with the current inputs. Hence, a DMU is labeled efficient if no input reductions or output increases are feasible. The set of efficient, i.e., non-dominated operating points, is called the efficient frontier. The DEA methodology uses an optimization model to compute an efficiency score and an efficient target for each DMU. There are different ways of carrying out the projection onto the efficient frontier and computing the corresponding efficiency scores, such as the radial efficiency approach ([3]), the multidirectional efficiency approach ([16]) or the potential efficiency approach ([22]), etc. These crisp DEA approaches require accurate measurement of both the inputs and outputs. However, the observed values of the input and output data in real-world problems are sometimes imprecise or vague, and, therefore, it is necessary to consider adequate rules and techniques to evaluate the objectives and expressions in which this type of data is involved.

Fuzzy sets appear as a suitable tool to manage imprecise quantities, and model incomplete and non-obtainable information, as discussed in [24]. Although we focus our attention in DEA, the applications and validity of fuzzy sets have been shown in many different fields, such as management science, decision theory, artificial intelligence, computer science, expert systems, fuzzy logic, fuzzy control, etc (see, for example, [30], [31].)

Fuzzy DEA (FDEA) refers to those DEA approaches that can handle fuzzy data. The range of application of Fuzzy DEA is wide, as in occupational safety, supplier evaluation and selection, health care centers, etc (see, for example, [5], and the bibliography therein). Hatami-Marbini et al. (2011a) [8] and Emrouznejad et al. (2014) [5] present a taxonomy that generally classifies the existing FDEA methods into α-level set approaches (e.g. Kao and Liu 2000 [12]), fuzzy ranking approaches (e.g. Ghasemi et al. 2015 [6]), possibility approaches (e.g. Wang and Chin 2011 [29]), fuzzy arithmetic approaches (e.g. Wang et al. 2009 [28]) and fuzzy random/type-2 fuzzy sets (e.g. Tavana et al. 2013 [27]).

FDEA approaches can also be classified according to other criteria, such as whether they use a multiplier or an envelopment formulation, whether they use radial or a non-radial metric, whether they consider a single process or a network DEA system, etc. In particular, FDEA approaches can be classified into two groups depending on whether or not all the variables are fuzzy. When the variables of the DEA optimization model (and not only the data) are fuzzy, the approach is labeled as fully fuzzy DEA (FFDEA). The first FFDEA approach was proposed in Hatami-Marbini et al. (2011b) [9], which uses a radial, input-oriented multiplier formulation to compute fuzzy efficiency scores. Kazemi and Alimi (2014) [13] use a fuzzy ranking approach and also consider a radial, input-oriented multiplier formulation. Puri and Yadav (2015) [19] use a fuzzy ranking approach and a radial, input-oriented, multiplier formulation that includes undesirable outputs. They also extend their approach to multi-component (i.e., parallel processes) systems. Puri and Yadav (2016) [20] use fuzzy ranking and an envelopment formulation to compute fuzzy cost and revenue efficiencies in the presence of undesirable outputs. Sotoudeh-Anvari et al. (2016) [23] and Namakin et al. (2018) [18] consider that the input and output data are given as Z-numbers, which are transformed into conventional fuzzy numbers so that the Hatami-Marbini et al. (2011b) [9] approach can be applied.

The above FFDEA approaches do not acknowledge the multiobjective optimization character of the FFDEA problem. To the best of our knowledge, there are only two papers that use a multiobjective optimization approach. One is Khaleghi et al. (2015) [14], which uses a radial, input-oriented multiplier formulation and transforms the corresponding fully fuzzy linear program (FFLP) into a multiobjective optimization model. This is solved using the weighted sum method. By contrast, Hatami-Marbini et al. (2017) [10] proposes a radial, input-oriented envelopment formulation, converting the corresponding FFLP into a multiobjective optimization model, which is solved using a lexicographic approach.

In this paper, we also map FFLP into a multiobjective optimization problem. Because it allows computing not only fuzzy efficiency scores but also fuzzy input and output targets, we have opted for an envelopment FFDEA formulation. Hence, the approach proposed in this paper is closer to Hatami-Marbini et al. (2017) [10]. However, we use a different solution approach that starts by defining the concept of fuzzy Pareto solutions. It is shown that the set of fuzzy Pareto solutions corresponds to the Pareto optimal solutions of an associated multiobjective optimization. The lexicographic weighted Tchebychef method is then used to compute a fuzzy Pareto solution. We are thus providing fuzzy input and output targets for each DMU, a fuzzy efficiency score, and fuzzy input and output slacks that can be used to classify each DMU as efficient, weakly efficient, partially efficient, or inefficient. The proposed FFDEA approach assumes that not only the input and output data but also the model variables are trapezoidal fuzzy numbers (TrFNs). The efficiency assessment is carried out in two phases: one that computes the fuzzy efficiency score using a radial, input-oriented approach, and another that maximizes the remaining input and output slacks in order to compute a fuzzy target.

The contributions of this paper are several, and following they are highlighted:

- The FFDEA technology considered is explicitly formulated, making clear the set of feasible fuzzy operating points that form the corresponding fuzzy Production Possibility Set. It is an innovative feature since all existing FFDEA approaches directly formulate the corresponding optimization model without first establishing the assumed Production Possibility Set.
- Using that FFDEA technology, a novel radial, input-oriented FFDEA model is formulated considering an LU-fuzzy partial order. This fuzzy partial order has not been considered before in the FDEA literature, al-

though it facilitates the formulation of the corresponding FFDEA models.

- The concept of fuzzy Pareto solutions is introduced, and the equivalence between the set of the Pareto solutions of FFDEA and the set of Pareto solutions of a related multiobjective linear problem is established. It is also a significant contribution of the proposed approach, i.e., the insight that the solution of the corresponding FFDEA model is not just a single fuzzy operating point but a whole set of fuzzy Pareto solutions.

- An algorithm to compute a fuzzy efficiency measure and a fuzzy target for each DMU based on the lexicographic weighted Tchebycheff method is proposed. The lexicographic weighted Tchebycheff method is commonly used in multiobjective optimization due to its attractive properties (see, e.g., Marler and Arora [17]). However, to the best of our knowledge, it has not been applied before in FFDEA nor even in FDEA.

The organization of the paper is the following. In Section 2, the necessary DEA and fuzzy sets concepts are introduced. The proposed FFDEA model and its associated multiobjective optimization problem are presented in Section 3. In Section 4, the proposed two-phase multiobjective optimization solution approach is explained in detail. Section 5 presents two numerical examples, a small one for illustrating the proposed approach and another one for comparing it with the FFDEA approach of Hatami-Marbini et al. [10]. Finally, Section 6 summarizes and concludes.

2. Preliminaries

In order to facilitate understanding of the proposed approach, this section presents a short review of some basic concepts of DEA and Fuzzy Sets.

2.1. Charnes-Cooper-Rhodes (CCR) model

Let us consider a set of N DMUs. For $n \in \{1, \dots, N\}$, each DMU_n has M inputs $X_n = (x_{1n}, \dots, x_{Mn}) \in \mathcal{R}^M$, and produces S outputs $Y_n = (y_{1m}, \dots, y_{Sm}) \in \mathcal{R}^S$. The

Charnes-Cooper-Rhodes (CCR) model (Charnes et al. [3]) was the first DEA model proposed. The first step in DEA is to use the observed data to infer the Production Possibility Set (PPS) (also called the technology), which represents all the operating points that are deemed feasible. This is done using some basic axioms and considering the minimum extrapolation principle (i.e., determining the smallest set that satisfies those axioms).

In particular, the PPS assumed by the CCR model, denoted by T, satisfies the following axioms:

(A1) Envelopment: $(X_n, Y_n) \in T$, for all $n \in \{1, \ldots, N\}$.

(A2) Free disposability: $(x, y) \in T$, $(x', y') \in \mathbb{R}^{M+S}$, $x' \geqq x$, $y' \leqq y \Rightarrow (x', y') \in T$.

(A3) Constant return to scale: $(x, y) \in T \Rightarrow (\lambda x, \lambda y) \in T$, for all $\lambda \in \mathbb{R}_+$.

(A4) Convexity: $(x, y), (x', y') \in T$, then $\lambda(x, y) + (1 - \lambda)(x', y') \in T$, for all $\lambda \in [0, 1]$.

Following the minimum extrapolation principle, the DEA PPS is the intersection of all sets that satisfies the axioms referred above (A1)-(A4). Assuming (A1)-(A4), the minimum extrapolation PPS can be stated as

$$T = \left\{ (x, y) \in \mathcal{R}_+^{M+S} : \ x \geq \sum_{n=1}^{N} \lambda_n X_n, \ \ y \leq \sum_{n=1}^{N} \lambda_n Y_n, \ \ \lambda_n \geq 0 \right\},$$

this is, the feasible input-output combinations. The interpretation of this PPS is that it is formed by all linear combinations of the observed DMUs (with non-negative combination weights) plus those operating points that consume more inputs and produce less outputs.

The basic purpose in DEA is to assess the efficiency of the DMUs. This is done by checking for each DMU_p in turn if there are feasible points within the PPS that dominate the DMU, i.e., that has more outputs and fewer inputs. Among those operating points that dominate the DMU, the one which represents the most significant efficiency improvement is selected, and an efficiency score that measures the extent of that efficiency improvement, together with the corresponding efficient input-output target, is provided.

In particular, the CCR model uses a radial efficiency approach. Radial DEA models can be input or output-oriented depending on whether the aim at (i) reducing all the inputs equi-proportionally without decreasing the outputs (input-oriented model);

$$E = E((x, y); T(\gamma)) = \min\{\theta_p \in \mathcal{R}_+ | (\theta_p x, y) \in T(\gamma)\}.$$

or (ii) expanding all the outputs equi-proportionally without increasing the inputs (output-oriented model):

$$F = F((x, y); T(\gamma)) = \max\{\theta_p \in \mathcal{R}_+ | (x, \theta_p y) \in T(\gamma)\}.$$

The relative efficiency of a given DMU_p, $1 \leq p \leq n$, can be computed by the following input-oriented CCR model, which was formulated by Charnes et al. [3] as,

$$\begin{aligned} \text{(CCR)} \quad \text{Min} \quad & \theta_p \\ \text{s.t.} \quad & \sum_{n=1}^{N} \lambda_n x_{mn} \leq \theta_p x_{mp}, \quad m = 1, \ldots, M, \\ & \sum_{n=1}^{N} \lambda_n y_{rn} \geq y_{rp}, \quad r = 1, \ldots, S, \\ & \theta_p, \lambda_n \geq 0, \quad n = 1, \ldots, N, \end{aligned} \tag{1}$$

where λ_n, $n = 1, \ldots, N$, are the weights on each DMU for making up the efficient facet of DMU_n. The variable θ_p that appears in the objective function represents the reduction factor that can be uniformly applied to all the input dimensions. A DMU is called *CCR-efficient* if and only if its optimal value is $\theta_p^* = 1$. Otherwise it is called *CCR-inefficient*.

The above CCR model can be adapted to deal with situations of uncertainty in the form of interval data (e.g., Hatami-Marbini et al. [11]). That leads to two related optimization problems (one for the lower and another for the upper limits of the efficiency estimate) that have a similar structure with respect to the constraints, as the CCR model [3]. Hatami-Marbini et al. [10] also consider uncertainty but in the form of trapezoidal fuzzy numbers, although not directly on the formulation of the CCR model given in [3], but on the following equivalent problem with slacks variables and equality constraints.

$$
\begin{aligned}
\text{(CCR2)} \quad \text{Min} \quad & \theta_p \qquad\qquad (2)\\
\text{s.t.} \quad & \sum_{n=1}^{N} \lambda_n x_{mn} + s_m^- = \theta_p x_{mp}, \quad m = 1, \dots, M,\\
& \sum_{n=1}^{N} \lambda_n y_{rn} = y_{rp} + s_r^+, \quad r = 1, \dots, S,\\
& \lambda_n \geq 0, \quad \theta_p free, \quad n = 1, \dots, N,\\
& s_m^-, s_r^+ \geq 0, \quad m = 1, \dots, M, \ \ r = 1, \dots, S.
\end{aligned}
$$

2.2. Mathematical framework: Fuzzy Numbers

We denote by $\mathcal{K}_C = \left\{ \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a} \right\}$ the family of all bounded closed intervals in $\mathbb{R}$. A fuzzy set on $\mathbb{R}^n$ is a mapping $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u, we denote its α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, and the support as $supp(u) = \{x \in \mathbb{R}^n \mid u(x) > 0\}$. The closure of $supp(u)$ defines the 0-level of u, .i.e. $[u]^0 = cl(supp(u))$ where $cl(M)$ means the closure of the subset $M \subset \mathbb{R}^n$. Recall that a fuzzy set u on $\mathbb{R}$ is said to be a fuzzy number if u is normal, upper semi-continuous function, $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}, x, y \in \mathbb{R}$, $\lambda \in [0, 1]$, and $[u]^0$ is compact. $\mathcal{F}_C$ denotes the family of all fuzzy numbers. The α-levels of a fuzzy number are given by $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right] \in \mathcal{K}_C$, $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0, 1]$. There are many parametrical families of fuzzy numbers that have been applied to measure imprecision in several situations. Among the most popular, we can find triangular and trapezoidal fuzzy numbers (TFN and TrFN, respectively), as well as polygonal, gaussian, quasi-gaussian, quasi-quadric, exponential, quasi-exponential, and singleton fuzzy numbers (see [7] for a complete description of these families). The representation of fuzzy numbers has also been deeply discussed in [24]. Two of the most used families of fuzzy numbers are triangular and trapezoidal fuzzy numbers, because of their easy modeling and interpretation. Let us recall that a TrFN can be represeneted by means of four real numbers. Thus, $\tilde{a} = (a^1, a^2, a^3, a^4)$ is a trapezoidal fuzzy number whose membership function is given by

$$\tilde{a}(x) = \begin{cases} \frac{x-a^1}{a^2-a^1}, & \text{if } a^1 \leq x < a^2, \\ 1, & \text{if } a^2 \leq x \leq a^3, \\ \frac{a^4-x}{a^4-a^3}, & \text{if } a^3 < x \leq a^4, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Its corresponding α-levels are determined by

$$[\tilde{a}]^\alpha = [a^1 + \alpha(a^2 - a^1), a^4 - \alpha(a^4 - a^3)]. \tag{4}$$

The set of all TrFNs is denoted by *TrFN* as well. A TrFN $\tilde{a}$ is a TFN if and only if $a^2 = a^3$. Therefore, only three parameters are really necessary to represent a TFN. A TrFN $\tilde{a} = (a^1, a^2, a^3, a^4)$ is said to be non-negative if $a^1 \geq 0$. The set of all non-negative TrFNs is denoted by $TrFN_+$. In the sequel, we denote as $\tilde{1}$ and $\tilde{0}$ the TrFNs whose four parameters are equal to 1 and 0, respectively. Since the framework in DEA, as regards the sign of the variables, is based on non-negative fuzzy numbers, we provide the following definition for arithmetic operations with them.

Definition 1. *Given two trapezoidal fuzzy numbers $\tilde{a} = (a^1, a^2, a^3, a^4) \in TrFN$ and $\tilde{b} = (b^1, b^2, b^3, b^4) \in TrFN$, it is defined the basic arithmetical operations as follows:*

(i) The addition,

$$\tilde{a} + \tilde{b} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4). \tag{5}$$

(ii) The multiplication by a scalar $\lambda \in \mathbb{R}$,

$$\lambda\tilde{a} = \begin{cases} (\lambda a^1, \lambda a^2, \lambda a^3, \lambda a^4) & \text{if } \lambda \geq 0, \\ (\lambda a^4, \lambda a^3, \lambda a^2, \lambda a^1) & \text{if } \lambda < 0. \end{cases} \tag{6}$$

(iii) The multiplication of two TrFN, $\tilde{a}\tilde{b} = \tilde{c} = (c^1, c^2, c^3, c^4)$, where

$$\begin{aligned} c^1 &= \min\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\}, \qquad c^4 = \max\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\}, \\ c^2 &= \min\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\}, \qquad c^3 = \max\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\}. \end{aligned} \tag{7}$$

In the particular case of non-negative TrFN $\tilde{a}$ and $\tilde{b}$, the multiplication is just $\tilde{a}\tilde{b} = (a^1b^1, a^2b^2, a^3b^3, a^4b^4)$.

In relation to the above definitions, let us provide the following comments. It is usual to use the Zadeh's principle to extend, for instance, the arithmetic operation $\circ \in \{+, \cdot\}$ between two fuzzy numbers $\tilde{a}, \tilde{b}$, and so the corresponding membership function for $\tilde{a} \circ \tilde{b}$ is given by

$$(\tilde{a} \circ \tilde{b})(x) = \sup_{y \circ z = x} \min\{\tilde{a}(y), \tilde{b}(z)\}. \tag{8}$$

In the case that $\tilde{a}$ and $\tilde{b}$ are two trapezoidal fuzzy numbers, the addition and multiplication by a scalar by (8) coincide with the arithmetic operations introduced in (i) and (ii) in Definition 1, respectively. However, the set of trapezoidal fuzzy numbers is not closed under the multiplication operation (8). This latter situation is shown in the examples in [32] applied to the subset of triangular fuzzy numbers. So, in the case of triangular and trapezoidal fuzzy numbers, it is usual to apply a different multiplication operation to avoid this situation (see, for instance, [2]), and (iii) in Definition 1 can be considered a natural extension to trapezoidal fuzzy numbers. Therefore, using the multiplication between trapezoidal fuzzy numbers introduced in Definition 1, it is guaranteed that the result is a trapezoidal fuzzy number, which can be considered as an approximation to the multiplication operation given in (8).

To formulate a Fully Fuzzy CCR model, we consider that all variables and parameters are trapezoidal fuzzy numbers and that the arithmetic operations between them are those established in Definition 1. However, before that, it is necessary to provide a partial order relationship between two trapezoidal fuzzy numbers. To this aim, we will use LU-fuzzy partial orders, which are well known in the literature (see, e.g., [25], and the references therein).

Definition 2. *Given two fuzzy numbers u, v, it is said that:*

(i) $\mu \leqq \nu$ *if and only if* $\underline{\mu}_\alpha \leq \underline{\nu}_\alpha$ *and* $\overline{\mu}_\alpha \leq \overline{\nu}_\alpha$*, for all* $\alpha \in [0, 1]$*,*

(ii) $\mu \prec \nu$ *if and only if* $\underline{\mu}_\alpha < \underline{\nu}_\alpha$ *and* $\overline{\mu}_\alpha < \overline{\nu}_\alpha$*, for all* $\alpha \in [0, 1]$*.*

The relationships $\mu \geqq \nu$ and $\mu \succ \nu$ means $\nu \leqq \mu$ and $\nu \prec \mu$, respectively. In [2], we can find a reformulation of the previous definition for two triangular fuzzy numbers by means of the relationship between their parameters. In what follows, we extend this result to trapezoidal fuzzy numbers.

Theorem 1. *Given two trapezoidal fuzzy numbers* $\tilde{a} = (a^1, a^2, a^3, a^4)$ *and* $\tilde{b} = (b^1, b^2, b^3, b^4)$ *and given a collection of* α*-levels* $\{\alpha_i : i = 0, 1 \ldots, k\}$*, it follows that*

(i) $\tilde{a} \leqq \tilde{b}$ *if and only if* $a^i \leq b^i$*, for all* $i = 1, 2, 3, 4$*,*

(ii) $\tilde{a} \prec \tilde{b}$ *if and only if* $a^i < b^i$*, for all* $i = 1, 2, 3, 4$*.*

Proof. The proof is similar to that given in [2]. □

3. Proposed FFDEA model and associated multiobjective optimization problem

Let us assume that all inputs and outputs are non-negative trapezoidal fuzzy numbers. That is, given a set of N DMUs, for $n \in \{1, \ldots, N\}$ each DMU_n has M inputs $\tilde{X}_n = (\tilde{x}_{1n}, \ldots, \tilde{x}_{Mn}) \in TrFN_+ \times \cdots \times TrFN_+ = (TrFN)_+^M$, and produces S outputs $\tilde{Y}_n = (\tilde{y}_{1n}, \ldots, \tilde{y}_{Sn}) \in (TrFN)_+^S$. It is necessary to define a technology to deal with fuzzy input and output data, as well as adequate modeling of the optimization problem using for computing the efficiency of the corresponding DMU. To this end, we consider the following fuzzy technology as a natural extension of that given in the classic Charnes et al. [3] CCR model:

$$T_{FDEA} = \left\{ (\tilde{x}, \tilde{y}) \in (TrFN)_+^{M+S} : \ \tilde{x} \geqq \sum_{n=1}^{N} \lambda_n \tilde{X}_n, \ \ \tilde{y} \leqq \sum_{n=1}^{N} \lambda_n \tilde{Y}_n, \ \ \lambda_n \geq 0, \forall n \right\}.$$

Under this technology, we propose the following Fully Fuzzy Linear Programming DEA formulation for projecting any DMU_p $p \in \{1, \dots, N\}$.

$$
\begin{aligned}
\text{(FFDEA)} \quad \text{Min} \quad & \tilde{\theta}_p && (9)\\
\text{s.t.} \quad & \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{x}_{mn} \leqq \tilde{\theta}_p \tilde{x}_{mp}, \quad m = 1, \dots, M, && (10)\\
& \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{y}_{rn} \geqq \tilde{y}_{rp}, \quad r = 1, \dots, S, && (11)\\
& \tilde{\theta}_p \leqq \tilde{1} \quad m = 1, \dots, M, && (12)\\
& \tilde{\theta}_p, \tilde{\lambda}_n \in (TrFN)_+, \quad n = 1, \dots, N, && (13)
\end{aligned}
$$

where all the elements of the problem, besides the inputs $\tilde{x}_{mn}$ and outputs $\tilde{y}_{rn}$, are also $(TrFN)_+$. This is,

$$
\begin{aligned}
\tilde{\theta}_p &= (\theta_p^1, \theta_p^2, \theta_p^3, \theta_p^4) \quad p \in \{1, \dots, N\},\\
\tilde{\lambda}_n &= (\lambda_n^1, \lambda_n^2, \lambda_n^3, \lambda_n^4), \quad n = 1, \dots, N,\\
\tilde{x}_{mn} &= (x_{mn}^1, x_{mn}^2, x_{mn}^3, x_{mn}^4), \quad m = 1, \dots, M, \ \ n = 1, \dots, N,\\
\tilde{y}_{rn} &= (y_{rn}^1, y_{rn}^2, y_{rn}^3, y_{rn}^4), \quad r = 1, \dots, S, \ \ n = 1, \dots, N.
\end{aligned}
$$

Observe that for proper modeling in the fuzzy extension of the crisp CCR model, the conditions $\tilde{\theta}_p \in (TrFN)_+$ and $\tilde{\theta} \leqq 1$ are included in the proposed FFDEA formulation. Note that in models (1) and (2) it was implicit that $0 \leq \theta_p \leq 1$ and hence including those constraints would be redundant. However, in the fuzzy case, it is necessary to explicitly impose that no value in the support of $\tilde{\theta}$ can be greater than 1. What it guarantees that the fuzzy input targets remain less than the observed fuzzy input data, i.e. $\tilde{X}_{mp}^{target} = \sum_{n=1}^{N} \tilde{\lambda}_j \tilde{x}_{mn} \leqq \tilde{x}_{mp}$, $m = 1, \dots, M$. Furthermore, we deal with a Fully Fuzzy Problem without any kind of ranking function.

Hatami-Marbini et al. [10] introduce the uncertainty with trapezoidal fuzzy numbers on the CCR formulation (2) and propose the following FFDEA model:

$$\begin{aligned} \text{(FF-CCR)} \quad \text{Min} \quad & \tilde{\theta}_p \\ \text{s.t.} \quad & \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{x}_{mn} + \tilde{s}_m^- = \tilde{\theta}_p \tilde{x}_{mp}, \quad m = 1, \ldots, M, \\ & \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{y}_{rn} = \tilde{y}_{rp} + \tilde{s}_r^+, \quad r = 1, \ldots, S, \\ & \tilde{\theta}_p, \tilde{\lambda}_n \in (TrFN)_+, \quad n = 1, \ldots, N, \\ & \tilde{s}_m^-, \tilde{s}_r^+ \in (TrFN)_+, \quad m = 1, \ldots, M, \ r = 1, \ldots, S. \end{aligned} \tag{14}$$

Note that the lexicographic optimization method proposed by Hatami-Marbini et al. [10] to solve model (FF-CCR) guarantees that $\tilde{\theta} \leqq 1$. However, in general, the feasible set of (14) is smaller than the technology set T_{FDEA}. This is because of the addition of the input/output slacks in (14) to form equality constraints. These issues with such addition operation are shown next with a simple example.

Example 1. *For any $\tilde{a}, \tilde{b} \in TrFN$ with $\tilde{a} \leqq \tilde{b}$ it is not guaranteed the existence of $\tilde{c} \in TrFN$ such that $\tilde{a} + \tilde{c} = \tilde{b}$.*

Let $\tilde{a} = (4, 6, 7, 9)$, and $\tilde{b} = (8, 10, 11, 12)$ for example. Given the definition (1) of the addition of two trapezoidal fuzzy numbers (5)*,*

$$\tilde{a} + \tilde{c} = (4 + c_0^-, 6 + c_1^-, 7 + c_1^+, 9 + c_0^+) = (8, 10, 11, 12) = \tilde{b}.$$

therefore, the only possible solution should be $\tilde{c} = (4, 4, 4, 3) \notin TrFN$, which is not a fuzzy number at all. On the other hand, the reverse statement is always true.

The above example means that

$$\sum_{n=1}^{N} \tilde{\lambda}_n \tilde{x}_{mn} + \tilde{s}_m^- = \tilde{\theta}_p \tilde{x}_{mp}, \quad m = 1, \ldots, M, \qquad \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{y}_{rn} = \tilde{y}_{rp} + \tilde{s}_r^+, \quad r = 1, \ldots, S, \tag{15}$$

are not really equivalent to (10) and (11). Actually, the correct expressions are

$$\sum_{n=1}^{N} \tilde{\lambda}_n \tilde{x}_{mn} + \tilde{s}_m^- \leqq \tilde{\theta}_p \tilde{x}_{mp}, \quad m = 1, \ldots, M, \qquad \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{y}_{rn} \leqq \tilde{y}_{rp} + \tilde{s}_r^+, \quad r = 1, \ldots, S. \tag{16}$$

This implies that the feasible set defined by the fully fuzzy DEA model (2) is smaller than T_{FDEA} technology used in the proposed fully fuzzy linear programming for the DEA given in (FFDEA) model (9)−(13), which therefore has more discriminant power than the approach in Hatami-Marbini et al. [10]. This is effectively confirmed in Section 5.2, in which applying the proposed approach to the same dataset as those authors, we obtain the same efficiency scores for all DMUs but two, for which we get a lower efficiency.

Based on the partial order defined previously at Section 2.2, we introduce the following definition of fuzzy non dominated solutions (i.e. fuzzy Pareto solutions) for (FFDEA).

Definition 3. *A feasible solution for (FFDEA) $(\tilde{\theta}_p^*, \tilde{\lambda}^*)$ is said to be a fuzzy Pareto solution if there does not exist any feasible solution of (FFDEA) $(\tilde{\theta}_p, \tilde{\lambda})$ such that $\tilde{\theta}_p \leqq \tilde{\theta}_p^*$ and $\tilde{\theta}_p \neq \tilde{\theta}_p^*$. In this case, $\tilde{\theta}_p^*$ is said to be a non-dominated score. We denote the set of all Fuzzy Pareto solutions of (FFDEA) for a DMU p as F_p, the set of all non-dominated scores for a DMU p as S_p.*

The proposed approach for solving (FFDEA) is through its associated Multiobjective Linear Programming (MOLP) problem. As all the data and variables are non-negative, we can rewrite the FFDEA model for the p−th DMU (9)−(13)

as the following Multiobjective Linear Problem.

$$
\begin{aligned}
\text{(MODEA)} \quad \text{Min} \quad & (\theta_p^1, \theta_p^2, \theta_p^3, \theta_p^4) && (17) \\
\text{s.t.} \quad & \sum_{n=1}^{N} \lambda_n^i x_{mn}^i \leq \theta_p^i x_{mp}^i, \quad i = 1,2,3,4, \;\; m = 1,\dots,M, && (18) \\
& \sum_{n=1}^{N} \lambda_n^i y_{sn}^i \geq y_{sp}^i, \quad i = 1,2,3,4, \;\; s = 1,\dots,S, && (19) \\
& \lambda_n^i - \lambda_n^{i+1} \leq 0, \quad i = 1,2,3, \;\; n = 1,\dots,N, && (20) \\
& \theta_p^i - \theta_p^{i+1} \leq 0, \quad i = 1,2,3, && (21) \\
& \theta_p^4 \leq 1, && (22) \\
& \lambda_n^i, \theta_p^i \geq 0, \quad i = 1,\dots,4, \;\; n = 1,\dots,N. && (23)
\end{aligned}
$$

As it was discussed above, (17) − (23) is a relaxed version of the MOLP presented by Hatami-Marbini et al. [10] (see models (7) or (8) in that paper and how we model the input and output constraints as inequalities). Precisely because the feasibility region of FFDEA is larger than that of Hatami-Marbini et al. [10], it is necessary to add constraint (22). These authors did not have to impose that constraint because the lexicographic approach they used for solving their MOLP ensures that $\tilde{\theta}_p \leqq \tilde{1}$.

4. Proposed multiobjective optimization solution approach

Arana [2] established a relationship between models similar to (FFDEA) and (MODEA) for the case of triangular fuzzy numbers. Those results can be extended to the case of trapezoidal fuzzy numbers is as follows.

Theorem 2. *Consider a DMU p and $(\tilde{\theta}_p, \tilde{\lambda})$ feasible for FFDEA. Then, $(\tilde{\theta}_p, \tilde{\lambda})$ is a fuzzy Pareto solution of (FFDEA), and hence $\tilde{\theta}_p$ is a non-dominated score for (FFDEA), if and only if $(\theta_p^1, \theta_p^2, \theta_p^3, \theta_p^4, \lambda_1^1, \lambda_1^2, \lambda_1^3, \lambda_1^4, \dots, \lambda_N^1, \lambda_N^2, \lambda_N^3, \lambda_N^4)$ is a Pareto solution of (MODEA).*

Proof. The proof is similar to that given in [2]. □

Thanks to this result, one can generate the whole set of fuzzy Pareto solutions of FFDEA by means of generating the set of Pareto solutions of MODEA. This can be done using different multiobjective optimization methods from the literature such as the weighted sum method, the ϵ-constraint method, etc. (see, for instance, [1, 4]). In particular, we propose using the *lexicographic weighted Tchebycheff method*, which not only is guaranteed to provide Pareto optimal solutions but also can be used to generate any Pareto optimal solution (Marler and Arora [17]). The method will be used for the two phases of the projection process: first, computing the radial fuzzy efficiency score and then maximizing the input and output slacks and determining the fuzzy targets.

4.1. PHASE I: computing the fuzzy efficiency score

The fuzzy efficiency score of each DMU p corresponds to the optimal value of the objective function of (FFDEA), i.e., the set of non-dominated scores S_p in Definition 3. In crisp DEA models, the set S_p is reduced to a number and then the efficiency of a DMU_p is just that value θ_p^*. However in the proposed fully fuzzy DEA (FFDEA) approach, S_p is a set of fuzzy numbers and not a single fuzzy number.

Hatami-Marbini et al. [10] proposed a lexicographic optimization method applied to an associated multi-objective problem to select a single fuzzy efficiency score. However, if the priority order of the objectives is changed in the application of the lexicographic method, then the obtained fuzzy efficiency score can be different. Similarly, if we applied the lexicographic optimization method to model (MODEA), depending on the ordering of the objective functions, the lexicographic method applied would lead to different non-dominated scores in S_p. That is why we discard using that approach.

In order to compute a unique fuzzy non-dominated efficiency score, we propose the introduction of a criterion to select the 'best score' among the non-dominated scores $\tilde{\theta}_p \in S_p$. In particular, we can use *compromise programming*, which is based on the idea of minimizing the distance between the solution and the utopia point in the criterion space ([33]). In fuzzy numbers, distances can be computed using the Pompeiu-Hausdorff distance, combined with the gH-difference (see, for instance, [21, 26]). Since we are working on the set of TrFNs, we can simply define the distance for two TrFNs as the distance in $\mathbb{R}^4$ of the vector of their corresponding four parameters. Thus, based on the l_p and l_∞ norms, and given $\tilde{a} = (a^1, a^2, a^3, a^4)$, $\tilde{b} = (b^1, b^2, b^3, b^4) \in TrFN$, we can define $D_p(\tilde{a}, \tilde{b}) = \left(\sum_{i=1}^{4} |b^i - a^i|^p\right)^{1/p}$ and $D_\infty(\tilde{a}, \tilde{b}) = \max_{i \in \{1,2,3,4\}} |b^i - a^i|$. It is not difficult to see that D_p and D_∞ verifies the non-negativity, identity, symmetry, and triangle inequality conditions. In this manner, we can directly work on model (MODEA) mapping model (FFDEA) solutions points to $\mathbb{R}^4$.

The *lexicographic weighted Tchebycheff method* corresponds to using l_∞ norm and obtains a unique solution that is guaranteed to be Pareto optimal. The first step is calculating the utopia objective function value of (FFDEA) (or utopia efficiency score) $\tilde{\theta}_p^{ut}$. Minimizing the l_∞ distance to the utopia score $\tilde{\theta}_p^{ut}$ is used to select among the set of non-dominated efficiency scores S_p, i.e. $\gamma^* = \min\{D_\infty(\tilde{\theta}_p^{ut}, \tilde{\theta}_p) : \tilde{\theta}_p \in S_p\}$. Note that the solution to this minimization problem is the same in the case that the feasible set S_p is extended to the set of all $\tilde{\theta}_p$ such that there exists $\tilde{\lambda}$ with $(\tilde{\theta}_p, \tilde{\lambda})$ feasible for (FFDEA). The last step of the lexicographic weighted Tchebycheff approach minimizes $D_1(\tilde{\theta}_p^{ut}, \tilde{\theta}_p)$ subject to $D_\infty(\tilde{\theta}_p^{ut}, \tilde{\theta}_p) \leq \gamma^*$, within the feasible set of (FFDEA). This is required to avoid weak Pareto optimality. These three steps, which work because of the relationship between models (FFDEA) and (MODEA), constitute Phase I of the proposed approach, which provides the non-dominated fuzzy efficiency score $\tilde{\theta}_p^*$. The process is summarized as follows.

Phase I

- **Step I.1**

  Compute the utopia efficiency score of DMU_p $\tilde{\theta}_p^{ut} = ({\theta^{ut}}^1_p, {\theta^{ut}}^2_p, {\theta^{ut}}^3_p, {\theta^{ut}}^4_p)$. This involves solving the following linear programming (LP) problem for each component $i = 1, 2, 3, 4$

$$
\begin{aligned}
{\theta^{ut}}^i_p = \quad & \text{Min} \quad \theta_p^i \\
& \text{s.t.} \quad (18) - (23),
\end{aligned}
\tag{24}
$$

- **Step I.2**

  Minimize the (l_∞) distance to the utopia efficiency score $\tilde{\theta}^{ut}_p$. This is done by solving

$$\min\left\{D_\infty(\tilde{\theta}^{ut}_p, \tilde{\theta}_p) : (\tilde{\theta}_p, \tilde{\lambda}) \text{ is feasible in (FFDEA)}\right\},$$

  which corresponds to the following LP problem

$$\gamma^* = \quad \text{Min} \quad \gamma \tag{25}$$

$$\text{s.t.} \quad \theta^i_p - {\theta^{ut}}^i_p \leq \gamma, \quad i = 1, \ldots, 4, \tag{26}$$

$$(18) - (23).$$

- **Step I.3**

  The last step minimizes $D_1(\tilde{\theta}^{ut}_p, \tilde{\theta}_p)$ subject to $D_\infty(\tilde{\theta}^{ut}_p, \tilde{\theta}_p) \leq \gamma^*$ within the feasible set of (FFDEA). This is equivalent to minimizing $\sum_{i=1}^4(\theta^i_p - {\theta^{ut}}^i_p)$ subject to $D_\infty(\tilde{\theta}^{ut}_p, \tilde{\theta}_p) \leq \gamma^*$ or, equivalently,

$$\text{Min} \quad \sum_{i=1}^4 \theta^i_p \tag{27}$$

$$\text{s.t.} \quad \theta^i_p - {\theta^{ut}}^i_p \leq \gamma^*, \quad i = 1, \ldots, 4, \tag{28}$$

$$(18) - (23).$$

  Let (θ^*_p, λ^*) be optimal solution of the above LP, and $(\tilde{\theta}^*_p, \tilde{\lambda}^*)$ the corresponding fuzzy solution.

The algorithm given in Phase I is well-defined and the solution in step I.3 is a fuzzy Pareto solution of (FFDEA) as Proposition 1 states.

Proposition 1. *(i)* $\tilde{\theta}^{ut}_p = ({\theta^{ut}}^1_p, {\theta^{ut}}^2_p, {\theta^{ut}}^3_p, {\theta^{ut}}^4_p) \in TrFN$, *where* ${\theta^{ut}}^i_p$ *is the solution of* (24) *for* $i = 1, 2, 3, 4$.

(ii) Consider the optimal solution of (27) (θ^*_p, λ^*)*, and its associated fuzzy solution* $(\tilde{\theta}^*_p, \tilde{\lambda}^*)$*. Then* $\tilde{\theta}^*_p$ *is unique,* $(\tilde{\theta}^*_p, \tilde{\lambda}^*) \in F_p$ *and* $\tilde{\theta}^*_p \in S_p$.

Proof. (i) By contradiction, suppose that $\tilde{\theta}_p^{ut} \notin TrFN$. This implies that there exist $i_0, i_1 \in \{1,2,3,4\}$, $i_0 < i_1$, such that ${\theta^{ut}}_p^{i_0} > {\theta^{ut}}_p^{i_1}$. But this is a contradiction with constraint (21) imposed in the computation of the utopia point in Step I.1.

(ii) Thanks to Theorem 2, there exists an equivalence relationship between fuzzy Pareto solutions of (FFDEA) and Pareto solutions of (MODEA). Taking this into account, Phase I is an application of the lexicographic weighted Tchebycheff method to (MODEA) thus leading to a unique Pareto solution (θ_p^*, λ^*) whose associated fuzzy solution $(\tilde{\theta}_p^*, \tilde{\lambda}^*)$ is a fuzzy Pareto solution for (FFDEA). Therefore, $\tilde{\theta}_p^*$ is unique, $(\tilde{\theta}_p^*, \tilde{\lambda}^*) \in F_p$ and $\tilde{\theta}_p^* \in S_p$. □

Since we have that $\tilde{\theta}_p^*$ is unique, we propose the following fuzzy efficiency measure.

Definition 4. *Consider DMU_p, with $\tilde{\theta}_p^*$ the solution obtained in Phase I above. Then, $\tilde{\theta}_p^*$ is the fuzzy efficiency measure of DMU_p.*

4.2. PHASE II: computing fuzzy targets and assessing the efficiency status

In order to assess the efficiency status of each DMU and to compute its target, we propose a Phase II that maximizes the remaining input excesses and output shortfalls that may remain after radial input contraction associated to the fuzzy efficiency score $\tilde{\theta}_p^*$ computed in Phase I. This means that we need to solve the following optimization problem.

$$\tilde{\sigma}(\tilde{\theta}_p^*) = \text{Max} \quad \sum_{m=1}^{M} \tilde{u}_m + \sum_{r=1}^{S} \tilde{v}_r \tag{29}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} \tilde{\lambda}_n \tilde{x}_{mn} + \tilde{u}_m \leqq \tilde{\theta}_p^* \tilde{x}_{mp}, \quad m = 1, \ldots, M, \tag{30}$$

$$\sum_{n=1}^{N} \tilde{\lambda}_n \tilde{y}_{rn} \geqq \tilde{y}_{rp} + \tilde{v}_r, \quad r = 1, \ldots, S, \tag{31}$$

$$\tilde{\lambda}_n, \tilde{u}_m, \tilde{v}_r \in (TrFN)_+, \quad n = 1, \ldots, N, m = 1, \ldots, M, r = 1, \ldots, S. \tag{32}$$

Applying again Theorem 2, this fuzzy optimization model can be solved by means of its associated MOLP problem associated

$$\text{Max} \quad \left(\sum_{m=1}^{M} u_m^1 + \sum_{r=1}^{S} v_r^1, \sum_{m=1}^{M} u_m^2 + \sum_{r=1}^{S} v_r^2, \sum_{m=1}^{M} u_m^3 + \sum_{r=1}^{S} v_r^3, \sum_{m=1}^{M} u_m^4 + \sum_{r=1}^{S} v_r^4 \right) \tag{33}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} \lambda_n^i x_{mn}^i + u_m^i \leq \theta^{*i}_{\ p} x_{mp}^i, \quad i = 1, 2, 3, 4, \;\; m = 1, \ldots, M, \tag{34}$$

$$\sum_{n=1}^{N} \lambda_n^i y_{rn}^i \geq y_{rp}^i + v_r^i, \quad i = 1, 2, 3, 4, \;\; r = 1, \ldots, S, \tag{35}$$

$$\lambda_n^i - \lambda_n^{i+1} \leq 0, \quad i = 1, 2, 3, \;\; n = 1, \ldots, N, \tag{36}$$

$$u_m^i - u_m^{i+1} \leq 0, \quad i = 1, 2, 3, \;\; m = 1, \ldots, M, \tag{37}$$

$$v_r^i - v_r^{i+1} \leq 0, \quad i = 1, 2, 3, \;\; r = 1, \ldots, S, \tag{38}$$

$$\lambda_n^i, u_m^i, v_r^i \geq 0, \quad i = 1, 2, 3, 4, \;\; n = 1, \ldots, N, m = 1, \ldots, M, r = 1, \ldots, S. \tag{39}$$

Proceeding as in Phase I, carrying out the corresponding three steps, we can obtain a unique non-dominated solution $\tilde{\sigma}(\tilde{\theta}_p^*)$ of model (29)-(32) as follows.

Phase II

- **Step II.1**

  Compute the utopia point of model (33)-(39), maximizing each of its components $k = 1, \ldots, 4$

$$\sigma^{ut^k}(\tilde{\theta}_p^*) = \quad \text{Max} \quad \sum_{m=1}^{M} u_m^k + \sum_{r=1}^{S} v_r^k, \qquad \text{s.t.} \quad (34) - (39). \tag{40}$$

  Therefore, the utopia solution for the problem (33)-(39) is

$$\sigma^{ut}(\tilde{\theta}_p^*) = \left(\sigma^{ut^1}(\tilde{\theta}_p^*), \sigma^{ut^2}(\tilde{\theta}_p^*), \sigma^{ut^3}(\tilde{\theta}_p^*), \sigma^{ut^4}(\tilde{\theta}_p^*) \right).$$

- **Step II.2**

  Compute the minimum l_∞ distance of a feasible solution of the multiobjective problem (33) to (39) to the utopia solution $\sigma^{ut}(\tilde{\theta}_p^*)$.

$$\delta^* = \min \quad \delta \tag{41}$$

$$\text{s.t.} \quad \sigma^{ut^k}(\tilde{\theta}_p^*) - \sum_{m=1}^{M} u_m^k - \sum_{r=1}^{S} v_r^k \leq \delta, \quad k = 1,2,3,4, \tag{42}$$

$$(34) - (39).$$

- **Step II.3**

  Minimize $D_1(\sigma^{ut}(\tilde{\theta}_p^*), \sum_{m=1}^{M} \tilde{u}_m + \sum_{r=1}^{S} \tilde{v}_r)$ subject to $D_\infty(\sigma^{ut}(\tilde{\theta}_p^*), \sum_{m=1}^{M} \tilde{u}_m + \sum_{r=1}^{S} \tilde{v}_r) = \delta^*$. This is equivalent to solving the following maximization problem.

$$\text{Max} \quad \sum_{k=1}^{4} \left(\sum_{m=1}^{M} u_m^k + \sum_{r=1}^{S} v_r^k \right), \tag{43}$$

$$\text{s.t.} \quad \sigma^{ut^k}(\tilde{\theta}_p^*) - \sum_{m=1}^{M} u_m^k - \sum_{r=1}^{S} v_r^k \leq \delta^*, \quad k = 1,2,3,4, \tag{44}$$

$$(34) - (39).$$

Let (λ^{**}, u^*, v^*) be the optimal solution of the above LP problem, (43)-(44),(34) - (39), and $(\tilde{\lambda}^{**}, \tilde{u}^*, \tilde{v}^*)$ its corresponding fuzzy solution. Similar to Phase I, the algorithm given in Phase II is well-defined and the solution $(\tilde{\lambda}^{**}, \tilde{u}^*, \tilde{v}^*)$ is Pareto optimal for $\tilde{\sigma}(\tilde{\theta}_p^*)$. Moreover, we can compute the fuzzy input and output targets of DMU_p as

$$\tilde{X}_{mp}^{target} = \sum_{n=1}^{N} \tilde{\lambda}_n^{**} \tilde{x}_{mn}, \quad m = 1, \ldots, M, \tag{45}$$

$$\tilde{Y}_{sp}^{target} = \sum_{n=1}^{N} \tilde{\lambda}_n^{**} \tilde{y}_{sn}, \quad s = 1, \ldots, S. \tag{46}$$

In addition, we can classify each DMU in terms of its efficiency status as follows.

Definition 5. *For each DMU p, $p \in \{1, \ldots, N\}$, consider the fuzzy efficiency measure $\tilde{\theta}_p^*$ computed in Phase I and the fuzzy Pareto solution $(\tilde{\lambda}^{**}, \tilde{u}^*, \tilde{v}^*)$ obtained in Phase II. We say that the DMU p is*

(i) efficient if $\tilde{\theta}_p^* = \tilde{1}$ and $(\tilde{u}^*, \tilde{v}^*) = \tilde{0}$,

(ii) weakly efficient if $\tilde{\theta}_p^* = \tilde{1}$ and $(\tilde{u}^*, \tilde{v}^*) \geqq \tilde{0}$, $(\tilde{u}^*, \tilde{v}^*) \neq \tilde{0}$,

(iii) partially efficient if $\tilde{\theta}_p^* \leqq \tilde{1}$, $\tilde{\theta}_p^* \neq \tilde{1}$ and $\tilde{\theta}_p^*(1) = 1$,

(iv) inefficient in other cases.

Note that given DMU p, only one of the four cases above applies, i.e. the classification given in the previous definition is well-constructed. Note also that if the DMU p is efficient then its corresponding fuzzy target given by (45) and (46) coincides with the own DMU p.

5. Computational application

5.1. Example 1

Let us first consider a simple example to illustrate the proposed approach. Let us have seven DMUs, with two fuzzy inputs $\tilde{x}_{nm} \in (TrFN)_+$, for $n = 1, \ldots, 7$ and $m = 1, 2$ $(N = 7, M = 2)$. There is a single output $(S = 1)$, with the same value for all DMUs, see Table 1. All the variables are given as triangular fuzzy numbers. The example will allow us to illustrate the proposed FFDEA approach and efficiency status classification. As regards the latter, among the seven DMUs considered, we will find all four efficiency status cases considered in Definition 5.

Below we report, using the first DMU (namely DMU a) as an example, the results of each of the steps for both phases introduced in the previous section.

Table 1: Data for Example 1

DMU	a	b	c	d	e	f	g
$\tilde{x}_1$	(1 , 1 , 1.5)	(0.5 , 1 , 1.5)	(2.7 , 3 , 3.3)	(3.6 , 4 , 4.4)	(5.75 , 6 , 6.25)	(7.7 , 8 , 8.3)	(8.7 , 9 , 9.3)
$\tilde{x}_2$	(9.5 , 10 , 10.5)	(7.5 , 8 , 8.5)	(2.7 , 3 , 3.3)	(5.6 , 6 , 6.4)	(3.75 , 4 , 4.25)	(0.7 , 1 , 1.3)	(0.7 , 1 , 1.3)
$\tilde{y}_1$	(1 , 1 , 1)	(1 , 1 , 1)	(1 , 1 , 1)	(1 , 1 , 1)	(1 , 1 , 1)	(1 , 1 , 1)	(1 , 1 , 1)

The corresponding (FFDEA) model, (9)−(13), associated with the first DMU $p = 1$ is as follows,

$$
\begin{aligned}
\text{Min}\quad & \tilde{\theta}_1 \\
\text{s.t.}\quad & \tilde{x}_{11}\tilde{\lambda}_1 + \tilde{x}_{12}\tilde{\lambda}_2 + \tilde{x}_{13}\tilde{\lambda}_3 + \tilde{x}_{14}\tilde{\lambda}_4 + \tilde{x}_{15}\tilde{\lambda}_5 + \tilde{x}_{16}\tilde{\lambda}_6 + \tilde{x}_{17}\tilde{\lambda}_7 \leqq \tilde{x}_{11}\tilde{\theta}_1, \quad (i) \\
& \tilde{x}_{21}\tilde{\lambda}_1 + \tilde{x}_{22}\tilde{\lambda}_2 + \tilde{x}_{23}\tilde{\lambda}_3 + \tilde{x}_{24}\tilde{\lambda}_4 + \tilde{x}_{25}\tilde{\lambda}_5 + \tilde{x}_{26}\tilde{\lambda}_6 + \tilde{x}_{27}\tilde{\lambda}_7 \leqq \tilde{x}_{21}\tilde{\theta}_1, \quad (ii) \\
& \tilde{y}_{11}\tilde{\lambda}_1 + \tilde{y}_{12}\tilde{\lambda}_2 + \tilde{y}_{13}\tilde{\lambda}_3 + \tilde{y}_{14}\tilde{\lambda}_4 + \tilde{y}_{15}\tilde{\lambda}_5 + \tilde{y}_{16}\tilde{\lambda}_6 + \tilde{y}_{17}\tilde{\lambda}_7 \geqq \tilde{y}_{11}, \quad (iii) \\
& \tilde{\theta}_1 \leqq \tilde{1}, \quad (iv) \\
& \tilde{\theta}_1, \tilde{\lambda}_n \in (TrFN)_+, \quad n = 1, \ldots, 7
\end{aligned}
$$

and its corresponding (MODEA) (17)−(23) Multiobjective Linear Problem is

$$
\begin{aligned}
\text{Min}\quad & (\theta_1^1, \theta_1^2, \theta_1^3) \\
\text{s.t.}\quad & \left.\begin{aligned}
\lambda_1^1 + 0.5\lambda_2^1 + 2.7\lambda_3^1 + 3.6\lambda_4^1 + 5.75\lambda_5^1 + 7.7\lambda_6^1 + 8.7\lambda_7^1 &\leq \theta_1^1, \\
\lambda_1^2 + \lambda_2^2 + 3\lambda_3^2 + 4\lambda_4^2 + 6\lambda_5^2 + 8\lambda_6^2 + 9\lambda_7^2 &\leq \theta_1^2, \\
1.5\lambda_1^3 + 1.5\lambda_2^3 + 3.3\lambda_3^3 + 4.4\lambda_4^3 + 6.25\lambda_5^3 + 8.3\lambda_6^3 + 9.3\lambda_7^3 &\leq 1.5\theta_1^3,
\end{aligned}\right\} \quad (i) \\
& \left.\begin{aligned}
9.5\lambda_1^1 + 7.5\lambda_2^1 + 2.7\lambda_3^1 + 5.6\lambda_4^1 + 3.75\lambda_5^1 + 0.7\lambda_6^1 + 0.7\lambda_7^1 &\leq 9.5\theta_1^1, \\
10\lambda_1^2 + 8\lambda_2^2 + 3\lambda_3^2 + 6\lambda_4^2 + 4\lambda_5^2 + \lambda_6^2 + \lambda_7^2 &\leq 10\theta_1^2, \\
10.5\lambda_1^3 + 8.5\lambda_2^3 + 3.3\lambda_3^3 + 6.4\lambda_4^3 + 4.25\lambda_5^3 + 1.3\lambda_6^3 + 1.3\lambda_7^3 &\leq 10.5\theta_1^3,
\end{aligned}\right\} \quad (ii) \\
& \left.\begin{aligned}
\lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 + \lambda_5^1 + \lambda_6^1 + \lambda_7^1 &\geq 1, \\
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 + \lambda_7^2 &\geq 1, \\
\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 + \lambda_5^3 + \lambda_6^3 + \lambda_7^3 &\geq 1,
\end{aligned}\right\} \quad (iii) \\
& \left.\begin{aligned}
\theta_1^i &\leq 1, \quad i = 1, 2, 3, \\
\theta_1^i - \theta_1^{i+1} &\leq 0, \quad 1 \leq i < 3,
\end{aligned}\right\} \quad (iv) \\
& \lambda_n^i - \lambda_n^{i+1} \leq 0, \quad 1 \leq i < 3, \;\; n = 1, \ldots, 7, \quad (v) \\
& \theta_1^i, \lambda_n^i \geq 0, \quad i = 1, 2, 3, \;\; n = 1, \ldots, 7. \quad (vi)
\end{aligned}
$$

Note how the constraints (i) to (iv) in the Fully Fuzzy DEA approach are transformed into their equivalent crisp constraints in the corresponding MOLP model. The first phase proceeds as follows,

Phase I

- **Step I.1** Compute the utopia efficiency score of DMU_1. This is, for $i = 1, 2, 3$ we solve the problems [(24),(18)-(23)],

$$\begin{aligned} {\theta^{ut}}^i_1 = \quad & \text{Min} \quad \theta^i_1 \\ & \text{s.t.} \quad (i) - (vi) \end{aligned}$$

  This gives the utopia efficiency score $\tilde{\theta}^{ut}_1 = \left({\theta^{ut}}^1_1 = 0.789, {\theta^{ut}}^2_1 = 1, {\theta^{ut}}^3_1 = 1\right)$. The corresponding results for the other DMUs are shown in Table 2.

- **Step I.2** Compute the minimum distance to the utopia efficiency score $\tilde{\theta}^{ut}_1$, among the feasible points in (FFDEA). This corresponds to the LP problem [(25),(26),(18)-(23)],

$$\begin{aligned} \gamma^* = \quad & \text{Min} \quad \gamma \\ & \text{s.t.} \quad \theta^1_1 - 0.789 \leq \gamma, \\ & \qquad\ \ \theta^2_1 - 1 \leq \gamma, \\ & \qquad\ \ \theta^3_1 - 1 \leq \gamma, \\ & \qquad\ \ (i) - (vi). \end{aligned}$$

  which leads, for this DMU, to an optimal value $\gamma^* = 0$.

- **Step I.3** Finally, we get the efficiency score $\tilde{\theta}^*_1$ for DMU_1, by minimizing $D_1(\tilde{\theta}^{ut}_1, \tilde{\theta}_1)$ subject to $D_\infty(\tilde{\theta}^{ut}_1, \tilde{\theta}1) \leq \gamma^* = 0$, within the feasible set of (FFDEA). This is equivalent to solve the problem [(27),(28),(18)-(23)],

$$\begin{aligned} & \text{Min} \quad \theta^1_1 + \theta^2_1 + \theta^3_1 \\ & \text{s.t.} \quad \theta^1_1 \leq 0.789, \\ & \qquad\ \ \theta^2_1 \leq 1, \\ & \qquad\ \ \theta^3_1 \leq 1, \\ & \qquad\ \ (i) - (vi). \end{aligned}$$

The optimal solution of the above LP problem is $(\boldsymbol{\theta}_1^*, \boldsymbol{\lambda}^*) = (\theta^{*1}_1 = 0.789$, $\theta^{*2}_1 = 1$, $\theta^{*3}_1 = 1$, $\lambda^{*1}_1 = 0$, $\lambda^{*2}_1 = 0$, $\lambda^{*3}_1 = 0$, $\lambda^{*1}_2 = 1$, $\lambda^{*2}_2 = 1$, $\lambda^{*3}_2 = 1$, $\lambda^{*1}_3 = 0$, $\lambda^{*2}_3 = 0$, $\lambda^{*3}_3 = 0$, $\lambda^{*1}_4 = 0$, $\lambda^{*2}_4 = 0$, $\lambda^{*3}_4 = 0$, $\lambda^{*1}_5 = 0$, $\lambda^{*2}_5 = 0$, $\lambda^{*3}_5 = 0$, $\lambda^{*1}_6 = 0$, $\lambda^{*2}_6 = 0$, $\lambda^{*3}_6 = 0$, $\lambda^{*1}_7 = 0$, $\lambda^{*2}_7 = 0$, $\lambda^{*3}_7 = 0)$

Therefore, the associated fuzzy solution of (FFDEA) is $(\tilde{\boldsymbol{\theta}}_1^*, \tilde{\boldsymbol{\lambda}}^*) = (\tilde{\theta}_1^* = (0.789, 1, 1)$, $\tilde{\lambda}_1^* = (0, 0, 0)$, $\tilde{\lambda}_2^* = (1, 1, 1)$, $\tilde{\lambda}_3^* = (0, 0, 0)$, $\tilde{\lambda}_4^* = (0, 0, 0)$, $\tilde{\lambda}_5^* = (0, 0, 0)$, $\tilde{\lambda}_6^* = (0, 0, 0)$, $\tilde{\lambda}_7^* = (0, 0, 0))$.

Given the efficiency score of DMU_1 $\tilde{\theta}_1^* = (0.789, 1, 1)$, obtained in Phase I, we can proceed to Phase II and obtain a unique non-dominated solution $\tilde{\sigma}(\tilde{\theta}_p^*)$ of model (29)-(32) as follows.

Phase II

- **Step II.1**
  Given $\tilde{\theta}_1^* = (0.789, 1, 1)$, compute the utopia point of model (33)-(39) maximizing each of its components $k = 1, 2, 3$,

$$
\sigma^{ut^k}(\tilde{\theta}_1^*) = \quad \text{Max} \;\; u_1^k + u_2^k + v_1^k, \qquad \text{s.t.}
$$

$$
\left.\begin{array}{rcl}
\lambda_1^1 + 0.5\lambda_2^1 + 2.7\lambda_3^1 + 3.6\lambda_4^1 + 5.75\lambda_5^1 + 7.7\lambda_6^1 + 8.7\lambda_7^1 + u_1^1 & \leq & \theta^{*1}_1 = 0.789, \\
\lambda_1^2 + \lambda_2^2 + 3\lambda_3^2 + 4\lambda_4^2 + 6\lambda_5^2 + 8\lambda_6^2 + 9\lambda_7^2 + u_1^2 & \leq & \theta^{*2}_1 = 1, \\
1.5\lambda_1^3 + 1.5\lambda_2^3 + 3.3\lambda_3^3 + 4.4\lambda_4^3 + 6.25\lambda_5^3 + 8.3\lambda_6^3 + 9.3\lambda_7^3 + u_1^3 & \leq & 1.5\theta^{*3}_1 = 1.5,
\end{array}\right\} \quad (vii)
$$

$$
\left.\begin{array}{rcl}
9.5\lambda_1^1 + 7.5\lambda_2^1 + 2.7\lambda_3^1 + 5.6\lambda_4^1 + 3.75\lambda_5^1 + 0.7\lambda_6^1 + 0.7\lambda_7^1 + u_2^1 & \leq & 9.5\theta^{*1}_1 = 7.5, \\
10\lambda_1^2 + 8\lambda_2^2 + 3\lambda_3^2 + 6\lambda_4^2 + 4\lambda_5^2 + \lambda_6^2 + \lambda_7^2 + u_2^2 & \leq & 10\theta^{*2}_1 = 10, \\
10.5\lambda_1^3 + 8.5\lambda_2^3 + 3.3\lambda_3^3 + 6.4\lambda_4^3 + 4.25\lambda_5^3 + 1.3\lambda_6^3 + 1.3\lambda_7^3 + u_2^3 & \leq & 10.5\theta^{*3}_1 = 10.5,
\end{array}\right\} \quad (viii)
$$

$$
\left.\begin{array}{rcl}
\lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 + \lambda_5^1 + \lambda_6^1 + \lambda_7^1 & \geq & v_1^1 + 1, \\
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 + \lambda_7^2 & \geq & v_1^2 + 1, \\
\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 + \lambda_5^3 + \lambda_6^3 + \lambda_7^3 & \geq & v_1^3 + 1,
\end{array}\right\} \quad (ix)
$$

$$
\left.\begin{array}{rcl}
u_m^i - u_m^{i+1} & \leq & 0, \quad i = 1, 2, \quad m = 1, 2 \\
v_1^i - v_1^{i+1} & \leq & 0, \quad i = 1, 2, \\
\lambda_n^i - \lambda_n^{i+1} & \leq & 0, \quad i = 1, 2, \;\; n = 1, \ldots, 7,
\end{array}\right\} \quad (x)
$$

$$
u_m^i, v_1^i, \lambda_n^i \geq 0, \quad i = 1, 2, 3, \;\; m = 1, 2, \; n = 1, \ldots, 7. \qquad (xi)
$$

Solving the three LP problems above leads to the utopia solution $\sigma^{ut}(\tilde{\theta}_1^*) = (0, 2, 2)$

- **Step II.2**

  Compute the minimum l_∞ distance of a feasible solution of the multiobjective problem (33) to (39) to the utopia solution $\sigma^{ut}(\tilde{\theta}_1^*) = (0, 2, 2)$.

$$\begin{aligned} \delta^* = \quad & \min \quad \delta \\ & \text{s.t.} \quad 0 - u_1^1 + u_2^1 + v_1^1 \leq \delta, \\ & \qquad\quad 2 - u_1^2 + u_2^2 + v_1^2 \leq \delta, \\ & \qquad\quad 2 - u_1^3 + u_2^3 + v_1^3 \leq \delta, \\ & \qquad\quad (vii) - (xi). \end{aligned}$$

  The corresponding optimal solution is $\delta^* = 0$.

- **Step II.3**

  This steps corresponds to minimizing $D_1(\sigma^{ut}(\tilde{\theta}_p^*), \sum_{m=1}^M \tilde{u}_m + \sum_{r=1}^S \tilde{v}_r)$ subject to $D_\infty(\sigma^{ut}(\tilde{\theta}_p^*), \sum_{m=1}^M \tilde{u}_m + \sum_{r=1}^S \tilde{v}_r) = \delta^*$. This is equivalent to solving the following maximization problem,

$$\begin{aligned} \text{Max} \quad & u_1^1 + u_2^1 + v_1^1 + u_1^2 + u_2^2 + v_1^2 + u_1^3 + u_2^3 + v_1^3, \\ \text{s.t.} \quad & 0 - u_1^1 + u_2^1 + v_1^1 \leq 0, \\ & 2 - u_1^2 + u_2^2 + v_1^2 \leq 0, \\ & 2 - u_1^3 + u_2^3 + v_1^3 \leq 0, \\ & (vii) - (xi). \end{aligned}$$

The optimal solution of the above LP problem is $(\boldsymbol{\lambda}^{**}, \boldsymbol{u}^*, \boldsymbol{v}^*) = (\lambda_1^1 = 0,$ $\lambda_1^2 = 0,$ $\lambda_1^3 = 0,$ $\lambda_2^1 = 1,$ $\lambda_2^2 = 1,$ $\lambda_2^3 = 1,$ $\lambda_3^1 = 0,$ $\lambda_3^2 = 0,$ $\lambda_3^3 = 0,$ $\lambda_4^1 = 0,$ $\lambda_4^2 = 0,$ $\lambda_4^3 = 0,$ $\lambda_5^1 = 0,$ $\lambda_5^2 = 0,$ $\lambda_5^3 = 0,$ $\lambda_6^1 = 0,$ $\lambda_6^2 = 0,$ $\lambda_6^3 = 0,$ $\lambda_7^1 = 0,$ $\lambda_7^2 = 0,$ $\lambda_7^3 = 0,$ $u_1^1 = 0,$ $u_1^2 = 0,$ $u_1^3 = 0,$ $u_2^1 = 0,$ $u_2^2 = 2,$ $u_2^3 = 2,$ $v_1^1 = 0,$ $v_1^2 = 0,$ $v_1^3 = 0)$, which corresponds to $(\tilde{\boldsymbol{\lambda}}^{**}, \tilde{\boldsymbol{u}}^*, \tilde{\boldsymbol{v}}^*) = (\tilde{\lambda}_1^{**} = (0, 0, 0), \tilde{\lambda}_2^{**} = (1, 1, 1), \tilde{\lambda}_3^{**} = (0, 0, 0), \tilde{\lambda}_4^{**} = (0, 0, 0), \tilde{\lambda}_5^{**} = (0, 0, 0), \tilde{\lambda}_6^{**} = (0, 0, 0), \tilde{\lambda}_7^{**} = (0, 0, 0), \tilde{u}_1^* = (0, 0, 0), \tilde{u}_2^* = (0, 2, 2), \tilde{v}_1^* = (0, 0, 0))$.

Table 2: Results for Phases I & II and DMU efficiency status classification for Example 1. For comparison purposes, last rows correspond with the results when Hatami-Marbini et al. [10] method is applied.

	DMU	a	b	c	d	e	f	g
Phase I	$\tilde{\theta}_p^{ut}$	$(0.789, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(0.639, 0.656, 0.672)$	$(0.625, 0.656, 0.684)$	$(1, 1, 1)$	$(1, 1, 1)$
	γ^*	0	0	0	0.003	0.005	0	0
	$\tilde{\theta}_p^*$	$(0.789, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(0.641, 0.657, 0.674)$	$(0.63, 0.658, 0.69)$	$(1, 1, 1)$	$(1, 1, 1)$
Phase II	$\tilde{\sigma}^{ut}(\tilde{\theta}_p^*)$	$(0, 2, 2)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0.016, 0.05)$	$(0, 0.024, 0.088)$	$(0, 0, 0)$	$(1, 1, 1)$
	δ^*	0	0	0	0	0	0	0
	$\tilde{u}_1$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0.024, 0.088)$	$(0, 0, 0)$	$(1, 1, 1)$
	$\tilde{u}_2$	$(0, 2, 2)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0.016, 0.05)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
	$\tilde{v}_1$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
	$\tilde{X}_1^{target}$	$(0.5, 1, 1.5)$	$(0.5, 1, 1.5)$	$(2.7, 3, 3.3)$	$(2.292, 2.629, 2.966)$	$(3.622, 3.922, 4.222)$	$(7.7, 8, 8.3)$	$(7.7, 8, 8.3)$
	$\tilde{X}_2^{target}$	$(7.5, 8, 8.5)$	$(7.5, 8, 8.5)$	$(2.7, 3, 3.3)$	$(3.59, 3.927, 4.264)$	$(2.331, 2.631, 2.931)$	$(0.7, 1, 1.3)$	$(0.7, 1, 1.3)$
	$\tilde{Y}_1^{target}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
	Status	Part. eff.	Eff.	Eff.	Ineff.	Ineff.	Eff	Weakly eff.
Hatami-Marbini et al.	$\tilde{\theta}_p^*$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(0.639, 0.659, 0.675)$	$(0.625, 0.661, 0.692)$	$(1, 1, 1)$	$(1, 1, 1)$
	$\tilde{X}_1^{target}$	$(1, 1, 1.5)$	$(0.5, 1, 1.5)$	$(2.7, 3, 3.3)$	$(2.299, 2.635, 2.972)$	$(3.593, 3.893, 4.193)$	$(7.7, 8, 8.3)$	$(7.7, 8, 8.3)$
	$\tilde{X}_2^{target}$	$(9.5, 10.0, 10.5)$	$(7.5, 8, 8.5)$	$(2.7, 3, 3.3)$	$(3.576, 3.912, 4.249)$	$(2.343, 2.643, 2.943)$	$(0.7, 1, 1.3)$	$(0.7, 1, 1.3)$
	$\tilde{Y}_1^{target}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
	Status	Eff.	Eff.	Eff.	Ineff.	Ineff.	Eff	Ineff.

Finally, we can compute the fuzzy input and output target of DMU_1 using (45) and (46),

$$
\begin{aligned}
\widetilde{X}_{1,1}^{target} &= \sum_{n=1}^{7} \tilde{\lambda}_n^{**} \tilde{x}_{1,n} = (0.5, 1, 1.5) \\
\widetilde{X}_{2,1}^{target} &= \sum_{n=1}^{7} \tilde{\lambda}_n^{**} \tilde{x}_{2,n} = (7.5, 8, 8.5) \\
\widetilde{Y}_{1,1}^{target} &= \sum_{n=1}^{7} \tilde{\lambda}_n^{**} \tilde{y}_{1n} = (1, 1, 1).
\end{aligned}
$$

The results of the two Phases of the proposed approach, as well as the corresponding efficiency status, for all the DMUS can be found in Table 2. Moreover, to illustrate the characterization of the DMUs and understand better their classification as efficient, weakly efficient, etc., given in Definition 5, Figure 1 shows the observed DMUs and their efficient targets. The representation of the fuzzy numbers used in this figure is the following. The support of

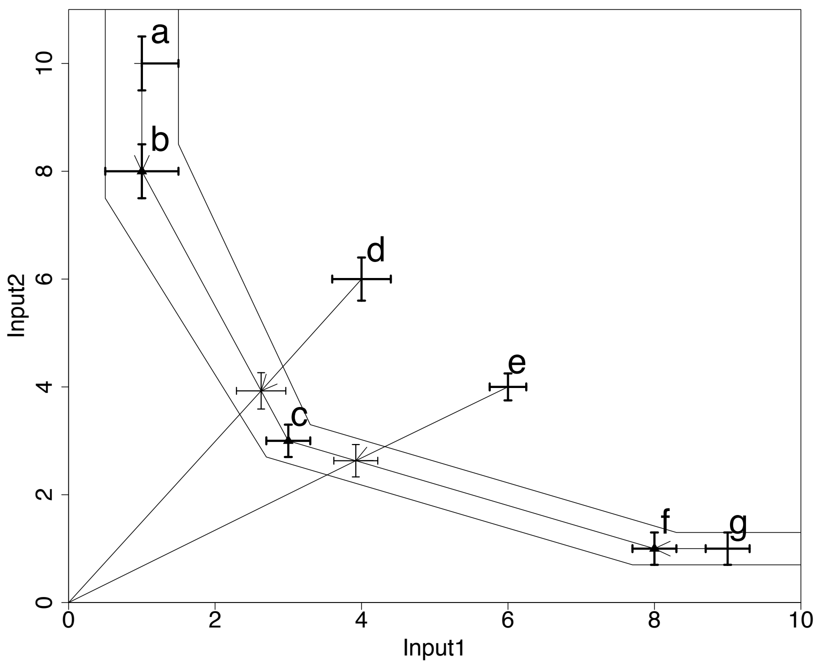

Figure 1: Observed DMU and computed targets for Example 5.1.

the two fuzzy inputs are represented as a horizontal (input 1) and a vertical (input 2) segment, respectively. And the middle point of these triangular fuzzy numbers is located at the intersection of the corresponding horizontal and vertical segments. DMUs *b*, *c* and *f* are efficient and define the efficient frontier. DMUs *d* and *e* are inefficient and are radially projected onto their corresponding efficient targets. DMU *g* is weakly efficient and is projected onto DMU *f*, while DMU *a* is partially efficient and is projected onto DMU *b*.

Figure 2 shows in detail the results of the proposed approach for some DMUs (namely DMUs *a*, *d* and *g*), comparing them with those obtained using the approach in Hatami-Marbini et al. [10] (also shown at the bottom of Table 2). Input data are represented with dots and filled lines, the targets computed by the proposed approach with squares and dashed lines, and the targets from

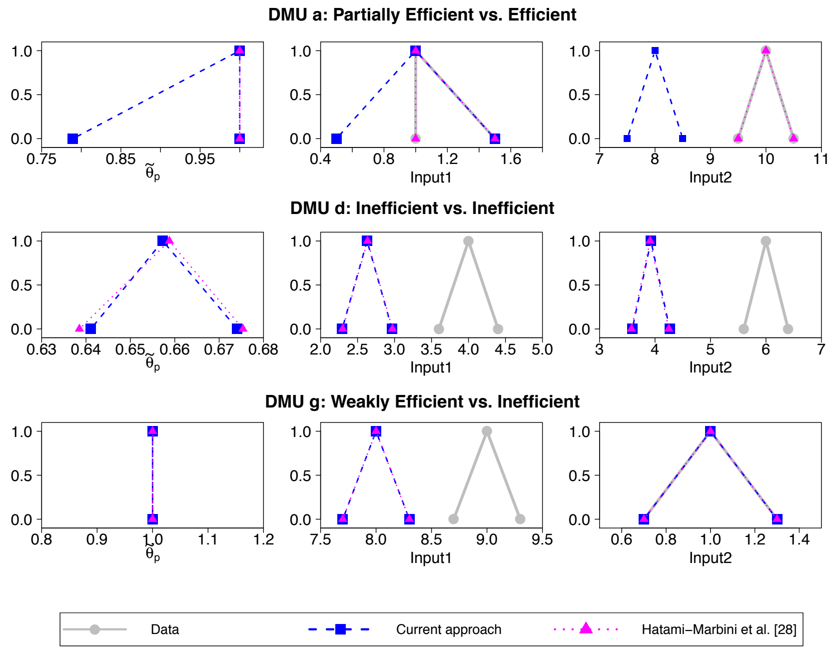

Figure 2: Observed input data and computed efficiency scores and targets for Example 5.1, for proposed approach and for Hatami-Marbini et al. Two DMUs with different classification in both approaches have been chosen, together with an inefficient DMU identified in both approaches.

Hatami-Marbini et al. [10] with triangles and dotted lines. Output data are not included since they remain equal to the observed value, as it can be checked in Tables 1 and 2.

Note that DMU a is classified as partially efficient since $\tilde{\theta}^*_a = (0.789, 1, 1) \leqq \tilde{1}$, $\tilde{\theta}^*_a \neq \tilde{1}$ and $\tilde{\theta}^*_a(1) = 1$. There are non-zero slacks for both inputs, and therefore the input targets improve the input data. In the case of Hatami-Marbini et al. [10] approach, DMU a is classified as efficient, with $\tilde{\theta}^*_a = \tilde{1}$ and zero slacks, so their input target coincides with the observed input.

DMU d is classified as inefficient in both approaches with slight differences in the efficiency scores or in the corresponding targets.

Finally, DMU g is classified as weakly efficient since although $\tilde{\theta}_g^* = \tilde{1}$, the input slack $\tilde{u}_1^* \neq \tilde{0}$, as it can be observed at the bottom panel of Figure 2. The Hatami-Marbini et al. [10] approach classifies this DMU directly as inefficient.

5.2. Example 2: Dataset from the literature

In this section, the proposed FFDEA is applied to the dataset used in Hatami-Marbini et al. [10], consisting of 26 suppliers of raw materials. There are four crisp inputs: the total cost of shipments (TC) and the number of shipments per month (NS), as economic criteria, and the eco-design cost (ED) and cost of work safety and labor health (CS), as social criteria. The two fuzzy outputs (given as triangular fuzzy numbers) are the number of shipments to arrive on time (NOT) and the number of bills received from the supplier without errors (NB).

We have compared our results with those from Hatami-Marbini et al. [10], which also apply a fully fuzzy DEA approach to this same dataset. The computations have been made in R[1] (version 3.3.2), and using the `lpSolve` package for solving Linear Programs. The codes are run on an Intel Core i7 macOS 10.14.3, 2.2 GHz, 8 GB RAM, 1600 MHz DDR3. The computing time required for evaluating all 26 DMUs was ≈ 0.4 seconds. Therefore, the computational cost of the proposed two-phase approach for this not so small problem is negligible.

In this dataset, all DMUs are classified as efficient or inefficient, i.e., there are no weakly or partially efficient DMUs as in the previous example. Moreover, except for DMUs 11 and 24, the efficiency scores and solutions of both methods are the same. In Table 3, we consider these two DMUs for which the results of the two approaches differ. Note that the efficiency scores computed by the proposed method are slightly lower than those calculated by [10]. This can also be seen in Figure 3 and, as indicated in Section 4, it is due to the fact that the feasible region considered in the proposed approach is larger than that of Hatami-Marbini et al. [10]. This confers the proposed approach more discriminant power.

[1] `https://www.r-project.org`

Table 3: Efficiency scores and targets computed by the proposed approach and by Hatami-Marbini et al. [10], for the inefficient DMUs 11 and 24. For the remaining DMUs the results coincide.

	DMU 11		DMU 24	
	Proposed approach	Hatami-Marbini et al. [10]	Proposed approach	Hatami-Marbini et al. [10]
$\tilde{\theta}_p$	(0.936 , 0.951 , 0.963)	(0.938, 0.952,0.963)	(0.875 , 0.892 , 0.906)	(0.875, 0.894,0.909)
$\tilde{X}_1^{target}$	(296.264 , 300.529 , 304.026)	(288.063,292.032,295.304)	(259.129 , 263.993 , 268.179)	(258.991,264.443,268.975)
$\tilde{X}_2^{target}$	(113.315 , 115.067 , 116.503)	(113.493,115.151,116.518)	(136.388 , 140.204 , 143.487)	(136.255,139.748,142.651)
$\tilde{X}_3^{target}$	(47.826 , 48.888 , 49.758)	(48.598, 49.379,50.023)	(35.297 , 36.211 , 36.998)	(35.265, 36.190,36.959)
$\tilde{X}_4^{target}$	(24.184 , 24.784 , 25.277)	(26.504,26.947,27.311)	(15.758 , 16.054 , 16.308)	(15.749,16.089,16.372)
$\tilde{Y}_1^{target}$	(125 , 145 , 165)	(125,145,165)	(112.104 , 132 , 152)	(112,132,152)
$\tilde{Y}_2^{target}$	(153 , 160 , 167)	(153,160,167)	(177 , 184.035 , 191.097)	(177,184.133,191.244)

Figure 3 shows the corresponding fuzzy efficiency scores, the input and output targets computed by both methods (squares and filled lines for proposed approach, and triangles and dashed lines for Hatami-Marbini et al.) and the observed inputs and output (dots and filled lines). Note that the output targets are equal or very similar to the observed values for both methods, and therefore superimposed in the graph. This happens often in input-oriented approaches in which the priority is reducing the inputs more than increasing the outputs. As regards the inputs, it must be remarked that although the observed values were crisp, the targets are fuzzy. In other words, the uncertainty present in some of the variables propagates to all other variables, an intrinsic feature of fully fuzzy DEA approaches.

6. Conclusions

In this paper, a new, two-phase, radial, input-oriented FFDEA approach that uses trapezoidal fuzzy numbers is proposed. Apart from explicitly formulating the fuzzy DEA technology considered, the novelty of the approach is the LU-fuzzy partial order used. Also, it is shown how the proposed FFLP model, through the definition of the fuzzy Pareto solutions, can be transformed into a multiobjective optimization problem that can be solved using the lexicographic weighted Tchebycheff method. In the end, a fuzzy efficiency measure and a fuzzy target are computed for each DMU. A classification of the efficiency status

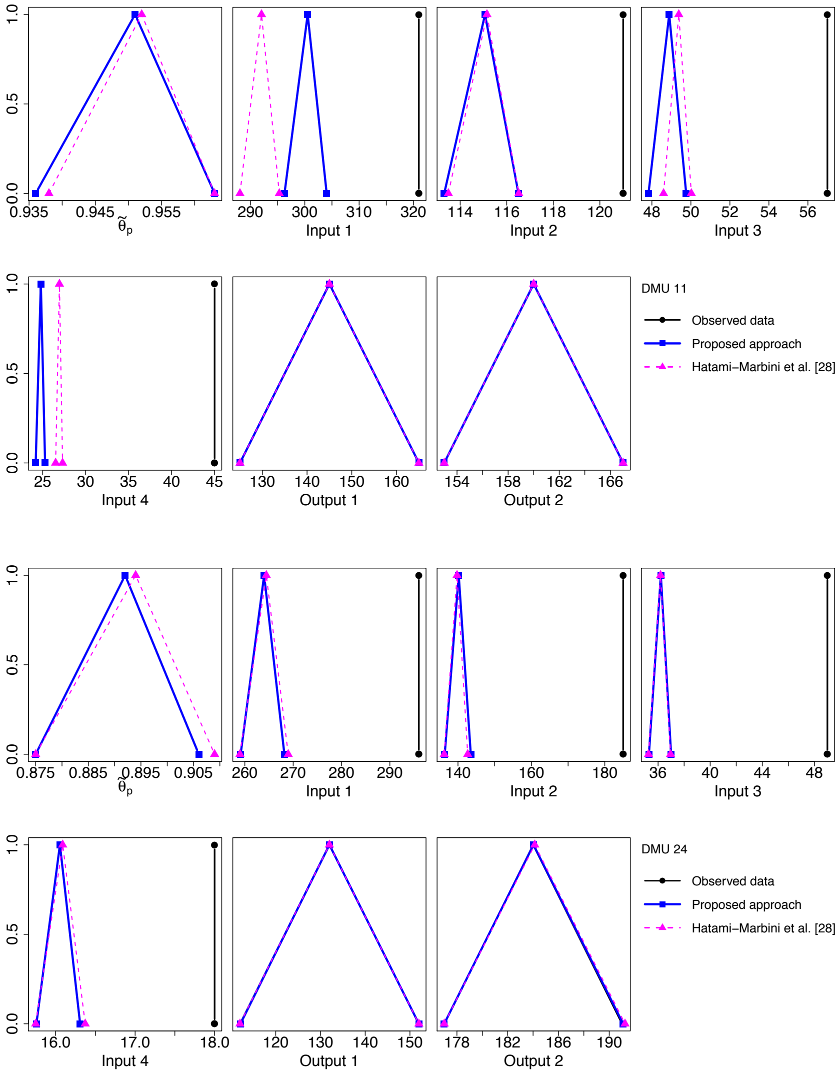

Figure 3: Efficieny scores and targets of proposed approach (squared and filled lines) and of Hatami-Marbini et al. [10] (triangles and dashed lines), for DMUs 11 and 24. Observed inputs and output are also shown (dots and filled lines). The corresponding numerical values are shown in Table 3. For the two outputs, both targets practically coincide with the data.

Table 4: Pareto optimal solution and efficiency status for Example 2 ($\tilde{\theta}^*_p$ from Phase I and $(\tilde{u}^*, \tilde{v}^*)$ from Phase II).

		Fuzzy input slacks				Fuzzy output slacks		
p	$\tilde{\theta}_p$	$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_4$	$\tilde{v}_1$	$\tilde{v}_2$	Eff. Status
1	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
2	(0.92 , 0.943 , 0.963)	(0 , 0 , 0)	(0 , 0 , 0)	(2.453 , 2.516 , 2.568)	(0 , 0 , 0)	(0 , 0 , 0)	(27.929 , 27.99 , 28.041)	Eff.
3	(0.936 , 0.936 , 0.936)	(0 , 0 , 0)	(6.748 , 6.748 , 6.748)	(44.997 , 44.997 , 44.997)	(0 , 0 , 0)	(0 , 1.828 , 3.656)	(0 , 0.64 , 1.279)	Ineff.
4	(0.836 , 0.842 , 0.847)	(0 , 0 , 0)	(32.45 , 32.706 , 32.916)	(1.801 , 1.839 , 1.871)	(5.089 , 5.128 , 5.16)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
5	(0.97 , 0.97 , 0.97)	(164.484 , 164.484 , 164.484)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 3.774 , 7.548)	(0 , 1.321 , 2.642)	Ineff.
6	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
7	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
8	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
9	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
10	(0.788 , 0.829 , 0.863)	(35.748 , 38.961 , 41.635)	(1.195 , 1.195 , 1.195)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0.029 , 0.053)	Ineff.
11	(0.936 , 0.951 , 0.963)	(4.349 , 4.731 , 5.044)	(0 , 0 , 0)	(5.124 , 5.124 , 5.124)	(17.958 , 18.009 , 18.051)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
12	(0.855 , 0.855 , 0.855)	(0 , 0 , 0)	(2.07 , 2.07 , 2.07)	(0 , 0 , 0)	(7.761 , 7.761 , 7.761)	(0 , 1.459 , 2.919)	(0 , 0.511 , 1.021)	Ineff.
13	(0.783 , 0.783 , 0.783)	(32.748 , 32.748 , 32.748)	(11.113 , 11.113 , 11.113)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 1.549 , 3.097)	(0 , 0.542 , 1.084)	Ineff.
14	(0.765 , 0.788 , 0.807)	(0 , 0 , 0)	(19.884 , 20.632 , 21.248)	(15.216 , 15.884 , 16.435)	(38.342 , 39.522 , 40.494)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
15	(0.489 , 0.53 , 0.567)	(0 , 0 , 0)	(22.848 , 24.228 , 26.131)	(0 , 0.247 , 0.585)	(7.128 , 7.569 , 8.112)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
16	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
17	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
18	(0.549 , 0.567 , 0.582)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(2.808 , 2.898 , 2.972)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
19	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
20	(0.567 , 0.59 , 0.609)	(0 , 0 , 0)	(1.395 , 1.395 , 1.395)	(9.156 , 9.156 , 9.156)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0.134 , 0.283)	Ineff.
21	(0.895 , 0.91 , 0.923)	(0 , 0 , 0)	(55.843 , 56.827 , 57.637)	(33.39 , 34.026 , 34.549)	(34.579 , 35.157 , 35.634)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
22	(1 , 1 , 1)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Eff.
23	(0.782 , 0.782 , 0.782)	(0 , 0 , 0)	(48.376 , 48.376 , 48.376)	(26.213 , 26.213 , 26.213)	(17.07 , 17.07 , 17.07)	(0 , 1.962 , 3.924)	(0 , 0.687 , 1.373)	Ineff.
24	(0.875 , 0.892 , 0.906)	(0 , 0 , 0)	(24.125 , 24.125 , 24.125)	(7.396 , 7.396 , 7.396)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0.035 , 0.097)	Ineff.
25	(0.583 , 0.629 , 0.667)	(75.539 , 83.429 , 89.97)	(39.575 , 42.378 , 44.702)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.
26	(0.796 , 0.796 , 0.796)	(0 , 0 , 0)	(11.171 , 11.171 , 11.173)	(0 , 0.001 , 0.002)	(3.268 , 3.268 , 3.269)	(0 , 0 , 0)	(0 , 0 , 0)	Ineff.

of the DMUs into efficient, weakly efficient, partially efficient, and inefficient is also presented.

The proposed FFDEA approach has been compared with that of Hatami-Marbini et al. [10]. It has been shown that the feasible region of the proposed FFLP approach is larger and contains the one used in Hatami-Marbini et al. [10], and hence the proposed approach has more discriminant power. Another difference with Hatami-Marbini et al. [10] is in the multiobjective method used, lexicographic optimization in the case of Hatami-Marbini et al. [10] and lexicographic weighted Tchebycheff in the proposed approach.

Among the limitations of the proposed fully fuzzy approach we can count the fact that a radial efficiency metric is used. Extending the approach to the radial output-oriented case is trivial. More challenging is extending the approach to non-radial, non-oriented efficiency measures like the hyperbolic graph or the slacks-based measure of efficiency. Applying the fully fuzzy approach to two-stage network DEA systems is also a worthy endeavour. Finally, another interesting line of research is using a lexicographic directional distance approach (see, [15] for the crisp data case).

7. Acknowledgements

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A two-phase, radial, input-oriented fully fuzzy DEA approach is proposed

Trapezoidal fuzzy numbers and fuzzy LU partial order are considered

Fully fuzzy LP are converted to equivalent multiobjetive optimization problems

A fuzzy Pareto solution is computed using lexicographic weighted Tchebycheff method

Proposed approach considers larger feasibility region and has more discriminant power

Efficiency assessment and target setting using a fully fuzzy DEA approach

AE

The paper received four review reports. On one hand, the reviewers found that the paper presents some interesting results that are publishable. On the other hand, however, the reviewers made a number of critical comments on the paper. Therefore, based on the reviewers' comments, the authors are encouraged to examine carefully the comments of the reviews and to consider making a major revision of the paper that can be submitted for further considerations in journal. In addition, the following comments should be also considered by authors:

1. *the readability and presentation of the work should be improved. And, all equations should be checked carefully.*
2. *the main contributions of the work should be clearly explained in both Theoretical and Practical aspects,*
3. *the dissemination of the results should be further explained and compared with existing results.*

As it can be checked in the answers to the respective referees, the above three points have been dealt with and improved.

1 Reviewer 1

This paper a new, two-phase, radial, input-oriented fully fuzzy data envelopment analysis (FFDEA) approach that uses trapezoidal fuzzy numbers. the novelty of the approach is the Lu-fuzzy partial order used. Some interesting results are derived. In the reviewer's opinion, although this paper presents a good results, there are still some concerns which should be addressed by the authors.

1. *Please be careful with the abbreviations. The authors should define all the abbreviations before using it. For instance, what is DEA at the beginning of the abstract part.*

   All the abbreviations have been checked and defined before using them.

2. *The major motivations of the conducted topic and new novelties can be further highlighted in Introduction and Conclusion parts.*

   The introduction section was modified in order to explain in detail the motivations and novelties of the present work. We include a new paragraph explaining the main differences in our work with respect to some existing in the literature. Another paragraph highlights the novel contributions, divided into four points.

3. *The designed procedure or formulate multiobjective linear problem process, and the main theorems need to better illustration, which is helpful for the readers to understand the details.*

   The procedure of multiobjective linear problem has been clarified. Now, each step in each phase of the procedure has been detailed. Furthermore, these steps have been illustrated, step by step, in the first example of the computational application section, section 5.1. The illustrations (new Figs. 2 and 3, and Table 2) have been modified in order to improve their visualization and to be more clear for the reader.

4. *The reference part should be modified by removing less relevant papers, since there are too many references now.*

   Thank you very much for the suggestion. We have removed all less relevant papers. In fact, by mistake, there were a number of references that appeared in the bibliography but that were not really cited. Those non-cited references have been deleted.

5. *There are some grammar errors and typos. The authors should check and correct them carefully.*

   The manuscript has been carefully revised, and a spell check has been carried out. All grammar errors and typos found have been corrected.

2 Reviewer 3

The submitted paper proposes a radial, input-oriented fully fuzzy DEA approach based on an LU-fuzzy partial order. The proposed approach involves a radial input contraction Phase I and an additive slacks maximization Phase II. Each phase is first formulated as a fully fuzzy linear programming (FFLP) and transformed into a multiobjective optimization problem which is then solved using the lexicographic weighted Tchebychef method. The proposed fully fuzzy DEA approach provides, for each unit, a fuzzy efficiency measure and a fuzzy target operating point. A classification of the efficiency status of the units is also presented. Computational experiences and comparison with other fuzzy DEA approaches are reported. In the reviewer's opinion there are some concerns or questions which should be addressed by the authors:

1. *Linguistics, readability of the paper should be improved and the authors should re-structure the paper in order to have a smooth transition among the sections.*

We have checked the readability as well as the English along the whole paper, improving it and correcting some mistakes. To make the transition among different sections smoother, we have added some introductory/explanatory paragraphs at the beginning or at the end of the sections. Moreover, Section 3 has been split into two sections and each of the two phases of the proposed method is presented as an independent subsection within the new Section 4.

2. *The state-of-the-part should be presented more deeper and efficient. What is the additive value of the current study? Detailed contributions should be discussed in the introduction part.*

   The introduction section has been modified in order to detail the novelties of the present work. In this regard, now a new paragraph contains the main differences in our work concerning others existing in the literature, and another paragraph highlights the novel contributions, divided into four points.

3. *The introduction can be improved by addressing the main feature of the work; more explanation on the cited references with a highlight on the differences. In the context of fuzzy techniques, there are many existing results to be considered to enrich the reference body, see; reliable fuzzy tracking control of near-space hypersonic vehicle using aperiodic measurement information, reliable intelligent path following control for a robotic airship against sensor faults.*

   The introduction has been modified and improved with the inclusion of new paragraphs. The justification of using fuzzy sets has been addressed, as well as their applications, with the incorporation of some new references.

4. *The benefits of the proposed fuzzy DEA method have been demonstrated clearly. What's the limitation of the method? Are there other ways that the results can be further improved? One or two remarks should be given to discuss it in detail.*

   Thank you for the suggestion. The main limitations of the proposed approach as well as some possible extensions and continuations of this research are discussed in the conclusions section.

5. *Detailed discussions of the CCR model are helpful to illustrate the results clearly.*

   Thank you for the suggestion. We have explained the CCR model more in detail.

6. *In simulation part, more design parameters and comparisons with some existing results are recommended. The visualization of figures needs to be improved.*

As suggested by the referee, in the first example of the computational section, we have included the application of Hatami-Marbini et al. approach for comparison purposes. Including these results in Table 2, and adding a new figure, now Figure 2. Furthermore, to make it easier to understand the implementation of the proposed two-phase approach, together with all its design parameters, we have also included explicitly the computation of the two phases, step-by-step. The old Fig. 2 and Fig. 3 have been improved, in the sense of their visualization, and combined as a unique figure, the new Figure 3.

7. *According to the topic of the paper, the authors may propose some interesting problem as future work in conclusion.*

   As suggested, some interesting extensions of the proposed approach and possible continuations of this research have been presented in the conclusions section.

3 Reviewer 4

I have read this paper very carefully and I can say that it is well and very clearly written. The results new, interesting, nice and significance.

In my opinion the paper has good motivation. I consider the paper a useful contribution in the this field. Hence, I recommend it for publication in your journal.

We really appreciate and thank for your very positive opinion on our manuscript.

4 Reviewer 8

This paper proposed a new, two phase, radial, input oriented FFDEA approach, where trapezoidal fuzzy numbers was used . The novelty of the approach is the LU fuzzy partial order used. It has been shown that the proposed FFLP model, through the definition of the fuzzy Pareto solutions, can be transformed into a multiobjective optimization problem . Computational applications have been validated. The topic in this paper is interesting, and the derivations seem correct. This paper can be considered for publication but subject to some necessary revisions:

1. *The novelty of the proposed results should be further elaborated*

   The introduction section has been modified in order to detail the novelties of the present work. In this regard, now a new paragraph contains the main differences in our work concerning others existing in the literature, and another paragraph highlights the novel contributions, divided into four points.