STRATIFICATIONS OF REAL VECTOR SPACES FROM CONSTRUCTIBLE SHEAVES WITH CONICAL MICROSUPPORT

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ABSTRACT. Interpreting the syzygy theorem for tame modules over posets in the setting of derived categories of subanalytically constructible sheaves proves two conjectures due to Kashiwara and Schapira concerning the existence of stratifications of real vector spaces that play well with sheaves having microsupport in a given cone or, equivalently, sheaves in the corresponding conic topology.

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1. INTRODUCTION

Overview and motivation. Persistent homology with multiple real parameters can be phrased in more or less equivalent ways using multigraded modules (e.g. [CZ09, Knu08, Mil20a]), or sheaves (e.g. [Cur14, Cur19]), or functors (e.g. [SCL⁺16]), or derived categories (e.g. [KS18]). All of these descriptions have in common an underlying partially ordered set indexing a family of vector spaces, and this is interpreted under increasing layers of abstraction.

The simplest objects at any level of abstraction are the *indicator* objects, which place a single copy of the ground field k at every point of an interval in the underlying poset Q, meaning an intersection of an upset of Q with a downset of Q. (The terminology is most clear when $Q = \mathbb{R}$, where "interval" has its usual meaning.) Among the indicator objects are those supported on the upsets and downsets themselves; over $Q = \mathbb{R}$ these objects are free and injective, respectively. Furthermore, when Q is totally ordered, every object is a direct sum of indicator objects [Cra13]. The theory for more general posets, including partially ordered real vector spaces, has in large part revolved around relating general objects as closely as possible to indicator objects, particularly where algorithmic computation is concerned.

Indeed, the foundations for the ideas in this paper, both from Kashiwara–Schapira [KS18, KS19] and the author [Mil20a, Mil20b, Mil20c] (see also [Mil17]), lies in algorithmic computation with persistent homology. To that end, effective methods demand concrete representatives of derived sheaves and stratifications of their support. Kashiwara and Schapira, in [KS19, Conjecture 3.20] (which they had previously stated as [KS18, Conjecture 4.19]) and [KS17, Conjecture 3.17], assert that derived sheaves in principle possess such concrete representatives. Corollaries 5.1 and 5.2 achieve more than mere existence: the engine behind their proofs, Theorem 4.5, produces concrete structures via the syzygy theorem for poset modules (Theorem 2.11), which is specifically designed to extract resolutions algorithmically [Mil20a].

Situations abound where concrete resolutions could be essential for algorithmic persistent homology. For example, bifiltration of a semialgebraic space by two semialgebraic functions yields bipersistent homology that is an \mathbb{R}^2 -module which should be tame and hence have finite indicator resolutions by upsets or downsets. (This statement requires proof and might be subtle or even false in the subanalytic instead of semialgebraic context.) This scenario is fundamental to motivating applications such as summarizing shape in biology [Mil15] or probability distributions in statistics [RS20]. Tameness in this biparameter setting connects to recent Morse-theoretic stratification perspectives by Budney and Kaczynski [BK21] as well as by Assif and Baryshnikov [AB21].

Conjectures and proofs. The conjectures of Kashiwara and Schapira are phrased in the most abstract derived setting. They posit, roughly speaking, that every object can be directly related to indicator objects, either by stratification of its support or more strongly—by resolution. More precisely, the first conjecture concerns the relation between, on one hand, constructibility of sheaves on real vector spaces in the derived category with microsupport restricted to a cone, and on the other hand, stratification of the vector space in a manner compatible with the cone [KS17, Conjecture 3.17]¹ (Corollary 5.2). The second concerns piecewise linear (PL) objects in this context, particularly existence of polyhedrally structured resolutions that, in principle, lend themselves to explicit or algorithmic computation [KS18, Conjecture 4.19] = [KS19, Conjecture 3.20] (Corollary 5.1).

This note uses the most elementary poset module setting [Mil20a] to prove these conjectures. Both follow immediately from Theorem 4.5 here, which translates the relevant real-vector-space special cases of the syzygy theorem for complexes of poset modules [Mil20a, Theorem 6.17] (reviewed in Section 2) into the language of derived categories of constructible sheaves with conic microsupport or under a conic topology (reviewed in Section 3).

The syzygy theorem [Mil20a, Theorems 6.12 and 6.17] leverages relatively weak topological framework into powerful homological structure: over any poset Q it enhances a constant subdivision—a partition of Q into finitely many regions over which the given module or complex is constant—to a more controlled subdivision (a finite encoding [Mil20a, §4]), and even to a finite resolution by upset modules and a finite resolution by downset modules, whose pieces play well with the ambient combinatorics. These resolutions are analogues over arbitrary posets of free and injective resolutions for modules over the poset \mathbb{Z}^n [GW78] (see [HM05] or [MS05, Chapter 11] for background, or [Mil20a, §5] for a treatment in the present context) or over the poset \mathbb{R} . Crucially, any available supplementary geometry—be it subanalytic, semialgebraic, or piecewise-linear, for instance—is preserved.

In the context of a partially ordered real vector space Q with positive cone Q_+ , the enhancement afforded by the syzygy theorem produces a Q_+ -stratification from an arbitrary subanalytic triangulation. If the triangulation is subordinate to a given constructible derived Q_+ -sheaf, meaning an object in the bounded derived category of constructible sheaves with microsupport contained in the negative polar cone of Q_+ , then this enhancement produces Q_+ -structured resolutions of the given sheaf. This makes the two conjectures into special cases of the syzygy theorem.

While sheaves with conical microsupport (see Section 3.4) are equivalent to the more elementary sheaves in the conic topology (see Section 3.3), as has been known from the outset [KS90] (see Theorem 3.15), the notion of constructibility has until now been available only on the microsupport side. The results here assert that constructibility can be detected entirely with the more rigid conic topology, via tameness, without appealing to subanalytic triangulations in the more flexible analytic topology, a point emphasized by Theorem 4.5'. More broadly, for applications in persistent homology

¹Bibliographic note: this conjecture appears in v3 (the version cited here) and earlier versions of the cited arXiv preprint. It does not appear in the published version [KS18], which is v6 on the arXiv. The published version is cited where it is possible to do so, and v3 [KS17] is cited otherwise.

the input is usually a sheaf in the conic topology induced by a given cone Q_+ instead of a sheaf in the ordinary topology with microsupport in the negative polar cone Q_+^{\vee} , so the main results, namely Theorem 4.5, Corollary 5.1, and Corollary 5.2, are restated using conic sheaves in Theorem 4.5', Corollary 5.1', and Corollary 5.2'.

Poset modules vs. constructible sheaves. The theory in [Mil20a, Mil20b, Mil20c] was developed simultaneously and independently from [KS18, KS19] (cf. [Mil17]). Having made the connection between these approaches, it is worth comparing them in detail.

The syzygy theorem [Mil20a, Theorems 6.12 and 6.17] and its combinatorial underpinnings involving poset encoding [Mil20a, §4] hold over arbitrary posets; see Section 2 here for indications toward this generality. When the poset is a real vector space, the constructibility encapsulated by topological tameness (Definition 2.3) has no subanalytic, algebraic, or piecewise-linear hypothesis, although these additional structures are preserved by the syzygy theorem transitions. For example, the upper boundary of a downset in the plane with the usual componentwise partial order could be the graph of any continuous weakly decreasing function, among other things, and could be present (i.e., the downset is closed) or absent (i.e., the downset is open), or somewhere in between (e.g., a Cantor set could be missing). The conic topology in [KS18] or [KS19] specializes at the outset to the case of a partially ordered real vector space, and it allows only subanalytic or polyhedral regions, respectively, with upsets having closed lower boundaries and downsets having open upper boundaries. The constructibility in [KS18, KS19] is otherwise essentially the same as tameness here (Theorem 4.5'), except that tameness requires constant subdivisions to be finite, whereas constructibility in the derived category allows constant subdivisions to be locally finite. That said, this agreement of constructibility with locally finite tameness that is subanalytic or PL, more or less up to boundary considerations, is visible in [KS17] or [KS19] only via conjectures, namely the ones proved here in Section 5 using the general poset methods.

The theory of primary decomposition in [Mil20b] requires the poset to be a partially ordered group whose positive cone has finitely many faces. These can be integer or real or something in between, but the finiteness is essential for primary decomposition in any of these settings; see [Mil20b, Example 5.9]. Local finiteness allowed by constructibility in [KS18] does not provide a remedy, although it is possible that the PL hypothesis in [KS19] does. In either the integer or real case, detailed understanding of the topology results in a stronger theory of primary decomposition than over an aribtrary polyhedral group, with much more complete supporting commutative algebra [Mil20c].

Most of the remaining differences between the developments in [Mil20a, Mil20b, Mil20c] and those in [KS18, KS19], beyond the types of allowed functions and the shapes of allowed regions, is the behavior allowed on boundaries of regions. That difference is accounted for by the transition between the conic topology and the Alexandrov topology, the distinction being that the Alexandrov topology has for its open sets all upsets, whereas the conic topology has only the upsets that are open in the usual topology.

This distinction is explored in detail by Berkouk and Petit [BP19]. It is intriguing that ephemeral modules are undetectable metrically [BP19, Theorem 4.22] but their presence here brings indispensable insight into homological behavior in the conic topology.

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2. Syzygy theorem for poset modules

This section recalls concepts surrounding modules over posets, concluding with a statement (Theorem 2.11) of the relevant special case of the syzygy theorem for complexes of poset modules [Mil20a, Theorem 6.17]. For reference, the definitions here correspond to [Mil20a, Definitions 2.1, 2.6, 2.11, 2.14, 2.15, 4.27, 3.1, 3.14, 6.1, and 6.16], sometimes special cases thereof.

2.1. Tame poset modules.

Definition 2.1. Let Q be a partially ordered set (*poset*) and \leq its partial order. A *module over* Q (or a Q-module) is

- a Q-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
- a homomorphism $M_q \to M_{q'}$ whenever $q \preceq q'$ in Q such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \preceq q' \preceq q''$.

A homomorphism $M \to N$ of Q-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms $M_q \to N_q$, that commute with the structure homomorphisms $M_q \to M_{q'}$ and $N_q \to N_{q'}$.

Definition 2.2. Fix a *Q*-module *M*. A constant subdivision of *Q* subordinate to *M* is a partition of *Q* into constant regions such that for each constant region *I* there is a single vector space M_I with an isomorphism $M_I \to M_i$ for all $\mathbf{i} \in I$ that has no monodromy: if *J* is some (perhaps different) constant region, then all comparable pairs $\mathbf{i} \preceq \mathbf{j}$ with $\mathbf{i} \in I$ and $\mathbf{j} \in J$ induce the same composite homomorphism $M_I \to M_i \to M_i \to M_j \to M_J$.

Definition 2.3. Fix a poset Q and a Q-module M.

- 1. A constant subdivision of Q is *finite* if it has finitely many constant regions.
- 2. The Q-module M is Q-finite if its components M_q have finite dimension over k.
- 3. The Q-module M is *tame* if it is Q-finite and Q admits a finite constant subdivision subordinate to M.

2.2. Real partially ordered groups.

Definition 2.4. An abelian group Q is *partially ordered* if it is generated by a submonoid Q_+ , called the *positive cone*, that has trivial unit group. The partial order is: $q \leq q' \Leftrightarrow q' - q \in Q_+$. A partially ordered group is

- 1. *real* if the underlying abelian group is a real vector space of finite dimension;
- 2. subanalytic if, in addition, Q_{+} is subanalytic, (see [KS90, §8.2] for the definition);
- 3. polyhedral if, in addition, Q_+ is a convex polyhedron: an intersection of finitely many half-spaces, each either closed or open.

Definition 2.5. A partition of a real partially ordered group Q into subsets is

- 1. *subanalytic* if the subsets are subanalytic sets, and
- 2. piecewise linear (PL) if the subsets are finite unions of convex polyhedra.

A module over a subanalytic or polyhedral real partially ordered group Q is *subanalytic* or PL, respectively, if the module is tamed by a subordinate finite constant subdivision of the corresponding type.

2.3. Complexes and resolutions of poset modules.

Definition 2.6. A homomorphism $\varphi : M \to N$ of Q-modules is *tame* if Q admits a finite constant subdivision subordinate to both M and N such that for each constant region I the composite homomorphism $M_I \to M_i \to N_i \to N_I$ does not depend on $i \in I$. The map φ is subanalytic or PL if this constant subdivision is.

Definition 2.7. The vector space $\mathbb{k}[Q] = \bigoplus_{q \in Q} \mathbb{k}$ that assigns \mathbb{k} to every point of the poset Q is a Q-module with identity maps on \mathbb{k} . More generally,

- 1. an upset (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in Q (so $U + Q_+ = U$, when Q is a partially ordered group) determines an *indicator submodule* or upset module $\Bbbk[U] \subseteq \Bbbk[Q]$; and
- 2. dually, a downset (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in Q (so $D-Q_+ = D$, when Q is a partially ordered group) determines an *indicator quotient module* or *downset module* $\Bbbk[Q] \twoheadrightarrow \Bbbk[D]$.

When Q is a subposet of a partially ordered real vector space, an indicator module of either sort is subanalytic or PL if the corresponding upset or downset is of the same type.

Definition 2.8. Let each of S and S' be a nonempty intersection of an upset in a poset Q with a downset in Q, so $\Bbbk[S]$ and $\Bbbk[S']$ are subquotients of $\Bbbk[Q]$. A homomorphism $\varphi : \Bbbk[S] \to \Bbbk[S']$ is *connected* if there is a scalar $\lambda \in \Bbbk$ such that φ acts as multiplication by λ on the copy of \Bbbk in degree q for all $q \in S \cap S'$.

Definition 2.9. Fix any poset Q and a Q-module M.

1. An upset resolution of M is a complex F_{\bullet} of Q-modules, each a direct sum of upset submodules of $\Bbbk[Q]$, whose differential $F_i \to F_{i-1}$ decreases homological degrees, has components $\Bbbk[U] \to \Bbbk[U']$ that are connected, and has only one nonzero homology $H_0(F_{\bullet}) \cong M$.

2. A downset resolution of M is a complex E^{\bullet} of Q-modules, each a direct sum of downset quotient modules of $\Bbbk[Q]$, whose differential $E^i \to E^{i+1}$ increases cohomological degrees, has components $\Bbbk[D'] \to \Bbbk[D]$ that are connected, and has only one nonzero cohomology $H^0(E^{\bullet}) \cong M$.

An upset or downset resolution is called an *indicator resolution* if the up- or down- nature is unspecified. The *length* of an indicator resolution is the largest (co)homological degree in which the complex is nonzero. An indicator resolution

- 3. is *finite* if the number of indicator module summands is finite,
- 4. dominates a constant subdivision of M if the subdivision or encoding is subordinate to each indicator summand, and
- 5. is *subanalytic* or PL if Q is a subposet of a real partially ordered group and the resolution dominates a constant subdivision of the corresponding type.

Definition 2.10. Fix a complex M^{\bullet} of modules over a poset Q.

- 1. M^{\bullet} is *tame* if its modules and morphisms are tame (Definitions 2.3 and 2.6).
- 2. A constant subdivision is *subordinate* to M^{\bullet} if it is subordinate to all of the modules and morphisms therein, and then M^{\bullet} is said to *dominate* the subdivision.
- 3. An upset resolution of M^{\bullet} is a complex of Q-modules in which each F^{i} is a direct sum of upset modules and the components $\Bbbk[U] \to \Bbbk[U']$ are connected, with a homomorphism $F^{\bullet} \to M^{\bullet}$ of complexes inducing an isomorphism on homology.
- 4. A downset resolution of M^{\bullet} is a complex of Q-modules in which each E^i is a direct sum of downset modules and the components $\Bbbk[D] \to \Bbbk[D']$ are connected, with a homomorphism $M^{\bullet} \to E^{\bullet}$ of complexes inducing an isomorphism on homology.

These resolutions are *finite*, or *dominate* a constant subdivision, or are *subanalytic* or PL as in Definition 2.9.

2.4. Syzygy theorem for complexes of poset modules.

Only certain aspects of the full syzygy theorem [Mil20a, Theorem 6.17] are required, so those are isolated here.

Theorem 2.11. A bounded complex M^{\bullet} of modules over a poset Q is tame if and only if it admits one, and hence all, of the following:

- 1. a finite constant subdivision of Q subordinate to M^{\bullet} ; or
- 2. a finite upset resolution; or
- 3. a finite downset resolution; or
- 4. a finite constant subdivision subordinate to any given one of items 2-3.

The statement remains true over any subposet of a real partially ordered group if "tame" and all occurrences of "finite" are replaced by "PL". Moreover, any tame or PL morphism $M^{\bullet} \to N^{\bullet}$ lifts to a similarly well behaved morphism of resolutions as in parts 2 and 3. All of these results hold in the subanalytic case if M^{\bullet} has compact support.

3. Stratifications, topologies, and cones

This section collects the relevant definitions and theorems regarding constructible sheaves from the literature. The sizeable edifice on which the subject is built makes it unavoidable that readers seeing some of these topics for the first time will need to consult the cited sources for additional background. The goal here is to bring readers as quickly as possible to a general statement (Theorem 4.5) while circumscribing the ingredients necessary for its proof in such a way that those familiar with the conjectures of Kashiwara and Schapira, specifically [KS17, Conjecture 3.17] and [KS19, Conjecture 3.20], can skip seamlessly to Section 4 after skimming Section 3 for terminology.

To avoid endlessly repeating hypotheses, and so readers can quickly identify when the same hypotheses are in effect, the blanket assumption henceforth is for Q to satisfy the following, where the positive cone Q_+ is *full* if it has nonempty interior.

Hypothesis 3.1. Q is a real partially ordered group with closed, full, subanalytic Q_+ .

Remark 3.2. Some basic notions are used freely without further comment.

- The notion of simplicial complex here is the one in [KS90, Definition 8.1.1]: a collection Δ of subsets (called *simplices*) of a fixed vertex set that is closed under taking subsets (called *faces*), contains every vertex, and is locally finite in the sense that every vertex of Δ lies in finitely many simplices of Δ. Any simplicial complex Δ has a realization |Δ| as a topological space, with each relatively open simplex |σ| being an open convex set in an appropriate affine space.
- 2. The notion of subanalytic set in an analytic manifold is as in [KS90, §8.2].
- 3. The term *sheaf* on a topological space here means a sheaf of k-vector spaces. Sometimes in the literature this word is used to mean an object in the bounded derived category of sheaves of k-vector spaces; for clarity here, the term *derived sheaf* is always used when an object in the derived category is intended.

3.1. Subanalytic triangulation.

Definition 3.3. Fix a real analytic manifold X.

- 1. A subanalytic triangulation of a subanalytic set $Y \subseteq X$ is a homeomorphism $|\Delta| \xrightarrow{\sim} Y$ such that the image in Y of the realization $|\sigma|$ of the relative interior of each simplex $\sigma \in \Delta$ is a subanalytic submanifold of X.
- 2. A subanalytic triangulation of Y is *subordinate* to a (derived) sheaf \mathscr{F} on X if Y contains the support of \mathscr{F} and (every homology sheaf of) \mathscr{F} restricts to a constant sheaf on the image in Y of every cell $|\sigma|$.

3.2. Subanalytic constructibility.

Definition 3.4. A (derived) sheaf on a real analytic manifold is *subanalytically weakly* constructible if there is a subanalytic triangulation subordinate to it. The word "weakly" is omitted if, in addition, the stalks have finite dimension as k-vector spaces.

Remark 3.5. Readers less familiar with constructibility can safely take Definition 3.4 at face value. For readers familiar with constructibility by other definitions, this characterization is a nontrivial theorem, which rests on the triangulability of subanalytic sets [KS90, Proposition 8.2.5] and other results concerning subanalytic stratification; see [KS90, §8.4] for the full proof of equivalence, especially Theorem 8.4.2, Definition 8.4.3, and part (a) of the proof of Theorem 8.4.5(i) there. Note that the modifier "subanalytically" in Definition 3.4 does not appear in [KS90], because the context there is subanalytic throughout. Also note that it makes no difference whether one takes constructible objects in the derived category or the derived category of constructible objects, since they yield the same result [KS90, Theorem 8.4.5]: every constructible derived sheaf is represented by a complex of constructible sheaves.

The reason to use subanalytic triangulation instead of arbitrary subanalytic stratification is the following, which is a step on the way to a constant subdivision.

Lemma 3.6 ([KS90, Proposition 8.1.4]). For a simplex σ in a subanalytic triangulation subordinate to a constructible sheaf \mathscr{F} , there is a natural isomorphism $\Gamma(|\sigma|, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}_x$ from the sections over $|\sigma|$ to the stalk at every point $x \in \sigma$.

The reason for specifically including the piecewise linear (PL) condition in Section 2 is for its application here, as one of the conjectures is in that setting. For this purpose, the sheaf version of this particularly strong type of constructibility is needed.

Definition 3.7. Fix *Q* satisfying Hypothesis 3.1.

- 1. A subanalytic subdivision (Definition 2.5.1) of Q is *subordinate* to a (derived) sheaf \mathscr{F} on Q if the restriction of \mathscr{F} to every *stratum* (meaning subset in the subdivision) is constant of finite rank.
- 2. If the subanalytic subdivision is PL (Definition 2.5.2) and Q is polyhedral (Definition 2.4.3), then \mathscr{F} is said to be *piecewise linear*, abbreviated *PL*.

Remark 3.8. Definition 3.7.2 is not verbatim the same as [KS19, Definition 2.3], which only requires Q to be a (nondisjoint) union of finitely polyhedra on which \mathscr{F} is constant. However, the notion of PL (derived) sheaf thus defined is the same, since any finite union of polyhedra can be refined to a finite union that is disjoint—that is, a partition. This refinement can be done, for example, by expressing Q as the union of (relatively open) faces in the arrangement of all hyperplanes bounding halfspaces defining the given polyhedra, of which there are only finitely many.

3.3. Conic and Alexandrov topologies.

Definition 3.9. Fix a real partially ordered group Q with closed positive cone Q_+ .

- 1. The *conic topology* on Q induced by Q_+ (or induced by the partial order) consists of the upsets that are open in the ordinary topology on Q.
- 2. The Alexandrov topology on Q induced by Q_+ (or induced by the partial order) consists of all the upsets in Q.

To avoid confusion when it might occur, write

- 1. Q^{con} for the set Q with the conic topology induced by Q_+ ,
- 2. Q^{ale} for the set Q with the Alexandrov topology induced by Q_+ , and
- 3. Q^{ord} for the set Q with its ordinary topology.

Remark 3.10. The conic topology in Definition 3.9 is also known as the γ -topology, where $\gamma = Q_+$ [KS90, KS18, KS19]. The Alexandrov topology makes just as much sense on any poset.

The type of stratification Kashiwara and Schapira specify [KS17, Conjecture 3.17] is not quite the same as subanalytic subdivision in Definition 2.5.1. To be precise, first recall two standard topological concepts.

Definition 3.11. A subset of a topological space Q is *locally closed* if it is the intersection of an open subset and a closed subset. A family of subsets of Q is *locally finite* if each compact subset of Q meets only finitely many members of the family.

Definition 3.12 ([KS17, Definition 3.15]). Fix Q satisfying Hypothesis 3.1.

- 1. A conic stratification of a closed subset $S \subseteq Q$ is a locally finite family of pairwise disjoint subanalytic subsets, called *strata*, which are locally closed in the conic topology and have closures whose union is S.
- 2. The stratification is *subordinate* to a (derived) sheaf \mathscr{F} on Q if S equals the support of \mathscr{F} and the restriction of (each homology sheaf of) \mathscr{F} to every stratum is locally constant of finite rank.

Remark 3.13. A conic stratification is called a γ -stratification in [KS17, Definition 3.15], with $\gamma = Q_+$. The only differences between conic stratification and subanalytic partition of a subset S in Definition 2.5.1 are that

- conic stratifications are only required to be locally finite, not necessarily finite;
- conic strata are required to be locally closed in the conic topology (that is, an intersection of an open upset in Q^{ord} with a closed downset in Q^{ord}); and
- the union need not actually equal all of S, because only the union of the stratum closures is supposed to equal S.

Proposition 3.14. Fix a real partially ordered group Q with closed positive cone Q_+ .

1. The identity on Q yields continuous maps of topological spaces

 $\iota: Q^{\mathrm{ord}} \to Q^{\mathrm{con}} \qquad and \qquad \jmath: Q^{\mathrm{ale}} \to Q^{\mathrm{con}}.$

2. Any sheaf \mathscr{F} on Q^{ord} pulled back from Q^{con} has natural maps

$$\mathscr{F}_q \to \mathscr{F}_{q'} \text{ for } q \preceq q' \text{ in } Q$$

on stalks that functorially define a Q-module $\bigoplus_{q \in Q} \mathscr{F}_q$.

3. Similarly, any sheaf \mathscr{G} on Q^{ale} has natural maps

$$\mathscr{G}_q \to \mathscr{G}_{q'}$$
 for $q \preceq q'$ in Q

on stalks that functorially define a Q-module $\bigoplus_{q \in Q} \mathscr{G}_q$. This functor from sheaves on Q^{ale} to Q-modules is an equivalence of categories.

- If sheaves \$\mathcal{F}\$ on \$Q^{\text{ord}}\$ and \$\mathcal{G}\$ on \$Q^{\text{ale}}\$ are both pulled back from the same sheaf \$\mathcal{E}\$ on \$Q^{\text{con}}\$, then the \$Q\$-modules in items \$2\$ and \$3\$ are the same.
- 5. The pushforward functor j_* is exact, and $j_*j^{-1}\mathscr{E} \cong \mathscr{E}$.

Proof. The maps in item 1 are continuous by definition: the inverse image of any open set is open because the ordinary topology refines each of the target topologies.

For item 2, if $\mathscr{F} = \iota^{-1} \mathscr{E}$ is pulled back to Q^{ord} from a sheaf \mathscr{E} on Q^{con} , then \mathscr{F} has the same stalks as \mathscr{E} (as a sheaf pullback in any context does), so the natural morphisms are induced by the restriction maps of \mathscr{E} from open neighborhoods of q to those of q'.

The result in 3 holds for arbitrary posets; for an exposition in a context relevant to persistence, see [Cur14, Theorem 4.2.10 and Remark 4.2.11] and [Cur19].

For item 4, the stalks $\mathscr{F}_q = \mathscr{E}_q = \mathscr{G}_q$ are the same.

For item 5, exactness is proved in passing in the proof of [BP19, Lemma 3.5], but it is also elementary to check that a surjection $\mathscr{G} \to \mathscr{G}'$ of sheaves on Q^{ale} yields a surjection of stalks for the pushforwards to Q^{con} because direct limits (filtered colimits) are exact. That $j_*j^{-1}\mathscr{E} \cong \mathscr{E}$ is because the natural morphism is the identity on stalks. \Box

3.4. Conic microsupport. The microsupport of a (derived) sheaf on an analytic manifold X is a certain closed conic isotropic subset of the cotangent bundle T^*X . The notion of microsupport is a central player in [KS90], to which the reader is referred for background on the topic. However, although the main result in this section (Theorem 4.5) is stated in terms of microsupport, the next theorem allows the reader to ignore it henceforth, as pointed out by Kashiwara and Schapira themselves [KS18, Remark 1.9], by immediately translating to the more elementary context of sheaves in the conic topology in Section 3.3.

Theorem 3.15 ([KS18, Theorem 1.5 and Corollary 1.6]). Fix Q satisfying Hypothesis 3.1. The pushforward ι_* of the map ι from Proposition 3.14.1 induces an equivalence from the category of sheaves with microsupport contained in the negative polar cone Q_+^{\vee} to the category of sheaves in the conic topology. The pullback ι^{-1} is a quasi-inverse. The same assertions hold for the bounded derived categories.

Remark 3.16. The pushforward ι_* and the pullback ι^{-1} have concrete geometric descriptions. Since ι is the identity on Q, the pushforward of a sheaf \mathscr{F} on Q has sections

$$\Gamma(U,\iota_*\mathscr{F}) = \Gamma(U,\mathscr{F})$$

for any open upset U, where "open upset" means the same things as "upset that is open in the usual topology" and "subset that is open in the conic topology". On the other hand, over any convex ordinary-open set \mathcal{O} , the pullback to the ordinary topology of a sheaf \mathscr{E} in the conic topology has sections

$$\Gamma(\mathcal{O}, \iota^{-1}\mathscr{E}) = \Gamma(\mathcal{O} + Q_+, \mathscr{E}),$$

namely the sections of \mathscr{E} over the upset generated by \mathcal{O} [KS90, (3.5.1)].

Remark 3.17. What Theorem 3.15 does in practice is allow a given (derived) sheaf with microsupport contained in the negative polar cone Q_+^{\vee} to be replaced with an isomorphic object that is pulled back from the conic topology induced by the partial order. The reason for mentioning the notion of microsupport at all is to emphasize that constructibility in the sense of Definition 3.4 requires the ordinary topology. This may seem a fine distinction, but the conjectures of Kashiwara and Schapira proved in Section 5 entirely concern the transition from the ordinary to the conic topology, so it is crucial to be clear on this point.

In view of Remark 3.17, discussion of constructibility for sheaves on conic topologies requires the following. The ad hoc nature of this definition is justified by Theorem 4.5'.

Definition 3.18. Fix Q satisfying Hypothesis 3.1. A constructible conic sheaf on Q is a sheaf in the conic topology Q^{con} whose pullback via ι^{-1} is subanalytically constructible.

4. Resolutions of constructible sheaves

Definition 4.1. Fix *Q* satisfying Hypothesis 3.1.

- 1. A subanalytic upset sheaf on Q is the extension by zero of the rank 1 constant sheaf on an open subanalytic upset in Q^{ord} .
- 2. A subanalytic downset sheaf on Q is the pushforward of the rank 1 locally constant sheaf on a closed subanalytic downset in Q^{ord} .
- 3. A subanalytic upset resolution of a complex \mathscr{F}^{\bullet} of sheaves on Q^{ord} is a homomorphism $\mathscr{U}^{\bullet} \to \mathscr{F}^{\bullet}$ of complexes inducing an isomorphism on homology, with each \mathscr{U}^{i} being a direct sum of subanalytic upset sheaves.

4. A subanalytic downset resolution of a complex \mathscr{F}^{\bullet} of sheaves on Q^{ord} is a homomorphism $\mathscr{F}^{\bullet} \to \mathscr{D}^{\bullet}$ of complexes inducing an isomorphism on homology, with each each \mathscr{D}^i being a direct sum of subanalytic downset sheaves.

Either type of resolution is

- *finite* if the total number of summands across all homological degrees is finite;
- *PL* if *Q* is polyhedral and the upsets or downsets are PL.

Proposition 4.2. Fix an upset U in a real partially ordered group Q with closed positive cone. If U° is the interior of U in Q^{ord} , then the sheaves on Q^{ale} corresponding to $\Bbbk[U]$ and $\Bbbk[U^{\circ}]$ push forward to the same sheaf on Q^{con} .

Proof. The stalk at q of any sheaf on Q^{con} is the direct limit over points $p \in q-Q_+^{\circ}$ of the sections over $p+Q_+^{\circ}$. In the case of the pushforward of the sheaf on Q^{ale} corresponding to an upset module, these sections are \Bbbk if p lies interior to the upset and 0 otherwise. The result holds because the upsets U and U° have the same interior, namely U° . \Box

Proposition 4.3. Fix a downset D in a real partially ordered group Q with closed positive cone. If \overline{D} is the closure of D in Q^{ord} , then the sheaves on Q^{ale} corresponding to $\Bbbk[D]$ and $\Bbbk[\overline{D}]$ push forward to the same sheaf on Q^{con} .

Proof. Calculating stalks as in the previous proof, in the case of the pushforward of the sheaf on Q^{ale} corresponding to a downset module, the sections over $p + Q^{\circ}_{+}$ are \Bbbk if p lies interior to the downset and 0 otherwise. The result holds because the downsets D and \overline{D} have the same interior.

Remark 4.4. The fundamental difference between Alexandrov and conic topologies reflected by Propositions 4.2 and 4.3 is explored in detail by Berkouk and Petit [BP19].

Here is the main result. It is little more than a restatement of the relevant part of Theorem 2.11 in the language of sheaves.

Theorem 4.5. Fix Q satisfying Hypothesis 3.1. If \mathscr{F}^{\bullet} is a complex of compactly supported subanalytically constructible sheaves on Q^{ord} with microsupport in the negative polar cone Q_{+}^{\vee} then \mathscr{F}^{\bullet} has a finite subanalytic upset resolution and a finite subanalytic downset resolution. If Q is polyhedral and \mathscr{F}^{\bullet} is PL, then \mathscr{F}^{\bullet} has PL such resolutions.

Proof. Using Theorem 3.15, assume that \mathscr{F}^{\bullet} is pulled back to Q^{ord} from Q^{con} , say $\mathscr{F}^{\bullet} = \iota^{-1} \mathscr{E}^{\bullet}$. Since \mathscr{F}^{\bullet} has compact support, any subordinate subanalytic triangulation (Definition 3.3) afforded by Definition 3.4 is necessarily finite because it is locally finite. The complex $F^{\bullet} = \bigoplus_{q \in Q} \mathscr{F}_q^{\bullet}$ of Q-modules that comes from Proposition 3.14.2 is tamed by the triangulation, which is a constant subdivision (Definition 2.2) because

• simplices are connected, so locally constant sheaves on them are constant, and

• $\Gamma(|\sigma_p|, \mathscr{F}^i) \to \mathscr{F}^i_p \to \mathscr{F}^i_q \leftarrow \Gamma(|\sigma_q|, \mathscr{F}^i)$ is locally constant—and hence constant, as simplices are connected—when $p \preceq q$ in Q. Here σ_x is the simplex containing x, the middle arrow is from Proposition 3.14.2, and the outer arrows are the natural isomorphisms from Lemma 3.6.

Hence the complex F^{\bullet} of Q-modules has resolutions of the desired sort by Theorem 2.11. Viewing any of these resolutions as a complex of sheaves on Q^{ale} via Proposition 3.14.3, push it forward from the Alexandrov topology to the conic topology via the exact functor j_* in Proposition 3.14.5. The resulting complex of sheaves on Q^{con} is a resolution of a complex isomorphic to \mathscr{E}^{\bullet} by Proposition 3.14.4 and 3.14.5. The upsets or downsets in the summands of the resolution may as well be assumed open or closed, respectively, by Propositions 4.2 or 4.3. The proof is concluded by pulling back the resolution from Q^{con} to Q^{ord} via the equivalence of Theorem 3.15.

Theorem 4.5'. Fix Q satisfying Hypothesis 3.1. If \mathscr{F}^{\bullet} is a complex of compactly supported sheaves in the conic topology Q^{con} then the following are equivalent.

- 1. \mathscr{F}^{\bullet} is constructible (Definition 3.18).
- 2. \mathscr{F}^{\bullet} has a finite subanalytic upset resolution.
- 3. \mathscr{F}^{\bullet} has a finite subanalytic downset resolution.

The implications $2 \Rightarrow 1$ and $3 \Rightarrow 1$ do not require compact support for \mathscr{F}^{\bullet} . If Q is polyhedral and \mathscr{F}^{\bullet} is PL, then all of these claims hold with "PL" in place of "subanalytic".

Proof. That $1 \Rightarrow 2$ and $1 \Rightarrow 3$ follows from Theorems 4.5 and 3.15. The opposite directions are by the definition and foundational results surrounding constructibility in Definition 3.4 and Remark 3.5.

Remark 4.6. While the notion of a sheaf with microsupport contained in the negative polar cone of Q_+ is equivalent to the notion of a sheaf in the conic topology, the notion of constructibility has until now only been available on the microsupport side, where simplices from arbitrary subanalytic triangulations achieve constancy of the sheaves in question. Theorem 4.5' makes precise the assertion that constructibility can be detected entirely with the more rigid conic topology, without the flexibility of appealing to arbitrary subanalytic triangulations.

Remark 4.7. Theorem 4.5 assumes compact support to get finite instead of locally finite subdivisions. The application in Section 5 to constructible sheaves without any assumption of compact support yields a locally finite subdivision by reducing to the case of compact support.

Remark 4.8. The final sentences of Theorems 4.5 and 4.5' are true with "polyhedral" and "PL" all replaced by "semialgebraic", with the same proofs, as long as the definitions of these semialgebraic concepts in the constructible sheaf setting are made appropriately. The semialgebraic constructible sheaf versions are not treated here because they are not relevant to the conjectures proved in Section 5.

5. Stratifications from constructible sheaves

Corollary 5.1 ([KS19, Conjecture 3.20]). Fix Q satisfying Hypothesis 3.1 with Q_+ polyhedral. If \mathscr{F}^{\bullet} is a PL object in the derived category of compactly supported constructible sheaves on Q^{ord} with microsupport contained in the negative polar cone Q_+^{\vee} then the isomorphism class of \mathscr{F}^{\bullet} is represented by a complex that is a finite direct sum of constant sheaves on bounded polyhedra that are locally closed in the conic topology.

Proof. The statement would directly be a special case of Theorem 4.5 were it not for the boundedness hypothesis on the polyhedra, since either a PL upset or PL downset resolution would satisfy the conclusion. That said, boundedness is easy to impose: since \mathscr{F}^{\bullet} has compact support, and the resolution has vanishing homology outside of the support of \mathscr{F}^{\bullet} , each upset or downset sheaf can be restricted to the support of \mathscr{F}^{\bullet} and extended by 0.

As in Theorem 4.5', Corollary 5.1 can be restated using constructible conic sheaves.

Corollary 5.1'. Fix Q satisfying Hypothesis 3.1 with Q_+ polyhedral. \mathscr{F}^{\bullet} is a PL object in the derived category of compactly supported constructible conic sheaves if and only if the isomorphism class of \mathscr{F}^{\bullet} is represented by a complex that is a finite direct sum of constant sheaves on bounded polyhedra that are locally closed in the conic topology.

Corollary 5.2 ([KS17, Conjecture 3.17]). Fix Q satisfying Hypothesis 3.1. If a compactly supported derived sheaf with microsupport in the negative polar cone Q_+^{\vee} is sub-analytically constructible, then its support has a subordinate conic stratification.

Proof. Part (ii) in the proof of [KS18, Theorem 3.17] reduces to the case where the support of the given derived sheaf is compact. The argument is presented in the case where Q is polyhedral and the derived sheaf is PL, but the argument works verbatim for Q satisfying Hypothesis 3.1, without any polyhedral or PL assumptions, because the requisite lemma, namely [KS18, Lemma 3.5]—and indeed, all of [KS18, §3.1]—is stated and proved in this non-polyhedral generality. So henceforth assume the given derived sheaf has compact support.

Remark 3.5 allows the assumption that the given derived sheaf is represented by a complex \mathscr{F}^{\bullet} of constructible sheaves. Theorem 4.5 produces a subanalytic indicator resolution, which for concreteness may as well be an upset resolution. Each upset that appears as a summand in the resolution partitions Q into the upset itself, which is open subanalytic, and its complement, which is a closed subanalytic downset. The common refinement of the partitions induced by the finitely many open subanalytic upsets in the resolution and their closed subanalytic downset complements is a partition of Q into finitely many strata such that

- each stratum is subanalytic and locally closed in the conic topology, and
- the restriction of \mathscr{F}^{\bullet} to each stratum has constant homology.

The strata with nonvanishing homology form the desired conic stratification.

As before, Corollary 5.2 can be restated in terms of constructible conic sheaves.

Corollary 5.2'. Fix Q satisfying Hypothesis 3.1. The support of any compactly supported constructible derived conic sheaf has a subordinate conic stratification.

Remark 5.3. The reference in [KS17, Conjecture 3.17] to a cone λ contained in the interior of the positive cone union the origin appears to be unnecessary, since (in the notation there) any γ -stratification is automatically a λ -stratification by [KS17, Definition 3.15] and the fact that $\lambda \subseteq \gamma$.

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