The reach of subsets of manifolds

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Abstract

Kleinjohann [1] and Bangert [2] extended the reach $\mathbf{rch}(\boldsymbol{\mathcal{S}})$ from subsets $\boldsymbol{\mathcal{S}}$ of Euclidean space to the reach $\mathbf{rch}_{\boldsymbol{\mathcal{M}}}(\boldsymbol{\mathcal{S}})$ of subsets $\boldsymbol{\mathcal{S}}$ of Riemannian manifolds $\boldsymbol{\mathcal{M}}$, where $\boldsymbol{\mathcal{M}}$ is smooth (we'll assume at least $\boldsymbol{C^3}$). Bangert showed that sets of positive reach in Euclidean space and Riemannian manifolds are very similar. In this paper we introduce a slight variant of Kleinjohann's and Bangert's extension and quantify the similarity between sets of positive reach in Euclidean space and Riemannian manifolds in a new way: Given $\boldsymbol{p} \in \boldsymbol{\mathcal{M}}$ and $\boldsymbol{q} \in \boldsymbol{\mathcal{S}}$, we bound the local feature size (a local version of the reach) of its lifting to the tangent space via the inverse exponential map ($\mathbf{exp}_{\boldsymbol{p}}^{-1}(\boldsymbol{\mathcal{S}})$) at $\boldsymbol{q}$, assuming that $\mathbf{rch}_{\boldsymbol{\mathcal{M}}}(\boldsymbol{\mathcal{S}})$ and the geodesic distance $\boldsymbol{d_{\mathcal{M}}(p, q)}$ are bounded. These bounds are motivated by the importance of the reach and local feature size to manifold learning, topological inference, and triangulating manifolds and the fact that intrinsic approaches circumvent the curse of dimensionality.

Keywords: Reach, Manifolds, surfaces, comparison theory

MSC Classification: 53-08 , 53A99 , 68U05 , 65D18

1 Introduction

Motivation. The reach of subsets of Euclidean space was introduced by Federer [3] and can be defined from the medial axis introduced later by Blum [4] (although it has also been studied by Federer). The medial axis of a set $\mathcal{S}$ in Euclidean space consists of the points for which there is no unique closest point on $\mathcal{S}$. The reach of $\mathcal{S}$, denoted $\mathrm{rch}(\mathcal{S})$, is defined as the distance between the medial axis and the original set. It is therefore the largest distance such that if a point $p \in \mathbb{R}^d$ is moved at most this distance from $\mathcal{S}$, then there exists a unique closest point on $\pi_{\mathcal{S}}(p) \in \mathcal{S}$. Here we write $\pi_{\mathcal{S}}$ for the closest point projection onto $\mathcal{S}$.

The reach is used to quantify the complexity of geometric sets of $\mathbb{R}^d$. In particular, from the reach, one can infer bounds on the local curvature [5] of submanifolds. The reach also provides more global information, such as how close different parts of a set $\mathcal{S}$ lie to one another. It has been extensively used to define sampling conditions on submanifolds of $\mathbb{R}^d$, to quantify how difficult it is to infer their topology (see for example [6–10]), to triangulate them (see for example [6, 11–14]), or learn them (see for example [15–19]).

Due to its importance and ubiquity, the reach has been redefined several times and appears under various names in the literature. It was called the condition number by Niyogi, Smale and Weinberger [20]. A local version of the reach, named the *local feature size*, was introduced in the computational geometry community by Amenta and Bern in their seminal work on surface reconstruction [21]. It is the distance of a point to the medial axis and was denoted $\mathrm{reach}(\mathcal{S}, p)$ by Federer [3, Definition 4.1].

Extending the notion of reach to subsets $\mathcal{S}$ of Riemannian manifolds $\mathcal{M}$ is a question that immediately shows up when one considers triangulating manifolds with boundary, submanifolds of Riemannian manifolds, or stratified manifolds, as required in dynamical systems, physics, and chemistry. Examples of stratified manifolds are the invariant sets that appear in dynamical systems [22] and the conformation spaces of molecules [23].

One particular motivation is the search for anisotropic triangulations for numerical partial differential equations, see e.g. [24–31], based on Riemannian metrics as in [32] but with interfaces or boundaries. To triangulate these complicated spaces one needs to understand the geometry of the interfaces and boundaries with respect to the Riemannian metric.

Another motivation comes from the reconstruction of submanifolds with boundary embedded in high dimensional Euclidean space. Working with such data one generally faces the curse of dimensionality, which could be avoided by ignoring the embedding completely and only working with the intrinsic geometry.

Contributions. In the early 1980s Kleinjohann [1] and Bangert [2] extended the definition of the reach to subsets $\mathcal{S}$ of Riemannian manifolds $\mathcal{M}$, see also [33]. The reach of a submanifold of a Riemannian manifold is similar to the definition in Euclidean space, which is roughly the distance to the medial axis, i.e. the set of points of $\mathcal{M}$ that have more than one closest point on $\mathcal{S}$. Kleinjohann was interested in (locally) convex subsets of Riemannian manifolds and their differential geometric properties, and Bangert studied the relation between the convex functions and (locally) convex sets. Their definition of the reach is not fully satisfactory for the applications mentionned above. Indeed, consider the cylinder $\mathbb{S}^1 \times \mathbb{R} = \mathcal{M}$ and the submanifold $\mathbb{S}^1 \times \{0\} = \mathcal{S}$. According to the definition of Kleinjohann and Bangert, the reach of $\mathbb{S}^1 \times \{0\}$ is infinite. However, at scales larger than 2π the 'balls' tangent to $\mathbb{S}^1 \times \{0\}$ are no longer topological balls, which is very inconvenient for practical applications.

We will introduce a slight variant of the reach for submanifolds of arbitrary smooth Riemannian manifolds. We add the condition that we look at neighbourhoods smaller than the (global) injectivity radius.[1] This assumption makes comparison results from both Riemannian geometry and computational geometry more straightforward to apply.

Our main result, Theorem 4.11, allows one to lift via the exponential map the results known on the reach of subsets of $\mathbb{R}^d$ to submanifolds of Riemannian manifolds. More specifically, given $p \in \mathcal{M}$ and $q \in \mathcal{S}$, we give explicit bounds on the local feature size (local reach) of its lifting to the tangent space via the inverse exponential map $(\exp_p^{-1}(\mathcal{S}))$ at q, assuming that $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ and the geodesic distance $d_{\mathcal{M}}(p,q)$ are bounded. We stress that the explicit bounds are of key importance, the fact that the local feature size is non-zero is easy to see, i.e. it is a direct consequence of smoothness. This quantified approach to the problem also distinguishes us from previous results in the literature [1, 2, 33] where the focus was on existence. Even though the expression for our bound remains non-trivial we provide closed expressions for the bounds. The computation of our bounds rests on two pillars: Comparison theory in particular the Rauch comparison theorem and the Kaul's bounds (see Section 4.2) and Federer's result on the behavior of the reach under an ambient diffeomorphism (Theorem 4.9). We note that our bounds could be improved (by using Kaul's bounds directly instead of a simplified version, see Section 4.2).

We see our bounds as a first major step in the long term quest of efficient algorithms to triangulate stratified Riemanniann manifolds. This will allow to extend the work in [14] by locally lifting a submanifold of a Riemannian manifold $\mathcal{M}$ to the tangent space of $\mathcal{M}$ using the exponential map, see Future work in Section 5. A similar but more complicated approach can be used for manifolds with boundary and stratifolds. The reach of $\mathcal{S}$ and $\exp^{-1}(\mathcal{S})$ will be used to quantify the conditions under which we obtain a triangulation.

2 The reach

Similar to the definition for manifolds embedded in Euclidean space we define the reach for compact subsets $\mathcal{S} \subset \mathcal{M}$, that is we make the global assumption that $\mathcal{S}$ is compact.

Definition 2.1 (The reach) *We let the medial axis $ax_{\mathcal{M}}(\mathcal{S})$ be the set of points in $\mathcal{M}$ that do not have a unique closest point on $\mathcal{S}$, with respect to the Riemannian metric. We denote the projection of a point x in $\mathcal{M}$ on the closest point on $\mathcal{S}$ by $\pi_{\mathcal{S}}(x)$. If*

[1] Recall that the injectivity radius is the largest radius such that the exponential map restricted to a ball of this radius is still a homeomorphism onto its image. The exponential map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ sends vectors to geodesics through p while preserving lengths and angles at p.

x lies in the medial axis this map is set valued. The pre-reach $\mathrm{prch}_{\mathcal{M}}(\mathcal{S})$ *is then the shortest distance between* $\mathrm{ax}_{\mathcal{M}}(\mathcal{S})$ *and* $\mathcal{S}$*. We now define the reach* $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ *to be*

$$\min\{\mathrm{prch}_{\mathcal{M}}(\mathcal{S}), \iota_{\mathcal{M}}\}, \tag{1}$$

where $\iota_{\mathcal{M}}$ *is the injectivity radius of* $\mathcal{M}$*.*

The injectivity radius is the largest radius r such that for any x, $\exp_x$ restricted to the open ball centred at x with radius r in $T_x\mathcal{M}$ is a diffeomorphism onto its image. Adding this bound to our definition of the reach is essential, because we would like to have that a tangent ball to $\mathcal{S}$ is indeed a topological ball.

We can also define a local version of the reach we call *local feature size* in accordance with the terminology used for subsets of Euclidean space [21]. For a point $y \in \mathcal{S}$, it is defined as the minimum of the injectivity radius at y and of the distance of y to $\mathrm{ax}_{\mathcal{M}}(\mathcal{S})$.

Remark 2.2 *In the mathematics literature, the closure of the medial axis is sometimes called the cut locus in the context of Riemannian manifolds, see [34, Section 2] or [35] (often the cut locus only refers to the medial axis of a single point in a Riemannian manifold). We have chosen to use the terminology and concept most common in the computational geometry community in view of future applications. Bangert [2] referred to what is here called the pre-reach as the reach. Kleinjohann seems to have been unaware of the work of Federer when writing [1].*

Throughout this paper we'll assume that $\mathcal{M}$ is smooth, by which we mean at least C^3. This is essential because we rely on the result of Kaul [36], see Section 4.2 below, which is based on bounds on the derivative of the Riemann curvature tensor. Because the Riemann curvature tensor involves second order derivatives the manifold has to be C^3. However, we conjecture that most of the results go through (with some minor adjustments) for $C^{2,1}$ manifolds, that is manifolds whose second order derivative is Lipschitz.

3 Smooth submanifolds have positive reach

In this section we restrict ourselves to a *smooth* submanifold $\mathcal{S}$ of a Riemannian manifold $\mathcal{M}$. It suffices for the submanifold $\mathcal{S}$ to be C^2, in this section. In Corollary 4.13 we'll see (by reduction to the Euclidean case) that this can be weakened to $C^{1,1}$, however this is far from trivial. We shall denote the normal space of $\mathcal{S}$ at a point x by $N_x\mathcal{S}$, and the bundle by $N\mathcal{S}$.

We now first need a counterpart of Theorem 4.8.8 of Federer [3] in the setting of smooth submanifolds $\mathcal{S}$ of Riemannian manifolds:

Theorem 3.1 (Tubular neighbourhood) *Let* $B_{N_p\mathcal{S}}(r)$*, be the ball of radius* r *centred at* p *in the normal space* $N_p\mathcal{S} \subset T_p\mathcal{M}$ *of a* C^2 *manifold with reach* $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$*, where* $r < \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$*. For every point* $x \in \exp_p(B_{N_p\mathcal{S}}(r))$*, we have* $\pi_{\mathcal{S}}(x) = p$*.*

Variants of this result are well known in differential topology and Riemannian geometry, but a proof is included for completeness. Kleinjohann [33, Section 3] proves related statements, but uses a somewhat different definition of normal. The proof of Theorem 3.1 is not so difficult and mainly follows Hirsch [37, Section 4.5] and Spivak [38, Appendix I of Chapter 9] with some variations.

We start with [38, Lemma 19 of Chapter 9]:

Lemma 3.2 *Let X be a compact metric space and $X_0 \subset X$ a closed subset. Let $f : X \to Y$ be a local homeomorphism such that $f\,|_{X_0}$ is injective. Then there exists a neighbourhood U of X_0 such that $f\,|_U$ is injective.*

With this lemma we can prove an embedding result, for which we have to make the following definition:

Definition 3.3 *Let NB_ϵ denote the ϵ-neighbourhood in $N\mathcal{S}$ of $\mathcal{S}$, that is the neighbourhood of all points closer than ϵ to $\mathcal{S}$, where we identify $\mathcal{S} \subset N\mathcal{S}$ via the zero section. Moreover write $f : NB_\epsilon \to \mathcal{M}$, for the map defined by sending $\mathcal{S} \subset N\mathcal{S}$ to $\mathcal{S}$ and fiberwise sending $N_p\mathcal{S}$ to $\exp_p(N_p\mathcal{S})$.*

Lemma 3.4 *There exists an $\epsilon > 0$ such that $f : NB_\epsilon \to \mathcal{M}$ is a (global) homeomorphism onto its image, that is an embedding.*

The proof combines arguments from Section 4.5 of [37] with Appendix I of Chapter 9 of [38], where small variations of this statement can be found.

Proof of Lemma 3.4 We first note that the fiberwise restriction of f to $B_{N_p\mathcal{S}}(r)$ is a diffeomorphism for each $r < \iota_{\mathcal{M}}$. Because $\mathcal{M}$ is smooth, f is smooth and we can consider the derivative $T_{(p,0)}f$ at a point $(p,0) \in N\mathcal{S}$. The tangent space splits as follows $T_{(p,0)}N\mathcal{S} = T_p\mathcal{S} \oplus N_p\mathcal{S}$. $T_{(p,0)}f$ is the identity if restricted to the tangent space of $\mathcal{M}$ as well as to a fiber. This gives that f is an immersion and in particular a local homeomorphism on its image because the codimension is zero. Due to Lemma 3.2, $f : NB_\epsilon \to \mathcal{M}$ is injective for some sufficiently small $\epsilon > 0$. $\square$

Note that in the proof of Lemma 3.4, we used the fact that $\mathcal{S}$ is C^2 because we considered the derivative of the normal (note that the tangent space is found by taking first derivatives and therefore the derivative of the normal involves second order differentiation).

We now define $N\mathcal{S}(r)$ fiberwise as those points in $N_p\mathcal{S}$ that are at a distance r from p. We refer to r as the radius of the tubular neighbourhood. For any smooth manifold $\mathcal{S}$ embedded in $\mathcal{M}$ and $x \in \mathcal{M}$, we know that the minimizing geodesic from x to $\pi_{\mathcal{S}}(x)$ is normal to $\mathcal{S}$ at $\pi_{\mathcal{S}}(x)$, as a direct consequence of the Gauss lemma (the Gauss lemma that refers to the exponential map). It follows that, for all ϵ such that $f : NB_\epsilon \to \mathcal{M}$ is a homeomorphism, each point in $f(N\mathcal{S}(r))$ is a distance r from $\mathcal{S}$, for all $0 < r < \epsilon$.

Because $\mathcal{S}$ is compact, so is $N\mathcal{S}(r)$ as is the closed r-neighbourhood $\overline{NB}_r$. Clearly f is continuous. In general, a continuous bijection from a compact to a Hausdorff space is a homeomorphism. This means that the map f from both $N\mathcal{S}(r)$ and the closed r-neighbourhood $\overline{NB}_r$ to their images are homeomorphisms, that is embeddings, if and only if f from both $N\mathcal{S}(r)$ and $\overline{NB}_r$ are injective.

We define the reach $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ of the submanifold $\mathcal{S} \subset \mathcal{M}$ to be the infimum of all radii r such that f from both $N\mathcal{S}(r)$ and $\overline{NB}_r$ are no longer injective. We have that $\mathrm{rch}_{\mathcal{M}}(\mathcal{S}) \geq \epsilon > 0$. By the same argument as above, the distance from a point in $N\mathcal{S}(r)$ to $\mathcal{S}$ is r for all r such that f restricted to $N\mathcal{S}(r)$ is injective. Or equivalently, the distance from a point in $f(N\mathcal{S}(r))$ to $\mathcal{S}$ is r, for all $r < \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$. Because the distance function to any closed set is continuous (it is even Lipschitz, see [3, Theorem 4.8(1)]), this in fact holds also for $r = \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$.

In summary we have,

Lemma 3.5 *The reach $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ is equal to the infimum over all radii r, with $r \leq \iota_{\mathcal{M}}$, such that f restricted to $N\mathcal{S}(r)$ and $\overline{NB}_r$ are no longer homeomorphisms to their images. For all $r \leq \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$, all points in $f(N\mathcal{S}(r))$ are a distance r from $\mathcal{S}$.*

In particular we have proven Theorem 3.1.

Here and in the following section we'll often speaking about (open or closed) ball $\mathcal{B}$ being tangent to a submanifold $\mathcal{S} \subset \mathcal{M}$. By this we mean that $\partial\overline{\mathcal{B}}$ and $\mathcal{S}$ have a point p in common such that the tangent spaces $T_p\mathcal{S} \subset T_p\mathcal{M}$ and $T_p\partial\overline{\mathcal{B}} \subset T_p\mathcal{M}$ satisfy $T_p\mathcal{S} \subset T_p\partial\overline{\mathcal{B}}$. Note that Theorem 3.1 immediately yields,

Corollary 3.6 *The interior of any ball $\mathcal{B} \subset \mathcal{M}$ that is tangent to $\mathcal{S}$ at $p \in \mathcal{S}$ and whose radius r satisfies $r \leq \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ does not intersect $\mathcal{S}$.*

Proof Let $r < \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$. Suppose that the intersection of $\mathcal{S}$ and the interior of the ball is not empty, then $\pi_{\mathcal{S}}(c) \neq p$, contradicting Theorem 3.1. The result for $r = \mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ now follows by taking the limit. □

4 Main result: Bounds on the local feature size

Our main result is to link our definition of the reach with the local feature size of submanifolds of Euclidean space in an explicit and quantified manner. Namely given a set of positive reach $\mathcal{S}$ in a Riemannian manifold $\mathcal{M}$, $p \in \mathcal{M}$ and $q \in \mathcal{S}$, we bound the (Euclidean) local feature size of $\exp_p^{-1}(\mathcal{S}) \subset T_p\mathcal{M}$ at q, assuming that the geodesic distance $d_{\mathcal{M}}(p, q)$, the curvature and derivatives of the curvature are bounded.

As a nice byproduct of the main result of this paper and [39], see also [40, 41], we see that a topologically embedded submanifold (with boundary) of a Riemannian manifold has positive reach (and a positive reach boundary) if and only if it is $C^{1,1}$ (with a $C^{1,1}$ boundary). Lytchak's results [40, 41] do include the Riemannian case, but the work by Lytchak [40, 41] relies in an essential manner on results from CAT(k)-theory, while the reduction in this paper requires only some minimal bounds on Christoffel symbols and is quantified.

4.1 Approach

The lower bounds on the reach of $\exp_p^{-1}(\mathcal{S})$, in terms of the reach of $\mathcal{S}$ and geometric properties of the manifold, follow by considering tangent balls to $\mathcal{S}$ that do not contain points of $\mathcal{S}$, from various viewpoints, namely from the manifold and from various tangent spaces. The argument will be explained in detail using Figure 1.

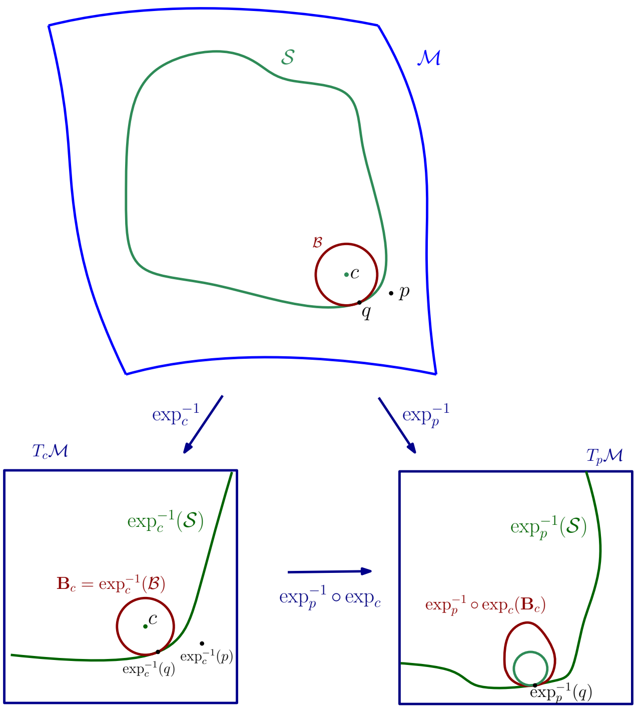

Fig. 1 An overview of the approach.

Let $\mathcal{B}$ be a geodesic ball in $\mathcal{M}$ centered at c, tangent to $\mathcal{S}$ at a point q and with radius the reach of $\mathcal{S}$. As in Euclidean space and by Corollary 3.6, $\mathcal{B}$ has an empty intersection with $\mathcal{S}$. $\mathcal{B}$ is indicated in red in the top of Figure 1.

The exponential map $\exp_c$ gives coordinates on a neighbourhood of the manifold, as does $\exp_p$. We will use the inverse of the exponential maps $\exp_c$

and $\exp_p$, and write $\mathbf{B}_c = \exp_c^{-1} \mathcal{B}$ and $\mathcal{B}_p = \exp_p^{-1} \mathcal{B}$ for the images (in $T_c\mathcal{M}$ and in $T_p\mathcal{M}$ respectively) of the geodesic ball $\mathcal{B}$. Note that $\mathbf{B}_c$ is an Euclidean ball which implies that its reach is equal to its radius, which is also the (geodesic) radius of $\mathcal{B}$. See bottom left part of Figure 1. Note also that $\mathbf{B}_c$ is tangent to $\mathcal{S}_c = \exp_c^{-1}(\mathcal{S})$ and that $\mathcal{B}_p$ is tangent to $\mathcal{S}_p = \exp_p^{-1}(\mathcal{S})$. The composition $\exp_p^{-1} \circ \exp_c$ gives a transformation between the two coordinate neighbourhoods, as indicated by the arrow from the bottom left to right in Figure 1.

We stress that we follow this convention for the notation of balls: geodesic balls on the manifold are denoted by calligraphic capital letter $\mathcal{B}$ without subscript. The image of a set under the inverse exponential map to a tangent space are denoted in the same way as the original set, but with an extra subscript indicating the point where the tangent space is taken. Euclidean balls in the tangent space will be denoted by lower case bold letters $\mathbf{b}_p$, with a subscript indicating the point where we take the tangent space. These balls are (except in very particular cases) not the image of a geodesic ball. The one exception to this rule is the ball $\mathbf{B}_c$, which is both the image of a geodesic ball and a Euclidean ball and therefore is denoted by a capital letter, like the goedesic balls, but bold like the Euclidean balls. All balls can be assumed to be open.

Thanks to the Toponogov or Rauch comparison theorem and a higher order variant (namely Kaul's bounds on the Christoffel symbols, see Section 4.2 below), we have bounds on the metric as well as on the derivatives of the metric, both expressed in the coordinates induced by the exponential maps $\exp_p$ and $\exp_c$. The bounds on the metric and its derivatives can then be used to give bounds on the first and second order derivatives of the transformation $\exp_p^{-1} \circ \exp_c$. Thanks to a result by Federer [3] one can find a bound on the reach after the transformation, based on the reach of the original. This gives bounds on the reach (in $T_p\mathcal{M}$) of $\mathcal{B}_p$, that is on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$.

Consider the Euclidean ball $\mathbf{b}_p \subseteq \mathcal{B}_p \subset T_p\mathcal{M}$ with radius $\mathrm{rch}(\mathcal{B}_p)$ that is tangent to $\partial\mathcal{B}_p$ at $\exp_p^{-1}(q)$, where ∂ indicates the boundary.[2] The Euclidean ball $\mathbf{b}_p$ is indicated in the bottom right of Figure 1, by the green Euclidean ball inside the red deformed ball. The Euclidean ball $\mathbf{b}_p$ is also tangent to $\mathcal{S}_p$ at $\exp_p^{-1}(q)$ and, since the interior of $\mathcal{B}$ does not intersect $\mathcal{S}$, $\mathcal{B}_p$ does not intersect $\mathcal{S}_p$ either nor does $\mathbf{b}_p$.

The reach of $\mathcal{S}_p$ is lower bounded by the minimum radius of any such Euclidean ball $\mathbf{b}_p$, that is

$$\mathrm{rch}(\mathcal{S}_p) \geq \min_{q\in\mathcal{S}} \min_{\mathcal{B}} \mathrm{rch}(\mathcal{B}_p) = \min_{q\in\mathcal{S}} \min_{\mathcal{B}} \mathrm{radius}(\mathbf{b}_p),$$

where the minimum over $\mathcal{B}$ is a minimum over geodesic balls in $\mathcal{M}$ of radius $\mathrm{rch}_{\mathcal{M}}(\mathcal{S})$ that are tangent to $\mathcal{S}$ at q. For the local feature size one takes q fixed instead of minimizing over $\mathcal{S}$.

[2]By boundary we mean the set minus its interior for a closed set, and the closure of the set minus the interior of the closure for an open set.

Outline. This section is subdivided as follows:

- In Section 4.2 we focus on Riemann normal coordinate systems. Thanks to standard comparison theorems such as the Toponogov or Rauch comparison theorem [42, 43], we are able to give bounds on the metric in Riemann normal coordinates. The Riemann normal coordinates of a neighbourhood of p are those coordinates that are found by lifting the metric to the tangent space at p via the exponential map. The work by Kaul [36] provides us bounds on the Christoffel symbols in the Riemann normal coordinate neighbourhood, and thus indirectly bounds on the derivative of the metric.
- In Section 4.3 we'll study the coordinates transformation $\exp_c^{-1} \circ \exp_p^{-1}$, the bottom arrow in Figure 1. In Section 4.3.1, we first see how we can go from bounds on the metric in the Riemann normal coordinates to bounds on the coordinate transformation $\exp_c^{-1} \circ \exp_p^{-1}$. In Section 4.3.2, we'll be applying a result by Federer [3] that will yield the reach.

4.2 Bounds on the metric and Kaul's bound on the Christoffel symbols

In this section we review the bounds on the metric and Kaul's bounds on the Christoffel symbols, see [36]. The expressions for these bounds have been simplified by Von Deylen [44, Section 6], at the cost of weakening the bounds. We shall make use of his simplification.

As is standard in Riemannian geometry, the (coordinates of) vectors (elements of the tangent space or bundle) are indicated by v^i (so that the vector v with coordinates v^i is given by $v = \sum_i v^i \partial_i$) and the (coordinates of) covectors (elements of the cotangent space or bundle) by ω_i (so that $\omega = \sum_i \omega_i \mathrm{d}x^i$). Similarly we follow the standard convention for all tensors, e.g. the (covariant) metric is denoted by g_{ij} (even though in some sense it would be better to only refer to $g = \sum_{ij} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ as the metric). Writing $g = \sum_{ij} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ is compatible with the fact that the metric at a point is a bilinear form on the tangent space of that point. The inverse metric tensor or contravariant metric tensor is denoted by g^{ij}, so that $\sum_j g_{ij} g^{jk} = \delta_j^k$. If we want to emphasize the point where to consider the (inverse) metric we write $g_{ij}(x)$ and $g^{ij}(x)$ respectively. We employ similar notation for all other tensors. The standard Euclidean metric is given by δ_{ij}. Informally one can think of tensors as generalizations of matrices (multi-linear maps), however in General Relativity and most of Riemannian geometry (covariant or contravariant) tensors also behave in a very specific way under coordinate transformations (using the Jacobian of the coordinate transformation and/or its inverse). We will encounter a number of multi-linear maps below, such as the Christoffel symbols, which are not tensors.

The norm with respect to the Riemannian metric is denoted by $|\cdot|_g$ while the norm with respect to the Euclidean metric is denoted by $|\cdot|_{\mathbb{E}}$. Distances on $\mathcal{M}$ will be denoted by $d_{\mathcal{M}}$. As before, ι_M is the injectivity radius.

The Christoffel symbols are

$$\Gamma^{\kappa}_{\mu\nu}(x) = \sum_{\lambda} \frac{1}{2} g^{\kappa\lambda}(x)(\partial_\mu g_{\lambda\nu}(x) + \partial_\nu g_{\lambda\mu}(x) - \partial_\lambda g_{\mu\nu}(x))$$

where ∂_μ denotes the partial derivative with respect to the coordinate x^μ. As mentioned, Christoffel symbols are themselves not tensors. The Christoffel symbols are used to express the covariant derivative of a vector field v^κ in local coordinates,

$$\nabla_\nu v^\kappa = \partial_\nu v^\kappa + \sum_\mu \Gamma^{\kappa}_{\mu\nu}(x) v^\mu.$$

The Riemann curvature tensor in local coordinates will be denoted by $R^{\sigma}_{\mu\nu\lambda}(x)$. We refer to Do Carmo [45] and Spivak [38], as some of the standard texts introducing these concepts.

We now consider the Riemann normal coordinates around p:

$$x : (x^1, \dots, x^d) \mapsto \exp_p(x^i E_i)$$

for some orthonormal basis E_i of $T_{\mathcal{M}}$.

We shall now assume that the curvature and its derivative are bounded, that is in any normal coordinate system,

$$|R^{\sigma}_{\mu\nu\lambda}(x)| \leq R_{\max} \tag{2}$$
$$|\nabla_\kappa R^{\sigma}_{\mu\nu\lambda}(x)| \leq R^{\nabla}_{\max}. \tag{3}$$

We will use the following version of the Toponogov or Rauch comparison theorem.

Lemma 4.1 (Weak Rauch theorem, Lemma 6.8 of [44]) *If $d_{\mathcal{M}}(p, x) \leq r$, with $R_{max} r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then $|g_{ij}(x) - \delta_{ij}| \leq R_{max} r^2$ in Riemann normal coordinates around p.*

Usually the Rauch comparison theorem is stated in terms of the sectional curvature, however for our setting reformulating the statement in terms of bounds on the Riemann curvature is more convenient. Moreover the bounds of Kaul [36] can be simplified (at the cost of weakening) to:

Lemma 4.2 (Weak Kaul lemma, Lemma 6.9 of [44]) *If $d_{\mathcal{M}}(p, x) \leq r$, with $R_{max} r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$\left| \sum_{\kappa,\lambda,\mu,\nu} g_{\kappa\lambda}(x) \Gamma^{\kappa}_{\mu\nu}(x) v^\mu w^\nu u^\lambda \right| \leq 10 R_{max} r + 5 R^{\nabla}_{max} r^2,$$

for all $v, u, w \in T_x\mathcal{M}$ such that $|u|_g = |v|_g = |w|_g = 1$ and using Riemann normal coordinates around p.

We now note that thanks to Lemma 4.1, we have that

$$\begin{aligned}
||v|_g^2 - |v|_{\mathbb{E}}^2| &= \left| \sum_{i,j} g_{ij}(x) v^i v^j - \delta_{ij} v^i v^j \right| \\
&\leq \sum_{ij} |g_{ij}(x) - \delta_{ij}||v^i v^j| \\
&\leq \sum_{ij} R_{\max} r^2 |v^i||v^j| \\
&= R_{\max} r^2 \left(\sum_i 1 |v^i| \right)^2 \\
&\leq R_{\max} r^2 \left(\sum_i 1 \right) \left(\sum_i |v^i|^2 \right), \\
&= d R_{\max} r^2 |v|_{\mathbb{E}}^2,
\end{aligned}$$

where we used the Cauchy-Schwartz inequality. Combining this with Lemma 4.2, we see that

$$\left| \sum_\kappa g_{\kappa\lambda}(x) \Gamma^\kappa_{\mu\nu}(x) \right| \leq \frac{10 R_{\max} r + 5 R^\nabla_{\max} r^2}{(1 - R_{\max} d\, r^2)^{3/2}}.$$

Using that

$$\partial_\nu g_{\kappa\mu}(x) = \sum \lambda g_{\kappa\lambda}(x) \Gamma^\lambda_{\mu\nu} + g_{\mu\lambda}(x) \Gamma^\lambda_{\kappa\nu}$$

and taking absolute values, we find the following corollary

Corollary 4.3 *If $d_{\mathcal{M}}(p, x) \leq r$, with $R_{max} r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$|\partial_\nu g_{\kappa\mu}(x)| \leq \frac{20 R_{max} r + 10 R^\nabla_{max} r^2}{(1 - R_{max} d\, r^2)^{3/2}},$$

using Riemann normal coordinates around p.

We now recall two results from linear algebra:

- Let E be a $d \times d$-matrix, then (see for example (2.3.8) of [46])

$$\max_{ij} |E_{ij}| \leq \|E\|_2 \leq d \max_{ij} |E_{ij}| \tag{4}$$

- If $G = I + E$, where G and E are $d \times d$-matrices, I denotes the identity matrix and $\|E\|_2 \leq 1$, then (see for example [47, Section 5.8])

$$\|I - G^{-1}\|_2 \leq \frac{\|E\|_2}{1 - \|E\|_2},$$

If now, $|E_{ij}| \leq c$, with $dc < 1$, then

$$\max_{i,j} |(I - G^{-1})_{ij}| \leq \frac{dc}{1 - dc}.$$

With this result and Lemma 4.1, in particular setting $g_{ij}(x) = G$ and recognizing that $|E_{ij}| = |g_{ij}(x) - \delta_{ij}| \leq R_{\max} r^2$, we have the following

Corollary 4.4 *If $d_{\mathcal{M}}(p, x) \leq r$, with $dR_{max}r^2 \leq 1$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, then*

$$|g^{ij}(x) - \delta^{ij}| \leq \frac{dR_{max}r^2}{1 - dR_{max}r^2},$$

using Riemann normal coordinates around p.

We'll also make use of the following result [47, Corollary 6.3.4], which we'll formulate as a lemma,

Lemma 4.5 *Let E be a $d \times d$-matrix, and $G = I + E$, with I the identity matrix. If λ is an eigenvalue of G, then $|\lambda - 1| \leq \|E\|_2$.*

Using (4) again now gives,

Corollary 4.6 *If $d_{\mathcal{M}}(p, x) \leq r$, with $R_{max}r^2 \leq \frac{\pi^2}{4}$ and $r \leq \frac{\iota_{\mathcal{M}}}{2}$, any eigenvalue λ of $g_{ij}(x)$ now satisfies*

$$|\lambda - 1| \leq dR_{max}r^2,$$

using Riemann normal coordinates around p.

4.3 From bounds on the metrics to bounds on the coordinate transformations

The starting point of this section is the following: We are given a metric in Riemann normal coordinates at two different points. We want to study the coordinate transformation between these coordinates systems, based on our knowledge of the metric in these two coordinate systems.

In fact, we assume we have bounds on the first and second order derivatives of the metric in both coordinate systems. These bounds yield bounds on the first and second derivatives of the coordinates transformation. This can be understood by considering the limit case: Suppose both metrics are the

Euclidean metric, then the transformation from one coordinate system to the other is a combination of a rotation and translation.

From bounds on the coordinate transformation, a result of Federer [3] gives bounds on the reach of $\exp_p^{-1}(B)$ where B is a geodesic ball centred at c with radius r, assuming that p and c are not too far from each other. Here we emphasize (as we noted before) that the radius of the ball B equals the radius of $\exp_c^{-1}(B)$.

We will consider a coordinate transformation from a coordinate system x to a coordinate system y. Because the emphasis is on coordinate transformations we'll follow a different convention in this section, and only this section, and use Latin indices. We'll use the indices a, b, c, e, f for y-coordinate system and the indices i, j, k, l, m for x and write

$$y^a = \sum_i T_i^a x^i + \sum_{i,j} Q_{ij}^a x^i x^j + \mathcal{O}(x^3).$$

Here we assumed that the coordinate systems are chosen such that the origins are mapped to one another, which can be done without loss of generality. Q_{ij}^a is symmetric in i and j.

Here and in Subsection 4.3.1 all tensors (g_{ij}, its inverse, its derivatives, etc.) will be evaluated at the origin (0) of the coordinate system unless specifically stated otherwise.

We have

$$\begin{aligned}
g_{ij}(x) &= \sum_{a,b} g_{ab}(y(x)) \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \\
g_{ij}(x) &= g_{ij} + \sum_k (\partial_k g_{ij}) x^k + \mathcal{O}(x^2) \\
g_{ab}(y) &= g_{ab} + \sum_c (\partial_c g_{ab}) y^c + \mathcal{O}(y)^2 \\
y^c &= \sum_m T_m^c x^m + \mathcal{O}(x^2) \\
\frac{\partial y^a}{\partial x^i} &= \sum_k T_i^a + Q_{ik}^a x^k + \mathcal{O}(x^2).
\end{aligned}$$

Combining these gives

$$\begin{aligned}
g_{ij} &= \sum_{i,j} g_{ab} T_i^a T_j^b, \qquad (5) \\
\partial_k g_{ij} &= \sum_{a,b,c} \partial_c g_{ab} T_k^c T_i^a T_j^b + \sum_{a,b} g_{ab} (Q_{ik}^a T_j^b + T_i^a Q_{jk}^b).
\end{aligned}$$

4.3.1 Bounds on the transformations

With the concepts and notations developed in the previous section, we can give bounds on the transformation in terms of bounds on the metric and its derivatives. We find the following.

Lemma 4.7 *Let x and y be two (smooth) coordinates systems for the same point on the manifold. By the smoothness assumption the metric and its derivatives in these coordinates systems are as follows*

$$g_{ij}(x) = g_{ij} + \sum_k (\partial_k g_{ij}) x^k + \mathcal{O}(x^2)$$
$$g_{ab}(y) = g_{ab} + \sum_c (\partial_c g_{ab}) y^c + \mathcal{O}(y^2).$$

We assume further that:

- *Any eigenvalue λ of g_{ij} is bounded by $|\lambda - 1| \leq A$.*
- *Any eigenvalue $\tilde{\lambda}$ of g_{ab} is bounded by $|\tilde{\lambda} - 1| \leq B$.*
- *The entries of g^{ef} are bounded from above by $1 + C$.*
- *For all i, j, k we also have that $|\partial_k g_{ij}| \leq \partial g_{max,x}$, and for all a, b, c, that $|\partial_a g_{bc}| \leq \partial g_{max,y}$.*

Now we have that the coordinate transformation between x and y,

$$y^a = \sum_i T^a_i x^i + \sum_{i,j} Q^a_{ij} x^i x^j + \mathcal{O}(x^3),$$

satisfies the following constraints:

- *The Lipschitz constants, or the metric distortion of the linear approximation T, are bounded by $\sqrt{1+A}/\sqrt{1-B}$ and the Lipschitz constant of its inverse T^{-1} by $\sqrt{1+B}/\sqrt{1-A}$.*
- *For all i, j, a, we have*

$$|Q^a_{ij}| \leq 3d^2 \partial g_{max,x} \frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}} + d^3 \partial g_{max,y} \frac{(1+A)(1+C)}{1-B}.$$

We do not assume we are using Riemann normal coordinates in the statement of the lemma, but the notation is chosen to be compatible with this. The metric is close to Euclidean metric whose metric (δ_{ij}) is often denoted by the identity matrix, but formally is a (covariant) two tensor. The derivatives of the metric ($g_{\max,x}$, $g_{\max,y}$) have been bounded in Riemann normal coordinate neighbourhoods in Corollary 4.3.

Proof of Lemma 4.7 We write G for the matrix $g_{ij}(0)$, and we assume that the eigenvalues λ_i, are not far from 1, that is $|\lambda_i - 1| \leq A$ for all i and some $A \geq 0$. Similarly, we write $\tilde{G}$ for g_{ab}, and assume that its eigenvalues $\tilde{\lambda}_i$ are bounded by $|\tilde{\lambda}_i - 1| \leq B$ for some $B \geq 0$. We'll also write $G = o^t_A D_A o_A$ with o_A the orthogonal matrix that diagonalizes G and D_A the diagonal matrix with the eigenvalues of G on the diagonal, that is $\mathrm{diag}(\lambda_i)$. We let S_A denote the matrix with square roots $\sqrt{\lambda_i}$ on the diagonal, that is $S_A = \mathrm{diag}(\sqrt{\lambda_i})$. And similarly, $\tilde{G} = o_B^t D_B o_B$ and $S_B = \mathrm{diag}(\sqrt{\tilde{\lambda}_i})$.

Now (5) gives

$$\begin{aligned}
G &= T^t \tilde{G} T \\
o_A^t S_A\, S_A o_A &= T^t o_B^t S_B\, S_B o_B T \\
o_A^t S_A^t\, S_A o_A &= T^t o_B^t S_B^t\, S_B o_B T \\
I &= S_A^{-t} o_A T^t o_B^t S_B^t\, S_B o_B T o_A^t S_A^{-1} \\
I &= o^t\, o,
\end{aligned}$$

that is $S_B o_B T o_A^t S_A^{-1} = o$, with o an orthogonal transformation.

This in turn implies that $T = o_B^t S_B^{-1} o S_A o_A$ is close to an orthogonal transformation if A and B are close to zero. The Lipschitz constant of a composition of function is the product of the Lipschitz constants and thus the Lipschitz constant of T is bounded by $(1-A)(1-B)$ and $(1+A)(1+B)$ respectively. Because we have that for any vector $|v^\mu| \leq |v|$, where the first $|\cdot|$ should be read as an absolute value and the second as the norm, and $A_{ij} = e_i^t A e_j$, where the e_i denote basis vectors, the entries of T are bounded by from above by $\sqrt{1+A}/\sqrt{1-B}$. By the same argument we have that the entries of T^{-1} are bounded by $\sqrt{1+B}/\sqrt{1-A}$.

We shall now consider the quadratic term. We start with

$$\partial_k g_{ij} = \sum_{a,b,c} \partial_c g_{ab} T_k^c T_i^a T_j^b + \sum_{a,b} g_{ab} (Q_{ik}^a T_j^b + T_i^a Q_{jk}^b).$$

Reshuffling and permuting the indices and changing the sign for the last equation gives

$$\begin{aligned}
\partial_k g_{ij} - \sum_{a,b,c} \partial_c g_{ab} T_k^c T_i^a T_j^b &= \sum_{a,b} g_{ab} Q_{ik}^a T_j^b + \sum_{a,b} g_{ab} T_i^a Q_{jk}^b \\
\partial_i g_{jk} - \sum_{a,b,c} \partial_c g_{ab} T_i^c T_j^a T_k^b &= \sum_{a,b} g_{ab} Q_{ji}^a T_k^b + \sum_{a,b} g_{ab} T_j^a Q_{ki}^b \\
-\partial_j g_{ki} + \sum_{a,b,c} \partial_c g_{ab} T_j^c T_k^a T_i^b &= -\sum_{a,b} g_{ab} Q_{kj}^a T_i^b - \sum_{a,b} g_{ab} T_k^a Q_{ij}^b .
\end{aligned}$$

Adding the terms yields

$$\begin{aligned}
&\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki} - \sum_{a,b,c} \partial_c g_{ab} T_k^c T_i^a T_j^b - \sum_{a,b,c} \partial_c g_{ab} T_i^c T_j^a T_k^b + \sum_{a,b,c} \partial_c g_{ab} T_j^c T_k^a T_i^b \\
&= \sum_{a,b} g_{ab} Q_{ik}^a T_j^b + \sum_{a,b} g_{ab} T_i^a Q_{jk}^b + \sum_{a,b} g_{ab} Q_{ji}^a T_k^b + \sum_{a,b} g_{ab} T_j^a Q_{ki}^b \\
&\quad - \sum_{a,b} g_{ab} Q_{kj}^a T_i^b - \sum_{a,b} g_{ab} T_k^a Q_{ij}^b \\
&= 2 \sum_{a,b} g_{ab} Q_{ik}^a T_j^b ,
\end{aligned}$$

and thus

$$\sum_{j,e} \Big(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki} \Big) (T^{-1})_e^j g^{ef}$$

$$
\begin{aligned}
&-\sum_{a,c,e}\partial_c g_{ae}T_k^cT_i^a g^{ef}-\sum_{b,c,e}\partial_c g_{eb}T_i^cT_k^b g^{ef}+\sum_{a,b,e}\partial_e g_{ab}T_k^aT_i^b g^{ef}\\
=&\sum_{j,e}\Bigg(\partial_k g_{ij}+\partial_i g_{jk}-\partial_j g_{ki}\\
&-\sum_{a,b,c}\partial_c g_{ab}T_k^cT_i^aT_j^b-\sum_{a,b,c}\partial_c g_{ab}T_i^cT_j^aT_k^b+\sum_{a,b,c}\partial_c g_{ab}T_j^cT_k^aT_i^b\Bigg)(T^{-1})_e^j g^{ef}\\
=&2\sum_{a,b,e,j} g_{ab}Q_{ik}^aT_j^b(T^{-1})_e^j g^{ef}\\
=&Q_{ik}^f,
\end{aligned}
$$

The idea now is the following: If we assume that the left hand side of the previous equation is close to zero (this is in line with Corollary 4.3 because we assume that the derivatives of the metric are not too large if the neighbourhood is not too big), g_{ab} is close to δ_{ab}, and T_j^b is close to a rotation, all entries of Q_{ij}^a have to be close to zero too.

Let us now assume that for all k, i, j we have that

$$|\partial_k g_{ij}| \leq \partial g_{\max,x},$$

and for all a, b, c, that

$$|\partial_a g_{bc}| \leq \partial g_{\max,y}.$$

We'll also assume that entries of g^{ef} are bounded in absolute value by $1 + C$. We will use that for a tensor $U_{\mu\nu}$, with $|U_{\mu\nu}| \leq U_{\max}$, and the coordinates of vectors v^μ, w^μ are bounded by $|v^\mu| \leq v_{\max}$ and $|w^\mu| \leq w_{\max}$, we have that

$$\left|\sum_{\mu,\nu} T_{\mu\nu}v^\mu w^\nu\right| \leq \sum_{\mu,\nu} T_{\max}v_{\max}w_{\max} = d^2 T_{\max}v_{\max}w_{\max}, \tag{6}$$

as well as its obvious generalization.

Thanks to the triangle inequality we now have

$$
\begin{aligned}
|Q_{ik}^f| =&\Bigg|\sum_{j,e}\left(\partial_k g_{ij}+\partial_i g_{jk}-\partial_j g_{ki}\right)(T^{-1})_e^j g^{ef}\\
&-\sum_{a,c,e}\partial_c g_{ae}T_k^cT_i^a g^{ef}-\sum_{b,c,e}\partial_c g_{eb}T_i^cT_k^b g^{ef}+\sum_{a,b,e}\partial_e g_{ab}T_k^aT_i^b g^{ef}\Bigg|\\
\leq&\left|\sum_{j,e}\left(\partial_k g_{ij}+\partial_i g_{jk}-\partial_j g_{ki}\right)(T^{-1})_e^j g^{ef}\right|+\left|\sum_{a,c,e}\partial_c g_{ae}T_k^cT_i^a g^{ef}\right|\\
&+\left|\sum_{b,c,e}\partial_c g_{eb}T_i^cT_k^b g^{ef}\right|+\left|\sum_{a,b,e}\partial_e g_{ab}T_k^aT_i^b g^{ef}\right|\\
\leq&3d^2\partial g_{\max,x}\frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}}+d^3\partial g_{\max,y}\frac{(1+A)(1+C)}{1-B}
\end{aligned}
$$

□

Remark 4.8 *In the previous lemma we have given a bound on the metric distortion (on the coordinates) of the coordinate transformation from x to y, as well as on the Lipschitz constant of the derivative of the transformation. In the formulation of the lemma we have chosen the coordinate systems such that the coordinates map the origin to the origin to keep the expressions short and relatively simple. However we could have chosen the map to go from an arbitrary x_0 to y_0. In other words, this choice of coordinates is not canonical, unlike the origin in Riemann normal coordinate systems which is canonical.*

4.3.2 Using Federer's estimates: from bounds on the coordinate transformation to bounds on the reach

In this section, we are finally able to give bounds on the reach by applying Theorem 4.19 of Federer [3]:

Theorem 4.9 (Federer) *Let $\mathcal{S}$ be a subset of $\mathbb{R}^d$ with $\mathrm{rch}(\mathcal{S}) > t > 0$, and $s > 0$. If*

$$\tilde{f} : \{x \mid d(x, \mathcal{S}) < s\} \to \mathbb{R}^d$$

is a C^2 diffeomorphism such that

$$|D\tilde{f}| \leq M \qquad |D\tilde{f}^{-1}| \leq N \qquad |D^2\tilde{f}| \leq P,$$

where D denotes the derivative, then

$$\mathrm{rch}(\tilde{f}(\mathcal{S})) \geq \min\{sN^{-1}, (Mt^{-1} + P)^{-1}N^{-2}\}.$$

We can now combine this result with the estimates of the previous sections. We want to investigate how an empty tangent ball to $\mathcal{S}$ transforms under the exponential map. Because a geodesic ball is also an Euclidean ball in the tangent space of its centre (lifted via the exponential map), this is equivalent to giving bounds on the reach of this ball under the map $\exp_p^{-1} \circ \exp_c$.

The bounds on the reach under the map $\exp_p^{-1} \circ \exp_c$ use almost all previous results in this paper: In particular the bounds on the metric and its derivatives are given in Lemma 4.1, and Corollaries 4.3, 4.4, and 4.6, while Lemma 4.7 tells us how to go from bounds on the metric to bounds on the coordinate transformation. Federer's result now gives us the reach after the transformation.

We can now give the two main results of the paper. The first gives a bound on the reach of a lifted (via the exponential map at a nearby point) geodesic sphere. The second uses this result to give a lower bound on the local feature size on the lifting of any set of positive reach.

Theorem 4.10 *Let $\mathcal{M}$ be a smooth d-dimensional Riemannian manifold whose curvatures are bounded as follows:*

$$|R^{\sigma}_{\mu\nu\lambda}| \leq R_{max} \tag{2}$$

$$|\nabla_\kappa R^{\sigma}_{\mu\nu\lambda}| \leq R^{\nabla}_{max}. \tag{3}$$

Suppose that r_p and r_c are the radii of geodesic balls centred at p and c respectively such that

- $B(c, 2r_c) \subset B(p, r_p)$, *where we have made the centres and radii explicit,*
- $\sqrt{d} R_{max} r_p^2 \leq 1$ *and* $r_p \leq \frac{\iota_{\mathcal{M}}}{2}$.

Then the reach of $\exp_p^{-1}(\partial B(c, r_c)) \subset T_p\mathcal{M}$ *is lower bounded. Specifically,*

$$\mathrm{rch}(\exp_p^{-1}(\partial B(c, r_c))) \geq \min\{r_c N^{-1}, (M r_c^{-1} + P)^{-1} N^{-2}\},$$

where

$$M = \frac{\sqrt{1+A}}{\sqrt{1-B}}$$
$$N = \frac{\sqrt{1+B}}{\sqrt{1-A}}$$
$$P = 12 \partial g_{max,x} \frac{(1+C)\sqrt{1+B}}{\sqrt{1-A}} + 8 \partial g_{max,y} \frac{(1+A)(1+C)}{1-B},$$

with

$$A = d R_{max} r_c^2 \qquad B = d R_{max} r_p^2. \qquad C = \frac{d R_{max} r_p^2}{1 - d R_{max} r_p^2},$$

and $\partial g_{max,x} = \partial g_{max}(r_c)$, $\partial g_{max,y} = \partial g_{max}(r_p)$, *with*

$$\partial g_{max}(r) = \frac{20 R_{max} r + 10 R^{\nabla}_{max} r^2}{(1 - R_{max} d\, r^2)^{3/2}}.$$

We now also find that

Theorem 4.11 *Suppose that* $\mathcal{M}$ *satisfies the same conditions as in Theorem 4.10 and let* $\mathcal{S}$ *be a subset of* $\mathcal{M}$ *of positive reach* $\mathrm{rch}(\mathcal{S})$. *Let* $q \in \mathcal{S}$ *and let* $B(p, r_p)$ *be a geodesic ball, whose radius satisfies*

- $10\,\rho < r_p$ *for some* $\rho \leq \mathrm{rch}(\mathcal{S})$,
- $d(q, p) < \frac{r_p}{2}$,
- $\sqrt{d} R_{max} r_p^2 \leq 1$ *and* $r_p \leq \frac{\iota_{\mathcal{M}}}{2}$ *(similarly to Proposition 4.10).*

Let $A, B, C, M, N, P, \partial g_{max,x}, \partial g_{max,y}$ *be as in Theorem 4.10, but with* r_c *replaced by* ρ. *The local feature size of* $\mathcal{S}_p = \exp_p^{-1}(\mathcal{S})$ *then satisfies* $\mathrm{lfs}_{\exp_p^{-1}(q)}(\mathcal{S}_p) \geq L$ *where*

$$L = \min\{\mathrm{rch}(\mathcal{S}) N^{-1}, (M\, \mathrm{rch}(\mathcal{S})^{-1} + P)^{-1} N^{-2}\}.$$

Proof The proof is more complicated than first appears because for general sets of positive reach we cannot directly work with the (classical) normal space.[3] To derive a contradiction assume that for some $L > \epsilon > 0$ there is a $p' \in B_{\mathbb{E}}(q, \epsilon) \subset T_p\mathcal{M}$, such that $\pi_{\exp_p^{-1}(\mathcal{S})}(p')$ consists of multiple points. The latter statement is equivalent to: there is a $L > r' > 0$ such that $B' \cap \mathcal{S}_p = \partial B' \cap \mathcal{S}_p$ contains more than one point, where $B' = B_{\mathbb{E}}(p', r') \subset T_p\mathcal{M}$. Let q' be such a point and consider $T_{q'}\partial B'$. We stress

[3]Even in Euclidean space, the normal spaces of sets of positive reach are not necessarily vector spaces. We'll see in Remark 4.14, that normal cones in the sense of Federer [3] are well defined, also for subsets of Riemannian manifolds.

that here $\partial B'$ is seen as a submanifold of $T_p\mathcal{M}$, which is (isometric to) a Euclidean space, and thus $T_{q'}\partial B' \subset T_p\mathcal{M}$.

Because the manifold $\mathcal{M}$ is smooth (C^3) and we work in a neighbourhood that is smaller than the injectivity radius $\exp_p$ is a smooth map whose differential maps $T_{q'}\partial B' \subset T_p\mathcal{M}$ to a subspace $ST_{q'}B$ of $T_{\exp_p(q')}\mathcal{M}$. Moreover, for the same reason, $\exp_p \partial B'$ is a C^2 manifold and therefore has positive reach by Theorem 3.1. It also has an interior and an exterior by the Jordan-Brouwer theorem, see for example [48] for a pedagogical proof of the Jordan-Brouwer theorem. We note that $ST_{q'}B$ is tangent to $\exp_p \partial B'$ and the interior of $\exp_p B'$ has an empty intersection with $\mathcal{S}$. Write ν for the interior unit normal vector of $\exp_p \partial B'$ and $ST_{q'}B$ at $\exp_p(q')$.

Let $\gamma_\nu(t)$ be the geodesic, parametrized by arc length, that starts at $\exp_p(q')$ and goes in the direction of ν. Using Theorem 3.1 once more, we see that for sufficiently small $\epsilon' > 0$ we have $\pi_{\exp_p \partial B'}(\gamma_\nu(t)) = \pi_{\mathcal{S}}(\gamma_\nu(t))$, for all $t' \leq \epsilon'$. Thanks to Kleinjohann [33, Satz 3.2 and 3.3], we have that this minimizing geodesic of length ϵ' can be extended to a geodesic of length $\mathrm{rch}(\mathcal{S})$, which is larger than ρ by assumption. Therefore, for each $r_c < \mathrm{rch}(\mathcal{S})$, the geodesic ball $B_c = B(\gamma_\nu(r_c), r_c)$ intersects $\mathcal{S}$ only in $\exp_p(q')$ and its boundary is tangent to $ST_{q'}B$ at $\exp_p(q')$. We note that $\exp_p^{-1}(\partial B_c)$ is tangent to $T_{q'}\partial B'$ in the point q'. We can now use Theorem 4.10 to see that the reach of $\exp_p^{-1}(\partial B_c)$ is lower bounded by $L_c := \min\{r_c N^{-1}, (M\, r_c^{-1} + P)^{-1} N^{-2}\}$, using the notation as defined in Theorem 4.10. If $r' < L_c$ for all $r_c < \rho$, then we have reached a contradiction, because $B' \cap \mathcal{S}_p = \partial B' \cap \mathcal{S}_p$ was supposed to contain more than one point.

We note that $r' < L_c$ for all $r_c < \rho$ is equivalent to $r' < L$. The result now follows. $\square$

Remark 4.12 *We have formulated Theorem 4.11 in terms of the local feature size, because we did not want to exclude subsets whose size is larger than the injectivity radius and therefore would not fit in its entirety in a single coordinate chart given by the exponential map.*

Because we know that a topologically embedded submanifold of $\mathbb{R}^d$ has positive reach if and only if it is $C^{1,1}$ embedded, see [39], we immediately have the following corollary:

Corollary 4.13 *A topologically embedded submanifold $\mathcal{S}$ of a C^3 manifolds $\mathcal{M}$ has positive reach if and only if $\mathcal{S}$ is $C^{1,1}$ embedded.*

As mentioned in the introduction of Section 4 this result was known to Lytchak [40, 41], but his work heavily relies on the theory of CAT(k) spaces, while [39] is more elementary.

We further notice

Remark 4.14 *In [3, Definitions 4.3 and 4.4 and Theorem 4.8] Federer shows that the generalized tangent space* Tan *and generalized normal space* Nor *of a set of positive reach or positive local feature size are convex cones. Theorem 4.11 says that the lifting* *(via* $\exp_p$*) set of positive reach has positive local feature size. Combining these two observations makes the following definition sensible*

$$\mathrm{Tan}(p,\mathcal{S}) := \mathrm{Tan}(p,\exp_p^{-1}\mathcal{S}) \qquad \mathrm{Nor}(p,\mathcal{S}) := \mathrm{Nor}(p,\exp_p^{-1}\mathcal{S}).$$

These spaces are convex cones (by definition), which is not obvious from [33].

5 Future work

As we mentioned in the introduction the main motivation of this work (at least for the authors) is the triangulation of stratified Riemannian manifolds. To make the importance of the bounds of the paper clear in this context we will discuss the approach for a Riemannian manifold with a submanifold inside (the simplest example of a stratified Riemannian manifold).

We are able to Delaunay triangulate the Riemannian manifold using intrinsic simplices [49]. If the intersection of the submanifold with every top dimensional intrinsic simplex in the Delaunay triangulation is nice, i.e. if the intersection after lifting the tangent space of a nearby point is a slightly deformed polytope, we can triangulate relatively straightforwardly using barycentric subdivision, in generalization of Whitney's approach [14, 50]. In the Euclidean setting one established this property by perturbing the triangulation in such a way that the (sub)manifold stays sufficiently far away from the $d-n-1$-skeleton of the ambient triangulation, where sufficiently far is a bound in terms of the reach. In the Riemannian setting one would like to apply (a variant of) this method after (locally) lifting to the tangent space (at a nearby point) of the Riemannian manifold. For this one needs two supporting results, namely to bound the reach of the lifted submanifold (which we discussed in this paper) and to understand the geometry of the faces of the Riemannian simplices in the ambient triangulation.

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Conflicts of interest/Competing interests

There are no conflicts of interest.

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