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q-Rung Orthopair Probabilistic Hesitant Fuzzy Rough Aggregation Information and Their Application in Decision Making

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Abstract The core contribution of this study is to develop a novel generalized idea of q-rung orthopair probabilistic hesitant fuzzy rough set (q-ROPHFRS) which is hybrid structure of the q-rung orthopair fuzzy set, probabilistic hesitant fuzzy set, and rough set. The q-ROPHFRS covers the positive and negative membership grades in the form of probabilistic hesitant fuzzy rough information to address the uncertainties in real-world decision-making problems. This paper proposes a list of novel q-rung orthopair probabilistic hesitant fuzzy rough averaging/geometric aggregation operators to handle the uncertainty effectively and reliably to aggregate the uncertain information under q-ROPHFRSs. Several interesting elementary properties have been investigated. Furthermore, a novel multi-attribute decision-making approach based on the proposed aggregation information is presented. Finally, a numerical application regarding selecting a medical oxygen supplier to the hospital is presented to illustrate the consistency of the developed decision-making technique. The comparison of the proposed technique with q-rung orthopair probabilistic hesitant fuzzy rough TOPSIS shows that the proposed multi-criteria decision-making methodology is reliable and effective in addressing the uncertainty in real-world decision-making problems.

Keywords q-rung orthopair probabilistic hesitant fuzzy rough sets · q-rung orthopair probabilistic hesitant fuzzy rough aggregation operators · Multi expert decision making · Oxygen supplier selection

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1 Introduction

Multi-criteria decision making (MCDM) is a systematic approach to solve real-world problems that leads to the best solution after a comprehensive review of the available choices. The MCDM approaches have grown in popularity, and they are commonly used in a broad range of fields, including medical sciences, architecture, economics, and many other branches of science and technology. The MCDM approach has recently become more challenging due to the prevalence of complexity and ambiguity in collected information, making it difficult for decision-makers to make the right decisions. Conventional MCDM strategies were rendered ineffective in the presence of such uncertain and ambiguous results. Zadeh [1] explored a solution to such problems by establishing the foundations of fuzzy set (FS) theory, in which each element is characterized by a membership degree ranging between 0 to 1. Atanassov [2] extended the idea of FS into intuitionistic fuzzy set (IFS) by introducing a non membership degree ($\Psi_F(x)$) to the membership degree ($\eta_F(x)$) of FS, with the restriction that $\eta_F(x) + \Psi_F(x) \leq 1$.

Yager [3] introduced the Pythagorean fuzzy (PF) set theory, which relaxes the previously mentioned IFS condition to

$(\eta_F(x))^2 + (\Psi_F(x))^2 \leq 1$. The PF expressions are undoubtedly raising the interest of many researchers, especially in terms of their application to decision making (DM). For example, Huang et al. [4] described a PF MULTIMOORA approach that utilizes a novel distance measure and a score function. They used this approach to evaluate disk productions and energy projects. Akram et al. [5] investigated risk assessment in failure modes and effects analysis (FMEA) using a hybrid TOPSIS and ELECTRE I framework with PF information. Zhang and Xu [6] established the TOPSIS approach in a Pythagorean fuzzy environment and employed it to assess the efficiency of private airline services. For more detail decision making study, we refer to [7–14].

In our universe, hesitancy is a natural phenomenon. Identifying one of the better alternatives with the same characteristics in real life is difficult. Due to the uncertainty and hesitancy of the results, experts are having difficulty in making decisions. To tackle hesitancy, Torra and Narukawa [15] developed the concept of hesitant fuzzy set (HFS), which allows an element to be a set of multiple possible values. The HFS can be used to solve a variety of DM problems. Khan et al. [16] introduced the concept of Pythagorean HFS (PyHFS). They established an evaluation method and identified operators for data aggregation. Xu and Zhou [17] identified a novel concept of probabilistic HF sets (PHFSs). Yager [18] established a new idea called "q-rung orthopair fuzzy sets" (q-ROFSs), in which the qth power of membership and non-membership is restricted to 1, i.e., $(\eta_F(x))^q + (\Psi_F(x))^q \leq 1$, and demonstrated that the q-ROFS are more general than IFS and PFS. The q-ROFSs provide a broader range of fuzzy information and are more versatile and appropriate approach to deal with unpredictable situations. Yager and Alajlan [19] explored the fundamental properties of these q-ROFSs and discussed that how they can be used in information representation. Subsequently, the authors in [20] put forward the notation of q-rung orthopair HF set (q-ROHFS) and discussed the operational laws which exist for any two q-ROHFSs. Wang et al. [21] investigated the Heronian mean operators in MADM in a q-ROHFS framework. They also proposed the Hamacher norm based aggregation operators (AOp) under dual hesitant q-ROFSs and discussed their usefulness in DM problems. Wang et al., [22] established the AOp based on Muirhead mean under dual hesitant q-rung orthopair fuzzy information. Yang [23] measured the entropy for HF information using the Hausdorff metric and the structure of HF TOPSIS. The TOPSIS is a useful information analysis tool developed by Hwang and Yoon [24]. It investigates the appropriate approach in terms of relative closeness based on their distances from the positive ideal solution (PIS) and the negative ideal solution (NIS), ensuring that the shortest distance from PIS and the farthest distance from NIS are satisfied. This analysis method effectively eliminates decision information uncertainty while maintaining the validity and precision of decision making by simply measuring the distance between PIS and NIS and ranking them accordingly [25]. TOPSIS method is straightforward and simple to understand and analyze as compared to ELECTRE method, VIKOR method, and other conventional methods, so it has been extensively studied and implemented by researchers.

In recent years, several authors have presented TOPSIS in various fuzzy information, (see [26,27] for more information). Boran et al. [28] used TOPSIS to identify the best supplier by using IF information. Chen et al. [29] suggested the TOPSIS technique based on interval-valued fuzzy information and addressed the experimental results. Li [30] proposed a TOPSIS-based nonlinear-programming technique for MADM with interval-valued IFs in order to deal with uncertainty in real-world decision making problems. The TOPSIS model for DM problems in an interval-valued IF information was introduced by Park et al. [31]. The Dombi-based aggregation operators for PF information were formulated in [9]. Barukab et al. [32] proposed the extended fuzzy TOPSIS method for spherical fuzzy information, which is based on the entropy measure. However, there are many research findings in applying the fuzzy TOPSIS method to solve MADM problems, the decision information used by these approaches is too old and restricted to manage increasingly challenging decision environments.

Pawlak [33] is pioneer who studied the dominant concept of rough sets theory. The classical set theory which deals with inconsistent and imprecise information is extended by rough set theory. In recent decades, research on the rough set has progressed significantly, both in terms of theoretical implementations and theory itself. In recent decades, research has demonstrated TOPSIS technique in a number of rough sets information. Khan et al. [34] implemented a rough set strategy and the TOPSIS method for selection of sites for food distribution. The concept of rough sets has been extended by several researchers around the globe in different directions. Using the fuzzy relation rather than the crisp binary relation, Dubois et al. [35] initiated the notion of fuzzy rough sets. The hybrid structure of IFSs and rough sets, intuitionistic fuzzy rough (IFR) sets introduced by Cornelis et al., [36]. By utilizing IFR approximations, Zhou and Wu [37] established a novel decision making technique under IFR environment to address their constrictive and axiomatic analysis in detail. Zhan et al. [38] presented the DM methodology under intuitionistic fuzzy rough environment and explored their applications in real word problems. Chinram et al. [39] established the algebraic norm based AOPs based on EDAS technique under IFR information and discussed their applications in MAGDM.

The q-rung orthopair probabilistic hesitant fuzzy rough sets, a hybrid intelligent structure of rough sets, and q-ROPHFS is an advanced classification strategy that has attracted researchers to address ambiguous and incomplete data. From the analysis, it is concluded that in decision-making, AOp play a significant role in aggregating the collective data from different sources to a single value. In accordance with the best available knowledge to date, the development of AOp with the hybridization of q-ROPHFS with a rough set is not observed in the q-ROF setting. As a result, the current q-ROPHF rough structure is inspired, and we define a list of algebraic aggregation operators depending on rough data, such as q -rung orthopair probabilistic hesitant fuzzy weighted averaging, order weighted averaging, hybrid weighted averaging, weighted geometric, ordered weighted geometric and hybrid weighted geometric aggregation operators, under the algebraic t-norm and t-conorm.

The rest of the manuscript is organized as follows: Section 2 briefly retrospect some basic concepts of q-ROFSs, HFSs and rough set theory. A novel notion of q-rung orthopair hesitant fuzzy sets, q-rung orthopair probabilistic hesitant fuzzy sets and a list of algebraic q-rung orthopair probabilistic hesitant fuzzy aggregation operators to aggregate the uncertain information in decision-making are presented in Section 3. Section 4 is devoted to a decision-making methodology based on the developed aggregation operators and presents the numerical illustration to find out the best oxygen supplier to hospital. Section 5 presents the validity and reliability test to check the supremacy of the developed methodology and establishes the q-ROHFR-TOPSIS methodology to validate the proposed aggregation operators based multi attribute decision making methodology. Section 6 concludes the manuscript.

2 Fundamental Concepts

In this section, we sort out the elementary terminologies i.e., q-rung orthopair probabilistic hesitant fuzzy rough set (q-ROPHFRS).

Definition 1 [39] Let £ be the universal set and $\propto \in IFS(£\times £)$ be a IF relation. Then

(1) $\propto$ is reflexive if $\mu_{\propto}(\ell,\ell)=1$ and $\nu_{\propto}(\ell,\ell)=0, \forall \ell\in £$;
(2) $\propto$ is symmetric if $\forall(\ell,a)\in £\times £$, $\mu_{\propto}(\ell,a)=\mu_{\propto}(a,\ell)$ and $\nu_{\propto}(\ell,a)=\nu_{\propto}(a,\ell)$;
(3) $\propto$ is transitive if $\forall(\ell,b)\in £\times £$,

$$\mu_{\propto}(\ell,b)=\bigvee_{a\in £}\left[\mu_{\propto}(\ell,a)\wedge\mu_{\propto}(a,b)\right];$$

and

$$\nu_{\propto}(\ell,b)=\bigwedge_{a\in £}\left[\nu_{\propto}(\ell,a)\wedge\nu_{\propto}(a,b)\right].$$

Definition 2 [40] Let £ be the universal set then any subset $\propto\in q-ROPHFS(£\times £)$ is said to be a q-rung probabilistic hesitant fuzzy relation. The pair $(£,\propto)$ is called q-ROPHF approximation space. If for any $\kappa\subseteq q-ROPHFS(£)$, then the upper and lower approximations of κ with respect to q-ROPHF approximation space $(£,\propto)$ are two q-ROPHFSs, which are denoted by $\overline{\propto}(\kappa)$ and $\underline{\propto}(\kappa)$ and defined as:

$$\overline{\propto}(\kappa)=\left\{\left\langle \ell,\eta_{h_{\overline{\propto}(\kappa)}}(\ell)/\eth_{h_{\overline{\propto}(\kappa)}},\Psi_{h_{\overline{\propto}(\kappa)}}(\ell)/\partial_{h_{\overline{\propto}(\kappa)}}\right\rangle|\ell\in £\right\};$$
$$\underline{\propto}(\kappa)=\left\{\left\langle \ell,\eta_{h_{\underline{\propto}(\kappa)}}(\ell)/\eth_{h_{\underline{\propto}(\kappa)}},\Psi_{h_{\underline{\propto}(\kappa)}}(\ell)/\partial_{h_{\underline{\propto}(\kappa)}}\right\rangle|\ell\in £\right\};$$

where

$$\eta_{h_{\overline{\propto}(\kappa)}}(\ell)/\eth_{h_\ell}=\bigvee_{k\in £}\left[\eta_{h_{\propto}}(\ell,k)\bigvee\eta_{h_\kappa}(k)\right]/\bigvee_{k\in £}\left[\eth_{h_{\propto}}(\ell,c)\bigvee\eth_{h_\kappa}(k)\right];$$
$$\Psi_{h_{\overline{\propto}(\kappa)}}(\ell)=\bigwedge_{k\in £}\left[\Psi_{h_{\propto}}(\ell,k)\bigwedge\Psi_{h_\kappa}(k)\right]/\bigwedge_{k\in £}\left[\partial_{h_{\propto}}(\ell,c)\bigwedge\partial_{h_\kappa}(k)\right];$$
$$\eta_{h_{\underline{\propto}(\kappa)}}(\ell)=\bigwedge_{k\in £}\left[\eta_{h_{\propto}}(\ell,k)\bigwedge\eta_{h_\kappa}(k)\right]/\bigwedge_{k\in £}\left[\eth_{h_{\propto}}(\ell,k)\bigwedge\eth_{h_\kappa}(k)\right];$$
$$\Psi_{h_{\underline{\propto}(\kappa)}}(\ell)=\bigvee_{k\in £}\left[\Psi_{h_{\propto}}(\ell,k)\bigvee\Psi_{h_\kappa}(k)\right]/\bigvee_{k\in £}\left[\partial_{h_{\propto}}(\ell,c)\bigvee\partial_{h_\kappa}(k)\right];$$

such that $0\leq\left(\max(\eta_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q+\left(\min(\Psi_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q\leq 1$ and $0\leq\left(\min(\eta_{h_{\underline{\propto}(\kappa)}}(\ell)\right)^q+\left(\max(\Psi_{h_{\underline{\propto}(\kappa)}}(\ell))\right)^q\leq 1$. As $\left(\overline{\propto}(\kappa),\underline{\propto}(\kappa)\right)$ are $q-ROPHFSs$, so $\overline{\propto}(\kappa),\underline{\propto}(\kappa):q-ROPHFS(£)\to q-RFS(£)$ are upper and lower approximation operators. The pair

$$\propto(\kappa)=\left(\underline{\propto}(\kappa),\overline{\propto}(\kappa)\right)=\left\{\left\langle\begin{array}{l}\ell,\left(\eta_{h_{\underline{\propto}(\kappa)}}(\ell)/\eth_{h_{\underline{\propto}(\kappa)}},\Psi_{h_{\underline{\propto}(\kappa)}}(\ell)/\partial_{h_{\underline{\propto}(\kappa)}}\right),\\ \left(\eta_{h_{\overline{\propto}(\kappa)}}(\ell)/\eth_{h_{\overline{\propto}(\kappa)}},\Psi_{h_{\overline{\propto}(\kappa)}}(\ell)/\partial_{h_{\overline{\propto}(\kappa)}}\right)\end{array}\right\rangle|\ell\in\kappa\right\}$$

will be called q-rung orthopair hesitent fuzzy rough set. For simplicity $\propto(\kappa)=\left\{\left\langle\ell,\left(\eta_{h_{\underline{\propto}(\kappa)}}(\ell)//\eth_{h_{\underline{\propto}(\kappa)}},\Psi_{h_{\underline{\propto}(\kappa)}}(\ell)/\partial_{h_{\underline{\propto}(\kappa)}}\right),\left(\eta_{h_{\overline{\propto}(\kappa)}}(\ell)//\eth_{h_{\overline{\propto}(\kappa)}},\Psi_{h_{\overline{\propto}(\kappa)}}(\ell)/\partial_{h_{\overline{\propto}(\kappa)}}\right)\right\rangle|\ell\in\kappa\right\}$ is represented as $\propto(\kappa)=\left((\underline{\xi}/\underline{\eth},\underline{\eta}/\underline{\partial}),(\overline{\eta}/\overline{\eth},\overline{\Psi}/\overline{\partial})\right)$ and is known as q-ROPHFRV.

Definition 3 Let $\propto(\kappa_1)=(\underline{\propto}(\kappa_1),\overline{\propto}(\kappa_1))$ and $\propto(\kappa_2)=(\underline{\propto}(\kappa_2),\overline{\propto}(\kappa_2))$ be two q-ROPHFRSs. Then

(1) $\propto(\kappa_1)\cup\propto(\kappa_2)=\{(\underline{\propto}(\kappa_1)\cup\underline{\propto}(\kappa_2)),(\overline{\propto}(\kappa_1)\cup\overline{\propto}(\kappa_2))\}$

(2) $\propto(\kappa_1) \cap \propto(\kappa_2) = \{(\underline{\propto}(\kappa_1) \cap \underline{\propto}(\kappa_2)), (\overline{\propto}(\kappa_1) \cap \overline{\propto}(\kappa_2))\}$.

Definition 4 The score function for q-ROPHFRV $\propto(\kappa) = (\underline{\propto}(\kappa), \overline{\propto}(\kappa)) = ((\underline{\xi}/\underline{\eth}, \underline{\eta}/\underline{\partial}), (\overline{\eta}/\overline{\eth}, \overline{\Psi}/\overline{\partial}))$ is given as:

$$\Delta(\propto(\kappa)) = \frac{1}{4}\left(\begin{array}{l} 2 + \frac{1}{M_{\mathcal{F}}} \sum\limits_{\underline{\mu_{h_i}} \in \eta_{h_{\underline{\propto}(\kappa)}}, \underline{\eth_i} \in \eth_{h_{\underline{\propto}(\kappa)}}} \left\{\underline{\mu_{h_i}} \times \underline{\eth_i}\right\} + \\ \frac{1}{N_{\mathcal{F}}} \sum\limits_{\overline{\mu_{h_i}} \in \eta_{h_{\overline{\propto}(\kappa)}}, \overline{\eth_{h_i}} \in \eth_{h_{\overline{\propto}(\kappa)}}} \left\{\overline{\mu_{h_i}} \times \overline{\eth_{h_i}}\right\} - \\ \frac{1}{M_{\mathcal{F}}} \sum\limits_{\underline{\nu_{h_i}} \in \Psi_{h_{\underline{\propto}(\kappa)}}, \underline{\partial_i} \in \partial_{h_{\underline{\propto}(\kappa)}}} (\underline{\nu_{h_i}} \times \underline{\partial_{h_i}}) - \\ \frac{1}{M_{\mathcal{F}}} \sum\limits_{\overline{\nu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\partial_{h_i}} \in \partial_{h_{\overline{\propto}(\kappa)}}} (\overline{\nu_{h_i}} \times \overline{\partial_{h_i}}) \end{array}\right),$$

The accuracy function for q-ROPHFRV $\propto(\kappa) = (\underline{\propto}(\kappa), \overline{\propto}(\kappa)) = ((\underline{\xi}/\underline{\eth}, \underline{\eta}/\underline{\partial}), (\overline{\eta}/\overline{\eth}, \overline{\Psi}/\overline{\partial}))$ is given as:

$$\mathbf{AC} \propto(\kappa) = \frac{1}{4}\left(\begin{array}{l} \frac{1}{M_{\mathcal{F}}} \sum\limits_{\underline{\mu_{h_i}} \in \eta_{h_{\underline{\propto}(\kappa)}}, \underline{\eth_i} \in \eth_{h_{\underline{\propto}(\kappa)}}} (\underline{\mu_{h_i}} \times \underline{\eth_i}) + \\ \frac{1}{M_{\mathcal{F}}} \sum\limits_{\overline{\mu_{h_i}} \in \eta_{h_{\overline{\propto}(\kappa)}}, \overline{\eth_{h_i}} \in \eth_{h_{\overline{\propto}(\kappa)}}} (\overline{\mu_{h_i}} \times \overline{\eth_{h_i}}) + \\ \frac{1}{M_{\mathcal{F}}} \sum\limits_{\underline{\nu_{h_i}} \in \Psi_{h_{\underline{\propto}(\kappa)}}, \underline{\partial_i} \in \partial_{h_{\underline{\propto}(\kappa)}}} (\underline{\nu_{h_i}} \times \underline{\partial_{h_i}}) + \\ \frac{1}{M_{\mathcal{F}}} \sum\limits_{\overline{\nu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\partial_{h_i}} \in \partial_{h_{\overline{\propto}(\kappa)}}} (\overline{\nu_{h_i}} \times \overline{\partial_{h_i}}) \end{array}\right),$$

where $M_{\mathcal{F}}$ and $N_{\mathcal{F}}$ represent the number of elements in η_{h_g} and Ψ_{h_g} respectively.

Definition 5 Suppose $\propto(\kappa_1) = (\underline{\propto}(\kappa_1), \overline{\propto}(\kappa_1))$ and $\propto(\kappa_2) = (\underline{\propto}(\kappa_2), \overline{\propto}(\kappa_2))$ are two q-ROPHFRVs. Then

(1) If $\Delta(\propto(\kappa_1)) > \Delta(\propto(\kappa_2))$, then $\propto(\kappa_1) > \propto(\kappa_2)$,
(2) If $\Delta(\propto(\kappa_1)) \prec \Delta(\propto(\kappa_2))$, then $\propto(\kappa_1) \prec \propto(\kappa_2)$,
(3) If $\Delta(\propto(\kappa_1)) = \Delta(\propto(\kappa_2))$, then

(a) If $\mathbf{AC} \propto(\kappa_1) > \mathbf{AC} \propto(\kappa_2)$ then $\propto(\kappa_1) > \propto(\kappa_2)$,
(b) If $\mathbf{AC} \propto(\kappa_1) \prec \mathbf{AC} \propto(\kappa_2)$ then $\propto(\kappa_1) \prec \propto(\kappa_2)$,
(c) If $\mathbf{AC} \propto(\kappa_1) = \mathbf{AC} \propto(\kappa_2)$ then $\propto(\kappa_1) = \propto(\kappa_2)$.

3 *q*-Rung Orthopair Probabilistic Hesitant Fuzzy Rough Aggregation Operators

Herein, we initiate a new idea of q-ROPHF rough aggregation operators by embedding the notions of rough sets and q-ROPHF aggregation operators to get aggregation concepts of q-ROPHFRWA, q-ROPHFROWA and q-ROPHFRHWA operators. Some fundamental properties of these notions are discussed.

3.1 *q*-Rung Orthopair Probabilistic Hesitant Fuzzy Rough Weighted Averaging Operator

Definition 6 Consider the collection $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))$ $(i = 1, 2, 3, 4, ..., n)$ of q-ROPHFRVs with weight vector $w = (w_1, w_2, ..., w_n)^T$ such that $\oplus_{i=1}^n w_i = 1$ and $0 \le w_i \le 1$ and probabilistic terms $\eth_{h_i}$ and ∂_i such that $\oplus_{i=1}^n \eth_{h_i} = 1$ and $\oplus_{i=1}^n \partial_i = 1$. The q-ROPHFRWA operator is determined as

$$q - ROPHFRWA(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) = \left(\oplus_{i=1}^n w_i \underline{\propto}(\kappa_i), \oplus_{i=1}^n w_i \overline{\propto}(\kappa_i)\right).$$

Theorem 1 *Let* $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))$ $(i = 1, 2, 3, 4, ..., n)$ *be the collection of q-ROPHFRVs with weight vector* $w = (w_1, w_2, ..., w_n)^T$. *Then the q-ROPHFRWA operator is defined as;*

$$\begin{aligned} & q - ROPHFRWA(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) \\ & = \left(\oplus_{i=1}^n w_i \underline{\propto}(\kappa_i), \oplus_{i=1}^n w_i \overline{\propto}(\kappa_i)\right) \\ & = \left[\begin{array}{l} \left(\begin{array}{l} \bigcup\limits_{\underline{\mu_{h_i}} \in \eta_{h_{\underline{\propto}(\kappa)}}, \underline{\eth_{hi}} \in \eth_{h_{\underline{\propto}(\kappa)}}} \left(\sqrt[q]{\left(1 - \bigotimes\limits_{i=1}^n \left(1 - \left(\underline{\mu_{h_i}}\right)^q\right)^{w_i}\right)} \Big/ \bigotimes\limits_{i=1}^n \underline{\eth}_{h_i}\right), \\ \bigcup\limits_{\underline{\nu_{h_i}} \in \Psi_{h_{\underline{\propto}(\kappa)}}, \underline{\partial_{h_i}} \in \partial_{h_{\underline{\propto}(\kappa)}}} \left(\bigotimes\limits_{i=1}^n \left(\underline{\nu_{h_i}}\right)^{w_i} \Big/ \bigotimes\limits_{i=1}^n \underline{\partial}_{h_i}\right) \end{array}\right) \\ \left(\begin{array}{l} \bigcup\limits_{\overline{\mu_{h_i}} \in \eta_{h_{\overline{\propto}(\kappa)}}, \overline{\eth_{h_i}} \in \eth_{h_{\overline{\propto}(\kappa)}}} \left(\sqrt[q]{\left(1 - \bigotimes\limits_{i=1}^n \left(1 - \left(\overline{\mu_{h_i}}\right)^q\right)^{w_i}\right)} \Big/ \bigotimes\limits_{i=1}^n \overline{\eth}_{h_i}\right), \\ \bigcup\limits_{\overline{\nu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\partial_{h_i}} \in \partial_{h_{\overline{\propto}(\kappa)}}} \left(\bigotimes\limits_{i=1}^n \left(\overline{\nu_{h_i}}\right)^{w_i} \Big/ \bigotimes\limits_{i=1}^n \overline{\partial}_{h_i}\right) \end{array}\right) \end{array}\right] \end{aligned}$$

Proof Utilizing mathematical induction to establish the proof. Using the operational law, it follows that

$$\propto(\kappa_1) \oplus \propto(\kappa_2) = \left[\underline{\propto}(\kappa_1) \oplus \underline{\propto}(\kappa_2), \overline{\propto}(\kappa_1) \oplus \overline{\propto}(\kappa_2)\right]$$

and

$$\gamma \propto(\kappa_1) = \left(\gamma \underline{\propto}(\kappa_1), \gamma \overline{\propto}(\kappa_1)\right)$$

If $n = 2$, then

$$
\begin{aligned}
&q-ROPHFRWA\left(\propto(\kappa_1),\propto(\kappa_2)\right)=\left(\oplus_{i=1}^{2}w_i\underline{\propto}(\kappa_i),\oplus_{i=1}^{2}w_i\overline{\propto}(\kappa_i)\right)\\
&=\begin{pmatrix}
\begin{pmatrix}
\bigcup\limits_{\underline{\mu_{h_i}}\in\eta_{h_{\underline{\propto}(\kappa)}},\underline{\eth_{h_i}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{2}\left(1-\left(\underline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{2}\underline{\eth_{h_i}}\right),\\
\bigcup\limits_{\underline{\nu_{h_i}}\in\Psi_{h_{\underline{\propto}(\kappa)}},\underline{\partial_{h_i}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{2}\left(\underline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{2}\underline{\partial_{h_i}}\right)
\end{pmatrix}\\
\begin{pmatrix}
\bigcup\limits_{\overline{\mu_{h_i}}\in\eta_{h_{\overline{\propto}(\kappa)}},\overline{\eth_{h_i}}\in\eth_{h_{\overline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{2}\left(1-\left(\overline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{2}\overline{\eth_{h_i}}\right),\\
\bigcup\limits_{\overline{\nu_{h_i}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\partial_{h_i}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{2}\left(\overline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{2}\overline{\partial_{h_i}}\right)
\end{pmatrix}
\end{pmatrix}
\end{aligned}
$$

Hence the result is true for $n=2$. Let it is true for $n=k$, that is,

$$
\begin{aligned}
&q-ROPHFRWA\left(\propto(\kappa_1),\propto(\kappa_2),\ldots\propto(\kappa_k)\right)\\
&=\left(\oplus_{i=1}^{k}w_i\underline{\propto}(\kappa_i),\oplus_{i=1}^{k}w_i\overline{\propto}(\kappa_i)\right)\\
&=\begin{pmatrix}
\begin{pmatrix}
\bigcup\limits_{\underline{\mu_{h_i}}\in\eta_{h_{\underline{\propto}(\kappa)}},\underline{\eth_{h_i}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k}\left(1-\left(\underline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k}\underline{\eth_{h_i}}\right),\\
\bigcup\limits_{\underline{\nu_{h_i}}\in\Psi_{h_{\underline{\propto}(\kappa)}},\underline{\partial_{h_i}}\in\partial_{h_{\underline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k}\left(\underline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k}\underline{\partial_{h_i}}\right)
\end{pmatrix}\\
\begin{pmatrix}
\bigcup\limits_{\overline{\mu_{h_i}}\in\eta_{h_{\overline{\propto}(\kappa)}},\overline{\eth_{h_i}}\in\eth_{h_{\overline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k}\left(1-\left(\overline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k}\overline{\eth_{h_i}}\right),\\
\bigcup\limits_{\overline{\nu_{h_i}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\partial_{h_i}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k}\left(\overline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k}\overline{\partial_{h_i}}\right)
\end{pmatrix}
\end{pmatrix}
\end{aligned}
$$

Now, we have to show that it is true for $n=k+1$, we have

$$
\begin{aligned}
&q-ROPHFRWA\left(\propto(\kappa_1),\propto(\kappa_2),\ldots\propto(\kappa_{k+1})\right)\\
&=\begin{pmatrix}\left(\oplus_{i=1}^{k}w_i\underline{\propto}(\kappa_i)\oplus w_{k+1}\underline{\propto}(\kappa_{k+1})\right),\\ \left(\oplus_{i=1}^{k}w_i\overline{\propto}(\kappa_i)\oplus w_{k+1}\overline{\propto}(\kappa_{k+1})\right)\end{pmatrix}\\
&=\begin{pmatrix}
\begin{pmatrix}
\bigcup\limits_{\underline{\mu_{h_i}}\in\eta_{h_{\underline{\propto}(\kappa)}},\underline{\eth_{h_i}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k+1}\left(1-\left(\underline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k+1}\underline{\eth_{h_i}}\right),\\
\bigcup\limits_{\underline{\nu_{h_i}}\in\Psi_{h_{\underline{\propto}(\kappa)}},\underline{\partial_{h_i}}\in\partial_{h_{\underline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k+1}\left(\underline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k+1}\underline{\partial_{h_i}}\right)
\end{pmatrix}\\
\begin{pmatrix}
\bigcup\limits_{\overline{\mu_{h_i}}\in\eta_{h_{\overline{\propto}(\kappa)}},\overline{\eth_{h_i}}\in\eth_{h_{\overline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k+1}\left(1-\left(\overline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k+1}\overline{\eth_{h_i}}\right),\\
\bigcup\limits_{\overline{\nu_{h_i}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\partial_{h_i}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k+1}\left(\overline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k+1}\overline{\partial_{h_i}}\right)
\end{pmatrix}
\end{pmatrix}.
\end{aligned}
$$

Thus the required result is true for $n=k+1$. Hence, the result is true for all $n\geq 1$.

From the above analysis $\underline{\propto}(\kappa)$ and $\overline{\propto}(\kappa)$ are q-ROPHFRVs. So, $\oplus_{i=1}^{k}w_i\underline{\propto}(\kappa_i)$ and $\oplus_{i=1}^{k}w_i\overline{\propto}(\kappa_i)$ are also q-ROPHFRVs. Therefore, q-ROPHFRWA $(\propto(\kappa_1),\propto(\kappa_2),\ldots\propto(\kappa_n))$ is a q-ROPHFRV under q-ROPHF approximation space $(£,\propto)$. □

Definition 7 Consider the collection $\propto(\kappa_i)=(\underline{\propto}(\kappa_i),\overline{\propto}(\kappa_i))$ $(i=1,2,3,4,\ldots,n)$ of q-ROPHFRVs with weight vector $w=(w_1,w_2,\ldots,w_n)^T$ such that $\oplus_{i=1}^{n}w_i=1$ and $0\leq w_i\leq 1$. Also a probabilistic terms $\eth_{h_i}$ and ∂_{h_i} such that $\oplus_{i=1}^{n}\eth_{h_i}=1$ and $\oplus_{i=1}^{n}\partial_{h_i}=1$ with the property that $0\leq\eth_{h_i},\partial_{h_i}\leq 1$. The q-ROHFROWA operator is determined as:

$$
\begin{aligned}
&q-ROHFROWA\left(\propto(\kappa_1),\propto(\kappa_2),\ldots,\propto(\kappa_n)\right)\\
&=\left(\oplus_{i=1}^{n}w_i\underline{\propto}(\kappa_i),\oplus_{i=1}^{n}w_i\overline{\propto_{\Omega_i}}(\kappa_i)\right).\\
&\begin{pmatrix}
\begin{pmatrix}
\bigcup\limits_{\underline{\mu_{h_i}}\in\eta_{h_{\underline{\propto}(\kappa)}},\underline{\eth_{h_i}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k+1}\left(1-\left(\underline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k+1}\underline{\eth_{h_i}}\right),\\
\bigcup\limits_{\underline{\nu_{h_i}}\in\Psi_{h_{\underline{\propto}(\kappa)}},\underline{\partial_{h_i}}\in\partial_{h_{\underline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k+1}\left(\underline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k+1}\underline{\partial_{h_i}}\right)
\end{pmatrix}\\
\begin{pmatrix}
\bigcup\limits_{\overline{\mu_{h_i}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\eth_{h_i}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{k+1}\left(1-\left(\overline{\mu_{h_i}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{k+1}\overline{\eth_{h_i}}\right),\\
\bigcup\limits_{\overline{\nu_{h_i}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\partial_{h_i}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{k+1}\left(\overline{\nu_{h_i}}\right)^{w_i}/\bigotimes_{i=1}^{k+1}\overline{\partial_{h_i}}\right)
\end{pmatrix}
\end{pmatrix}
\end{aligned}
$$

Theorem 2 *Let $\propto(\kappa_i)=(\underline{\propto}(\kappa_i),\overline{\propto}(\kappa_i))$ $(i=1,2,3,\ldots,n)$ be the collection of q-ROPHFRVs with weights vector $w=(w_1,w_2,\ldots,w_n)^T$. Then q-ROHFROWA operator is defined as*

$$
\begin{aligned}
&q-ROHFROWA\left(\propto(\kappa_1),\propto(\kappa_2),\ldots,\propto(\kappa_n)\right)\\
&=\left(\oplus_{i=1}^{n}w_i\underline{\propto}(\kappa_i),\oplus_{i=1}^{n}w_i\overline{\propto_{\Omega_i}}(\kappa_i)\right)\\
&=\begin{bmatrix}
\begin{pmatrix}
\bigcup\limits_{\underline{\mu_{h_{\Omega_i}}}\in\eta_{h_{\underline{\propto}(\kappa)}},\underline{\eth_{h_{\Omega_i}}}\in\eth_{h_{\underline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{n}\left(1-\left(\underline{\mu_{h_{\Omega_i}}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{n}\underline{\eth_{h_{\Omega_i}}}\right),\\
\bigcup\limits_{\underline{\nu_{h_{\Omega_i}}}\in\Psi_{h_{\underline{\propto}(\kappa)}},\underline{\partial_{h_{\Omega_i}}}\in\partial_{h_{\underline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{n}\left(\underline{\nu_{h_{\Omega_i}}}\right)^{w_i}/\bigotimes_{i=1}^{n}\underline{\partial_{h_{\Omega_i}}}\right)
\end{pmatrix}\\
\begin{pmatrix}
\bigcup\limits_{\overline{\mu_{h_{\Omega_i}}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\eth_{h_{\Omega_i}}}\in\eth_{h_{\overline{\propto}(\kappa)}}}\left(\sqrt[q]{\left(1-\bigotimes_{i=1}^{n}\left(1-\left(\overline{\mu_{h_{\Omega_i}}}\right)^q\right)^{w_i}\right)}/\bigotimes_{i=1}^{n}\overline{\eth_{h_{\Omega_i}}}\right),\\
\bigcup\limits_{\overline{\nu_{h_{\Omega_i}}}\in\Psi_{h_{\overline{\propto}(\kappa)}},\overline{\partial_{h_{\Omega_i}}}\in\partial_{h_{\overline{\propto}(\kappa)}}}\left(\bigotimes_{i=1}^{n}\left(\overline{\nu_{h_{\Omega_i}}}\right)^{w_i}/\bigotimes_{i=1}^{n}\overline{\partial_{h_{\Omega_i}}}\right)
\end{pmatrix}
\end{bmatrix},
\end{aligned}
$$

where $\propto_{\Omega}(\kappa_i)=(\underline{\propto}(\kappa_i),\overline{\propto_{\Omega_i}}(\kappa_i))$ depicts the superior value of permutation from the collection of q-ROPHFRVs.

Proof The proof is similar to the proof of Theorem 1. □

3.2 q-Rung Orthopair Probabilistic Hesitant Fuzzy Rough Geometric Aggregation Operator

In this section, we discuss q-ROPHFR geometric aggregation operator by employing the idea of rough sets into q-ROPHF geometric operators. The important characteristics of the developed operators are illustrated.

Definition 8 Let $\propto(\kappa_i)=(\underline{\propto}(\kappa_i),\overline{\propto}(\kappa_i))$ $(i=1,2,3,4,\ldots,n)$ be the collection of q-ROPHFRVs with weight vector $w=(w_1,w_2,\ldots,w_n)^T$ such that $\oplus_{i=1}^{n}w_i=1$ and $0\leq w_i\leq 1$ and probabilistic terms $\eth_{h_i}$ and ∂_i such that $\oplus_{i=1}^{n}\eth_{h_i}=1$ and $\oplus_{i=1}^{n}\partial_i=1$. Then q-ROPHFRWGA operator is determined as:

$$
\begin{aligned}
&q-ROPHFRWG\left(\propto(\kappa_1),\propto(\kappa_2),\ldots,\propto(\kappa_n)\right)\\
&=\left(\oplus_{i=1}^{n}\left(\underline{\propto}(\kappa_i)\right)^{w_i},\oplus_{i=1}^{n}\left(\overline{\propto}(\kappa_i)\right)^{w_i}\right).
\end{aligned}
$$

Based on the aforementioned definition the aggregated result for q-ROPHFRWGA operator is presented in the following theorem.

Theorem 3 *Let* $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))\,(i = 1, 2, 3, ..., n)$ *be the collection of q-ROPHFRVs with weight vector* $w = (w_1, w_2, ..., w_n)^T$ *such that* $\oplus_{i=1}^{n} w_i = 1$ *and* $0 \le w_i \le 1$. *Then q-ROPHFRWG operator is described as:*

$$
\begin{aligned}
& q-ROPHFRWG\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) \\
& = \left(\oplus_{i=1}^{n}\left(\underline{\propto}(\kappa_i)\right)^{w_i}, \oplus_{i=1}^{n}\left(\overline{\propto}(\kappa_i)\right)^{w_i}\right) \\
& = \left(\begin{array}{c}
\left\{\begin{array}{c}
\bigcup\limits_{\underline{\mu_{h_i}} \in \eta_{h_{\underline{\propto}(\kappa)}}, \underline{\eth_{h_i}} \in \eth_{h_{\underline{\propto}(\kappa)}}} \left(\bigotimes_{i=1}^{n}\left(\underline{\mu_{h_i}}\right)^{w_i} / \bigotimes_{i=1}^{n} \underline{\eth_{h_i}}\right) \\
\bigcup\limits_{\underline{\nu_{h_i}} \in \Psi_{h_{\underline{\propto}(\kappa)}}, \underline{\partial_{h_i}} \in \partial_{h_{\underline{\propto}(\kappa)}}} \sqrt[q]{1 - \bigotimes_{i=1}^{n}\left(1 - \left(\underline{\nu_{h_i}}\right)^{q}\right)^{w_i}} / \bigotimes_{i=1}^{n} \underline{\partial_{h_i}}
\end{array}\right\}, \\
\left\{\begin{array}{c}
\bigcup\limits_{\overline{\mu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\eth_{h_i}} \in \eth_{h_{\overline{\propto}(\kappa)}}} \bigotimes_{i=1}^{n}\left(\overline{\mu_{h_i}}\right)^{w_i} / \bigotimes_{i=1}^{n} \overline{\eth_{h_i}} \\
\bigcup\limits_{\overline{\nu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\partial_{h_i}} \in \partial_{h_{\overline{\propto}(\kappa)}}} \sqrt[q]{1 - \bigotimes_{i=1}^{n}\left(1 - \left(\overline{\nu_{h_i}}\right)^{q}\right)^{w_i}} / \bigotimes_{i=1}^{n} \overline{\partial_{h_i}}
\end{array}\right\},
\end{array}\right).
\end{aligned}
$$

Proof The proof is similar to the proof of Theorem 1. □

Definition 9 Let $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))\,(i = 1, 2, 3, ..., n)$ be the collection of q-ROPHFRVs with weight vector $w = (w_1, w_2, ..., w_n)^T$ such that $\oplus_{i=1}^{n} w_i = 1$ and $0 \le w_i \le 1$ and probabilistic terms $\eth_{h_i}$ and ∂_i such that $\oplus_{i=1}^{n} \eth_{h_i} = 1$ and $\oplus_{i=1}^{n} \partial_i = 1$. Then q-ROPHFROWG operator is described as:

$$
\begin{aligned}
& q-ROPHFROWG\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) \\
& = \left(\oplus_{i=1}^{n}\left(\underline{\propto_{\Omega}}(\kappa_i)\right)^{w_i}, \oplus_{i=1}^{n}\left(\overline{\propto_{\Omega}}(\kappa_i)\right)^{w_i}\right).
\end{aligned}
$$

Theorem 4 *Let* $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))\,(i = 1, 2, 3, ..., n)$ *be the collection of q-ROPHFRVs with weight vector' $w = (w_1, w_2, ..., w_n)^T$ such that $\oplus_{i=1}^{n} w_i = 1$ and $0 \le w_i \le 1$ and probabilistic terms $\eth_{h_i}$ and ∂_i such that $\oplus_{i=1}^{n} \eth_{h_i} = 1$ and $\oplus_{i=1}^{n} \partial_i = 1$. Then q-ROPHFROWG operator is a mapping defined as:*

$$
\begin{aligned}
& q-ROPHFROWG\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) \\
& = \left(\oplus_{i=1}^{n}\left(\underline{\propto_{\Omega}}(\kappa_i)\right)^{w_i}, \oplus_{i=1}^{n}\left(\overline{\propto_{\Omega}}(\kappa_i)\right)^{w_i}\right) \\
& = \left(\begin{array}{c}
\left\{\begin{array}{c}
\bigcup\limits_{\underline{\mu_{h_{\Omega_i}}} \in \eta_{h_{\underline{\propto}(\kappa)}}, \underline{\eth_{h_{\Omega_i}}} \in \eth_{h_{\underline{\propto}(\kappa)}}} \bigotimes_{i=1}^{n}\left(\underline{\mu_{h_{\Omega_i}}}\right)^{w_i} / \bigotimes_{i=1}^{n} \underline{\eth_{h_{\Omega_i}}}, \\
\bigcup\limits_{\underline{\nu_{h_{\Omega_i}}} \in \Psi_{h_{\underline{\propto}(\kappa)}}, \underline{\partial_{h_{\Omega_i}}} \in \partial_{h_{\underline{\propto}(\kappa)}}} \sqrt[q]{1 - \bigotimes_{i=1}^{n}\left(1 - \left(\underline{\nu_{h_{\Omega_i}}}\right)^{q}\right)^{w_i}} / \bigotimes_{i=1}^{n} \underline{\partial_{h_{\Omega_i}}}
\end{array}\right\} \\
\left\{\begin{array}{c}
\bigcup\limits_{\overline{\mu_{h_{\Omega_i}}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\eth_{h_{\Omega_i}}} \in \eth_{h_{\overline{\propto}(\kappa)}}} \bigotimes_{i=1}^{n}\left(\overline{\mu_{h_{\Omega_i}}}\right)^{w_i} / \bigotimes_{i=1}^{n} \overline{\eth_{h_{\Omega_i}}}, \\
\bigcup\limits_{\overline{\nu_{h_i}} \in \Psi_{h_{\overline{\propto}(\kappa)}}, \overline{\partial_{h_i}} \in \partial_{h_{\overline{\propto}(\kappa)}}} \sqrt[q]{1 - \bigotimes_{i=1}^{n}\left(1 - \left(\overline{\nu_{h_{\Omega_i}}}\right)^{q}\right)^{w_i}} / \bigotimes_{i=1}^{n} \overline{\partial_{h_{\Omega_i}}}
\end{array}\right\}
\end{array}\right),
\end{aligned}
$$

where $\propto_{\Omega}(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto_{\Omega_i}}(\kappa_i))$ *depicts the superior value of permutation from the collection of q-ROPHFRVs.*

Proof The proof is similar to the proof of Theorem 1. □

All the proposed aggregation operators satisfy the following properties, which are as follows:

Theorem 5 *Consider the collection* $\propto(\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))$ $(i = 1, 2, 3, 4, ..., n)$ *of q-ROPHFRVs with weight vector* $w = (w_1, w_2, ..., w_n)^T$ *such that* $\oplus_{i=1}^{n} w_i = 1$ *and* $0 \le w_i \le 1$. *Also a probabilistic terms* $\eth_{h_i}$ *and* ∂_i *such that* $\oplus_{i=1}^{n} \eth_{h_i} = 1$ *and* $\oplus_{i=1}^{n} \partial_{h_i} = 1$ *with the property that* $0 \le \eth_{h_i}, \partial_{h_i} \le 1$. *Then q-ROPHFRW/G operator satisfy the following properties:*

(1) Idempotency: *If* $\propto(\kappa_i) = \mathrm{F}(\kappa)$ *for* $\left(i = 1, 2, 3, 4, ..., n\right)$, *where* $\mathrm{F}(\kappa) = \left(\underline{\mathrm{F}}(\kappa), \overline{\mathrm{F}}(\kappa)\right) = \left((\underline{b_{h(x)}}/\underline{\eth_{h(x)}}, \underline{d_{h(x)}}/\underline{\partial_{h(x)}}), (\overline{b_{h(x)}}/\overline{\eth_{h(x)}}, \overline{d_{h(x)}}/\overline{\partial_{h(x)}})\right)$, *then*

$$
q-ROPHFRWA/G\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)) = \mathrm{F}(\kappa).
$$

(2) Boundedness: *Let* $(\propto(\kappa))^{-} = \left(\min\limits_{i} \underline{\propto}(\kappa_i), \max\limits_{i} \overline{\propto}(\kappa_i)\right)$ *and* $(\propto(\kappa))^{+} = \left(\max\limits_{i} \underline{\propto}(\kappa_i), \min\limits_{i} \overline{\propto}(\kappa_i)\right)$. *Then*

$$
\begin{aligned}
(\propto(\kappa))^{-} \le q-ROPHFRWA/G\Big(& \propto(\kappa_1), \propto(\kappa_2), ..., \\
& \propto(\kappa_n)\Big) \le (\propto(\kappa))^{+}.
\end{aligned}
$$

(3) Monotonicity: *Suppose* $\mathrm{F}(\kappa) = \left(\underline{\mathrm{F}}(\kappa_i), \overline{\mathrm{F}}(\kappa_i)\right)(i = i, 2, ..., n)$ *is another collection of q-ROPHFRVs such that* $\underline{\mathrm{F}}(\kappa_i) \le \underline{\propto}(\kappa_i)$ *and* $\overline{\mathrm{F}}(\kappa_i) \le \overline{\propto}(\kappa_i)$. *Then*

$$
\begin{aligned}
& q-ROPHFRWA/G\,(\mathrm{F}(\kappa_1), \mathrm{F}(\kappa_2), ..., \mathrm{F}(\kappa_n)) \\
& \le q-ROPHFRWA/G\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)).
\end{aligned}
$$

(4) Shiftinvariance: *Consider another q-ROPHFRV* $\mathrm{F}(\kappa) = \left(\underline{\mathrm{F}}(\kappa), \overline{\mathrm{F}}(\kappa)\right) = \left((\underline{b_{h(x)}}/\underline{\eth_{h(x)}}, \underline{d_{h(x)}}/\underline{\partial_{h(x)}}), (\overline{b_{h(x)}}/\overline{\eth_{h(x)}}, \overline{d_{h(x)}}/\overline{\partial_{h(x)}})\right)$. *Then*

$$
\begin{aligned}
& q-ROPHFRWA/G\Big(\propto(\kappa_1) \oplus \mathrm{F}(\kappa), \propto(\kappa_2) \oplus \mathrm{F}(\kappa) \\
& \quad , ..., \propto(\kappa_n) \oplus \mathrm{F}(\kappa)\Big) = q \\
& -ROPHFRWA/G\Big(\propto(\kappa_1), \propto(\kappa_2), ..., \\
& \quad \propto(\kappa_n)\Big) \oplus \mathrm{F}(\kappa).
\end{aligned}
$$

(5) Homogeneity: *For any real number* $\gamma > 0$;

$$
\begin{aligned}
& q-ROPHFRWA/G\,(\gamma \propto(\kappa_1), \gamma \propto(\kappa_2), ..., \gamma \propto(\kappa_n)) \\
& = \gamma \cdot q-ROPHFRWA/G\,(\propto(\kappa_1), \propto(\kappa_2), ..., \propto(\kappa_n)).
\end{aligned}
$$

(6) Commutativity: *Suppose* $\propto' (\kappa_i) = \left(\underline{\propto}(\kappa_i), \overline{\propto'}(\kappa_i)\right)$ *and* $\propto (\kappa_i) = (\underline{\propto}(\kappa_i), \overline{\propto}(\kappa_i))$, $(i = 1, 2, 3, 4, ..., n)$ *is a collection of q-ROPHFRVs. Then*

$$q - ROPHFRWA/G\,(\propto (\kappa_1), \propto (\kappa_2), ..., \propto (\kappa_n))$$
$$= q - ROPHFRWA/G\left(\propto' (\kappa_1), \propto' (\kappa_2), ..., \propto' (\kappa_n)\right).$$

Proof The proof is straightforward and similar to the proof of Theorem 1. □

4 Multi-attribute Decision Making Methodology

Here, we develope an algorithm for addressing uncertainty in MAGDM under q-ROHFR information. Consider a DM problem with a set $\{\ell_1, \ell_2, ..., \ell_n\}$ of n alternatives and a set of n attributes $\{c_1, c_2, ..., c_n\}$ with $(w_1, w_2, ..., w_n)^T$ the weights, that is, $w_i \in [0, 1]$, $\oplus_{i=1}^n w_i = 1$. Also a probabilistic terms $\eth_{h_i}$ and ∂_{h_i} such that $\oplus_{i=1}^n \eth_{h_i} = 1$ and $\oplus_{i=1}^n \partial_{h_i} = 1$ with the property that $0 \leq \eth_{h_i}, \partial_{h_i} \leq 1$. To test the reliability of kth alternative ℓ_i under the the attribute c_i, let $\left\{\mathring{D}_1, \mathring{D}_2, ..., \mathring{D}_{\hat{j}}\right\}$ be a set of decision makers (DMs) and $(\rho_1, \rho_2, ..., \rho_n)^T$ be DMs weights such that $\rho_i \in [0, 1]$, $\oplus_{i=1}^n \rho_i = 1$. The expert evalution matrix is described as:

$$M = \left[\left(\overline{\propto}(\kappa_{ij}^{\hat{j}}), \underline{\propto}(\kappa_{ij}^{\hat{j}})\right)\right]_{m\times n}$$

such that $0 \leq \left(\max(\eta_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q + \left(\min(\Psi_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q \leq 1$ and $0 \leq \left(\min(\eta_{h_{\underline{\propto}(\kappa)}}(\ell)\right)^q + \left(\max(\Psi_{h_{\underline{\propto}(\kappa)}}(\ell))\right)^q \leq 1$ are the q-ROPHF rough values. The main steps for MAGDM are as follows:

Step 1 Construct the experts evaluation matrix as

$$(E)^{\hat{j}} = \begin{bmatrix} \left(\overline{\propto}(\kappa_{11}^{\hat{j}}), \underline{\propto}(\kappa_{11}^{\hat{j}})\right) & \left(\overline{\propto}(\kappa_{12}^{\hat{j}}), \underline{\propto}(\kappa_{12}^{\hat{j}})\right) & \cdots & \left(\overline{\propto}(\kappa_{1j}^{\hat{j}}), \underline{\propto}(\kappa_{1j}^{\hat{j}})\right) \\ \left(\overline{\propto}(\kappa_{21}^{\hat{j}}), \underline{\propto}(\kappa_{21}^{\hat{j}})\right) & \left(\overline{\propto}(\kappa_{22}^{\hat{j}}), \underline{\propto}(\kappa_{22}^{\hat{j}})\right) & \cdots & \left(\overline{\propto}(\kappa_{2j}^{\hat{j}}), \underline{\propto}(\kappa_{2j}^{\hat{j}})\right) \\ \left(\overline{\propto}(\kappa_{31}^{\hat{j}}), \underline{\propto}(\kappa_{31}^{\hat{j}})\right) & \left(\overline{\propto}(\kappa_{32}^{\hat{j}}), \underline{\propto}(\kappa_{32}^{\hat{j}})\right) & \cdots & \left(\overline{\propto}(\kappa_{3j}^{\hat{j}}), \underline{\propto}(\kappa_{3j}^{\hat{j}})\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\overline{\propto}(\kappa_{i1}^{\hat{j}}), \underline{\propto}(\kappa_{i1}^{\hat{j}})\right) & \left(\overline{\propto}(\kappa_{i2}^{\hat{j}}), \underline{\propto}(\kappa_{i2}^{\hat{j}})\right) & \cdots & \left(\overline{\propto}(\kappa_{ij}^{\hat{j}}), \underline{\propto}(\kappa_{ij}^{\hat{j}})\right) \end{bmatrix}$$

where $\hat{j}$ shows the number of expert.

Step 2 Evaluate the normalized experts matices $(N)^{\hat{j}}$, as

$$(N)^{\hat{j}} = \begin{cases} \propto (\kappa_{ij}) = \left(\underline{\propto}(\kappa_{ij}), \overline{\propto}(\kappa_{ij})\right) & \text{if For benefit} \\ (\propto (\kappa_{ij}))^c = \left((\underline{\propto}(\kappa_{ij}))^c, \left(\overline{\propto}(\kappa_{ij})\right)^c\right) & \text{if For cost} \end{cases}$$

Step 3 Compute the collected q-rung orthopair hesitant fuzzy rough information of decision makers using the q-ROPHFRWA aggregation operators.

Step 4 Evaluate the aggregated q-ROPHFRVs for each considered alternative with respect to the given list of criteria/attributes by utilizing the proposed aggregation information.

Step 5 Find the ranking of alternatives based on score function.

Step 6 Rank all the alternative scores in descending order. The alternative having larger value will be superior/best.

4.1 Numerical Example

To validate our established operators, we consider a numerical MCGDM example of selecting the best medical oxygen supplier (Fig. 1).

Case Study (Medical Oxygen Supplier)

In the ongoing battle against Coronavirus, oxygen therapy is an effective treatment choice for patients exhibiting virus symptoms. Nowadays, having access to oxygen is a matter of life and death. The world can and must do more to improve access to medical oxygen. COVID-19 was declared a pandemic by the World Health Organization (WHO) a year ago. There is currently no effective antiviral treatment for COVID-19. However, the public health sector has since made substantial strides, such as the successful advancement of COVID-19 testing, and vaccines, which is a collective achievement that should be appreciated. Additionally, COVID-19 vaccine doses have been distributed to more than 50 low- and middle-income countries (LMICs) and territories so far, with intentions for a larger distribution in the future. Despite these advancements, too many societies continue to suffer one critical component: proper oxygen access. Oxygen is an important treatment for COVID-19 patients who are unable to breathe, but oxygen availability remains a challenge in low- and middle-income countries due to a bundle of interconnected issues. During this pandemic situation, access to medical oxygen could be the matter of life and death. Oxygen therapy can save millions of lives. Approximately 15% of all COVID-19 patients need oxygen support. Every day, an estimated half a million people in LMICs need 1 million cylinders of oxygen. However, health-care services are unable to control the supply. On a regular basis, countries around the world including India, Brazil, Jordan, Nigeria, Pakistan, South Africa, Zimbabwe, and others report oxygen shortages. So far, oxygen should be identified as a critical utility, comparable to electricity or water. The global community has an opportunity to introduce structural solutions that will save lives now and increase long-term oxygen access. It will necessitate a comprehensive set of alternatives, ranging from resolving market inefficiencies that impact supply shortages to creating supportive policy environments and offering advanced training for health workers. Recognizing the central importance of oxygen as a treatment for COVID-19, the World Health Organization declared that oxygen therapy

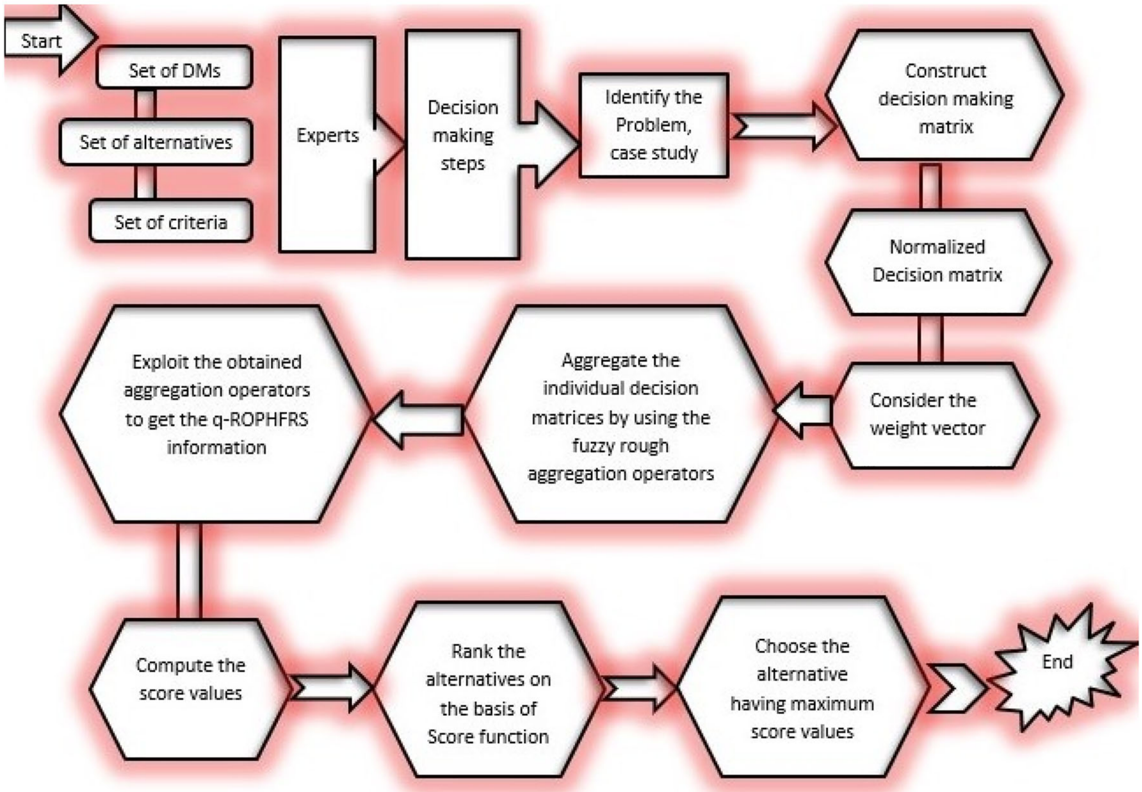

Fig. 1 Algorithmic steps

will be added to the therapeutics component of its Access to COVID-19 Tools (ACT) accelerator. Along with this step, WHO established the COVID-19 Oxygen Emergency Taskforce, which brings together key players, including path, to assess need, increase funding, promote increased supply, and raise public awareness.

The supply of oxygen has been a major issue and the shortage of oxygen has been a significant factor to increase exponentially in COVID-19 deaths in numerous hospitals. To overwhelmed on the aforementioned challenge to tackle an acute shortage of medical oxygen, supplier evaluation and selection is one of the most important components of an effective supply. The main objective of this research are to identify the criteria for selecting proper medical oxygen suppliers and to demonstrate strategies for choosing the best medical oxygen supplier.

Suppose a hospital wants to purchase a significant quantity of medical oxygen. They invite a panel of experts to analyze a number of suppliers. Let $\{\ell_1, \ell_2, \ell_3, \ell_4\}$ be the set of supplier and $\{c_1, c_2, c_3, c_4\}$ be the selection criteria defined by the supplier evaluation team established in the hospital's purchasing department be Product Volume (c_1), Delivery Time (c_2), Supply Variety (c_3), and Geographical Location (c_4). To determine the best supplier of medical oxygen based on the list of criteria c_1, c_2, c_3, and c_4, the hospital's purchasing department invites a group of industry professionals to examine the four evaluation criteria. Because of the uncertainty, the DMs' selection information is presented as q- ROPHFR information. The weight vector for criteria is $w = (0.13, 0.27, 0.29, 0.31)^T$. To solve the MCDM problem using the developed methodology for evaluating alternatives, the following calculations are performed:

Step 1 The informations of professional expert is given in Table 1(a)-1(d) in the form of q-ROPHFRS.

Step 2 The expert information are benefit type. So in this case, we do not need to normalize the q-ROPHFRVs.

Step 3 In this problem, only one expert is consider for collection of uncertain information. So, we do not need to find the collected information.

Step 4 Aggregation information of the alternative under the given list of attributes are evaluted using proposed aggregation operators are as follows:

Case-1: Aggregation information using weighted averaging operator is shown in Table 2.

Case-2: Aggregation information using q-ROHFRWG operator presented in Table 3.

Table 1 Expert information

	c_1	c_2
(a)		
ℓ_1	$\begin{pmatrix}\begin{pmatrix}(0.1/0.3, 0.2/0.5, 0.5/0.2)\,,\\ (0.3/0.6, 0.4/0.4)\end{pmatrix},\\ \begin{pmatrix}(0.8/1)\,,\\ (0.4/0.5, 0.6/0.5)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.5/0.4, 0.7/0.6)\,,\\ (0.5/0.7, 0.6/0.3)\end{pmatrix},\\ \begin{pmatrix}(0.5/0.3, 0.4/0.7)\,,\\ (0.7/0.3, 0.9/0.7)\end{pmatrix}\end{pmatrix}$
ℓ_2	$\begin{pmatrix}\begin{pmatrix}(0.6/0.7, 0.6/0.3)\,,\\ (0.7/0.5, 0.9/0.5)\end{pmatrix},\\ \begin{pmatrix}(0.3/0.2, 0.5/0.8)\,,\\ (0.6/1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.2/0.2, 0.4/0.1, 0.5/0.7)\,,\\ (0.5/1.0)\end{pmatrix},\\ \begin{pmatrix}(0.6/0.3, 0.7/0.7)\,,\\ (0.3/1)\end{pmatrix}\end{pmatrix}$
(b)		
ℓ_3	$\begin{pmatrix}\begin{pmatrix}(0.4/0.3, 0.5/0.6, 0.6/0.1)\,,\\ (0.6/0.1, 0.7/0.9)\end{pmatrix},\\ \begin{pmatrix}(0.9/1)\,,\\ (0.5/1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.1/1.0)\,,\\ (0.5/0.5, 0.6/0.5)\end{pmatrix},\\ \begin{pmatrix}(0.4/0.3, 0.6/0.4, 0.7/0.3)\,,\\ (0.5/0.2, 0.7/0.8)\end{pmatrix}\end{pmatrix}$
ℓ_4	$\begin{pmatrix}\begin{pmatrix}(0.4/1)\,,\\ (0.5/0.5, 0.6/0.5)\end{pmatrix},\\ \begin{pmatrix}(0.3/0.7, 0.4/0.3)\,,\\ (0.8/1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.4/0.4, 0.5/0.6)\,,\\ (0.4/1)\end{pmatrix},\\ \begin{pmatrix}(0.1/0.6, 0.2/0.4)\,,\\ (0.2/0.2, 0.3/0.8)\end{pmatrix}\end{pmatrix}$
	c_3	c_4
(c)		
ℓ_1	$\begin{pmatrix}\begin{pmatrix}(0.4/1)\,,\\ (0.3/0.2, 0.7/0.8)\end{pmatrix},\\ \begin{pmatrix}(0.5/1)\,,\\ (0.9/1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.6/1)\,,\\ (0.7/1)\end{pmatrix},\\ \begin{pmatrix}(0.6/0.4, 0.8/0.2, 0.9/0.4)\,,\\ (0.7/0.7, 0.9/0.3)\end{pmatrix}\end{pmatrix}$
ℓ_2	$\begin{pmatrix}\begin{pmatrix}(0.8/1)\,,\\ (0.4/0.6, 0.5/0.2, 0.7/0.2)\end{pmatrix},\\ \begin{pmatrix}(0.2/0.6, 0.5/0.4)\,,\\ (0.4/0.3, 0.5/0.7)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.8/1)\,,\\ (0.5/1)\end{pmatrix},\\ \begin{pmatrix}(0.7/1.0)\,,\\ (0.1/0.5, 0.3/0.3, 0.4/0.2)\end{pmatrix}\end{pmatrix}$
(d)		
ℓ_3	$\begin{pmatrix}\begin{pmatrix}(0.3/1)\,,\\ (0.7/0.6, 0.8/0.4)\end{pmatrix},\\ \begin{pmatrix}(0.7/0.6, 0.8/0.4)\,,\\ (0.1/0.7, 0.4/0.2, 0.7/0.1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.3/0.2, 0.6/0.8)\,,\\ (0.8/1)\end{pmatrix},\\ \begin{pmatrix}(0.7/1)\,,\\ (0.3/1)\end{pmatrix}\end{pmatrix}$
ℓ_4	$\begin{pmatrix}\begin{pmatrix}(0.3/1)\,,\\ (0.7/0.7, 0.8/0.3)\end{pmatrix},\\ \begin{pmatrix}(0.7/1)\,,\\ (0.6/1)\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}(0.6/0.2, 0.7/0.4, 0.9/0.4)\,,\\ (0.3/0.3, 0.4/0.7)\end{pmatrix},\\ \begin{pmatrix}(0.2/0.2, 0.7/0.8)\,,\\ (0.7/0.4, 0.8/0.2, 0.9/0.4)\end{pmatrix}\end{pmatrix}$

Step 5 Score values of all alternatives under developed aggregation operators presented in Table 4.

Step 6 Rank the alternatives $\ell_k (k = 1, 2, 3, 4)$ is enclosed in Table 5:

From the above computational process, we concluded that alternative ℓ_2 is the finest alternative among others, and therefore it is highly recommended.

5 Comparison Analysis

5.1 TOPSIS Methodology Based on q-Rung Orthopair Probabilistic Hesitant Fuzzy Rough information

Hwang and Yoon (1981) proposed the TOPSIS technique for Ideal Solution (TOPSIS), which allows policymakers to compare the Ideal positive solution and Ideal negative solution. TOPSIS is based on the assumption that the best alternative would be the closest to the ideal and the furthest away from the perfect negative solution [6,41]. The main parts of the method are as follows:

Table 2 Aggregated information using q-ROPHFRWA

(a)	
ℓ_1	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.4968/0.12, 0.5728/0.18, 0.4979/0.2, 0.5735/0.3, 0.5163/0.08, 0.5866/0.12\},\\ \left\{\begin{array}{c}0.4478/0.084, 0.5725/0.336, 0.4704/0.036, 0.6014/0.144,\\ 0.4649/0.056, 0.5944/0.224, 0.4883/0.024, 0.6243/0.096\end{array}\right\}\end{array}\right),\\ \left(\begin{array}{c}\{0.6001/0.12, 0.6863/0.06, 0.7583/0.12, 0.5865/0.28, 0.6774/0.14, 0.7523/0.28\},\\ \left\{\begin{array}{c}0.7001/0.105, 0.7568/0.045, 0.7492/0.245, 0.8100/0.105,\\ 0.7380/0.105, 0.7978/0.045, 0.7898/0.245, 0.8538/0.105\end{array}\right\}\end{array}\right)\end{array}\right)$
ℓ_2	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.7188/0.14, 0.7251/0.07, 0.7321/0.49, 0.7188/0.06, 0.7251/0.03, 0.7321/0.21\},\\ \{0.4896/0.03, 0.5224/0.1, 0.5759/0.1, 0.5059/0.3, 0.5397/0.1, 0.5950/0.1\}\end{array}\right),\\ \left(\begin{array}{c}\left\{\begin{array}{c}0.5675/0.036, 0.5962/0.024, 0.6044/0.084, 0.6288/0.056,\\ 0.5788/0.144, 0.6062/0.096, 0.6140/0.336, 0.6374/0.224\end{array}\right\},\\ \{0.2539/0.15, 0.3569/0.09, 0.3901/0.06, 0.2708/0.35, 0.3807/0.21, 0.4162/0.21\}\end{array}\right)\end{array}\right)$
(b)	
ℓ_3	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.2923/0.6, 0.4450/0.24, 0.3223/0.12, 0.4580/0.48, 0.3613/0.02, 0.4775/0.08\},\\ \left\{\begin{array}{c}0.6530/0.03, 0.6788/0.02, 0.6860/0.03, 0.7130/0.02,\\ 0.6662/0.27, 0.6925/0.18, 0.6998/0.27, 0.7275/0.18\end{array}\right\}\end{array}\right),\\ \left(\begin{array}{c}\{0.7086/0.18, 0.7423/0.12, 0.7280/0.24, 0.7586/0.18, 0.7456/0.12, 0.7735/0.12\},\\ \{0.2676/0.14, 0.4381/0.04, 0.4705/0.02, 0.2931/0.56, 0.4381/0.16, 0.5153/0.08\}\end{array}\right)\end{array}\right)$
ℓ_4	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.4703/0.08, 0.5334/0.16, 0.7083/0.16, 0.4935/0.12, 0.5508/0.24, 0.7160/0.24\},\\ \left\{\begin{array}{c}0.4430/0.105, 0.4843/0.245, 0.4605/0.045, 0.5035/0.105,\\ 0.4536/0.105, 0.4959/0.245, 0.4715/0.045, 0.5155/0.105\end{array}\right\}\end{array}\right),\\ \left(\begin{array}{c}\left\{\begin{array}{c}0.4936/0.084, 0.6089/0.336, 0.4959/0.056, 0.6102/0.224,\\ 0.4996/0.036, 0.6124/0.144, 0.5018/0.024, 0.6137/0.144\end{array}\right\},\\ \{0.4857/0.08, 0.5062/0.04, 0.5250/0.08, 0.5418/0.32, 0.5647/0.16, 0.5857/0.32\}\end{array}\right)\end{array}\right)$

Let $\kappa = \{\ell_1, \ell_2, \ell_3, ..., \ell_m\}$ be the set of alternatives and $C = \{c_1, c_2, c_3, ..., c_n\}$ be a set of criteria. The decision matrix of the expert is presented as:

$$M = \left[\underline{\propto}(\kappa_{ij}^{\hat{j}}), \overline{\propto}(\kappa_{ij}^{\hat{j}})\right]_{m\times n}$$

where $\overline{\propto}(\kappa_{ij}) = \left\langle \ell, \eta_{h_{\overline{\propto}(\kappa)}}(\ell)/\eth_{h_{\overline{\propto}(\kappa)}}, \Psi_{h_{\overline{\propto}(\kappa)}}(\ell)/\partial_{h_{\overline{\propto}(\kappa)}} \right\rangle$ and $\underline{\propto}(\kappa) = \left\langle \ell, \eta_{h_{\underline{\propto}(\kappa)}}(\ell)/\eth_{h_{\underline{\propto}(\kappa)}}, \Psi_{h_{\underline{\propto}(\kappa)}}(\ell)/\partial_{h_{\underline{\propto}(\kappa)}} \right\rangle$ such that $0 \le \left(\max(\eta_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q + \left(\min(\Psi_{h_{\overline{\propto}(\kappa)}}(\ell))\right)^q \le 1$ and $0 \le \left(\min(\eta_{h_{\underline{\propto}(\kappa)}}(\ell)\right)^q + \left(\max(\Psi_{h_{\underline{\propto}(\kappa)}}(\ell))\right)^q \le 1$ are the q-ROHF rough values. Also $\eth_{h_i} \in [0, 1]$, $\oplus_{i=1}^{s}\eth_{h_i} = 1$ and $\partial_i \in [0, 1]$, $\oplus_{i=1}^{s}\partial_i = 1$ (s is a positive number represents the number of elements contained in q-ROPHFRS).

Step 1 First, we collect information from decision makers in the form of q-ROPHFRNs.

Step 2 Evaluate the normalized experts matrix $(N)^{\hat{j}}$, as

$$(N)^{\hat{j}} = \begin{cases} \propto(\kappa_{ij}) = \left(\underline{\propto}(\kappa_{ij}), \overline{\propto}(\kappa_{ij})\right) & \text{if For benefit} \\ \left(\propto(\kappa_{ij})\right)^c = \left(\left(\underline{\propto}(\kappa_{ij})\right)^c, \left(\overline{\propto}(\kappa_{ij})\right)^c\right) & \text{if For cost} \end{cases}$$

Step 3 Based on the score value, we determine the positive ideal solution and the negative ideal solution. Herein, the positive ideal solutions and negative ideal solution are denoted as $I^+ = \left(\varpi_1^+, \varpi_2^+, \varpi_3^+, ..., \varpi_n^+\right)$ and $I^- = \left(\varpi_1^-, \varpi_2^-, \varpi_3^-, ..., \varpi_n^-\right)$ respectively. For positive ideal solution I^+, it can be computed by the formulla:

$$I^+ = \left(\varpi_1^+, \varpi_2^+, \varpi_3^+, ..., \varpi_n^+\right) = \Big(\max_i score(\varpi_{i1}), \max_i score\varpi_{i2}, \max_i score\varpi_{i3}, ..., \max_i score\varpi_{in}.\Big)$$

Likewise, the negative ideal solution calculated by the formula as follows:

$$I^- = \left(\varpi_1^-, \varpi_2^-, \varpi_3^-, ..., \varpi_n^-\right) = \Big(\min_i score\varpi_{i1}, \min_i score\varpi_{i2}, \min_i score\varpi_{i3}, ..., \min_i score\varpi_{in}.\Big)$$

Afterward, find the geometric distance between all the alternatives and positive ideal I^+ as follows:

$$d(\alpha_{ij}, I^+) = \frac{1}{8}\left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{\#h}\sum_{s=1}^{\#h}\left|\left(\underline{\mu}_{ij(s)} \times \underline{\eth}_{\epsilon_{ij(s)}}\right)^2 - \left(\underline{\mu}_i^+ \times \underline{\eth}^+_{\epsilon_i(s)}\right)^2\right| \\ +\left|\left(\overline{\mu}_{ij(s)} \times \overline{\eth}^+_{\epsilon_{i(s)}}\right)^2 - \left(\overline{\mu}^+_{i(s)} \times \overline{\eth}^+_{\epsilon_i(s)}\right)^2\right|\end{array}\right) \\ +\left(\begin{array}{c}\frac{1}{\#g}\sum_{s=1}^{\#g}\left|\left(\underline{\nu}_{ij(s)} \times \underline{\partial}_{\epsilon_{ij(s)}}\right)^2 - \left(\underline{\nu}^+_{i(s)} \times \underline{\partial}^+_{\epsilon_i(s)}\right)^2\right| \\ +\left|\left(\overline{\nu_{h_{ij}}} \times \overline{\partial}^+_{\epsilon_i(s)}\right)^2 - \left(\overline{\nu_{h_i}}^+ \times \overline{\partial}^+_{\epsilon_i(s)}\right)^2\right|\end{array}\right)\end{array}\right),$$

where $i = 1, 2, 3, ..., n$, and $j = 1, 2, 3, ..., m$.

Analogously, the geometric distance between all the alternatives and negative ideal I^- as follows:

Table 3 Aggregated information using q -ROPHFRWG

(a)	
ℓ_1	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.4023/0.12, 0.4406/0.18, 0.4402/0.2, 0.4821/0.3, 0.4959/0.08, 0.5431/0.12\},\\ \left\{\begin{array}{c}0.5461/0.084, 0.6324/0.336, 0.5722/0.036, 0.6501/0.144,\\ 0.5508/0.056, 0.6356/0.224, 0.5763/0.024, 0.6530/0.096\end{array}\right\}\end{array}\right),\\ \left(\begin{array}{c}\{0.5624/0.12, 0.6149/0.06, 0.6377/0.12, 0.5295/0.28, 0.5789/0.14, 0.6004/0.28\},\\ \left\{\begin{array}{c}0.7763/0.105, 0.8414/0.045, 0.8345/0.245, 0.8801/0.105,\\ 0.7830/0.105, 0.8457/0.045, 0.8390/0.245, 0.8832/0.105\end{array}\right\}\end{array}\right)\end{array}\right)$
ℓ_2	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.5300/0.14, 0.6391/0.07, 0.6788/0.49, 0.5300/0.06, 0.6391/0.03, 0.6788/0.21\},\\ \{0.5197/0.03, 0.5395/0.1, 0.6075/0.1, 0.6161/0.3, 0.6288/0.1, 0.6758/0.1\}\end{array}\right),\\ \left(\begin{array}{c}\left\{\begin{array}{c}0.4182/0.036, 0.5455/0.024, 0.4360/0.084, 0.5687/0.056,\\ 0.4469/0.144, 0.5830/0.096, 0.4659/0.336, 0.6077/0.224\end{array}\right\},\\ \{0.3845/0.15, 0.4011/0.09, 0.4230/0.06, 0.4219/0.35, 0.4355/0.21, 0.4540/0.21\}\end{array}\right)\end{array}\right)$
(b)	
ℓ_3	$\left(\begin{array}{c}\left(\begin{array}{c}\left\{\begin{array}{c}0.2315/0.6, 0.2870/0.24, 0.2383/0.12,\\ 0.2954/0.48, 0.2440/0.02, 0.3025/0.08\end{array}\right\},\\ \left\{\begin{array}{c}0.6963/0.03, 0.7321/0.02, 0.7094/0.03, 0.7430/0.02,\\ 0.7065/0.27, 0.7406/0.18, 0.7189/0.27, 0.7510/0.18\end{array}\right\}\end{array}\right),\\ \left(\begin{array}{c}\left\{\begin{array}{c}0.6218/0.18, 0.6464/0.12, 0.6938/0.24,\\ 0.7212/0.18, 0.7232/0.12, 0.7518/0.12\end{array}\right\},\\ \left\{\begin{array}{c}0.3921/0.14, 0.5272/0.04, 0.5516/0.02,\\ 0.5069/0.56, 0.5272/0.16, 0.6125/0.08\end{array}\right\}\end{array}\right)\end{array}\right)$
ℓ_4	$\left(\begin{array}{c}\left(\begin{array}{c}\{0.4173/0.08, 0.4377/0.16, 0.4732/0.16, 0.4432/0.12, 0.4649/0.24, 0.5025/0.24\}\\ \left\{\begin{array}{c}0.5343/0.105, 0.5459/0.245, 0.6060/0.045, 0.6143/0.105,\\ 0.5480/0.105, 0.5589/0.245, 0.6159/0.045, 0.6238/0.105\end{array}\right\},\end{array}\right),\\ \left(\begin{array}{c}\left\{\begin{array}{c}0.2514/0.084, 0.3708/0.336, 0.3032/0.056, 0.4471/0.224,\\ 0.2610/0.036, 0.3849/0.144, 0.3147/0.024, 0.4641/0.144\end{array}\right\},\\ \{0.6353/0.08, 0.6853/0.04, 0.7576/0.08, 0.6385/0.32, 0.6878/0.16, 0.7593/0.32\}\end{array}\right)\end{array}\right)$

Table 4 Score values

Operators	$\Delta(\ell_1)$	$\Delta(\ell_2)$	$\Delta(\ell_3)$	$\Delta(\ell_4)$
q-ROPHFRWA	0.5090	0.5184	0.5182	0.5062
q-ROPHFRWG	0.4979	0.5053	0.5009	0.4851

Table 5 Ranking of the alternatives

Operators	Score	Best Alternative
q-ROPHFRWA	$\Delta(\ell_2) > \Delta(\ell_3) > \Delta(\ell_1) > \Delta(\ell 4)$	ℓ_2
q-ROPHFRWG	$\Delta(\ell_2) > \Delta(\ell_3) > \Delta(\ell_1) > \Delta(\ell_4)$	ℓ_2

$$d(\alpha_{ij}, I^-) = \frac{1}{8}\left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{\#h}\sum_{s=1}^{\#h}\left|\left(\underline{\mu}_{ij(s)} \times \underline{\eth}_{\epsilon_{ij(s)}}\right)^2 - \left(\underline{\mu}^-_{i(s)} \times \underline{\eth}^-_{\epsilon_{i(s)}}\right)^2\right|\\ +\left|\left(\overline{\mu}_{ij(s)} \times \overline{\eth}_{\epsilon_{ij(s)}}\right)^2 - \left(\overline{\mu}^-_{i(s)} \times \overline{\eth}^-_{\epsilon_{i(s)}}\right)^2\right|\end{array}\right)\\ +\left(\begin{array}{c}\frac{1}{\#g}\sum_{s=1}^{\#g}\left|\left(\underline{\nu}_{ij(s)} \times \underline{\partial}_{\epsilon_{ij(s)}}\right)^2 - \left(\underline{\nu}^-_{i(s)} \times \underline{\partial}^-_{\epsilon_{i(s)}}\right)^2\right|\\ +\left|\left(\overline{\nu_{h}}_{ij} \times \overline{\partial}_{\epsilon_{ij(s)}}\right)^2 - \left(\overline{\nu_{h}}^-_{i} \times \overline{\partial}^-_{\epsilon_{i(s)}}\right)^2\right|\end{array}\right)\end{array}\right),$$

where $i = 1, 2, 3, ...n$, and $j = 1, 2, 3, ...m$.

Step 4 The relative closeness indices for all DMe of the alternatives are calculated as follows:

Table 6 Score value of Table 1(a) -1(b)

	c_1	c_2	c_3	c_4
ℓ_1	0.6142	0.4600	0.4225	0.4433
ℓ_2	0.3825	0.4196	0.6412	0.7317
ℓ_3	0.5538	0.4213	0.5567	0.4675
ℓ_4	0.3725	0.4400	0.5087	0.5254

$$RC(\alpha_{ij}) = \frac{d(\alpha_{ij}, I^+)}{d(\alpha_{ij}, I^-) + d(\alpha_{ij}, I^+)}$$

Step 5 The ranking orders of alternatives can be determined and choose the most desirable alternative having minimum distance.

5.2 Numerical Example

A numerical example relevant to "selection of oxygen supplier" is given below to demonstrate the validity of our approach:

Step 1 The decision maker information in the form of q-ROPHFRNs is given in Table 1(a)-1(b).

Step 2 The given information is benefit type, therefore, no need to normalize.

Table 7 Ideal solutions

Criteria	I^+	I^-
c_1	$\left(\begin{array}{c}\left(\begin{array}{c}(0.1/0.3, 0.2/0.5, 0.5/0.2),\\(0.3/0.6, 0.4/0.4)\end{array}\right),\\ \left(\begin{array}{c}(0.8/1),\\(0.4/0.5, 0.6/0.5)\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\begin{array}{c}(0.4/1),\\(0.5/0.5, 0.6/0.5)\end{array}\right),\\ \left(\begin{array}{c}(0.3/0.7, 0.4/0.3),\\(0.8/1)\end{array}\right)\end{array}\right)$
c_2	$\left(\begin{array}{c}\left(\begin{array}{c}(0.5/0.4, 0.7/0.6),\\(0.5/0.7, 0.6/0.3)\end{array}\right),\\ \left(\begin{array}{c}(0.5/0.3, 0.4/0.7),\\(0.7/0.3, 0.9/0.7)\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\begin{array}{c}(0.2/0.2, 0.4/0.1, 0.5/0.7),\\(0.5/1.0)\end{array}\right),\\ \left(\begin{array}{c}(0.6/0.3, 0.7/0.7),\\(0.3/1)\end{array}\right)\end{array}\right)$
c_3	$\left(\begin{array}{c}\left(\begin{array}{c}(0.8/1),\\(0.4/0.6, 0.5/0.2, 0.7/0.2)\end{array}\right),\\ \left(\begin{array}{c}(0.2/0.6, 0.5/0.4),\\(0.4/0.3, 0.5/0.7)\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\begin{array}{c}(0.4/1),\\(0.3/0.2, 0.7/0.8)\end{array}\right),\\ \left(\begin{array}{c}(0.5/1),\\(0.9/1)\end{array}\right)\end{array}\right)$
c_4	$\left(\begin{array}{c}\left(\begin{array}{c}(0.8/1),\\(0.5/1)\end{array}\right),\\ \left(\begin{array}{c}(0.7/1.0),\\(0.1/0.5, 0.3/0.3, 0.4/0.2)\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\begin{array}{c}(0.6/1),\\(0.7/1)\end{array}\right),\\ \left(\begin{array}{c}(0.6/0.4, 0.8/0.2, 0.9/0.4),\\(0.7/0.7, 0.9/0.3)\end{array}\right)\end{array}\right)$

Step 3 Positive and negative ideal solution are computed in Table 7, as follows

Compute the distance measure of positive ideal solution (PIS) and negative ideal solution (NIS)

0.3799	0.2158	0.4174	0.4971

and

0.4038	0.4301	0.5054	0.2434

Step 4 The relative closeness indices for all DMe of the alternatives are calculated as follows:

0.4848	0.3341	0.4523	0.6713

Step 5 From ranking of alternative it could be seen that ℓ_2 has the minimum distance. Hence ℓ_2 is the best alternative.

6 Conclusion

To provide greater freedom for decision makers in MAGDM problems by qualitatively describing the evaluation values, the concept of q-ROPHFRS has been developed under q-rung orthopair fuzzy set, hesitant fuzzy set, and rough set environment. In comparison to classical fuzzy models, a q-ROPHFR model has the capability to solve real world problems with ambiguity, imprecision, and incompleteness. We introduced the concept of q-ROPHFRWA and q-ROPHFRWG aggregation operators which is an effective and flexible operator to address MCGDM problems with uncertainty. Moreover, an example of supplier selection for medical oxygen to hospitals is presented to demonstrate the validity and efficacy of the technique. Comparison analysis has been conducted, in the final ranking and optimal decision making of the medical oxygen supplier by the proposed techniques with q-ROHFR Topsis method and the superiorities have been illustrated.

It is observed that in terms of potential future works that are good enough to justify, some essential topics still remain. In the future, our research will be extended to: (1) q-rung orthopair probabilistic hesitant fuzzy rough ordered weighted averaging operator; (2) q-rung orthopair probabilistic hesitant fuzzy rough hybrid weighted averaging operator; (3) q-rung orthopair probabilistic hesitant fuzzy rough ordered weighted geometric operator; (4) q-rung orthopair probabilistic hesitant fuzzy rough hybrid weighted geometric operator.

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