Constructible sheaves and functions up to infinity

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Abstract

We introduce the category of b-analytic manifolds, a natural tool to define constructible sheaves and functions up to infinity. We study with some details the operations on these objects and also recall the Radon transform for constructible functions.

Contents

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Introduction

Sheaf theory is a mathematical tool to treat the dichotomy local/global and it is not surprising that it appears now as essential in topological data analysis (TDA), its use in this field appearing first in Justin Curry's thesis [Cur13]. Of course, sheaves have to be treated in their derived version. To illustrate this point, see Example 1.1 below.

On the other-hand, sheaf theory is a very general theory, perhaps too general for applications. In TDA one essentially encounters sheaves associated to subsets which are topologically "reasonable" and there is a perfectly suited framework for such sheaves, namely that of constructible sheaves or sometimes, on real vector spaces, piecewise linear (PL) sheaves (see [KS21a]). The triangulated category of constructible sheaves over a commutative Noetherian ring $\mathbf{k}$ on a real analytic manifold plays an increasing role in various fields of mathematics and is well understood.

To an abelian or a triangulated category, one naturally associates its Grothendieck group: any function defined on the objects of the this category, additive with respect to exact sequences or to distinguished triangles and with values in a commutative group, factorizes uniquely through the Grothendieck group and, in some sense, this group contains all the additive informations of the category. When $\mathbf{k}$ is a field of characteristic 0, the Grothendieck group of the triangulated category of constructible sheaves is known to be isomorphic to the group of constructible functions as well as to that of Lagrangian cycles.

Recall that constructible functions and Lagrangian cycles first appeared in the complex analytic setting with Masaki Kashiwara [Kas73] and in the algebraic setting with Robert MacPherson [Mac74]. In the complex setting, Lagrangian cycles were studied for their functorial properties by several people and in particular by Victor Ginsburg [Gin86] and Claude Sabbah [Sab85]. The real case was first treated in [Kas85]. See also [KS90, Ch. IX, Notes] for an history of the subject. Lagrangian cycles are not so easy to describe, contrarily to constructible functions and we shall not study them here.

The Euler calculus of constructible functions has been introduced independently by Oleg Viro (see [Vir88]) in the complex analytic setting and by the author in the subanalytic setting (see [Sch89]). It has many applications, particularly to tomography *i.e.,* real Radon transform, see [Sch95] (see also Lars Ernström [Ern94] for complex projective duality) and more generally in TDA where it appears in particular in sensing (see [CGR12] for a survey) and shape analysis [CMT18] and also in the study of persistence modules through their rank invariants and their local Euler characteristic also known as Betti curve in the community [Ume17].

A constructible function φ on a real analytic manifold X is mathematically very simple: it is a $\mathbb{Z}$-valued function, the sets $\varphi^{-1}(m)$ ($m \in \mathbb{Z}$) being all subanalytic and the family of such sets being locally finite. It is not difficult (with the tools of subanalytic geometry at hands) to check that the set $\mathscr{CF}(X)$ of constructible functions on X is a commutative unital algebra and that the inverse image (*i.e.,* composition) of such a function by a morphism of real analytic manifolds $f\colon Z \to X$ is again constructible.

Things become more unusual when looking at direct images, in particular integration. Assume that φ has compact support. One may write φ as a finite sum $\sum_{i\in I} c_i \mathbf{1}_{K_i}$

where $c_i \in \mathbb{Z}$, K_i is a compact subanaytic subset of X and for $S \subset X$, $\mathbf{1}_S$ is the characteristic function of S. Then one defines the integral of φ by the formula

$$\int_X \varphi = \sum_{i \in I} c_i \cdot \chi(K_i)$$

where $\chi(K_i)$ denotes the Euler-Poincaré index of K_i. (This definition does not depend on the decomposition of φ–see the comments after (3.11).) For a morphism $f\colon X \to Y$ of real analytic manifolds, one defines the integral along f of a function $\varphi \in \mathscr{CF}(X)$ whose support is proper with respect to f by setting for $y \in Y$,

$$(\int_f \varphi)(y) = \int_X \varphi \cdot \mathbf{1}_{f^{-1}(y)},$$

and one checks that one obtains a constructible function on Y. This integral has all properties of classical integrals (linearity and Fubini theorem–that is, functoriality), except that it is not positive (the integral of $\mathbf{1}_{(0,1)}$ is -1) and a set reduced to one point has integral 1. In fact, one easily translates all operations on constructible sheaves to operations on constructible functions. In particular duality makes sense for constructible functions and commutes with direct images.

Constructible sheaves and functions cause problems at infinity. For example, the set $\mathbb{N}$ is subanalytic in $\mathbb{R}$ (contrarily to the set $\{1/n; n \in \mathbb{N}\}$) but of course no finiteness properties may be obtained in this case. Hence, we shall define the notion of being "constructible up to infinity", as mentioned in the title. For that purpose we introduce the category of b-analytic manifolds. An object X_∞ is an open embedding $X \subset \widehat{X}$ of smooth real analytic manifolds with X subanalytic and relatively compact in $\widehat{X}$, and a morphism $f\colon X_\infty \to Y_\infty$ is a real analytic map $f\colon X \to Y$ such that the graph of f is subanalytic in $\widehat{X} \times \widehat{Y}$. Then a subset of X is "subanalytic up to infinity"–we shall also say "b-subanalytic", for short– if it is subanalytic in $\widehat{X}$. (As a non-example, $\mathbb{N}$ is not subanalytic up to infinity in $\mathbb{R}$ whatever the choice of $\mathbb{R}_\infty$.) As we shall see, this notion is much more natural than the usual one and makes calculations easier. For example, the direct and inverse images for sheaves commute now with duality (see below for a precise statement), non proper convolution becomes associative, etc.

The notion of being subanalytic up to infinity is closely related to that of definable sets and of *o*-minimal structures, well known from the specialists (see in particular [VdD98, VdDM96]) and constructible sheaves and functions in this framework have already been defined in [Sch03, EP20]. Nevertheless, our approach for sheaves, based on the notion of micro-support, is of a different nature and provides a convenient setup to use microlocal sheaf theory while benefiting of the finiteness properties enjoyed in the framework of *o*-minimal structure (see for instance [CCG$^+$].

Sections 1 and 3 are detailed reviews on (derived) sheaves and constructible functions, posted here for the reader's convenience.

In **Section 2** we define a derived sheaf constructible up to infinity on X as a constructible sheaf whose micro-support is subanalytic in the cotangent bundle $T^*\widehat{X}$. We shall also say, for short, that such a sheaf is "b-constructible". This is equivalent to saying that its (proper or non proper) direct image in $\widehat{X}$ is again constructible. Note that such a property already appeared in [KS21b]. We briefly study the six operations on the triangulated category of b-constructible sheaves. Contrarily to the classical constructible case, the two inverse images f^{-1} and $f^!$ are exchanged by duality, the two direct images $\mathrm{R}f_*$ and $\mathrm{R}f_!$ are constructible without any properness assumptions and, again, are exchanged by duality. As a nice application, we find that non proper convolution on a real vector space $\mathbb{V}$ is well defined on constructible sheaves up to infinity and is associative. Such a non proper convolution appears when using the so-called γ-topology, associated with a closed convex proper cone γ of $\mathbb{V}$. This topology, already introduced in [KS90], plays an increasing role in TDA. For example, Betti curves and surfaces of multi-parameters persistence modules are examples of constructible functions for the γ-topology.

In **Section 4**, we define the space $\mathscr{CF}(X_\infty)$ of constructible functions up to infinity and study with some care the operations on such functions. Contrarily to the classical case, we have now two kind of integrals, proper and non proper, and, as for sheaves, these operations are defined without any properness hypothesis. Moreover, they are exchanged by duality. On a real vector space $\mathbb{V}$, we study with some care constructible functions for the γ-topology.

In **Section 5**, posted here for easier accessibility, we recall (and adapt) the main results of [Sch95] in which we obtain an inversion formula for the Radon transform of constructible functions. This formula asserts that one can recover a constructible function on a real vector space $\mathbb{V}$ from the knowledge of the Euler-Poincaré index of its restriction to all affine hyperplanes. For example, if $\dim \mathbb{V} = 3$, one can reconstruct a compact subanalytic subset from the knowledge of the number of connected components and holes of the restriction of the compact set to all slices (affine planes).

To conclude this introduction, let us recall that the Euler calculus of constructible functions already had many applications in TDA, especially under the impulse of Robert Ghrist and his collaborators (see in particular [CGR12]). Very recently, in a paper partly based on some results exposed here, Vadim Lebovici [Leb21] introduces the very promising idea of hybrid transform of constructible functions, a transform which combines classical Lebesgue integration and the Euler calculus. This new idea generalizes and unifies several previous results of specialists of TDA.

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Convention. In this paper, $\mathbf{k}$ denotes a commutative unital Noetherian ring with finite global dimension (see *e.g.* [KS90, exe. I. 28]). From Section 3 and until the end of the paper, $\mathbf{k}$ is a field of characteristic 0.

1 A short review on sheaves

In this section, we shall give a very brief overview of sheaf theory in its derived setting. We shall assume that the reader has some basic notions on sheaves. In particular, we do not recall the definitions of presheaves and sheaves, neither the fundamental result which asserts that the forgetful functor, from sheaves to presheaves, admits a left adjoint. We denote by $\mathrm{PSh}(\mathbf{k}_X)$ the abelian category of presheaves on X with values in $\mathrm{Mod}(\mathbf{k})$ and by $\mathrm{Mod}(\mathbf{k}_X)$ the full abelian subcategory consisting of sheaves. Hence, by definition, a morphism of sheaves is a morphism of the underlying presheaves. We refer to [KS90] for a detailed exposition.

Some notations

Recall that a topological space is *good* if it is Hausdorff, locally compact, countable at infinity (that is, countable union of compact subsets) and of finite flabby dimension. This last condition means that there exists an integer d such that any sheaf admits a resolution of length $\leq d$ by flabby sheaves. (Recall that a sheaf is flabby if any section on an open subset extends to the whole space.) It is satisfied for example by C^0-manifolds of dimension $\leq d-1$.

For a space X, we denote by Δ_X the diagonal of $X \times X$ and if $f\colon X \to Y$ is a map, we denote by Γ_f its graph in $X \times Y$. We denote by pt the space consisting of a single element and by $a_X\colon X \to \mathrm{pt}$ the unique map from X to pt.

Given topological spaces X_i $(i = 1, 2, 3)$ we set $X_{ij} = X_i \times X_j$, $X_{123} = X_1 \times X_2 \times X_3$. We denote by $q_i\colon X_{ij} \to X_i$ and $q_{ij}\colon X_{123} \to X_{ij}$ the projections.

(1.1)
$$\begin{tikzcd} & X_{12} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X_1 & & X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_{123} \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ X_{12} & X_{13} & X_{23} \end{tikzcd}$$

For $A \subset X_{12}$ and $B \subset X_{23}$, one sets

$$A \times_2 B = A \times_{X_2} B = q_{12}^{-1} A \cap q_{23}^{-1} B, \quad A \mathbin{\underset{2}{\circ}} B = q_{13}(A \times_2 B). \tag{1.2}$$

Basic operations on sheaves

We consider a commutative unital Noetherian ring $\mathbf{k}$ of finite global dimension. However, assuming that $\mathbf{k}$ is a field would be sufficient for most applications.

Let us first only consider sheaves, passing to the derived categories later.

Given two sheaves F and G on X, one defines their **tensor product** $F \otimes G$ as the sheaf associated to the presheaf $U \mapsto F(U) \otimes G(U)$, ($U$ open in X).

The **internal hom**, denoted $\mathcal{H}om$, is the presheaf $U \mapsto \mathrm{Hom}\,(F|_U, G|_U)$ where Hom is taken in the category $\mathrm{PSh}(\mathbf{k}_U)$ and it appears that this presheaf is a sheaf as soon as G is a sheaf. One proves ([KS90, Prop. 2.2.9]) that $(\otimes, \mathcal{H}om)$ is a pair of adjoint functors, that is, for three sheaves F, G, H

$$\mathrm{Hom}\,(F \otimes G, H) \simeq \mathrm{Hom}\,(F, \mathcal{H}om\,(G, H)),$$

functorially in F, G, H.

Now consider a continuous map $f\colon X \to Y$. If F is a sheaf on X, its **direct image** denoted $f_* F$ is the presheaf on Y which, to V open in Y, associates $F(f^{-1}V)$. It is easily checked that this presheaf is a sheaf.

The **inverse image** is more delicate. If G is a sheaf on Y, one first defines it inverse image as a presheaf, $f^\dagger G$, as follows. For U open in X, $f^\dagger G(U) = \operatorname{colim} G(V)$ where V ranges through the family of open subset of Y such that $U \subset f^{-1}V$. Then the inverse image $f^{-1}G$ is the sheaf associated with the presheaf $f^\dagger G$. One proves ([KS90, Prop. 2.3.3]) that (f^{-1}, f_*) is a pair of adjoint functors, that is

$$\operatorname{Hom}(f^{-1}G, F) \simeq \operatorname{Hom}(G, f_*F),$$

functorially in F, G. Hence f^{-1} is right exact and f_* is left exact. In fact, f^{-1} is exact.

As a combination of these functors we get the external product. In the situation of (1.1), for $F_i \in \operatorname{Mod}(\mathbf{k}_{X_i})$, $i = 1, 2$, one sets

$$F_1 \boxtimes F_2 := q_1^{-1}F_1 \otimes q_2^{-1}F_2. \tag{1.3}$$

One denotes by $\mathbf{k}_X$ the **constant sheaf** on X with stalk $\mathbf{k}$. It is defined as $\mathbf{k}_X = a_X^{-1}\mathbf{k}$, after having identified $\operatorname{Mod}(\mathbf{k})$ and $\operatorname{Mod}(\mathbf{k}_{\mathrm{pt}})$. The sheaf $\mathbf{k}_X$ is also the sheaf of locally constant functions on X with values in $\mathbf{k}$. One defines similarly the sheaf M_X for $M \in \operatorname{Mod}(\mathbf{k})$.

Consider a closed subset S of X and denote by $j_S \colon S \hookrightarrow X$ the embedding. One sets $\mathbf{k}_{XS} := j_{S*}\mathbf{k}_S$. This is the sheaf on X of functions with values in $\mathbf{k}$ and which are locally constant on S and 0 elsewhere. Now set $U = X \setminus S$. One defines the sheaf $\mathbf{k}_{XU}$ by the exact sequence

$$0 \to \mathbf{k}_{XU} \to \mathbf{k}_X \to \mathbf{k}_{XS} \to 0.$$

A locally closed set Z is the (non unique) intersection of a closed set T and an open set V. One sets $\mathbf{k}_{XZ} := \mathbf{k}_{XT} \otimes \mathbf{k}_{XV}$, this last sheaf depending uniquely on Z. One often writes $\mathbf{k}_Z$ instead of $\mathbf{k}_{XZ}$, especially when Z is closed in X. For a sheaf F on X, one then sets

$$F_Z := F \otimes \mathbf{k}_{XZ}.$$

Note that the functor $\bullet \otimes \mathbf{k}_{XZ}$ is exact. Moreover, if U is open in Z and S is closed in Z, there are natural morphisms $F_U \to F_Z$ and $F_Z \to F_S$.

Assuming that X and Y are good topological spaces, there is also a notion of **proper direct image** denoted $f_!F$. It is defined as follows, for F a sheaf on X:

$$f_!F = \operatorname*{colim}_U f_*F_U$$

where U ranges over the family of open subsets of X such that the map f is proper on $\overline{U}$. Hence, $f_!F$ is a subsheaf of f_*F. In particular, if f is proper on X (or better, on $\operatorname{supp}(F)$), then $f_!F \xrightarrow{\sim} f_*F$.

One checks ([KS90, Prop. 2.5.4]) that if Z is locally closed in X, denoting by $j_Z \colon Z \hookrightarrow X$ the embedding, then $F_Z \simeq j_{Z!}j_Z^{-1}F$.

The six Grothendieck operations

Sheaf theory takes its full strength when treated in the derived setting, the preceding functors being replaced with their derived version. We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on X and simply calls an object of this category "a sheaf". An object of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ may be represented by a bounded complex of sheaves $F^\bullet$ and a quasi-isomorphism $u\colon F^\bullet \to G^\bullet$ becomes an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. (A quasi-isomorphism is a morphism which induces isomorphisms on the cohomology objects.) Note that morphisms of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ are not easy to describe.

The bifunctor $\otimes$ being right exact, one has to replace it with its left derived functor $\overset{\mathrm{L}}{\otimes}$, and similarly with the functor $\boxtimes$ that one replaces with its left derived functor $\overset{\mathrm{L}}{\boxtimes}$. . By the hypothesis that the ring $\mathbf{k}$ has finite global dimension, the derived functor applied to objects of the bounded derived category takes its values in this category.

The functors f_*, $f_!$ and the bifunctor $\mathscr{H}om$ being left exact, one has to replace them with their right derived versions, $\mathrm{R}f_*$, $\mathrm{R}f_!$ and $\mathrm{R}\mathscr{H}om$. To calculate a right derived functor, for example $\mathrm{R}f_*F$, the recipe is to represent F by a complex of injective sheaves and to apply f_* to this complex.

Let us illustrate the strength of the derived approach with an example.

Example 1.1. Consider a real finite dimensional vector space $\mathbb{V}$ and a closed proper cone γ with vertex at 0. Denote by γ° the polar cone in $\mathbb{V}^*$. This last cone is convex and only allows us to recover the convex hull of γ. However, if one replaces γ with the sheaf $\mathbf{k}_\gamma$ and replaces the polar cone with the Fourier-Sato transform (see [KS90, § 3.7]) of $\mathbf{k}_\gamma$, a transform which uses the six Grothendieck operations, then no information is lost and one recovers $\mathbf{k}_\gamma$, hence the initial cone γ, even if this cone is not convex.

Let us come back to the non derived operations described above. Taking the derived functors we get two pairs of adjoint functors

$$(\overset{\mathrm{L}}{\otimes}, \mathrm{R}\mathscr{H}om), \quad (f^{-1}, \mathrm{R}f_*).$$

The functor $f_!$ does not have an adjoint but the functor $\mathrm{R}f_!$ has a right adjoint (see [KS90, § 3.1])

$$f^!\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$$

and we get the pair of adjoint functors (in the derived categories)

$$(\mathrm{R}f_!, f^!).$$

On a topological manifold X, the dualizing complex ω_X is defined by $\omega_X := a_X^!\mathbf{k}_{\{\mathrm{pt}\}}$. One proves (see [KS90, § 3.3]) that

$$\omega_X \simeq \mathrm{or}_X\,[\dim X]$$

where or_X is the orientation sheaf on X, $\dim X$ is the dimension of X and $\mathrm{or}_X\,[\dim X]$ is the shifted object. We shall encounter the duality functors

$$\mathrm{D}'_X(\,\bullet\,) = \mathrm{R}\mathscr{H}om(\,\bullet\,, \mathbf{k}_X), \quad \mathrm{D}_X = \mathrm{R}\mathscr{H}om(\,\bullet\,, \omega_X).$$

Kernels

For good topological spaces X_i's as above, one often calls an object $K_{ij} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$ *a kernel*. One defines as usual the convolution (one also says "composition") of kernels

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}(q_{12}^{-1}K_{12} \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_{23}). \tag{1.4}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

It is easily checked, and well known, that convolution is associative, namely given three kernels $K_{ij} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$, $i = 1, 2, 3$, $j = i + 1$ one has an isomorphism

$$(K_{12} \circ K_{23}) \circ K_{34} \simeq K_{12} \circ (K_{23} \circ K_{34}), \tag{1.5}$$

this isomorphism satisfying natural compatibility conditions that we shall not make here explicit.

Of course, this construction applies in the particular case where $X_i = \mathrm{pt}$ for some i. In this case, let us change our notations to $X_1 = X$ and $X_2 = Y$. If $K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times Y})$ and $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one usually sets $\Phi_K(F) = F \circ K$. Hence

$$\Phi_K(F) = F \circ K = \mathrm{R}q_{2!}(q_1^{-1}F \overset{\mathrm{L}}{\otimes} K). \tag{1.6}$$

We shall also use the right adjoint of the functor $\Phi_K(\cdot)$, namely the functor $\Psi_K(\cdot)$ (see [KS90, § 3.6]), defined for $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ by:

$$\Psi_K(G) = \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(K, q_2^!G). \tag{1.7}$$

Hence:

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)}(\Phi_K(F), G) \simeq \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)}(F, \Psi_K(G)).$$

For $K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X\times Y})$, set $K^v = v_*K$ where v is the map $X \times Y \xrightarrow{\sim} Y \times X$, $(x, y) \mapsto (y, x)$.

Lemma 1.2. *Let $f \colon X \to Y$, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$. Set for short $K_f = \mathbf{k}_{\Gamma_f}$. Then*

$$f^{-1}G \simeq K_f \circ G = \Phi_{K_f^v}G, \quad \mathrm{R}f_*F \simeq \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(K_f, q_1^!F) = \Psi_{K_f^v}F,$$
$$\mathrm{R}f_!F \simeq F \circ K_f = \Phi_{K_f}F, \quad f^!G \simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(K_f, q_2^!G) = \Psi_{K_f}G.$$

Proof. The first and third isomorphisms are obvious (identify X with Γ_f). The two others follow by adjunction. □

Remark 1.3. One may also define the non-proper convolution of kernels by the formula below, similar to (1.4)

$$K_{12} \overset{\mathrm{np}}{\underset{2}{\circ}} K_{23} := \mathrm{R}q_{13*}(q_{12}^{-1}K_{12} \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_{23}). \tag{1.8}$$

However, one should be aware that, in general, this operation is no more associative.

Let $f\colon X \to Y$ be as above and denote by $j\colon \Gamma_f \hookrightarrow X \times Y$ the embedding of the graph of f. By remarking that the composition $q_1 \circ j\colon \Gamma_f \to X$ is an isomorphism, we get:

$$\begin{aligned}\mathrm{R}f_*F &\simeq \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(K_f, q_1^!F) \simeq \mathrm{R}q_{2*}j_!j^!q_1^!F \\ &\simeq \mathrm{R}q_{2*}j_!j^{-1}q_1^{-1}F \simeq \mathrm{R}q_{2*}(q_1^{-1}F\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f}) \simeq F \overset{\mathrm{np}}{\circ} K_f.\end{aligned}$$

Micro-support

Now assume that X is a real manifold of class C^∞ and denote by $\pi_X\colon T^*X \to X$ its cotangent bundle. To $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one associates its *micro-support* $\mathrm{SS}(F)$ (also called *singular support*), a closed $\mathbb{R}^+$-conic subset of T^*X and this set is co-isotropic (in a sense that we do not recall here). See [KS90, Th. 6.5.4].

Subanalytic subsets

From now on and unless otherwise specified, we work on real analytic manifolds. However, almost all results extend to the case of subanalytic spaces for the definition of which we refer to [KS16, § 2.4].

We shall not review here the history of subanalytic geometry, which takes its origin in the work of Lojasiewicz, simply mentioning the names of Gabrielov and Hironaka. References are made to [BM88].

Let X be a real analytic manifold. Denote by $\mathscr{S}_X$ the family of subanalytic subsets of X. Then $\mathscr{S}_X$ is a Boolean algebra which contains the family of semi-analytic subsets (those locally defined by analytic inequalities) and is closed under taking the closure and the interior. If $f\colon X \to Y$ is subanalytic, $A \in \mathscr{S}_X$, $B \in \mathscr{S}_Y$, then $f^{-1}(B) \in \mathscr{S}_X$ and if f is proper on the closure of A, then $f(A) \in \mathscr{S}_Y$.

Moreover, to be subanalytic in X is a local property on X. More precisely, given $X = \bigcup_{a\in A} U_a$ an open covering, a subset $Z \subset X$ is subanalytic in X if and only if $Z \cap U_a$ is subanalytic in U_a for all $a \in A$.

Note that if Z is a locally closed subanalytic subset of X, then there exist an open set U and a closed subset S both subanalytic in X such that $Z = U \cap S$. Indeed, set $Y = \overline{Z} \setminus Z$. Then Y is closed since Z is locally closed. Choose $S = \overline{Z}$ and $U = X \setminus Y$.

A subanalytic stratification of X is a locally finite partition $X = \bigsqcup_{a\in A} X_a$ where each X_a is a smooth locally closed real analytic submanifold of X subanalytic in X, and for all $a, b \in A$, $X_a \cap \overline{X_b} \neq \varnothing$ implies $X_a \subset \overline{X_b}$.

Constructible sheaves

A sheaf $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ is weakly $\mathbb{R}$-constructible if there exists a subanalytic stratification $X = \bigsqcup_{a\in A} X_a$ such that for all $j \in \mathbb{Z}$, $H^j(F)|_{X_a}$ is locally constant. If moreover, these locally constant sheaves are finitely generated (recall that $\mathbf{k}$ is Noetherian), then F is $\mathbb{R}$-constructible. By the results of [KS90, Ch. VIII], F is weakly $\mathbb{R}$-constructible if and only if $\mathrm{SS}(F)$ is contained in a closed conic subanalytic isotropic subvariety of T^*X and this implies that $\mathrm{SS}(F)$ is equal to a closed conic subanalytic Lagrangian subvariety.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of $\mathbb{R}$-constructible sheaves. The categories of constructible sheaves are closed under the six Grothendieck operations with the exception of direct images which should be proper on the supports of the constructible sheaves.

2 Constructible sheaves up to infinity

2.1 Subanalytic subsets up to infinity

In order to define subanalytic subsets up to infinity, we introduce the category of b-analytic manifolds, inspired by (but rather different from) that of bordered space of [DK16]. As mentioned in the introduction, the notion of being subanalytic up to infinity is a particular case of that of definable set, well known from the specialists (see [VdD98, VdDM96]), and constructible sheaves in this framework have already been defined in [Sch03, EP20]. However, our approach is direct and quite different since it is based on the notion of micro-support.

Definition 2.1. The category of *b-analytic manifolds* is the category defined as follows.

(a) An object X_∞ is a pair $(X, \widehat{X})$ with $X \subset \widehat{X}$ an open embedding of real analytic manifolds such that X is relatively compact and subanalytic in $\widehat{X}$.

(b) A morphism $f\colon X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ of b-analytic manifolds is a morphism of real analytic manifolds $f\colon X \to Y$ such that the graph Γ_f of f in $X \times Y$ is subanalytic in $\widehat{X} \times \widehat{Y}$.

(c) The composition $(X, \widehat{X}) \xrightarrow{f} (Y, \widehat{Y}) \xrightarrow{g} (Z, \widehat{Z})$ is given by $g \circ f\colon X \to Z$ and the identity $\mathrm{id}_{(X,\widehat{X})}$ is given by id_X (see Lemma 2.3 below).

If there is no risk of confusion, we shall often denote by $j_X\colon X \hookrightarrow \widehat{X}$ the open embedding.

Remark 2.2. Instead of requiring $\widehat{X}$ to be a smooth real analytic manifold and X relatively compact in it, one could ask $\widehat{X}$ to be a compact subanalytic space in the sense of [KS16, § 2.4]. However, Definition 2.8 below should be modified by using uniquely properties (c) and (d) of Lemma 2.7. One could also define the notion of a b-subanalytic space.

Remark that in general, contrarily to the case of bordered spaces, neither (X, X) nor $(\widehat{X}, \widehat{X})$ are b-analytic manifolds. However, if X is compact, (X, X) is a b-analytic manifold.

Lemma 2.3. (a) *The identity* $\mathrm{id}_{(X,\widehat{X})}$ *is a morphism of b-analytic manifolds.*

(b) *Let $f\colon (X,\widehat{X}) \to (Y,\widehat{Y})$ and $g\colon (Y,\widehat{Y}) \to (Z,\widehat{Z})$ be morphisms of b-analytic manifolds. Then the composition $g\circ f$ is a morphism of b-analytic manifolds.*

Proof. (a) Since X is subanalytic in $\widehat{X}$, $X\times X$ is subanalytic in $\widehat{X}\times\widehat{X}$, and $\Delta_X = X\times X\cap\Delta_{\widehat{X}}$ is subanalytic in $\widehat{X}\times\widehat{X}$.

(b) By the hypothesis, Γ_g is subanalytic and relatively compact in $\widehat{Y}\times\widehat{Z}$ and Γ_f is subanalytic and relatively compact in $\widehat{X}\times\widehat{Y}$. It follows that $\Gamma_f\times_{\widehat{Y}}\Gamma_g$ is subanalytic and relatively compact in $\widehat{X}\times\widehat{Y}\times\widehat{Z}$. Therefore, its projection $\Gamma_f\circ\Gamma_g$ is subanalytic in $\widehat{X}\times\widehat{Z}$. Since $\Gamma_f\circ\Gamma_g=\Gamma_{g\circ f}$, the proof is complete. (Note that one could also have applied Proposition 2.6 below.) □

Definition 2.4. Let $X_\infty = (X,\widehat{X})$ be a b-analytic manifold and let Z be a subset of X. We say that Z is subanalytic up to infinity if Z is subanalytic in $\widehat{X}$. We shall also say for short that Z is b-subanalytic.

Note the following remarks.

- The property of being subanalytic up to infinity depends on the choice of X_∞ and such a choice is supposed to have been made when using this terminology.
- Given X, there does not always exist X_∞. As an example (of non-existence), choose $X=\mathbb{N}$, a real analytic manifold of dimension 0.
- The family of subsets subanalytic up to infinity inherits all of the properties of the family of subanalytic subsets with the exception that this property is no more local (but it is local for finite coverings). In particular, this family is closed under interior, closure, complement, finite unions and finite intersections and X itself is subanalytic up to infinity (once X_∞ exists).

On a real analytic manifold X, the subanalytic topology and the site X_{sa} are defined in [KS01].

Definition 2.5. Let $X_\infty = (X,\widehat{X})$ be a b-analytic manifold.

(a) We shall denote by $\mathrm{Op}_{X_{\infty\mathrm{sa}}}$ the category of open subsets of X subanalytic up to infinity, the morphisms being the inclusions.

(b) We endow $\mathrm{Op}_{X_{\infty\mathrm{sa}}}$ with a Grothendieck topology as follows. A family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{X_{\infty\mathrm{sa}}}$ is a covering of $U\in\mathrm{Op}_{X_{\infty\mathrm{sa}}}$ if $U_i\subset U$ for all $i\in I$ and there exists $J\subset I$ with J finite such that $U=\bigcup_{j\in J}U_j$.

(c) We denote by $X_{\infty\mathrm{sa}}$ the site so obtained.

Note that the category $\mathrm{Op}_{X_{\infty\mathrm{sa}}}$ is closed under product of two elements (namely, the intersection of two open subsets) and admits a terminal object, namely X. This makes the study of sheaves on $X_{\infty\mathrm{sa}}$ particularly easy.

In the sequel, for $U\in\mathrm{Op}_{X_{\infty\mathrm{sa}}}$, we shall denote by U_∞ the b-analytic manifold $(U,\widehat{X})$ where the embedding $j_U\colon U\hookrightarrow\widehat{X}$ is the composition of j_X and the embedding $U\hookrightarrow X$.

Proposition 2.6. *Let $X_{i\infty} = (X_i, \widehat{X}_i)$ $(i = 1, 2, 3)$ be three b-analytic manifolds.*

(a) *Setting $\widehat{X}_{12} = \widehat{X}_1 \times \widehat{X}_2$, the pair $(X_{12}, \widehat{X}_{12})$ is a b-analytic manifold. Moreover, if S_1 and S_2 are two b-subanalytic subsets of X_1 and X_2 respectively, then $S_1 \times S_2$ is b-subanalytic in X_{12}.*

(b) *Let S_1 and S_2 be two b-subanalytic subsets of X_{12} and X_{23} respectively, then $S_1 \underset{2}{\circ} S_2$ is b-subanalytic in X_{13}.*

(c) *In particular, let $f\colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds. If $Z \subset Y$ is b-subanalytic, then $f^{-1}(Z)$ is b-subanalytic in X and if $S \subset X$ is b-subanalytic, then $f(S)$ is b-subanalytic in Y.*

We shall denote by $(X \times Y)_\infty$ the b-analytic manifold $(X \times Y, \widehat{X} \times \widehat{Y})$.

Proof. (a) is obvious.
(b) $S_1 \times_{X_2} S_2$ is subanalytic and relatively compact in $\widehat{X}_{123}$. Therefore, its image by q_{13} is subanalytic and relatively compact in $\widehat{X}_{13}$.
(c) By the hypothesis, Γ_f is subanalytic up to infinity in $\widehat{X} \times \widehat{Y}$. By (b), $f^{-1}(Z) = \Gamma_f \underset{Y}{\circ} Z$ is subanalytic up to infinity in X and $f(S) = S \underset{X}{\circ} \Gamma_f$ is subanalytic up to infinity in Y. □

2.2 Constructible sheaves up to infinity

Constructible sheaves up to infinity can be regarded as a generalization of the notion of tame multiparameter persistence modules. In this section, we consider b-analytic manifolds $X_\infty = (X, \widehat{X})$ and $Y_\infty = (Y, \widehat{Y})$.

Definitions

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. Recall that the micro-support $\mathrm{SS}(F)$ of F is a closed $\mathbb{R}^+$-conic subanalytic Lagrangian subset of T^*X.

Lemma 2.7 (See [KS21b, Th.2.2])**.** *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. The following conditions are equivalent.*

(a) *The micro-support $\mathrm{SS}(F)$ is subanalytic in $T^*\widehat{X}$.*

(b) *The micro-support $\mathrm{SS}(F)$ is contained in a locally closed $\mathbb{R}^+$-conic subanalytic isotropic subset of $T^*\widehat{X}$.*

(c) $j_{X!}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$.

(d) $\mathrm{R}j_{X*}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$.

Proof. For the reader's convenience, we recall the proof of loc. cit, a proof which uses the notion of a μ-stratification (see [KS90, Def. 8.3.19]). Note that in loc. cit. the statement was formulated slightly differently.

(a)⇒(b) is obvious.

(c)⇒(a) and (d)⇒(a) follow from the fact that T^*X is subanalytic in $T^*\widehat{X}$. Indeed, set either $\Lambda = \mathrm{SS}(j_{X!}F)$ or $\Lambda = \mathrm{SS}(\mathrm{R}j_{X*}F)$. Then Λ is subanalytic in $T^*\widehat{X}$ and $\mathrm{SS}(F) = \Lambda \cap T^*X$ is still subanalytic in $T^*\widehat{X}$.

(b)⇒(c). Assume that $\mathrm{SS}(F)$ is contained in a locally closed $\mathbb{R}^+$-conic subanalytic isotropic subset Λ of $T^*\widehat{X}$. By [KS90, Cor. 8.3.22], there exists a μ-stratification $\widehat{X} = \bigsqcup_{a\in A} Y_a$ such that $\Lambda \subset \bigsqcup_{a\in A} T^*_{Y_a}\widehat{X}$.

Set $X_a = X \cap Y_a$. Then $X = \bigsqcup_{a\in A} X_a$ is a μ-stratification and one can apply loc. cit. Prop. 8.4.1. Hence, for each $a \in A$, $F|_{X_a}$ is locally constant of finite rank. Hence $(j_{X!}F)|_{X_a}$ as well as $(j_{X!}F)_{\widehat{X}\setminus X} \simeq 0$ is locally constant of finite rank. Hence $j_{X!}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$.

(c)⇒(d). Using the implication (b)⇒(c), we get that $\mathrm{R}j_{X!}\mathbf{k}_X$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$. Set $G = j_{X!}F$. Then $\mathrm{R}j_{X*}F \simeq \mathrm{R}\mathscr{H}om(\mathrm{R}({}_!j_X)\mathbf{k}_X, G)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$ by [KS90, Prop. 8.4.10]. □

Definition 2.8. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. One says that F is constructible up to infinity if it satisfies one of the equivalent conditions in Lemma 2.7. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ consisting of sheaves constructible up to infinity.

We shall also say, for short, that F is "b-constructible" instead of "constructible up to infinity".

It follows that if $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$, then $j_X^{-1}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$.

Example 2.9. Piecewise linear sheaves (PL sheaves) on a real vector space $\mathbb{V}$ are defined in [KS21a, Def. 2.3]. Clearly, PL-sheaves are constructible up to infinity.

Operations

Proposition 2.10. *Let X_∞ and Y_∞ be two b-analytic manifolds.*

(i) *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{Y_\infty})$. Then $F \overset{\mathrm{L}}{\boxtimes} G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{(X\times Y)_\infty})$.*

(ii) *Let F_1 and F_2 belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$. Then $F_1 \overset{\mathrm{L}}{\otimes} F_2$ and $\mathrm{R}\mathscr{H}om(F_1, F_2)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$. In particular, the dual $\mathrm{D}_X F$ of $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$.*

Proof. All the statements follow from the similar ones for usual constructible sheaves and the isomorphisms:

$$\begin{aligned} j_{X\times Y!}(F \overset{\mathrm{L}}{\boxtimes} G) &\simeq j_{X!}F \overset{\mathrm{L}}{\boxtimes} j_{Y!}G, \\ j_{X!}(F_1 \overset{\mathrm{L}}{\otimes} F_2) &\simeq j_{X!}F_1 \overset{\mathrm{L}}{\otimes} j_{X!}F_2, \\ j_{X!}\mathrm{R}\mathscr{H}om(F_1, F_2) &\simeq \mathrm{R}\mathscr{H}om(j_{X!}F_1, j_{X!}F_2). \end{aligned}$$

The proof of the first isomorphism is left as an exercise. The second one follows from the projection formula:

$$j_{X!}F_1\overset{\mathrm{L}}{\otimes}j_{X!}F_2\simeq j_{X!}(F_1\overset{\mathrm{L}}{\otimes}j_X^{-1}j_{X!}F_2)\simeq j_{X!}(F_1\overset{\mathrm{L}}{\otimes}F_2).$$

The third isomorphism follows by applying $j_{X!}$ to the isomorphism

$$\mathrm{R}\mathscr{H}om\,(F_1,F_2)\simeq j_X^!\mathrm{R}\mathscr{H}om\,(j_{X!}F_1,j_{X!}F_2),$$

using the fact that, j_X being an open immersion, $j_X^!\circ j_{X!}\simeq\mathrm{id}$. □

Proposition 2.11. *Let $f\colon X_\infty\to Y_\infty$ be a morphism of b-analytic manifolds.*

(i) *Let $G\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{Y_\infty})$. Then $f^{-1}(G)$ and $f^!G$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$.*

(ii) *Let $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$. Then $\mathrm{R}f_!F$ and $\mathrm{R}f_*F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{Y_\infty})$.*

Proof. Let $K_f=\mathbf{k}_{\Gamma_f}$. Then $K_f\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{(X\times Y)_\infty})$. By Proposition 2.10 and Lemma 1.2, we are reduced to prove that
(a) if $H\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{(X\times Y)_\infty})$, then $\mathrm{R}q_{1!}H$ and $\mathrm{R}q_{1*}H$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$,
(b) if $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$, then $q_1^{-1}F$ and $q_1^!F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{(X\times Y)_\infty})$.
The assertion (b) follows from Proposition 2.10 since $q_1^{-1}F\simeq F\boxtimes\mathbf{k}_Y$ and $q_1^!F\simeq F\boxtimes\omega_Y$. To prove (a), denote by $\widehat{q_1}$ the projection $\widehat{X}\times\widehat{Y}\to\widehat{X}$. Then $\mathrm{R}q_{1!}H\simeq j_X^{-1}\mathrm{R}\widehat{q_1}_!\mathrm{R}j_{X\times Y!}H$ and similarly $\mathrm{R}q_{1*}H\simeq j_X^{-1}\mathrm{R}\widehat{q_1}_*\mathrm{R}j_{X\times Y*}H$. □

Remark 2.12. We see in Proposition 2.11 an important difference between constructible sheaves and constructible sheaves up to infinity. Indeed, for usual constructible sheaves, the (proper or non proper) direct image is no more constructible in general.

Corollary 2.13. *Let $f\colon X_\infty\to Y_\infty$ be a morphism of b-analytic manifolds and let $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and $G\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{Y_\infty})$. Then $\mathrm{R}f_*F\simeq\mathrm{D}_Y\mathrm{R}f_!\mathrm{D}_XF$ and $f^!G\simeq\mathrm{D}_Xf^{-1}\mathrm{D}_YG$.*

Proof. (i) Both D_XF and $\mathrm{R}f_!\mathrm{D}_XF$ are $\mathbb{R}$-constructible. Then apply [KS90, Exe. VIII.3].
(ii) Similarly, both D_YG and $f^{-1}\mathrm{D}_YG$ are $\mathbb{R}$-constructible. Then apply loc. cit. □

Consider b-analytic manifolds $X_{i\infty}=(X_i,\widehat{X}_i)$, $(i=1,2,3)$, and kernels $K_{ij}\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_{ij\infty}})$, $i=1,2$, $j=i+1$. We have already defined in (1.4) the convolution of kernels $K_{12}\underset{2}{\circ}K_{23}$.

Applying Propositions 2.10 and 2.11, we get:

Corollary 2.14. *In the preceding situation, $K_{12}\underset{2}{\circ}K_{23}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_{13\infty}})$.*

Recall that the convolution of kernels is associative (see (1.5)).

Base change formula and projection formula

Consider two morphisms $f\colon X_\infty \to Z_\infty$ and $g\colon Y_\infty \to Z_\infty$ of b-analytic manifolds and consider[1] a Cartesian square of topological spaces

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd} \tag{2.1}$$

Recall that the square is Cartesian means that W is isomorphic to the space $\{(x,y) \in X \times Y; f(x) = g(y)\}$. We consider W as a closed subanalytic subset of $X \times Y$.

The classical base change formula for sheaves (see for example [KS90, Prop. 2.6.7]) together with Proposition 2.11 gives:

Proposition 2.15. *Consider the Cartesian square (2.1) and let $F \in \mathrm{D^b}(\mathbf{k}_{X_\infty})$. Then*

$$g^{-1}\mathrm{R}f_!F \simeq \mathrm{R}f'_!g'^{-1}F \text{ in } \mathrm{D^b}(\mathbf{k}_{Y_\infty}). \tag{2.2}$$

Remark that the left hand-side of this isomorphism belongs to $\mathrm{D^b}(\mathbf{k}_{Y_\infty})$ by the preceding results and this implies that the same is true for the right hand-side.

Similarly, the classical projection formula together with Proposition 2.11 gives:

Proposition 2.16. *Let $f\colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds, let $F \in \mathrm{D^b}(\mathbf{k}_{X_\infty})$ and $G \in \mathrm{D^b}(\mathbf{k}_{Y_\infty})$. Then*

$$\mathrm{R}f_!(F \overset{\mathrm{L}}{\otimes} f^{-1}G) \simeq \mathrm{R}f_!F \overset{\mathrm{L}}{\otimes} G \text{ in } \mathrm{D^b}(\mathbf{k}_{Y_\infty}). \tag{2.3}$$

2.3 Convolution and γ-topology

In this subsection, we consider a real n-dimensional vector space $\mathbb{V}$. We consider its projective compactification $\mathbb{P} = (\mathbb{V} \oplus \mathbb{R} \setminus \{0\})/\mathbb{R}^\times$. The pair $(\mathbb{V}, \mathbb{P})$ is a b-analytic manifold and we set

$$\mathbb{V}_\infty = (\mathbb{V}, \mathbb{P}). \tag{2.4}$$

If there is no risk of confusion, we simply write $\mathbb{V}$ instead of $\mathbb{V}_\infty$.

Convolution

We denote by s the addition map.

$$s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}, \quad (x,y) \mapsto x+y.$$

[1] In [Sch20, v1, v2] it was made reference to the notion of a Cartesian square in the category of b-analytic manifolds, a notion which should have been defined more precisely and that we avoid here.

Clearly, s is a morphism of b-analytic manifolds.

We define the convolution and the non-proper convolution as follows. For $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$, we set

$$F \star G := \mathrm{R}s_!(F \boxtimes G), \quad F \overset{\mathrm{np}}{\star} G := \mathrm{R}s_*(F \boxtimes G).$$

By Propositions 2.11 and 2.10, both $F \star G$ and $F \overset{\mathrm{np}}{\star} G$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{V}_\infty)$. One checks easily that both convolution operations are commutative and that usual (proper) convolution is associative. Note that, denoting by $\mathbb{V}_i$ $(i = 1, 2)$ two copies of $\mathbb{V}$ one has $F_1 \star F_2 \simeq (F_1 \boxtimes F_2) \underset{12}{\circ} \mathbf{k}_{\Gamma_s}$ where Γ_s is the graph of s in $\mathbb{V}_{12} \times \mathbb{V}$.

Proposition 2.17. *Let $F_i \in \mathrm{D}^{\mathrm{b}}(\mathbb{V}_\infty)$, $i = 1, 2, 3$. Then*

$$F_1 \overset{\mathrm{np}}{\star} F_2 \simeq \mathrm{D}_{\mathbb{V}}(\mathrm{D}_{\mathbb{V}} F_1 \star \mathrm{D}_{\mathbb{V}} F_2), \quad (F_1 \overset{\mathrm{np}}{\star} F_2) \overset{\mathrm{np}}{\star} F_3 \simeq F_1 \overset{\mathrm{np}}{\star} (F_2 \overset{\mathrm{np}}{\star} F_3).$$

Proof. (i) The first isomorphism follows from Corollary 2.13.

(ii) The second isomorphism follows from the first one and the associativity of the usual convolution. ☐

Remark 2.18. Proposition 2.17 is remarkable since, as already mentioned, the operation $\overset{\mathrm{np}}{\star}$ is not associative in general.

γ-topology

References to the γ-topology and its links with sheaf theory are made to [KS90, KS18]. We consider a real n-dimensional vector space $\mathbb{V}$. We set $\dot{\mathbb{V}} = \mathbb{V} \setminus \{0\}$ and we recall that $\mathbb{V}_\infty$ is defined in (2.4). Clearly, the antipodal map $a \colon \mathbb{V} \to \mathbb{V}$, $x \mapsto -x$, is a morphism of b-analytic manifolds. For a subset A of $\mathbb{V}$, we denote by A^a its image by the antipodal map.

A subset γ of $\mathbb{V}$ is called a cone if $\mathbb{R}_{>0}\gamma = \gamma$. A closed convex cone γ is proper if $\gamma \cap \gamma^a = \{0\}$.

We consider a cone $\gamma \subset \mathbb{V}$ and we assume:

(2.5) *γ is a closed convex proper subanalytic cone with non-empty interior.*

Lemma 2.19. *Let $\gamma \subset \dot{\mathbb{V}}$ be a cone, subanalytic in $\dot{\mathbb{V}}$. Then γ is subanalytic up to infinity.*

Proof. (a) The set γ is subanalytic in $\mathbb{V}$ by [KS90, Prop. 8.3.8 (i)].

(b) Choose a subanalytic norm $\| \cdot \|$ on $\mathbb{V}$ and consider the real analytic isomorphism $f \colon \dot{\mathbb{V}} \to \dot{\mathbb{V}}$, $f(x) = x/\|x\|^2$. The map f defines an automorphism of the b-analytic manifold $\mathbb{V}_\infty$. It is thus enough to check that $f(\gamma)$ is subanalytic in $\mathbb{V}$. Since this set is a subanalytic cone, this follows from (a). ☐

The family of γ-invariant open subsets U of $\mathbb{V}$ (that is, satisfying $U = U + \gamma$) defines a topology, which is called the *γ-topology* on $\mathbb{V}$. One denotes by $\mathbb{V}_\gamma$ the space $\mathbb{V}$ endowed with the γ-topology and one denotes by

$$\varphi_\gamma \colon \mathbb{V} \to \mathbb{V}_\gamma \tag{2.6}$$

the continuous map associated with the identity. Note that the closed sets for this topology are the γ^a-invariant closed subsets of $\mathbb{V}$ and that a subset is γ-locally closed if it is the intersection of a γ-closed subset and a γ-open subset.

Lemma 2.20. *Let $A \subset \mathbb{V}$. The conditions below are equivalent:*

(a) *$A = (U + \gamma) \cap (\overline{U + \gamma^a})$ with U open and subanalytic up to infinity.*

(b) *A is the intersection of a γ-closed subset S and a γ-open subset U, both S and U being subanalytic up to infinity.*

(c) *A is γ-locally closed and A is subanalytic up to infinity.*

Proof. (a)$\Rightarrow$(b). It is enough to check that U being subanalytic up to infinity, $U + \gamma$ is subanalytic up to infinity. This set is the image of the set $U \times \gamma$ by the map $s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $(x, y) \mapsto x + y$. Hence, the result follows from Proposition 2.6.
(b)$\Rightarrow$(c) is obvious.
(c)$\Rightarrow$(a). By [KS18, Prop. 3.4], we may write $A = (U + \gamma) \cap (\overline{U + \gamma^a})$ with $U = \mathrm{Int}(A)$. Therefore, U is subanalytic up to infinity. ∎

Definition 2.21. Let A be a subset of $\mathbb{V}$. One says that A is *b-subanalytic γ-locally closed* if A satisfies one of the equivalent conditions in Lemma 2.20.

Let γ be a cone satisfying (2.5). Recall that one denotes by $\gamma^\circ \subset \mathbb{V}^*$ the polar cone.:

$$\gamma^\circ = \{y \in \mathbb{V}^*; \langle x, y\rangle \geq 0 \text{ for all } x \in \gamma\}.$$

γ-constructible sheaves

Consider the full triangulated subcategories of the category $\mathrm{D^b}(\mathbf{k}_\mathbb{V})$:

$$\begin{cases} \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_\mathbb{V}) := \{F \in \mathrm{D^b}(\mathbf{k}_\mathbb{V}); \mathrm{SS}(F) \subset \mathbb{V} \times \gamma^{\circ a}\}, \\ \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty}) := \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_\mathbb{V}). \end{cases} \tag{2.7}$$

We call an object of the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$ a γ-constructible sheaf.

Theorem 2.22. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$. Then there exists a finite partition $\mathbb{V} = \bigsqcup_{a \in A} Z_a$ where the Z_a's are b-subanalytic γ-locally closed and $F|_{Z_a}$ is constant.*

Proof. This result is proved by Ezra Miller in [Mil20b], using the tools of [Mil20a]. If we make the extra hypothesis that F is PL (piecewise linear) and the cone γ is polyhedral, then this result is proved in [KS18, Th. 3.18]. Note that in loc. cit. the notion of being subanalytic up to infinity is not used and the partition (which is called a stratification there) is only locally finite. However, in our situation, the fact that the partition is finite is implicit in the first part of the proof. ∎

Lemma 2.23. *The endofunctor* $\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star}$ *of* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ *defines a projector* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty}) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$.

Denoting by ι the embedding $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty}) \hookrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$ and by p the functor $\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star}$, we mean that $p \circ \iota$ is an equivalence.

Proof. We know by [KS90, Prop. 5.2.3] that the functor $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*} \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ is a projector and we know by [KS90, Prop. 3.5.4] that the two functors $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}$ and $\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star}$ are isomorphic. Moreover, the functor $\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$ to itself by Proposition 2.11. $\square$

Remark 2.24. In general, non proper convolution is not defined on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ and, in particular, even if γ is subanalytic, the functor $\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star}$ does not send $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ to itself.

3 A short review on constructible functions

From now on and until the end of this paper, we assume that $\mathbf{k}$ is a field of characteristic 0.

In this section, we recall without proofs the main constructions and results on constructible functions. References are made to [Sch91] and [KS90, § 9.7].

3.1 From constructible sheaves to constructible functions

Definition 3.1. Let X be a real analytic manifold. A function $\varphi\colon X \to \mathbb{Z}$ is constructible if:

(i) for all $m \in \mathbb{Z}$, $\varphi^{-1}(m)$ is subanalytic in X,

(ii) the family $\{\varphi^{-1}(m)\}_{m\in\mathbb{Z}}$ is locally finite.

Notation 3.2. For a locally closed subanalytic subset $S \subset X$, we denote by $\mathbf{1}_S$ the characteristic function of S (with values 1 on S and 0 elsewhere). For $a \in X$ we also set $\delta_a = \mathbf{1}_{\{a\}}$.

The next result is well known. Note that the implication (b)$\Rightarrow$(d) follows from the triangulation theorem for compact subanalytic subsets (see [Har76]).

Lemma 3.3. *Let φ be a $\mathbb{Z}$-valued function on X. The conditions below are equivalent.*

(a) *φ is constructible,*

(b) *there exist a locally finite family of subanalytic locally closed subsets $\{Z_i\}_{i\in I}$ and $c_i \in \mathbb{Z}$ such that $\varphi = \sum_i c_i \mathbf{1}_{Z_i}$,*

(c) *there exist a subanalytic stratification $\{Z_i\}_{i\in I}$ and $c_i \in \mathbb{Z}$ such that $\varphi = \sum_i c_i \mathbf{1}_{Z_i}$,*

(d) *same as* (b) *assuming moreover each Z_i compact and contractible.*

Notation 3.4. One denotes by $\mathscr{CF}(X)$ the group of constructible functions on X and by $\mathscr{CF}_X$ the presheaf $U \mapsto \mathscr{CF}(U)$.

Proposition 3.5. *The presheaf $\mathscr{CF}_X$ is a sheaf on X.*

Proof. (i) Clearly, the presheaf $U \mapsto \mathscr{CF}(U)$ is separated.
(ii) Let $X = \bigcup_{a \in A} U_a$ be an open covering of X and let φ be a $\mathbb{Z}$-valued function on X such that $\varphi|_{U_a}$ is constructible on U_a. Since X is paracompact, one may assume that the covering is locally finite. For $m \in \mathbb{Z}$, set $Z_m := \varphi^{-1}(m)$ and $Z_{m,a} = Z_m \cap U_a$. Each $Z_{m,a}$ is subanalytic in U_a, which implies that Z_m is subanalytic in X. Moreover, the family $\{Z_{m,a}\}_m$ being locally finite in U_a, the family $\{Z_m\}_m$ is locally finite in X. Hence, φ is constructible on X. The same argument holds when replacing X with an open subset $U \subset X$. $\square$

Recall now that if $V \in \mathrm{D^b}(\mathbf{k})$ has the property that all its cohomology objects are finite dimensional, one defines its Euler-Poincaré index by

$$\chi(V) = \sum_i (-1)^i \dim H^i(V). \tag{3.1}$$

For a constructible sheaf F, one defines its local Euler-Poincaré index at $x \in X$ by

$$\chi_{\mathrm{loc}}(F)(x) = \sum_i (-1)^i \dim H^i(F_x).$$

Clearly, the function $x \mapsto \chi_{\mathrm{loc}}(F)(x)$ is constructible and we get a map:

$$\chi_{\mathrm{loc}} \colon \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)) \to \mathscr{CF}(X). \tag{3.2}$$

Denote by $\mathbb{K}(\mathscr{C})$ the Grothendieck group of either an abelian or a triangulated category $\mathscr{C}$, and recall that if $\mathscr{C}$ is abelian then $\mathbb{K}(\mathscr{C}) \xrightarrow{\sim} \mathbb{K}(\mathrm{D^b}(\mathscr{C}))$. Recall that if $F \colon \mathscr{C} \to \mathscr{C}'$ is a triangulated functor (of triangulated categories), then it defines a linear map $\mathbb{K}(\mathscr{C}) \to \mathbb{K}(\mathscr{C}')$.

In the sequel, we set for short

$$\mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X) := \mathbb{K}(\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)).$$

The tensor product on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ defines a ring structure on $\mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, with unit the image of the constant sheaf $\mathbf{k}_X$. The next theorem clarifies the notion of constructible function.

Theorem 3.6 ([KS90, Th. 9.7.1])**.** *Let X be a real analytic manifold. Then the map χ_{loc} defines an isomorphism of commutative unital algebras (we keep the same notation) $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X) \xrightarrow{\sim} \mathscr{CF}(X)$.*

Note that if $\chi_{\mathrm{loc}}(F) = \varphi$ and $S := \mathrm{supp}(\varphi)$, then $\chi_{\mathrm{loc}}(F_S) = \varphi$. Hence, given $\varphi \in \mathscr{CF}(X)$, we may always represent φ with a constructible sheaf of same support. We have the general "principle" that we shall make explicit in the sequel:

> *The operations on constructible functions are the image by the local Euler-Poincaré index χ_{loc} of the corresponding operations on constructible sheaves.*

In the sequel, we shall also encounter the global Euler-Poincaré indices of a sheaf F (assuming that these indices are finite):

$$\chi(F) = \chi(\mathrm{R}\Gamma(X;F)), \quad \chi_c(F) = \chi(\mathrm{R}\Gamma_c(X;F)). \tag{3.3}$$

In particular, for a locally closed subanalytic subset Z of X, we set

$$\chi(\mathbf{k}_Z) = \chi(\mathrm{R}\Gamma(Z;\mathbf{k}_Z)), \quad \chi_c(\mathbf{k}_Z) = \chi(\mathrm{R}\Gamma_c(Z;\mathbf{k}_Z)). \tag{3.4}$$

Classically, the Euler-Poincaré index of a compact subanalytic set K is defined by

$$\chi(K) = \chi(\mathbb{Q}_K). \tag{3.5}$$

Recall that, denoting by $j\colon Z \hookrightarrow X$ the embedding, $\mathbf{k}_{XZ} = j_!\mathbf{k}_Z$. Hence, $\mathrm{R}\Gamma_c(Z;\mathbf{k}_Z) \simeq \mathrm{R}\Gamma_c(X;\mathbf{k}_{XZ})$ and if Z is closed, $\mathrm{R}\Gamma(Z;\mathbf{k}_Z) \simeq \mathrm{R}\Gamma(X;\mathbf{k}_{XZ})$ since $\mathbf{k}_{XZ} \simeq j_*\mathbf{k}_Z$ in this case. However, $\mathrm{R}\Gamma(Z;\mathbf{k}_Z) \simeq \mathrm{R}\Gamma(X;\mathrm{R}j_*\mathbf{k}_Z) \neq \mathrm{R}\Gamma(X;\mathbf{k}_{XZ})$ in general.

Remark 3.7. Recall that $\mathbf{k}$ be a field of characteristic 0. Let Z be a locally closed subanalytic subset of X. Applying the projection formula, we get the isomorphism $\mathrm{R}\Gamma_c(Z;\mathbb{Q}_Z) \otimes \mathbf{k} \xrightarrow{\sim} \mathrm{R}\Gamma_c(Z;\mathbf{k}_Z)$. Hence

$$\chi_c(\mathbf{k}_Z) = \chi_c(\mathbb{Q}_Z). \tag{3.6}$$

3.2 Operations

Internal operations

The sum on $\mathscr{CF}(X)$ is the image by χ_{loc} of the direct sum for sheaves, the unit $\mathbf{1}_X$ is the image of the constant sheaf $\mathbf{k}_X$, the map $\varphi \mapsto -\varphi$ corresponds to the shift $F \mapsto F[+1]$ and the usual product on $\mathscr{CF}(X)$ is the image of the tensor product.

External product

For two real analytic manifolds X and Y, one defines the morphism

$$\boxtimes\colon \mathscr{CF}_X \boxtimes \mathscr{CF}_Y \to \mathscr{CF}_{X\times Y}, \quad (\varphi \boxtimes \psi)(x,y) = \varphi(x)\psi(y). \tag{3.7}$$

Inverse image or composition

Let $f\colon X \to Y$ be a morphism of real analytic manifolds. One defines the inverse image morphism

$$f^*\colon f^{-1}\mathscr{CF}_Y \to \mathscr{CF}_X, \quad (f^*\psi)(x) = \psi(f(x)) \text{ for } \psi \in \mathscr{CF}(Y). \tag{3.8}$$

(Recall that a morphism $f^{-1}\mathscr{CF}_Y \to \mathscr{CF}_X$ is nothing but a morphism $\mathscr{CF}_Y \to f_*\mathscr{CF}_X$.)

Inverse images are functorial, that is, if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of manifolds, then:

$$f^* \circ g^* = (g \circ f)^*.$$

Direct image or integral

Recall that, if K is a subanalytic compact subset of X, then the Euler-Poincaré index $\chi(K)$ is defined in (3.5). In particular, if K is contractible, then $\chi(\mathbf{k}_K) = 1$ and one sets in this case

$$\int_X \mathbf{1}_K = 1. \tag{3.9}$$

If φ has compact support, one may assume that the sum in Lemma 3.3 (d) is finite, and one checks (using either Theorem 3.6 or the triangulation theorem for subanalytic sets) that the integer $\sum_i c_i$ depends only on φ, not on its decomposition. One sets:

$$\int_X \varphi = \sum_i c_i.$$

In particular, if Z is locally closed relatively compact and subanalytic in X, then (see (3.4)):

$$\int_X \mathbf{1}_Z = \chi_c(\mathbf{k}_Z). \tag{3.10}$$

By (3.6), this integer does not depend on the choice of $\mathbf{k}$ as soon as $\mathbf{k}$ has characteristic 0.

One should be aware that the integral is not positive, that is

$$\varphi \geq 0 \text{ does not imply } \int_X \varphi \geq 0.$$

For example, take $X = \mathbb{R}$ and $\varphi = \mathbf{1}_{(-1,1)}$. Hence, $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi = -1$.

Let $f\colon X \to Y$ be a morphism of real analytic manifolds. One defines the direct image morphism

$$\int_X \colon f_!\mathscr{CF}_X \to \mathscr{CF}_Y, \quad \Big(\int_f \varphi\Big)(y) = \int_X \mathbf{1}_{f^{-1}(y)} \cdot \varphi. \tag{3.11}$$

Recall that a section of $f_!\mathscr{CF}_X$ on an open subset $V \subset Y$ is a section of $\mathscr{CF}_X(f^{-1}V)$ such that f is proper on its support. Hence the integral makes sense as a function but it is not obvious that it is a constructible function. This follows for example from the corresponding result for direct images of constructible sheaves. Indeed, let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ be such that $\chi_{\mathrm{loc}}(F) = \varphi$ and $\mathrm{supp}(F) = \mathrm{supp}(\varphi)$. Then $\int_f \varphi = \chi_{\mathrm{loc}}(\mathrm{R}f_!F)$.

Direct images are functorial, that is, if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of manifolds, then:

$$\int_g \circ \int_f = \int_{g \circ f}.$$

Duality

On X, the dual of a constructible function is the image by χ_{loc} of the duality functor D_X for sheaves. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and $x_0 \in X$, one has

$$(\mathrm{D}_X F)_{x_0} \simeq (\mathrm{R}\Gamma_{\{x_0\}}(F))^*,$$

where * denotes the duality functor for $\mathbf{k}$-vector spaces. Since F is constructible, there exists a local chart and $\varepsilon_0 > 0$ such that, denoting by $B_\varepsilon(x_0)$ the open ball with center x_0 and radius $\varepsilon > 0$ in this chart, one has for $0 < \varepsilon \leq \varepsilon_0$:

$$\mathrm{R}\Gamma_{\{x_0\}}(F) \simeq \mathrm{R}\Gamma_c(B_\varepsilon(x_0); F) \simeq \mathrm{R}a_{X!}(F \otimes \mathbf{k}_{B_\varepsilon(x_0)}).$$

Hence, one defines the dual of a constructible function φ on X as follows. Let $x_0 \in X$, and choose a local chart in a neighborhood of x_0 and $\varepsilon > 0$ as above. One sets

$$(\mathrm{D}_X\varphi)(x_0) = \int_X \varphi \cdot \mathbf{1}_{B_\varepsilon(x_0)}. \tag{3.12}$$

The integral $\int_X \varphi \cdot \mathbf{1}_{B_\varepsilon(x_0)}$ neither depends on the local chart nor on ε, for $0 < \varepsilon \leq \varepsilon_0$, for some $\varepsilon_0 > 0$ depending on x_0.

We get a morphism of sheaves $\mathrm{D}_X \colon \mathscr{CF}_X \to \mathscr{CF}_X$ and this morphism is an involution, that is,

$$\mathrm{D}_X \circ \mathrm{D}_X \simeq \mathrm{id}_X .$$

Moreover, duality commutes with integration. Assuming that f is proper on the support of φ, one has:

$$\mathrm{D}_Y(\int_f \varphi) = \int_f \mathrm{D}_X(\varphi). \tag{3.13}$$

By mimicking a classical formula for constructible sheaves, one sets

$$\mathit{hom}(\varphi, \psi) := \mathrm{D}_X(\mathrm{D}_X\psi \cdot \varphi). \tag{3.14}$$

Example 3.8. Let Z be a closed subanalytic subset of X and assume that Z is a C^0-manifold of dimension d with boundary ∂Z. Set $A = Z \setminus \partial Z$. Hence, locally on X, $Z \subset X$ is topologically isomorphic to $\overline{U} \subset \mathbb{R}^n$ where U is a convex open subset of $\mathbb{R}^d \subset \mathbb{R}^n$ and $A \simeq U$. We thus have

$$\mathrm{D}_X \mathbf{1}_Z = (-1)^d \mathbf{1}_A \tag{3.15}$$

Moreover

$$\int_X \mathbf{1}_{\partial Z} = \int_X \mathbf{1}_Z - \int_X \mathbf{1}_A = (1 - (-1)^d) \int_X \mathbf{1}_Z.$$

When Z is a closed convex polyhedron, one recovers the classical Euler formula.

Other operations

In fact, most (if not all) operations on constructible sheaves admit a counterpart in the language of constructible functions. In [KS90, Def. 9.7.8] one defines the specialization ν_M along a submanifold M, its Fourier-Sato transform, the microlocalization μ_M and μhom:

$$\nu_M\colon \mathscr{CF}(X) \to \mathscr{CF}_{\mathbb{R}^+}(T_M X), \quad \mu_M\colon \mathscr{CF}(X) \to \mathscr{CF}_{\mathbb{R}^+}(T^*_M X)$$
$$\mu hom\colon \mathscr{CF}(X) \times \mathscr{CF}(X) \to \mathscr{CF}_{\mathbb{R}^+}(T^* X).$$

Here, for a vector bundle $E \to M$, one denotes by $\mathscr{CF}_{\mathbb{R}^+}(E)$ the subspace of $\mathscr{CF}(E)$ consisting of functions constant on the orbits of the $\mathbb{R}^+$-action.

One can also define the micro-support of $\varphi \in \mathscr{CF}(X)$ by setting

$$\mathrm{SS}(\varphi) = \mathrm{supp}(\mu hom(\varphi, \varphi)). \tag{3.16}$$

4 Constructible functions up to infinity

4.1 Definitions

Definition 4.1. Let $X_\infty = (X, \widehat{X})$ be a b-analytic manifold.

(a) A function $\varphi\colon X \to \mathbb{Z}$ is constructible up to infinity, or b-constructible for short, if:

  (i) for all $m \in \mathbb{Z}, \varphi^{-1}(m)$ is subanalytic up to infinity,

  (ii) the family $\{\varphi^{-1}(m)\}_{m\in\mathbb{Z}}$ is finite.

(b) We denote by $\mathscr{CF}(X_\infty)$ the space of functions on X constructible up to infinity.

(c) For any function φ on X, we denote by $j_{X!}\varphi$ the function on $\widehat{X}$ obtained as the function φ on X extended by 0 on $\widehat{X} \setminus X$.

Lemma 4.2. *Let $\varphi \in \mathscr{CF}(X)$. The conditions below are equivalent.*

(a) *The function φ is constructible up to infinity,*

(b) *The function $j_{X!}\varphi$ belongs to $\mathscr{CF}(\widehat{X})$.*

(c) *There exists $\psi \in \mathscr{CF}(\widehat{X})$ such that $\varphi = \psi|_X$.*

(d) *There exist a finite family of locally closed b-subanalytic subsets $\{Z_i\}_{i\in I}$ and $c_i \in \mathbb{Z}$ such that $\varphi = \sum_i c_i \mathbf{1}_{Z_i}$.*

Proof. (a)⇒(b). By the hypothesis, one may write $\varphi = \sum_i c_i \mathbf{1}_{Z_i}$ where the sum is finite and the Z_i's are subanalytic up to infinity. Therefore, $\mathbf{1}_{Z_i} \in \mathscr{CF}(\widehat{X})$ and the result follows from Lemma 3.3.
(b)⇒(c) is obvious.

(c)⇒(d) and (c)⇒(a). By definition, for each $m \in \mathbb{Z}$, $Z_m := \psi^{-1}(m)$ is subanalytic in $\widehat{X}$ and the family $\{\mathbb{Z}_m\}_m$ is locally finite. Therefore, $Z_m \cap X$ is subanalytic in X and X being relatively compact, the family $\{X \cap Z_m\}_m$ is finite.
(d)⇒(b) is obvious. □

Clearly, $\mathscr{CF}(X_\infty)$ is a subalgebra of $\mathscr{CF}(X)$.

Let us denote by $\mathscr{CF}_{X_\infty}$ the presheaf on $X_{\infty\mathrm{sa}}$ given by $U \mapsto \mathscr{CF}(U_\infty)$.

Proposition 4.3. *The presheaf $\mathscr{CF}_{X_\infty}$ is a sheaf on $X_{\infty\mathrm{sa}}$.*

The proof is straightforward.

Recall Theorem 3.6 and denote now by $\mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ the Grothendieck group of the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$.

Theorem 4.4. *The isomorphism of commutative unital algebras $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X) \xrightarrow{\sim} \mathscr{CF}(X)$ induces an isomorphism $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty}) \xrightarrow{\sim} \mathscr{CF}(X_\infty)$.*

Proof. (i) The map χ_{loc} takes its values in $\mathscr{CF}(X_\infty)$. Indeed, for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$, $\chi_{\mathrm{loc}}(F) = j_X^*(\chi_{\mathrm{loc}}(j_{X!}F))$.
(ii) The map $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty}) \to \mathscr{CF}(X_\infty)$ is injective by the same arguments as in the proof of [KS90, Th. 9.7.1].
(iii) The map χ_{loc} is surjective since for Z locally closed and subanalytic up to infinity, $\mathbf{1}_Z = \chi_{\mathrm{loc}}(\mathbf{k}_Z)$ and $\mathbf{k}_Z$ is constructible up to infinity. □

4.2 Operations

Lemma 4.5. *If $\varphi \in \mathscr{CF}(X_\infty)$, then $\mathrm{D}_X\varphi \in \mathscr{CF}(X_\infty)$.*

Proof. The result follows from Lemma 4.2 (c) since duality commutes with restriction to an open subset. □

Let $\varphi \in \mathscr{CF}(X_\infty)$. One sets

$$j_{X*}\varphi = \mathrm{D}_{\widehat{X}} j_{X!} \mathrm{D}_X \varphi. \tag{4.1}$$

The next result follows from the corresponding result for sheaves.

Lemma 4.6. *If $\varphi \in \mathscr{CF}(X_\infty)$ has compact support in X, then $j_{X*}\varphi = j_{X!}\varphi$.*

Proposition 4.7. *Let X_∞ and Y_∞ be two b-analytic manifolds.*

(a) *Let $\varphi \in \mathscr{CF}(X_\infty)$ and $\psi \in \mathscr{CF}(Y_\infty)$. Then the function $\varphi \boxtimes \psi$, defined by $(\varphi \boxtimes \psi)(x, y) = \varphi(x)\psi(y)$, belongs to $\mathscr{CF}((X \times Y)_\infty)$.*

(b) *Let $f \colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds and let $\psi \in \mathscr{CF}(Y_\infty)$. Then the function $f^*\psi$ defined by $f^*\psi(x) = \psi(f(x))$ belongs to $\mathscr{CF}(X_\infty)$.*

In other words we have extended the morphisms (3.8) and (3.7) to b-analytic manifolds.

Proof. (a) Apply condition (b) of Lemma 4.2.
(b) Apply Proposition 2.6 together with Definition 4.1. ☐

Although we shall not use it, let us mention that one can also define the internal hom and the exceptional inverse image by the formulas

$$
\begin{aligned}
&hom(\varphi,\psi) := \mathrm{D}_X(\mathrm{D}_X\psi\cdot\varphi), \quad \varphi,\psi\in\mathscr{CF}(X_\infty),\\
&f^!\psi := \mathrm{D}_X f^*(\mathrm{D}_Y\psi), \psi\in\mathscr{CF}(Y_\infty).
\end{aligned}
\tag{4.2}
$$

Now we study the integrals of constructible functions up to infinity. One can define two integrals of $\varphi\in\mathscr{CF}(X_\infty)$. One sets

$$
\int_X \varphi := \int_{\widehat{X}} j_{X!}\varphi, \quad \int_X^{\mathrm{np}} \varphi := \int_{\widehat{X}} j_{X*}\varphi.
\tag{4.3}
$$

Recall notations (3.4).

Lemma 4.8. (a) *One has* $\int_X^{\mathrm{np}}\varphi = \int_X \mathrm{D}_X\varphi$.

(b) *Let Z be a locally closed b-subanalytic subset of X. Then* [2] $\int_X \mathbf{1}_Z = \chi_c(\mathbf{k}_Z)$.

(c) *The integrals $\int_X\varphi$ and $\int_X^{\mathrm{np}}\varphi$ do not depend on the choice of $\widehat{X}$.*

Proof. (a) follows from

$$
\int_{\widehat{X}} j_{X*}\varphi = \int_{\widehat{X}} \mathrm{D}_{\widehat{X}} j_{X!}\mathrm{D}_X\varphi = \int_{\widehat{X}} j_{X!}\mathrm{D}_X\varphi = \int_X \mathrm{D}_X\varphi.
$$

where the second equality follows from (3.13) applied with $Y = \mathrm{pt}$.
(b) Recall (3.10). Also recall that a_Z is the map $Z\to\mathrm{pt}$ and similarly with $a_{\widehat{X}}$. Denoting by j_Z the embedding $Z\hookrightarrow\widehat{X}$, we have

$$
\int_X \mathbf{1}_Z = \chi(\mathrm{R}a_{\widehat{X}!}\mathrm{R}j_{X!}\mathbf{k}_{XZ}) = \chi(\mathrm{R}a_{\widehat{X}!}\mathrm{R}j_{Z!}\mathbf{k}_Z) = \chi(\mathrm{R}a_{Z!}\mathbf{k}_Z) = \chi_c(\mathbf{k}_Z).
$$

(c) follows from (b) and (a). ☐

Example 4.9. Let $X=\mathbb{R}$. Then:

(i) One has $\int_{\mathbb{R}}\mathbf{1}_{\mathbb{R}} = -1$, $\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_{\mathbb{R}} = 1$.
(ii) Let $U=(-\infty,b)$ with $-\infty<b<\infty$. Then $\int_{\mathbb{R}}\mathbf{1}_U=-1$, $\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_U=0$.
(iii) Let $Z=(-\infty,b]$ with $-\infty<b<+\infty$. Then $\int_{\mathbb{R}}\mathbf{1}_Z=0$, $\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_Z=1$.
(iv) Let $S=[a,b]$ with $-\infty<a\le b<+\infty$. Then $\int_{\mathbb{R}}\mathbf{1}_S=\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_S=1$.
(v) Let $Z=(a,b)$ with $-\infty<a<b<+\infty$. Then $\int_{\mathbb{R}}\mathbf{1}_Z=\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_Z=-1$.
(vi) Let $Z=[a,b)$ with $-\infty<a\le b<+\infty$. Then $\int_{\mathbb{R}}\mathbf{1}_Z=\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_Z=0$.

Indeed, (i) is obvious. Let U be as in (ii). Then U is topologically isomorphic to $\mathbb{R}$ and we get $\int_{\mathbb{R}}\mathbf{1}_U=-1$. By the additivity of the integral, we deduce that for Z as in (iii), $\int_{\mathbb{R}}\mathbf{1}_Z=0$. By Lemma 4.8 (a), we get $\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_U=0$ and by additivity, $\int_{\mathbb{R}}^{\mathrm{np}}\mathbf{1}_Z=1$. Finally, (iv), (v) and (vi) are obvious.

[2] In [Sch20, v3], it was written $\int_X^{\mathrm{np}}\mathbf{1}_Z=\chi(Z)$, which is not correct.

Let $f\colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds and let $\varphi \in \mathscr{CF}(X_\infty)$. Similarly as in (3.11), one sets for $y \in Y$:

$$(4.4)\qquad (\int_f \varphi)(y) = \int_X \mathbf{1}_{f^{-1}(y)} \cdot \varphi.$$

Of course, when $Y = \mathrm{pt}$, one recovers (4.3).

Lemma 4.10. *The function $\int_f \varphi$ defined by (4.4) belongs to $\mathscr{CF}(Y_\infty)$.*

Proof. Let us choose $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_\infty})$ such that $\chi_{\mathrm{loc}}(F) = \varphi$. Then $(\int_f \varphi)(y) = \chi_{\mathrm{loc}}(\mathrm{R}f_!F)$ and $\mathrm{R}f_!F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{Y_\infty})$. □

Hence, we have constructed a morphism

$$\int_f \colon f_*\mathscr{CF}_{X_\infty} \to \mathscr{CF}_{Y_\infty}, \quad \varphi \mapsto \int_f \varphi.$$

We also define

$$(4.5)\qquad \int_f^{\mathrm{np}} \colon f_*\mathscr{CF}_{X_\infty} \to \mathscr{CF}_{Y_\infty}, \quad \int_f^{\mathrm{np}} \varphi := \mathrm{D}_Y \int_f \mathrm{D}_X \varphi.$$

The next results are easily checked.

- If f is proper on $\mathrm{supp}(\varphi)$, then $\int_f \varphi = \int_f^{\mathrm{np}} \varphi$.
- If $\varphi = \chi_{\mathrm{loc}}(F)$ for some $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$, then $\int_f \varphi = \chi_{\mathrm{loc}}(\mathrm{R}f_!F)$ and $\int_f^{\mathrm{np}} \varphi = \chi_{\mathrm{loc}}(\mathrm{R}f_*F)$.
- Let $g\colon Y_\infty \to Z_\infty$ be another morphism of b-analytic manifolds. Then
$$\int_{g\circ f} \varphi = \int_g \int_f \varphi, \quad \int_{g\circ f}^{\mathrm{np}} \varphi = \int_g^{\mathrm{np}} \int_f^{\mathrm{np}} \varphi.$$

Base change formula and projection formula

Proposition 4.11. *Consider the Cartesian square (2.1) and let $\varphi \in \mathscr{CF}(X_\infty)$. Then $\int_{f'}(g'^*\varphi)$ is well defined, belongs to $\mathscr{CF}(Y_\infty)$ and*

$$(4.6)\qquad g^* \int_f \varphi = \int_{f'} (g'^*\varphi).$$

Proof. Choose $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ such that $\chi_{\mathrm{loc}}(F) = j_{X!}\varphi$. Then apply the base change formula for sheaves (Proposition 2.15). □

Proposition 4.12. *Let $f\colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds, let $\varphi \in \mathscr{CF}(X_\infty)$ and $\psi \in \mathscr{CF}(Y_\infty)$. Then*

$$(4.7)\qquad \int_f (\varphi \cdot f^*\psi) = \psi \int_f \varphi.$$

Proof. Choose $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ such that $\chi_{\mathrm{loc}}(F) = j_{X!}\varphi$ and choose $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{Y_\infty})$ such that $\chi_{\mathrm{loc}}(G) = j_{Y!}\psi$. Then apply the projection formula for sheaves (Proposition 2.16).. □

Example 4.13. Equality (4.7) is no longer true when replacing $\int_f$ with $\int_f^{\mathrm{np}}$. Set $X = \mathbb{R}^2$ with coordinates (y,t) and $Y = \mathbb{R}$, f being the first projection. Let $\varphi = \mathbf{1}_S$ with $S = \{(y,t); t = 1/(1-y^2), -1 < y < 1\}$ and let $\psi = \mathbf{1}_Z$ with $Z = (-1,1)$. One checks easily that φ is subanalytic up to infinity when choosing for example for $\widehat{X}$ the projective compactification of $\mathbb{R}^2$. We have $\mathbf{1}_S \cdot f^*\mathbf{1}_Z = \mathbf{1}_S$, $\int_f \mathbf{1}_S = \mathbf{1}_Z$ and $\mathrm{D}_X\mathbf{1}_S = -\mathbf{1}_S$ (see Example 3.8). Hence,

$$\int_f^{\mathrm{np}} \mathbf{1}_S \cdot f^*\mathbf{1}_Z = \int_f^{\mathrm{np}} \mathbf{1}_S = \mathrm{D}_Y \int_f \mathrm{D}_X \mathbf{1}_S = -\mathrm{D}_Y\mathbf{1}_Z = \mathbf{1}_{[-1,1]},$$
$$\mathbf{1}_Z \cdot \int_f^{\mathrm{np}} \mathbf{1}_S = \mathbf{1}_Z \cdot \mathbf{1}_{[-1,1]} = \mathbf{1}_{(-1,1)}.$$

Convolution of kernels

Recall Diagram 1.1 when replacing the manifolds X_i with b-analytic manifolds $X_{i\infty}$ ($i = 1,2,3$). Let $\lambda_{12} \in \mathscr{CF}(X_{12\infty})$ and $\lambda_{23} \in \mathscr{CF}(X_{23\infty})$. It follows from Proposition 4.7 that the function

$$\lambda_{12} \underset{2}{\circ} \lambda_{23} := \int_{q_{13}} q_{12}^*\lambda_{12} \cdot q_{23}^*\lambda_{23}. \tag{4.8}$$

is well-defined and belongs to $\mathscr{CF}(X_{13\infty})$. Moreover

Theorem 4.14. *Let $\lambda_{ij} \in \mathscr{CF}(X_{ij\infty})$ ($i = 1,2,3,4$, $j = i+1$). One has*

$$(\lambda_{12} \underset{2}{\circ} \lambda_{23}) \underset{3}{\circ} \lambda_{34} = \lambda_{12} \underset{2}{\circ} (\lambda_{23} \underset{3}{\circ} \lambda_{34}) \in \mathscr{CF}(X_{14\infty}).$$

One can prove this theorem by mimicking the classical proof for sheaves, using now Propositions 4.11 and 4.12. One can also prove this result by replacing each λ_{ij} with a kernel $K_{ij} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_{ij\infty}})$.

4.3 γ-constructible functions

As already mentioned in the introduction, γ-constructible functions appear naturally in TDA (see [CGR12, Leb21, KM21] among others).

Let $\mathbb{V}$ and $\mathbb{V}_\infty$ be as in § 2.3. We define the convolution and the non-proper convolution similarly as for sheaves (see Proposition 2.17). For $\varphi, \psi \in \mathscr{CF}(\mathbb{V}_\infty)$, we set

$$\varphi \star \psi := \int_s \varphi \boxtimes \psi, \quad \varphi \overset{\mathrm{np}}{\star} \psi := \int_s^{\mathrm{np}} \varphi \boxtimes \psi.$$

By the preceding results, both $\varphi \star \psi$ and $\varphi \overset{\mathrm{np}}{\star} \psi$ belong to $\mathscr{CF}(\mathbb{V}_\infty)$. Note that

$$\varphi \star \psi = \psi \star \varphi, \quad \varphi \overset{\mathrm{np}}{\star} \psi = \psi \overset{\mathrm{np}}{\star} \varphi.$$

Lemma 4.15. *Let $\varphi_i \in \mathscr{CF}(\mathbb{V}_\infty)$, $i = 1, 2, 3$. Then*

$$\varphi_1 \overset{\mathrm{np}}{\star} \varphi_2 = \mathrm{D}_X(\mathrm{D}_X\varphi_1 \star \mathrm{D}_X\varphi_2), \quad (\varphi_1 \overset{\mathrm{np}}{\star} \varphi_2) \overset{\mathrm{np}}{\star} \varphi_3 = \varphi_1 \overset{\mathrm{np}}{\star} (\varphi_2 \overset{\mathrm{np}}{\star} \varphi_3).$$

Proof. The first equality follows from the definition of $\int^{\mathrm{np}}$ (see (4.5)) and the second equality follows from the first one. □

We consider a cone $\gamma \subset \mathbb{V}$ and we assume (2.5), that is, γ is a closed convex proper subanalytic cone with non-empty interior. Recall that $\mathbf{k}_\gamma$ is then constructible up to infinity.

Definition 4.16. Let $\varphi \in \mathscr{CF}(\mathbb{V}_\infty)$. We say that φ is γ-constructible if there exists a finite covering $\mathbb{V} = \bigcup_a Z_a$ such that $\varphi = \sum_a c_a \mathbf{1}_{Z_a}$ and the Z_a's are b-subanalytic γ-locally closed subsets of $\mathbb{V}$. We denote by $\mathscr{CF}(\mathbb{V}_\gamma)$ the space of γ-constructible functions on $\mathbb{V}$.

By construction, we have $\mathscr{CF}(\mathbb{V}_\gamma) \subset \mathscr{CF}(\mathbb{V}_\infty)$.

Recall notations (2.7) and denote by $\mathbb{K}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$ the Grothendieck group of the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$.

Theorem 4.17. *The isomorphism of commutative unital algebras $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_\mathbb{V}) \xrightarrow{\sim} \mathscr{CF}(\mathbb{V})$ induces an isomorphism $\chi_{\mathrm{loc}} \colon \mathbb{K}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty}) \xrightarrow{\sim} \mathscr{CF}(\mathbb{V}_\gamma)$.*

Proof. (i) It follows from Theorem 2.22 that the map χ_{loc} takes its values in $\mathscr{CF}(\mathbb{V}_\gamma)$.
(ii) The map χ_{loc} is injective by Lemma 2.23. Indeed, if $\mathscr{A}$ is a full triangulated subcategory of a triangulated category $\mathscr{T}$ and if there is a projector $P \colon \mathscr{T} \to \mathscr{A}$, then P induces a projector $\mathbb{K}(P) \colon \mathbb{K}(\mathscr{T}) \to \mathbb{K}(\mathscr{A})$. In particular, $\mathbb{K}(\mathscr{A})$ is a subgroup of $\mathbb{K}(\mathscr{T})$.
(iii) The map χ_{loc} is surjective since for Z subanalytic γ-locally closed, $\mathbf{1}_Z = \chi_{\mathrm{loc}}(\mathbf{k}_Z)$ and $\mathbf{k}_Z \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{V}_\infty)$. Moreover, $\mathrm{SS}(\mathbf{k}_Z) \subset \mathbb{V} \times \gamma^{\circ a}$ by [KS18, Cor. 1.8]. □

The projector of Lemma 2.23 allows us to construct a projector $\mathscr{CF}(\mathbb{V}_\infty) \to \mathscr{CF}(\mathbb{V}_\gamma)$.

Proposition 4.18. (a) *Let $\varphi \in \mathscr{CF}(\mathbb{V}_\infty)$. Then $\varphi \overset{\mathrm{np}}{\star} \mathbf{1}_{\gamma^a}$ belongs to $\mathscr{CF}(\mathbb{V}_\gamma)$.*

(b) *If $\varphi \in \mathscr{CF}(\mathbb{V}_\gamma)$, then $\varphi \overset{\mathrm{np}}{\star} \mathbf{1}_{\gamma^a} = \varphi$.*

Proof. The result follows from Theorem 4.17, Lemma 2.23 and the fact that the operation $\overset{\mathrm{np}}{\star}$ commutes with χ_{loc}. □

5 Correspondences for constructible functions

This section is a variation on [Sch95] in which we replace some properness hypotheses with that of being constructible up to infinity.

5.1 Correspondences

Consider the situation of Diagram (1.1) when replacing the manifolds X_i with b-analytic manifolds $X_{i\infty}$ $(i = 1, 2, 3)$. Assume to be given two locally closed subsets subanalytic up to infinity:

$$S_1 \subset X_{12}, \quad S_2 \subset X_{23}.$$

We set, for $\varphi \in \mathscr{CF}(X_{1\infty})$

$$\mathscr{R}_{S_1}(\varphi) = \varphi \circ \mathbf{1}_{S_1} = \int_{q_2} q_1^* \varphi \cdot \mathbf{1}_{S_1}.$$

Set

$$\lambda := \mathbf{1}_{S_1} \underset{2}{\circ} \mathbf{1}_{S_2} \in \mathscr{CF}(X_{13\infty}). \tag{5.1}$$

Applying Theorem 4.14, we get that λ is well defined and moreover

$$\mathscr{R}_{S_2} \circ \mathscr{R}_{S_1}(\varphi) = \varphi \circ \lambda. \tag{5.2}$$

Now we assume that $X_1 = X_3$ and we change our notations, setting

$$X_1 = X_3 = X, \quad X_2 = Y.$$

For $(x, x') \in X \times X$, let

$$S_{12}(x, x') = \{y \in Y; (x, y) \in S_1, (y, x') \in S_2\} = (S_1 \times_Y S_2) \cap q_{13}^{-1}(x, x'). \tag{5.3}$$

Then

$$\lambda(x, x') = \int_{q_{13}} \mathbf{1}_{S_1 \times_Y S_2} \cdot \mathbf{1}_{\{q_{13}^{-1}(x,x')\}} = \int_Y \mathbf{1}_{S_{12}(x,x')}. \tag{5.4}$$

We now consider the hypothesis

$$\begin{cases} \text{there exists } a, b \in \mathbb{Z} \text{ such that, for } (x, x') \in X \times X\text{:} \\ \lambda(x, x') = \begin{cases} a & \text{if } x \neq x', \\ b & \text{if } x = x'. \end{cases} \end{cases} \tag{5.5}$$

Writing $\lambda(x, x') = (b - a)\mathbf{1}_\Delta + a\mathbf{1}_{X \times X}$, we get:

Corollary 5.1 ([Sch95, Th. 3.1])**.** *Assume* (5.5)*. Let* $\varphi \in \mathscr{CF}(X)$*. Then:*

$$\mathscr{R}_{S_2} \circ \mathscr{R}_{S_1}(\varphi) = (b - a)\varphi + a \int_X \varphi.$$

Here, $a \int_X \varphi \in \mathbb{Z}$ is identified with the constant function $(a \int_X \varphi) \cdot \mathbf{1}_X$.

Application to flag manifolds

Let $\mathbb{W}$ be a real $(n+1)$-dimensional vector space (with $n \geq 2$) and denote by $F_{n+1}(p,q)$, with $1 \leq p \leq q \leq n$, the set of pairs $\{(l,h)\}$ of linear subspaces of $\mathbb{W}$ with $l \subset h$ and $\dim l = p$, $\dim h = q$. One sets $F_{n+1}(p) = F_{n+1}(p,p)$ and denotes as usual by q_1 and q_2 the two projections defined on $F_{n+1}(p) \times F_{n+1}(q)$. Then $F_{n+1}(p,q)$ is a real compact submanifold of $F_{n+1}(p) \times F_{n+1}(q)$, called the incidence relation. We denote by $F_{n+1}(q,p)$ its image by the map $F_{n+1}(p) \times F_{n+1}(q) \to F_{n+1}(q) \times F_{n+1}(p), (x,y) \mapsto (y,x)$. In the sequel, we set

$$X = F_{n+1}(p), \quad Y = F_{n+1}(q), \quad S = F_{n+1}(p,q) \subset X \times Y, \quad S' = F_{n+1}(q,p) \subset Y \times X.$$

Now we shall assume $p = 1$ and $q > 1$. Recall that $F_{n+1}(1) = \mathbb{P}_n$, the n-dimensional real projective space.

In order to apply Corollary 5.1, it is enough to calculate $\lambda_{12}(x,x')$ given by (5.4) and (5.3) with $S_1 = S$ and $S_2 = S'$. Set

$$\mu_{n+1}(q) = \chi(F_{n+1}(q)).$$

Proposition 5.2. *Let $\varphi \in \mathscr{CF}(\mathbb{P}_n)$. Then:*

$$\mathscr{R}_{(n+1;q,1)} \circ \mathscr{R}_{(n+1;1,q)}(\varphi) = (\mu_n(q-1) - \mu_{n-1}(q-2))\varphi + \mu_{n-1}(q-2) \int_{\mathbb{P}_n} \varphi.$$

Proof. Let us represent x and x' by lines in $\mathbb{W}$ and $y \in F_{n+1}(q)$ by a q-dimensional linear subspace. Then the set $S_{12}(x,x')$ is the set of q-dimensional linear subspaces of $\mathbb{W}$ containing both the line x and the line x'. This set is isomorphic to $F_{n-1}(q-2)$ if $x \neq x'$ and to $F_n(q-1)$ if $x = x'$. $\square$

Of course, this formula is interesting only when $\mu_n(q-1) \neq \mu_{n-1}(q-2))$.

5.2 Application: the Radon transform

This section is extracted from [Sch95]. Recall that $n \geq 2$.

One can roughly describe the Radon transform as follows. How to reconstruct a function (say with compact support) on a real vector space $\mathbb{V}$ from the knowledge of its integral along all affine hyperplanes? Since the family of these hyperplanes (including the hyperplane at infinity) is given by the dual projective space $\mathbb{P}^*$, where $\mathbb{P}$ is the projective compactification of $\mathbb{V}$, it is natural to replace $\mathbb{V}$ with $\mathbb{P}$.

We have $F_{n+1}(1) = \mathbb{P}_n$, the n-dimensional projective space and $F_{n+1}(n) = \mathbb{P}_n^*$, the dual projective space. The Radon transform thus corresponds to the case $p = 1$, $q = n$.

With the preceding notations, the incidence relation S is given by

$$S = F_{n+1}(1,n) = \{(x,y) \in \mathbb{P}_n \times \mathbb{P}_n^*; \langle x,y \rangle = 0\}.$$

The Radon transform of $\varphi \in \mathscr{CF}(\mathbb{P}_n)$, an element of $\mathscr{CF}(\mathbb{P}_n^*)$, is defined by

$$\mathscr{R}_{(n+1;1,n)}(\varphi) = \int_{\mathbb{P}_n} \mathbf{1}_S \cdot q_1^* \varphi = \varphi \circ \mathbf{1}_S. \tag{5.6}$$

For $y \in \mathbb{P}_n^*$, we shall denote by h_y its image in $\mathbb{P}_n$ by the incidence relation:

$$h_y = \{x \in \mathbb{P}_n, \langle x, y\rangle = 0\}.$$

Therefore,

$$\mathscr{R}_{(n+1;1,n)}(\varphi)(y) = \int_{\mathbb{P}_n} \varphi \cdot \mathbf{1}_{h_y}.$$

Recall that the Euler-Poincaré index of $\mathbb{P}_n$ is given by the formula:

$$\chi(\mathbb{P}_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{5.7}$$

Applying Proposition 5.2 together with (5.7), we get:

Corollary 5.3. *Let $\varphi \in \mathscr{CF}(\mathbb{P}_n)$. Then:*

$$\mathscr{R}_{(n+1,n,1)} \circ \mathscr{R}_{(n+1;1,n)}(\varphi) = \begin{cases} \varphi & \textit{if } n \textit{ is odd,} \\ -\varphi + \int_{\mathbb{P}_n} \varphi & \textit{if } n \textit{ is even.} \end{cases}$$

Now assume $\dim \mathbb{V} = 3$ and let us calculate the Radon transform of the characteristic function $\mathbf{1}_K$ of a compact subanalytic subset K of $\mathbb{V}$ (see (3.9)). First, consider a compact subanalytic subset L of a two dimensional affine vector space W. By Poincaré's duality, there is an isomorphism $\mathrm{H}^1_L(W; \mathbb{Q}_W) \simeq \mathrm{H}^1(L; \mathbb{Q}_L)$ and moreover there is a short exact sequence:

$$0 \to \mathrm{H}^0(W; \mathbb{Q}_W) \to \mathrm{H}^0(W \setminus L; \mathbb{Q}_W) \to \mathrm{H}^1_L(W; \mathbb{Q}_W) \to 0,$$

from which one deduces that:

$$\mathrm{b}_1(L) = \mathrm{b}_0(W \setminus L) - 1,$$

where b_i is the i-th Betti number. Note that $\mathrm{b}_0(W \setminus L)$ is the number of connected components of $W \setminus L$, hence $\mathrm{b}_1(L)$ is the "number of holes" of the compact set L. We may sumarize:

Corollary 5.4. *The value at $y \in \mathbb{P}_3^*$ of the Radon transform of $\mathbf{1}_K$ is the number of connected components of $K \cap h_y$ minus the number of its holes.*

The inversion formula of the Radon transform tells us how to reconstruct the set K from the knowledge of the number of connected components and holes of all its affine slices.

Conflict of interest
The author declares that he has no conflict of interest.

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