On rooted k-connectivity problems in quasi-bipartite digraphs

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Abstract We consider the directed MIN-COST ROOTED SUBSET k-EDGE-CONNECTION problem: given a digraph G = (V, E) with edge costs, a set $T \subseteq V$ of terminals, a root node r, and an integer k, find a min-cost subgraph of G that contains k edge disjoint rt-paths for all $t \in T$. The case when every edge of positive cost has head in T admits a polynomial time algorithm due to Frank [9], and the case when all positive cost edges are incident to r is equivalent to the k-MULTICOVER problem. Chan et al. [2] gave an LP-based $O(\ln k \ln |T|)$ -approximation algorithm for quasi-bipartite instances, when every edge in G has an end (tail or head) in $T \cup \{r\}$. We give a simple combinatorial algorithm with the same ratio for a more general problem of covering an arbitrary T-intersecting supermodular set function by a minimum cost edge set, and for the case when only every positive cost edge has an end in $T \cup \{r\}$.

Keywords min-cost rooted k-edge-connection \cdot quasi-bipartite digraphs \cdot T-intersecting supermodular set functions \cdot approximation algorithms

1 Introduction

All graphs considered here are directed, unless stated otherwise. We consider the following problem (a.k.a. k-EDGE-CONNECTED DIRECTED STEINER TREE):

MIN-COST ROOTED SUBSET k-EDGE-CONNECTION Input: A directed (multi-)graph G = (V, E) with edge costs $\{c(e) : e \in E\}$, a set $T \subset V$ of terminals, a root node $r \in V \setminus T$, and an integer k. Output: A min-cost subgraph that has k edge disjoint rt-paths for all $t \in T$.

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Z. Nutov The Open University of Israel E-mail: nutov@openu.ac.il The case when every edge of positive cost has head in T admits a polynomial time algorithm due to Frank [9]. When all positive cost edges are incident to r we get the MIN-COST MULTICOVER problem. The case when all positive cost edges are incident to the same node admits approximation ratio $O(\ln n)$ [24]. More generally, a graph (or an edge set) is called **quasi-bipartite** if every edge has at least one end (tail or head) in $T \cup \{r\}$.

In the augmentation version of the problem – MIN-COST ROOTED SUBSET (k_0, k) -EDGE-CONNECTION AUGMENTATION, the input graph G contains a subgraph $G_0 = (V, E_0)$ of cost zero that has k_0 -edge disjoint rt-paths for all $t \in T$. Recently, Chan, Laekhanukit, Wei, & Zhang [2] obtained approximation ratio $O(\ln(k - k_0 + 1) \ln |T|)$ for the case when G is quasi-bipartite. We provide a simple proof for a more general setting.

An integer valued set function f on a groundset V is intersecting supermodular if any $A, B \subseteq V$ that intersect satisfy the supermodular inequality $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$; if this holds whenever $A \cap B \cap T \neq \emptyset$ for a given set $T \subseteq V$ of terminals, then f is T-intersecting supermodular. We say that $A \subseteq V$ is an f-positive set if f(A) > 0. f is positively T-intersecting supermodular if the supermodular inequality holds whenever $A \cap B \cap T \neq \emptyset$ and f(A), f(B) > 0. A typical way to create a positively intersecting supermodular function is to take the "non-negative part" of an intersecting supermodular one, which means replacing each negative value by zero; namely, if g is T-intersecting supermodular then $f(A) = \max\{g(A), 0\}$ is positively T intersecting supermodular, see [9].

An edge *e* covers a set *A* if it enters *A*, namely, if its head is in *A* and tail is not in *A*. For an edge set/graph *J* let $d_J(A)$ denote the number of edges in *J* that cover *A*. We say that *J* covers *f* or that *J* is a cover of *f* if $d_J(A) \ge f(A)$ for all $A \subseteq V$. We consider the following generic problem.

<u>MIN-COST SET FUNCTION EDGE COVER</u> Input: A digraph G = (V, E) with edge costs and a set function f on V. Output: A min-cost edge subset $J \subseteq E$ that covers f.

Here f may not be given explicitly, and for a polynomial time implementation of algorithms we need that certain queries related to f can be answered in polynomial time. For an edge set I, the **residual function** f^{I} of f is defined by $f^{I}(A) = \max\{f(A) - d_{I}(A), 0\}$. It is known that if f is positively T-intersecting supermodular then so is f^{I} , c.f. [9]; to see this, note that $g(A) = f(A) - d_{I}(A)$ is positively T-intersecting supermodular (since g(A) > 0implies f(A) > 0 and since -d(A) is supermodular), and thus the positive part $\max\{g(A), 0\}$ of g is also positively T-intersecting supermodular.

Let $\max(f) = \max\{f(A) : A \subseteq V\}$ denote the maximum f-value taken over all sets. An inclusion minimal member of a set-family \mathcal{F} is called an \mathcal{F} **core**, or simply a **core**, if \mathcal{F} is clear from the context. Let $\mathcal{C}_{\mathcal{F}}$ denote the family of \mathcal{F} -cores. We will assume the following.

Assumption 1. The cores of the set family $\mathcal{F} = \{A : f^{I}(A) = \max(f^{I})\}$ can be found in polynomial time for any edge set I.

Given a set function f on V and a set $T \subseteq V$ of terminals, we say that a graph G = (V, E) is f-quasi-bipartite if every its edge has an end (tail or head) v such that $v \in T$ or such that v does not belong to any f-positive set. Let E_0 be the set of zero cost edges of G. By Menger's Theorem, MIN-COST ROOTED SUBSET k-EDGE-CONNECTION AUGMENTATION is equivalent to the problem of finding a min-cost edge set $J \subseteq E \setminus E_0$ that covers the function f defined by

$$f(A) = \begin{cases} \max\{k - d_{G_0}(A), 0\} & \text{if } A \cap T \neq \emptyset, r \notin A \\ 0 & \text{otherwise} \end{cases}$$

This f is positively T-intersecting supermodular, see [9]. Since r does not belong to any f-positive set, if G is quasi-bipartite then $G \setminus E_0$ is f-quasibipartite. Assumption 1 holds for this f, since the cores as in Assumption 1 can be found by computing for every $t \in T$ the closest to t minimum rt-cut of $G_0 + I$, c.f. [9,28]. Under Assumption 1, we prove the following.

Theorem 1 The MIN-COST SET FUNCTION EDGE COVER problem with positively T-intersecting supermodular f and f-quasi-bipartite G admits approximation ratio $4H(\max(f)) \cdot (1 + \ln |T|)$, where $H(k) = \sum_{i=1}^{k} 1/i$ denotes the kth Harmonic number.

Theorem 1 implies the following extension of the result of Chan et al. [2].

Corollary 1 The MIN-COST ROOTED SUBSET (k_0, k) -EDGE-CONNECTION AUGMENTATION problem admits approximation ratio $4H(k - k_0) \cdot (1 + \ln |T|)$ if the set of positive cost edges of G is quasi-bipartite.

As far as we can see, Corollary 1 cannot be deduced from the work of Chan et al. [2]. Our approach is motivated by an earlier result of Frank [9], who showed that MIN-COST ROOTED SUBSET k-EDGE-CONNECTION can be solved in polynomial time provided that every positive cost edge has head in T. For this, he proved that MIN-COST SET FUNCTION EDGE COVER with positively T-intersecting supermodular f can be solved in polynomial time provided that every positive cost edge has head in T. While our approximation ratio is asymptotically similar to the one of $[2] - O(\ln k \cdot \ln |T|)$, our constant hidden in the $O(\cdot)$ term is smaller and the proof (of a more general result) is substantially simpler. Moreover, our algorithm is combinatorial and thus is much faster than the one of [2], that repeatedly solves linear programs and rounds LP solutions. Chan et al. [2] do not specify how the LPs are solved, but one can easily see that they can be solved using the ellipsoid algorithm.

We use a method initiated by the author in [28], that extends the Klein-Ravi [21] algorithm for the NODE WEIGHTED STEINER TREE problem, to high connectivity problems. It was applied later in [29,30] also for node weighted problems, and the same method is used in [2]; a restricted version of this method appeared earlier in [22] and later in [7]. The method was further developed by Fukunaga [11] and Chekuri, Ene, and Vakilian [4] for prize-collecting connectivity problems.

In the rest of this section we briefly survey some literature on rooted connectivity problems. The DIRECTED STEINER TREE problem admits approximation ratio $O(\ell^3 |T|^{2/\ell})$ in time $O(|T|^{2\ell}n^\ell)$ for any integer ℓ , see [33,3,25,18], and also a tight quasi-polynomial time approximation $O(\log^2 |T|/\log \log |T|)$ [16,13]; see also a survey in [6]. For similar results for MIN-COST ROOTED SUBSET 2-EDGE-CONNECTION see [15]. DIRECTED STEINER TREE is $\Omega(\log^2 n)$ hard to approximate even on very special instances [17] that arise from the GROUP STEINER TREE problem on trees; the latter problem admits a tight approximation ratio $O(\log^2 n)$ [12]. The (undirected) STEINER TREE problem was also studied extensively, c.f. [1,14] and the references therein. The study of quasi-bipartite instances was initiated for undirected graphs in the 90's [32], while the directed version was shown to admits approximation ratio $O(\ln |T|)$ in [10,19].

Rooted k-connectivity problems were studied for both directed and undirected graphs, edge-connectivity and node-connectivity, and various types of graphs and costs; c.f. a survey [31]. For undirected graphs the problem admits approximation ratio 2 [20], but for digraphs it has approximation threshold max{ $k^{1/2-\epsilon}$, $|T|^{1/4-\epsilon}$ } [26]. For the undirected node connectivity version, the currently best known approximation ratio is $O(k \ln k)$ [30] and threshold max{ $k^{0.1-\epsilon}$, $|T|^{1/4-\epsilon}$ } [26]. However, the augmentation version when any edge can be added by a cost of 1 is just SET COVER hard and admits approximation ratios $O(\ln |T|)$ for digraphs and min{ $O(\ln |T|, O(\ln^2 k)$ } for graphs [23]; a similar result holds when positive cost edges form a star [24].

In digraphs, node connectivity can be reduced to edge-connectivity by a folklore reduction of "splitting" each node v into two nodes $v^{\text{in}}, v^{\text{out}}$. However, this reduction does not preserve quasi-bipartiteness. The reductions of [27] that transfers undirected connectivity problems into directed ones, and a reduction of [5] that reduces general connectivity requirements to rooted requirements, also do not preserve quasi-bipartiteness.

2 Covering *T*-intersecting supermodular functions (Theorem 1)

A set family \mathcal{F} is a *T*-intersecting family if $A \cap B, A \cup B \in \mathcal{F}$ whenever $A \cap B \cap T \neq \emptyset$. It is known that if *f* is (positively) *T*-intersecting supermodular then the family $\mathcal{F} = \{A \subseteq V : f(A) = \max(f)\}$ is *T*-intersecting, see [9]. We say that an edge set *I* covers \mathcal{F} if $d_I(A) \geq 1$ for all $A \in \mathcal{F}$. Recall that inclusion minimal members of \mathcal{F} are called \mathcal{F} -cores, and that $\mathcal{C}_{\mathcal{F}}$ denotes the family of \mathcal{F} -cores. For $C \in \mathcal{C}_{\mathcal{F}}$ let $\mathcal{F}(C)$ denote the family of sets in \mathcal{F} that contain no core distinct from *C*; for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ let $\mathcal{F}(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$.

An analogue of the following lemma was proved in [28, Lemma 3.3] for intersecting families, and the proof for T-intersecting families is similar.

Lemma 1 Let \mathcal{F} be a *T*-intersecting family. If an edge set *S* covers $\mathcal{F}(\mathcal{C})$ for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ then $\nu(\emptyset) - \nu(S) \ge |\mathcal{C}|/2$, where $\nu(S)$ denotes the number of cores of the residual family $\mathcal{F}^S = \{A \in \mathcal{F} : d_S(A) = 0\}.$

Proof The \mathcal{F}^S -cores are T-disjoint, and each of them contains some \mathcal{F} -core. Every \mathcal{F}^S -core that contains a core from \mathcal{C} contains at least two \mathcal{F} -cores. Thus the number of \mathcal{F}^S -cores that contain exactly one \mathcal{F} -core is at most $\nu(\emptyset) - |\mathcal{C}|/2$.

Consider an instance of the MIN-COST SET FUNCTION EDGE COVER problem with positively *T*-intersecting supermodular f and f-quasi-bipartite G, and optimal solution value τ_f . Let $\mathcal{F} = \{A \subseteq V : f(A) = \max(f)\}$, and for $I \subseteq E$ let $\nu_f(I)$ denote the number of \mathcal{F}^I -cores. In the next section we will prove the following.

Lemma 2 There exists a polynomial time algorithm that finds $\emptyset \neq C \subseteq C_F$ and a cover $S \subseteq E$ of $\mathcal{F}(C)$ such that

$$\frac{c(S)}{|\mathcal{C}|} \le \frac{2}{\max(f)} \cdot \frac{\tau_f}{|\mathcal{C}_{\mathcal{F}}|} = \frac{2}{\max(f)} \cdot \frac{\tau_f}{\nu_f(\emptyset)}$$

Now let $I \subseteq E$ be an edge set such that $\nu_f(I) \geq 1$, and note that then $\max(f^I) = \max(f)$. Applying Lemmas 1 and 2 on the residual function $g = f^I$ we get that we can find in polynomial time an edge set $S \subseteq E \setminus I$ such that

$$\frac{c(S)}{\nu_g(\emptyset) - \nu_g(S)} \le \frac{c(S)}{|\mathcal{C}|/2} \le \frac{4}{\max(g)} \cdot \frac{\tau_g}{\nu_g(\emptyset)} \ .$$

Observing that $\nu_g(\emptyset) = \nu(I), \ \nu_g(S) = \nu_f(I \cup S)$, and $\tau_g \leq \tau_f$ we get:

Corollary 2 There exists a polynomial time algorithm that given $I \subseteq E$ with $\nu_f(I) \ge 1$ finds an edge set $S \subseteq E \setminus I$ such that

$$\frac{c(S)}{\nu_f(I) - \nu_f(I \cup S)} \le \frac{4}{\max(f)} \cdot \frac{\tau_f}{\nu_f(I)}$$

From Corollary 2 it is a routine to deduce the following corollary, c.f. [21] and [29, Theorem 3.1]; we provide a proof for completeness of exposition.

Corollary 3 There exists a polynomial time algorithm that computes a cover I of $\mathcal{F} = \{A \subseteq V : f(A) = \max(f)\}$ of cost $c(I) \leq \frac{4}{\max(f)} \cdot (1 + \ln \nu_f(\emptyset)) \cdot \tau_f$.

Proof Start with $I = \emptyset$ an while $\nu_f(I) \ge 1$ add to I an edge set S as in Corollary 2. Let I_j be the partial solution at the end of iteration j, where $I_0 = \emptyset$, and let S_j be the set added at iteration j; thus $I_j = I_{j-1} \cup S_j$, $j = 1, \ldots, q$. Let $\nu_j = \nu_f(I_j)$, so $\nu_0 = \nu_f(\emptyset)$, $\nu_q = 0$, and $\nu_{q-1} \ge 1$. Let $\rho = \frac{4}{\max(f)}$. Then

$$\frac{c_j}{\nu_{j-1}-\nu_j} \le \rho \cdot \frac{\tau_f}{\nu_{j-1}} \qquad j=1,\ldots,q \; .$$

This implies $c_q \leq \rho \tau_f$ and

$$\nu_j \le \nu_{j-1} \left(1 - \frac{c_j}{\rho \tau_f} \right) \qquad j = 1, \dots, q \ .$$

Unraveling we get

$$\frac{\nu_{q-1}}{\nu_0} \le \prod_{j=1}^{q-1} \left(1 - \frac{c_j}{\rho \tau_f} \right)$$

Taking natural logarithms and using the inequality $\ln(1+x) \leq x$, we obtain

$$\rho \cdot \tau_f \cdot \ln\left(\frac{\nu_0}{\nu_{q-1}}\right) \ge \sum_{j=1}^{q-1} c_j \ .$$

Since $c_q \leq \rho \tau_f$ and $\nu_{q-1} \geq 1$, we get $c(I) \leq c_q + \sum_{j=1}^{q-1} c_j \leq \rho \tau_f (1 + \ln \nu_0)$. \Box

To see that Corollary 3 implies Theorem 1, consider the following algorithm that uses the so called "backward augmentation" method.

Algorithm 1: Backward-Augmentation $(f, G = (V, E), c)$
1 $I \leftarrow \emptyset$
2 for $\ell = \max(f)$ downto 1 do
3 Compute a cover I_{ℓ} of $\mathcal{F}_{\ell} = \{A \subseteq V : f^{I}(A) = \ell\}$ as in Corollary 3
$4 \left[\begin{array}{c} I \leftarrow I \cup I_{\ell} \end{array} \right]$
5 return I

At iteration ℓ we have $c(I_{\ell})/\tau_f \leq 4(1+\ln|T|)/\ell$, hence the overall approximation ratio is $4(1+\ln|T|) \cdot \sum_{\ell=\max(f)}^{1} 1/\ell = 4H(\max(f)) \cdot (1+\ln|T|)$, as required in Theorem 1. It remains only to prove Lemma 2, which is done in the next section, where we also describe a simple polynomial time implementation of our algorithm.

3 Proof of Lemma 2

Let $\langle G = (V, E), c, T, f \rangle$ be an instance of MIN-COST SET FUNCTION EDGE COVER with positively *T*-intersecting supermodular *f* and *f*-quasi-bipartite *G*, and an optimal solution value $\tau = \tau_f$. Let us denote $p = \max(f)$ and let $\mathcal{F} = \{A \subseteq V : f(A) = p\}$. Recall that $\mathcal{F}(C)$ denotes the family of sets in \mathcal{F} that contain no core distinct from *C*, and that $\mathcal{F}(C) = \bigcup_{C \in \mathcal{C}} \mathcal{F}(C)$ for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$. We need to show that there exists a subfamily of cores $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ and a cover $S \subseteq E$ of $\mathcal{F}(C)$ such that

$$\frac{c(S)}{|\mathcal{C}|} \le \frac{1}{p} \cdot \frac{\tau}{|\mathcal{C}_{\mathcal{F}}|} . \tag{1}$$

We also need to design a polynomial time algorithm that finds such \mathcal{C}, S .



Fig. 1 (a) A 3-cover I of $\mathcal{F}(\{C, C', C''\})$; here $e_{C''}^2, e_{C''}^3$ have cost 3 each, $e, e_C^2, e_{C'}^2$ have cost 2 each, and all other edges have cost 1. (b) The auxiliary graph \mathcal{H} . The star $S_{\mathcal{H}}$ with center e and leaf set $\mathcal{C} = \{C, C', C''\}$ has ratio $\frac{c(S_{\mathcal{H}})}{|\mathcal{C}|} = \frac{6}{3} = 2$ (the same ratio 2 is achieved by the star $S_{\mathcal{H}} \setminus \{C\}$). The edge subset S of I that corresponds to $S_{\mathcal{H}}$ is $I_C^3 \cup I_{C'}^1 \cup I_{C''}^1$. Here c(I) = 26 and $c(\mathcal{H}) = 29$. Note that $e_{C'}^1 = e = e_{C''}^1$, and that $e \in I_C^3$ but $e \neq e_C^3$.

3.1 Roadmap of the proof

Here is a roadmap of the proof of Lemma 2. To make this roadmap a complete proof we just need to describe a polynomial time implementation and to prove formally three Lemmas 4, 5, and 6 mentioned in this roadmap; this is done in Sections 3.2 and 3.3, respectively.

We say that $I \subseteq E$ is a *p*-cover of \mathcal{F} if $d_I(A) \geq p$ for all $A \in \mathcal{F}$, and I is \mathcal{F} -quasi-bipartite if every edge in I has an end (tail or head) v such that $v \in T$ or such that v does not belong to any set in \mathcal{F} . Fix an optimal solution $I \subseteq E$, so I is a cover of f of cost $c(I) = \tau$. Note that I is a *p*-cover of \mathcal{F} (since f(A) = p for all $A \in \mathcal{F}$) and that I is \mathcal{F} -quasi-bipartite (since G is f-quasi-bipartite and since $I \subseteq E$).

- (A) For every $C \in C_{\mathcal{F}}$ fix some inclusion minimal *p*-cover $I_C \subseteq I$ of $\mathcal{F}(C)$. In Lemma 4 we show the following:
 - (i) Each I_C partitions into p inclusion minimal 1-covers I_C^1, \ldots, I_C^p of $\mathcal{F}(C)$.
 - (ii) Each $\mathcal{F}(C)$ has a unique inclusion maximal set M_C and each I_C^j has a unique edge e_C^j that covers M_C , which we call the **prime edge of** I_C^j .
- (B) In Lemma 5 we show that for distinct C, C' ∈ C_F and any 1 ≤ j, j' ≤ p, if I^j_C ∩ I^{j'}_{C'} ≠ Ø then I^j_C ∩ I^{j'}_{C'} = {e^j_C} or I^j_C ∩ I^{j'}_{C'} = {e^{j'}_{C'}}, see Fig. 1(a); this property is since I is F-quasi-bipartite. Consequently, for every e ∈ I there is at most one set I^j_C such that e ∈ I^j_C and e ≠ e^j_C.
- (C) Construct an auxiliary bipartite graph \mathcal{H} with node- and edge-costs as follows, see Fig. 1(b) The node parts of \mathcal{H} are the prime edges and $\mathcal{C}_{\mathcal{F}}$. Each node e of \mathcal{H} that is a prime edge inherits its cost c(e) in G, and is connected to each $C \in \mathcal{C}_{\mathcal{F}}$ such that $e \in I_C^j$ for some j by an edge of cost $c(I_C^j) - c(e)$ (this edge represents the set I_C^j). Since for every $e \in I$ at most one set I_C^j contains e as a non-prime edge, and since the sets I_C^j are pairwise disjoint, the total cost of \mathcal{H} is at most 2 times the cost of I.

- (D) Every node $C \in C_{\mathcal{F}}$ of \mathcal{H} has at least p neighbors in \mathcal{H} (the prime edges of the sets I_C^1, \ldots, I_C^p). In **Lemma 6** we show that \mathcal{H} contains a star $S_{\mathcal{H}}$ with leaf set $\mathcal{C} \subseteq C_{\mathcal{F}}$ such that $\frac{c(S_{\mathcal{H}})}{|\mathcal{C}|} \leq \frac{1}{p} \cdot \frac{c(\mathcal{H})}{|\mathcal{C}_{\mathcal{F}}|} \leq \frac{2}{p} \cdot \frac{\tau}{|\mathcal{C}_{\mathcal{F}}|}$. Then the edge subset $S \subseteq I$ that corresponds to $S_{\mathcal{H}}$ covers $\mathcal{F}(\mathcal{C})$, and S, \mathcal{C} satisfy inequality (1).
- (E) To find $\emptyset \neq C \subseteq C_{\mathcal{F}}$ and a cover S of $\mathcal{F}(\mathcal{C})$ that satisfies (1), we make a similar construction: now \mathcal{H} has node set $E \cup \mathcal{C}$, every node $e \in E$ of \mathcal{H} has cost equal to the cost of e in G, and in \mathcal{H} each node $C \in C_{\mathcal{F}}$ is connected to each node $e \in E$ by an edge of cost being the minimum cost of an edge set S such that $S \cup \{e\}$ covers $\mathcal{F}(C)$. In such a graph \mathcal{H} we can find a star $S_{\mathcal{H}}$ with leaf set \mathcal{C} that minimizes $\frac{c(S_{\mathcal{H}})}{|\mathcal{C}|}$ using the method of Klein & Ravi [21]; see also step 3 of the implementation discussed in the next section.

3.2 Implementation

Here we briefly discuss a simple implementation of the entire algorithm. We start with the particular case of the MIN-COST ROOTED SUBSET (k_0, k) -EDGE-CONNECTION AUGMENTATION problem. In what follows let n = |V| and m = |E|. As a pre-processing step, we assign unit capacities to edges in E and compute a k_0 -flow from the root r to each $t \in T$. This can be done in O(km|T|) time using the Ford-Fulkerson algorithm. Let us consider iteration ℓ of Algorithm 1, when $\max(f) = k - \ell$. We will assume that we already have a flow on zero cost edges of value $k - \ell - 1$ to each $t \in T$, and perform the following steps.

- 1. We increase the flow by 1 to each $t \in T$, and discard terminals for which the flow can be further increased by 2. This can be done in O(m|T|) time.
- 2. To compute the cost of an edge of \mathcal{H} between nodes C and e, we add a "dummy" edge of cost 0 from r to some terminal in every core distinct from C, set the cost of e to 0, and compute a minimum cost edge set that increases the rC-flow by 1; the later problem admits a linear time reduction to the shortest path problem and thus can be implemented in $O(n^2)$ time. The number of edges in \mathcal{H} is O(m|T|), hence \mathcal{H} can be constructed in $O(n^2m|T|)$ time.
- 3. We can sort the edges of \mathcal{H} by increasing cost in $O(m|T|\log n)$ time. Then finding a (nontrivial) star S^e in \mathcal{H} with a specific center e that minimizes $\frac{c(S^e)}{|\mathcal{C}|}$ can be done in time linear in the degree of e in \mathcal{H} as follows. We take the lowest cost edge incident to e into S^e and then add edges incident to e one by one in increasing cost order until reaching a local minimum of $\frac{c(S^e)}{|\mathcal{C}|}$; see [21]. The overall time for computing all stars S^e is $O(mn \log n)$, which is dominated by the time $O(n^2m|T|)$ of the construction of \mathcal{H} .
- 4. At iteration ℓ we need to construct the graph \mathcal{H} at most |T| times, hence the overall time per iteration ℓ is $O(n^2m|T|^2)$. And since we have $k k_0$ iterations, the overall running time is $(k k_0) \cdot O(n^2m|T|^2) = O(kn^6)$.

We note that while the running time of the described implementation is somewhat high, it is still much lower than that of Chan et al. [2]. The implementation of steps 1, 3, 4 for the MIN-COST SET FUNCTION EDGE COVER problem under Assumption 1 is similar. For step 2, for any $C \in C_{\mathcal{F}}$ and $e \in E \setminus I$ we need to find in polynomial time a min-cost edge set S = S(e, C) such that $S \cup \{e\}$ covers $\mathcal{F}(C)$. For this, it is sufficient to find a min-cost cover of $\mathcal{F}(C)$ after resetting the cost of e to zero. The family $\mathcal{F}(C)$ is a T-intersecting family that has a unique core; such a family is called a **ring**. It is known that a min-cost edge-cover of a ring can be found in polynomial time under Assumption 1 (c.f. [9,28]), by a standard primal dual algorithm.

3.3 Proofs of Lemmas

Now we turn to formal proofs of Lemmas 4,5 and 6 mentioned in our roadmap. At each step we will specify the part of our roadmap that is proved.

A *T*-intersecting family \mathcal{R} that has a unique core *C* is called a **ring**. Then *C* is the intersection of all sets in \mathcal{R} , and \mathcal{R} also has a unique inclusion maximal set *M* which is the union of all sets in \mathcal{R} . The following lemma is a folklore.

Lemma 3 If \mathcal{F} is a *T*-intersecting family then $\mathcal{F}(C)$ is a ring family for any $C \in \mathcal{C}_{\mathcal{F}}$; thus $\mathcal{F}(C)$ also has a unique inclusion maximal set M_C . Furthermore, $M_C \cap M_{C'} \cap T = \emptyset$ for any distinct $C, C' \in \mathcal{C}_{\mathcal{F}}$.

The next lemma gives two additional known properties of rings; c.f. [8] for the first property and [28, Lemma 2.6 and Corollary 2.7] for the second. These two properties imply part (\mathbf{A}) .

Lemma 4 Let \mathcal{R} be a ring with minimal member C and maximal member M.

- (i) Any p-cover of \mathcal{R} is a union of p edge disjoint covers of \mathcal{R} .
- (ii) Let I be an inclusion minimal cover of R. Then there is an ordering e₁, e₂,..., e_q of I and a nested family C = C₁ ⊂ C₂... ⊂ C_q = M of sets in R such that for every j = 1,...,q, e_j is the unique edge in I that enters C_j (namely, e_j has head in C_j and tail not in C_j).

Lemmas 3 and 4(i) imply the following lemma that implies parts (\mathbf{B}, \mathbf{C}) .

Lemma 5 Let I be an \mathcal{F} -quasi-bipartite cover of a T-intersecting family \mathcal{F} . For $C \in C_{\mathcal{F}}$ let $I_C \subseteq I$ be an inclusion minimal cover of $\mathcal{F}(C)$, and let e_C be the unique (by Lemma 4(i)) edge in I_C that covers M_C . Let $C, C' \in C_{\mathcal{F}}$ be distinct and let $e \in I_C \cap I_{C'}$. Then $e = \{e_C\}$ or $e = \{e_{C'}\}$.

Proof Suppose that $e \neq e_C$ and we will show that then $e = e_{C'}$. Note that e does not cover M_C , hence e has both ends in M_C , by the minimality of I_C and Lemma 4(ii). Since I is \mathcal{F} -quasi-bipartite, e has an end t in $M_C \cap T$. By Lemma 3, $t \notin M_{C'}$, hence by the minimality of $I_{C'}$ we must have $e = e_{C'}$. \Box

The next lemma implies part (D).

Lemma 6 Let $H = (A \cup B, E)$ be a bipartite graph with edge- and node- costs $\{c(e) : e \in E\} \cup \{c(a) : a \in A\}$ and let S be the set of stars in H with center in A and leaves in B. If the degree of every $b \in B$ is at least p then there is $S^* \in S$ such that $\frac{c(S^*)}{|L(S^*)|} \leq \frac{1}{p} \cdot \frac{c(G)}{|B|}$, where $L(S^*)$ is the set of leaves of S^* .

Proof For $S \in S$ let c_S denote the cost of S and let $\mathbf{c} = \{c_S : S \in S\}$ be a vector of costs of the stars. For an integer q let $\mathcal{L}(q)$ be the following set of linear constraints:

$$\sum_{\substack{L(S) \ni b \\ 0 \le x_S \le 1}} x_S \ge q \quad \forall b \in B$$

Note that the characteristic vector \mathbf{x} of the inclusion maximal stars in S satisfies the set of constraints $\mathcal{L}(p)$ and that $\mathbf{c} \cdot \mathbf{x} = c(H)$. Thus the vector $\mathbf{y} = \mathbf{x}/p$ satisfies $\mathcal{L}(1)$ and $\mathbf{c} \cdot \mathbf{y} = c(H)/p$. Let $S^* = \arg \max_{S \in S} \frac{|L(S)|}{c(S)}$. Then

$$\frac{|L(S^*)|}{c(S^*)}(\mathbf{c}\cdot\mathbf{y}) \ge \sum_{S\in\mathcal{S}} \frac{|L(S)|}{c_S} c_S y_S = \sum_{S\in\mathcal{S}} |L(S)| y_S = \sum_{b\in B} \sum_{L(S)\ni b} y_S \ge \sum_{b\in B} 1 = |B|.$$

The first inequality is by the choice of S^* and the second inequality is since **y** satisfies $\mathcal{L}(1)$.

From this we get that
$$\frac{|L(S^*)|}{c(S')} \ge \frac{|B|}{\mathbf{c} \cdot \mathbf{y}}$$
, so $\frac{c(S^*)}{|L(S^*)|} \le \frac{\mathbf{c} \cdot \mathbf{y}}{|B|} = \frac{\mathbf{c} \cdot \mathbf{x}/p}{|B|} = \frac{1}{p} \cdot \frac{c(H)}{|B|}$.

This concludes the proof of Lemma 2, and thus also the proofs Theorem 1 and Corollary 1 are complete.

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