# On rooted $k$-connectivity problems in quasi-bipartite digraphs 

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Received: date / Accepted: date


#### Abstract

We consider the directed Min-Cost Rooted Subset $k$-EdgeConnection problem: given a digraph $G=(V, E)$ with edge costs, a set $T \subseteq V$ of terminals, a root node $r$, and an integer $k$, find a min-cost subgraph of $G$ that contains $k$ edge disjoint $r t$-paths for all $t \in T$. The case when every edge of positive cost has head in $T$ admits a polynomial time algorithm due to Frank [9, and the case when all positive cost edges are incident to $r$ is equivalent to the $k$-Multicover problem. Chan et al. [2] gave an LP-based $O(\ln k \ln |T|)$ approximation algorithm for quasi-bipartite instances, when every edge in $G$ has an end (tail or head) in $T \cup\{r\}$. We give a simple combinatorial algorithm with the same ratio for a more general problem of covering an arbitrary $T$ intersecting supermodular set function by a minimum cost edge set, and for the case when only every positive cost edge has an end in $T \cup\{r\}$.


Keywords min-cost rooted $k$-edge-connection • quasi-bipartite digraphs • $T$-intersecting supermodular set functions • approximation algorithms

## 1 Introduction

All graphs considered here are directed, unless stated otherwise. We consider the following problem (a.k.a. $k$-Edge-Connected Directed Steiner Tree):

```
Min-Cost Rooted Subset }k\mathrm{ -Edge-Connection
Input: A directed (multi-)graph G=(V,E) with edge costs {c(e):e\inE },
a set T\subsetV of terminals, a root node r 
Output: A min-cost subgraph that has k edge disjoint rt-paths for all t\inT.
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Preliminary version in CSR 2021: 339-348.
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The case when every edge of positive cost has head in $T$ admits a polynomial time algorithm due to Frank [9]. When all positive cost edges are incident to $r$ we get the Min-Cost Multicover problem. The case when all positive cost edges are incident to the same node admits approximation ratio $O(\ln n)$ [24]. More generally, a graph (or an edge set) is called quasi-bipartite if every edge has at least one end (tail or head) in $T \cup\{r\}$.

In the augmentation version of the problem - Min-Cost Rooted Subset $\left(k_{0}, k\right)$-Edge-Connection Augmentation, the input graph $G$ contains a subgraph $G_{0}=\left(V, E_{0}\right)$ of cost zero that has $k_{0}$-edge disjoint $r t$-paths for all $t \in T$. Recently, Chan, Laekhanukit, Wei, \& Zhang [2] obtained approximation ratio $O\left(\ln \left(k-k_{0}+1\right) \ln |T|\right)$ for the case when $G$ is quasi-bipartite. We provide a simple proof for a more general setting.

An integer valued set function $f$ on a groundset $V$ is intersecting supermodular if any $A, B \subseteq V$ that intersect satisfy the supermodular inequality $f(A)+f(B) \leq f(A \cap B)+f(A \cup B)$; if this holds whenever $A \cap B \cap T \neq \emptyset$ for a given set $T \subseteq V$ of terminals, then $f$ is $T$-intersecting supermodular. We say that $A \subseteq V$ is an $f$-positive set if $f(A)>0 . f$ is positively $T$-intersecting supermodular if the supermodular inequality holds whenever $A \cap B \cap T \neq \emptyset$ and $f(A), f(B)>0$. A typical way to create a positively intersecting supermodular function is to take the "non-negative part" of an intersecting supermodular one, which means replacing each negative value by zero; namely, if $g$ is $T$-intersecting supermodular then $f(A)=\max \{g(A), 0\}$ is positively $T$ intersecting supermodular, see [9].

An edge $e$ covers a set $A$ if it enters $A$, namely, if its head is in $A$ and tail is not in $A$. For an edge set/graph $J$ let $d_{J}(A)$ denote the number of edges in $J$ that cover $A$. We say that $J$ covers $f$ or that $J$ is a cover of $f$ if $d_{J}(A) \geq f(A)$ for all $A \subseteq V$. We consider the following generic problem.

## Min-Cost Set Function Edge Cover <br> Input: A digraph $G=(V, E)$ with edge costs and a set function $f$ on $V$. <br> Output: A min-cost edge subset $J \subseteq E$ that covers $f$.

Here $f$ may not be given explicitly, and for a polynomial time implementation of algorithms we need that certain queries related to $f$ can be answered in polynomial time. For an edge set $I$, the residual function $f^{I}$ of $f$ is defined by $f^{I}(A)=\max \left\{f(A)-d_{I}(A), 0\right\}$. It is known that if $f$ is positively $T$-intersecting supermodular then so is $f^{I}$, c.f. [9]; to see this, note that $g(A)=f(A)-d_{I}(A)$ is positively $T$-intersecting supermodular (since $g(A)>0$ implies $f(A)>0$ and since $-d(A)$ is supermodular), and thus the positive part $\max \{g(A), 0\}$ of $g$ is also positively $T$-intersecting supermodular.

Let $\max (f)=\max \{f(A): A \subseteq V\}$ denote the maximum $f$-value taken over all sets. An inclusion minimal member of a set-family $\mathcal{F}$ is called an $\mathcal{F}$ core, or simply a core, if $\mathcal{F}$ is clear from the context. Let $\mathcal{C}_{\mathcal{F}}$ denote the family of $\mathcal{F}$-cores. We will assume the following.

Assumption 1. The cores of the set family $\mathcal{F}=\left\{A: f^{I}(A)=\max \left(f^{I}\right)\right\}$ can be found in polynomial time for any edge set $I$.

Given a set function $f$ on $V$ and a set $T \subseteq V$ of terminals, we say that a graph $G=(V, E)$ is $f$-quasi-bipartite if every its edge has an end (tail or head) $v$ such that $v \in T$ or such that $v$ does not belong to any $f$-positive set. Let $E_{0}$ be the set of zero cost edges of $G$. By Menger's Theorem, Min-Cost Rooted Subset $k$-Edge-Connection Augmentation is equivalent to the problem of finding a min-cost edge set $J \subseteq E \backslash E_{0}$ that covers the function $f$ defined by

$$
f(A)= \begin{cases}\max \left\{k-d_{G_{0}}(A), 0\right\} & \text { if } A \cap T \neq \emptyset, r \notin A \\ 0 & \text { otherwise }\end{cases}
$$

This $f$ is positively $T$-intersecting supermodular, see 9 . Since $r$ does not belong to any $f$-positive set, if $G$ is quasi-bipartite then $G \backslash E_{0}$ is $f$-quasibipartite. Assumption 1 holds for this $f$, since the cores as in Assumption 1 can be found by computing for every $t \in T$ the closest to $t$ minimum rt-cut of $G_{0}+I$, c.f. 9, 28. Under Assumption 1, we prove the following.

Theorem 1 The Min-Cost Set Function Edge Cover problem with positively $T$-intersecting supermodular $f$ and $f$-quasi-bipartite $G$ admits approximation ratio $4 H(\max (f)) \cdot(1+\ln |T|)$, where $H(k)=\sum_{i=1}^{k} 1 / i$ denotes the $k$ th Harmonic number.

Theorem 1 implies the following extension of the result of Chan et al. [2].
Corollary 1 The Min-Cost Rooted Subset ( $k_{0}, k$ )-Edge-Connection AugMEntation problem admits approximation ratio $4 H\left(k-k_{0}\right) \cdot(1+\ln |T|)$ if the set of positive cost edges of $G$ is quasi-bipartite.

As far as we can see, Corollary 1 cannot be deduced from the work of Chan et al. [2]. Our approach is motivated by an earlier result of Frank [9, who showed that Min-Cost Rooted Subset $k$-Edge-Connection can be solved in polynomial time provided that every positive cost edge has head in $T$. For this, he proved that Min-Cost Set Function Edge Cover with positively $T$-intersecting supermodular $f$ can be solved in polynomial time provided that every positive cost edge has head in $T$. While our approximation ratio is asymptotically similar to the one of [2] - $O(\ln k \cdot \ln |T|)$, our constant hidden in the $O(\cdot)$ term is smaller and the proof (of a more general result) is substantially simpler. Moreover, our algorithm is combinatorial and thus is much faster than the one of [2], that repeatedly solves linear programs and rounds LP solutions. Chan et al. [2] do not specify how the LPs are solved, but one can easily see that they can be solved using the ellipsoid algorithm.

We use a method initiated by the author in [28], that extends the KleinRavi [21] algorithm for the Node Weighted Steiner Tree problem, to high connectivity problems. It was applied later in [29,30] also for node weighted problems, and the same method is used in [2]; a restricted version of this method appeared earlier in [22] and later in [7]. The method was further developed by Fukunaga 11 and Chekuri, Ene, and Vakilian 4 for prizecollecting connectivity problems.

In the rest of this section we briefly survey some literature on rooted connectivity problems. The Directed Steiner Tree problem admits approximation ratio $O\left(\ell^{3}|T|^{2 / \ell}\right)$ in time $O\left(|T|^{2 \ell} n^{\ell}\right)$ for any integer $\ell$, see [33|3, 25, 18, and also a tight quasi-polynomial time approximation $O\left(\log ^{2}|T| / \log \log |T|\right)$ [16, 13]; see also a survey in [6. For similar results for Min-Cost Rooted Subset 2-Edge-Connection see [15]. Directed Steiner Tree is $\Omega\left(\log ^{2} n\right)$ hard to approximate even on very special instances [17] that arise from the Group Steiner Tree problem on trees; the latter problem admits a tight approximation ratio $O\left(\log ^{2} n\right)$ [12]. The (undirected) Steiner Tree problem was also studied extensively, c.f. [1,14] and the references therein. The study of quasi-bipartite instances was initiated for undirected graphs in the 90's 32, while the directed version was shown to admits approximation ratio $O(\ln |T|)$ in (10, 19 .

Rooted $k$-connectivity problems were studied for both directed and undirected graphs, edge-connectivity and node-connectivity, and various types of graphs and costs; c.f. a survey [31. For undirected graphs the problem admits approximation ratio 2 [20], but for digraphs it has approximation threshold $\max \left\{k^{1 / 2-\epsilon},|T|^{1 / 4-\epsilon}\right\}$ [26]. For the undirected node connectivity version, the currently best known approximation ratio is $O(k \ln k)$ 30 and threshold $\max \left\{k^{0.1-\epsilon},|T|^{1 / 4-\epsilon}\right\}[26]$. However, the augmentation version when any edge can be added by a cost of 1 is just Set Cover hard and admits approximation ratios $O(\ln |T|)$ for digraphs and $\min \left\{O\left(\ln |T|, O\left(\ln ^{2} k\right)\right\}\right.$ for graphs 23]; a similar result holds when positive cost edges form a star [24].

In digraphs, node connectivity can be reduced to edge-connectivity by a folklore reduction of "splitting" each node $v$ into two nodes $v^{\text {in }}, v^{\text {out }}$. However, this reduction does not preserve quasi-bipartiteness. The reductions of [27] that transfers undirected connectivity problems into directed ones, and a reduction of [5] that reduces general connectivity requirements to rooted requirements, also do not preserve quasi-bipartiteness.

## 2 Covering $T$-intersecting supermodular functions (Theorem 1)

A set family $\mathcal{F}$ is a $T$-intersecting family if $A \cap B, A \cup B \in \mathcal{F}$ whenever $A \cap B \cap T \neq \emptyset$. It is known that if $f$ is (positively) $T$-intersecting supermodular then the family $\mathcal{F}=\{A \subseteq V: f(A)=\max (f)\}$ is $T$-intersecting, see [9]. We say that an edge set $I$ covers $\mathcal{F}$ if $d_{I}(A) \geq 1$ for all $A \in \mathcal{F}$. Recall that inclusion minimal members of $\mathcal{F}$ are called $\mathcal{F}$-cores, and that $\mathcal{C}_{\mathcal{F}}$ denotes the family of $\mathcal{F}$-cores. For $C \in \mathcal{C}_{\mathcal{F}}$ let $\mathcal{F}(C)$ denote the family of sets in $\mathcal{F}$ that contain no core distinct from $C$; for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ let $\mathcal{F}(\mathcal{C})=\cup_{C \in \mathcal{C}} \mathcal{F}(C)$.

An analogue of the following lemma was proved in [28, Lemma 3.3] for intersecting families, and the proof for $T$-intersecting families is similar.

Lemma 1 Let $\mathcal{F}$ be a $T$-intersecting family. If an edge set $S$ covers $\mathcal{F}(\mathcal{C})$ for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ then $\nu(\emptyset)-\nu(S) \geq|\mathcal{C}| / 2$, where $\nu(S)$ denotes the number of cores of the residual family $\mathcal{F}^{S}=\left\{A \in \mathcal{F}: d_{S}(A)=0\right\}$.

Proof The $\mathcal{F}^{S}$-cores are $T$-disjoint, and each of them contains some $\mathcal{F}$-core. Every $\mathcal{F}^{S}$-core that contains a core from $\mathcal{C}$ contains at least two $\mathcal{F}$-cores. Thus the number of $\mathcal{F}^{S}$-cores that contain exactly one $\mathcal{F}$-core is at most $\nu(\emptyset)-|\mathcal{C}| / 2$. Consequently, $\nu(S) \leq \nu(\emptyset)-|\mathcal{C}| / 2$.

Consider an instance of the Min-Cost Set Function Edge Cover problem with positively $T$-intersecting supermodular $f$ and $f$-quasi-bipartite $G$, and optimal solution value $\tau_{f}$. Let $\mathcal{F}=\{A \subseteq V: f(A)=\max (f)\}$, and for $I \subseteq E$ let $\nu_{f}(I)$ denote the number of $\mathcal{F}^{I}$-cores. In the next section we will prove the following.

Lemma 2 There exists a polynomial time algorithm that finds $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ and a cover $S \subseteq E$ of $\mathcal{F}(\mathcal{C})$ such that

$$
\frac{c(S)}{|\mathcal{C}|} \leq \frac{2}{\max (f)} \cdot \frac{\tau_{f}}{\left|\mathcal{C}_{\mathcal{F}}\right|}=\frac{2}{\max (f)} \cdot \frac{\tau_{f}}{\nu_{f}(\emptyset)}
$$

Now let $I \subseteq E$ be an edge set such that $\nu_{f}(I) \geq 1$, and note that then $\max \left(f^{I}\right)=\max (f)$. Applying Lemmas 1 and 2 on the residual function $g=f^{I}$ we get that we can find in polynomial time an edge set $S \subseteq E \backslash I$ such that

$$
\frac{c(S)}{\nu_{g}(\emptyset)-\nu_{g}(S)} \leq \frac{c(S)}{|\mathcal{C}| / 2} \leq \frac{4}{\max (g)} \cdot \frac{\tau_{g}}{\nu_{g}(\emptyset)} .
$$

Observing that $\nu_{g}(\emptyset)=\nu(I), \nu_{g}(S)=\nu_{f}(I \cup S)$, and $\tau_{g} \leq \tau_{f}$ we get:
Corollary 2 There exists a polynomial time algorithm that given $I \subseteq E$ with $\nu_{f}(I) \geq 1$ finds an edge set $S \subseteq E \backslash I$ such that

$$
\frac{c(S)}{\nu_{f}(I)-\nu_{f}(I \cup S)} \leq \frac{4}{\max (f)} \cdot \frac{\tau_{f}}{\nu_{f}(I)}
$$

From Corollary 2 it is a routine to deduce the following corollary, c.f. [21] and [29, Theorem 3.1]; we provide a proof for completeness of exposition.

Corollary 3 There exists a polynomial time algorithm that computes a cover $I$ of $\mathcal{F}=\{A \subseteq V: f(A)=\max (f)\}$ of cost $c(I) \leq \frac{4}{\max (f)} \cdot\left(1+\ln \nu_{f}(\emptyset)\right) \cdot \tau_{f}$.

Proof Start with $I=\emptyset$ an while $\nu_{f}(I) \geq 1$ add to $I$ an edge set $S$ as in Corollary 2, Let $I_{j}$ be the partial solution at the end of iteration $j$, where $I_{0}=\emptyset$, and let $S_{j}$ be the set added at iteration $j$; thus $I_{j}=I_{j-1} \cup S_{j}$, $j=1, \ldots, q$. Let $\nu_{j}=\nu_{f}\left(I_{j}\right)$, so $\nu_{0}=\nu_{f}(\emptyset), \nu_{q}=0$, and $\nu_{q-1} \geq 1$. Let $\rho=\frac{4}{\max (f)}$. Then

$$
\frac{c_{j}}{\nu_{j-1}-\nu_{j}} \leq \rho \cdot \frac{\tau_{f}}{\nu_{j-1}} \quad j=1, \ldots, q
$$

This implies $c_{q} \leq \rho \tau_{f}$ and

$$
\nu_{j} \leq \nu_{j-1}\left(1-\frac{c_{j}}{\rho \tau_{f}}\right) \quad j=1, \ldots, q
$$

Unraveling we get

$$
\frac{\nu_{q-1}}{\nu_{0}} \leq \prod_{j=1}^{q-1}\left(1-\frac{c_{j}}{\rho \tau_{f}}\right)
$$

Taking natural logarithms and using the inequality $\ln (1+x) \leq x$, we obtain

$$
\rho \cdot \tau_{f} \cdot \ln \left(\frac{\nu_{0}}{\nu_{q-1}}\right) \geq \sum_{j=1}^{q-1} c_{j} .
$$

Since $c_{q} \leq \rho \tau_{f}$ and $\nu_{q-1} \geq 1$, we get $c(I) \leq c_{q}+\sum_{j=1}^{q-1} c_{j} \leq \rho \tau_{f}\left(1+\ln \nu_{0}\right)$.
To see that Corollary 3implies Theorem 1, consider the following algorithm that uses the so called "backward augmentation" method.

```
Algorithm 1: Backward-Augmentation \((f, G=(V, E), c)\)
    \(I \leftarrow \emptyset\)
    for \(\ell=\max (f)\) downto 1 do
        Compute a cover \(I_{\ell}\) of \(\mathcal{F}_{\ell}=\left\{A \subseteq V: f^{I}(A)=\ell\right\}\) as in Corollary 3
        \(I \leftarrow I \cup I_{\ell}\)
    return \(I\)
```

At iteration $\ell$ we have $c\left(I_{\ell}\right) / \tau_{f} \leq 4(1+\ln |T|) / \ell$, hence the overall approximation ratio is $4(1+\ln |T|) \cdot \sum_{\ell=\max (f)}^{1} 1 / \ell=4 H(\max (f)) \cdot(1+\ln |T|)$, as required in Theorem1. It remains only to prove Lemma 2 which is done in the next section, where we also describe a simple polynomial time implementation of our algorithm.

## 3 Proof of Lemma 2

Let $\langle G=(V, E), c, T, f\rangle$ be an instance of Min-Cost Set Function Edge Cover with positively $T$-intersecting supermodular $f$ and $f$-quasi-bipartite $G$, and an optimal solution value $\tau=\tau_{f}$. Let us denote $p=\max (f)$ and let $\mathcal{F}=\{A \subseteq V: f(A)=p\}$. Recall that $\mathcal{F}(C)$ denotes the family of sets in $\mathcal{F}$ that contain no core distinct from $C$, and that $\mathcal{F}(\mathcal{C})=\cup_{C \in \mathcal{C}} \mathcal{F}(C)$ for $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$. We need to show that there exists a subfamily of cores $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ and a cover $S \subseteq E$ of $\mathcal{F}(\mathcal{C})$ such that

$$
\begin{equation*}
\frac{c(S)}{|\mathcal{C}|} \leq \frac{1}{p} \cdot \frac{\tau}{\left|\mathcal{C}_{\mathcal{F}}\right|} \tag{1}
\end{equation*}
$$

We also need to design a polynomial time algorithm that finds such $\mathcal{C}, S$.


Fig. 1 (a) A 3-cover $I$ of $\mathcal{F}\left(\left\{C, C^{\prime}, C^{\prime \prime}\right\}\right)$; here $e_{C^{\prime \prime}}^{2}, e_{C^{\prime \prime}}^{3}$ have cost 3 each, $e, e_{C}^{2}, e_{C^{\prime}}^{2}$ have cost 2 each, and all other edges have cost 1 . (b) The auxiliary graph $\mathcal{H}$. The star $S_{\mathcal{H}}$ with center $e$ and leaf set $\mathcal{C}=\left\{C, C^{\prime}, C^{\prime \prime}\right\}$ has ratio $\frac{c\left(S_{\mathcal{H}}\right)}{|\mathcal{C}|}=\frac{6}{3}=2$ (the same ratio 2 is achieved by the star $S_{\mathcal{H}} \backslash\{C\}$ ). The edge subset $S$ of $I$ that corresponds to $S_{\mathcal{H}}$ is $I_{C}^{3} \cup I_{C^{\prime}}^{1} \cup I_{C^{\prime \prime}}^{1}$. Here $c(I)=26$ and $c(\mathcal{H})=29$. Note that $e_{C^{\prime}}^{1}=e=e_{C^{\prime \prime}}^{1}$, and that $e \in I_{C}^{3}$ but $e \neq e_{C}^{3}$.

### 3.1 Roadmap of the proof

Here is a roadmap of the proof of Lemma 2. To make this roadmap a complete proof we just need to describe a polynomial time implementation and to prove formally three Lemmas 4, 5, and 6 mentioned in this roadmap; this is done in Sections 3.2 and 3.3, respectively.

We say that $I \subseteq E$ is a $p$-cover of $\mathcal{F}$ if $d_{I}(A) \geq p$ for all $A \in \mathcal{F}$, and $I$ is $\mathcal{F}$-quasi-bipartite if every edge in $I$ has an end (tail or head) $v$ such that $v \in T$ or such that $v$ does not belong to any set in $\mathcal{F}$. Fix an optimal solution $I \subseteq E$, so $I$ is a cover of $f$ of $\operatorname{cost} c(I)=\tau$. Note that $I$ is a $p$-cover of $\mathcal{F}$ (since $f(A)=p$ for all $A \in \mathcal{F}$ ) and that $I$ is $\mathcal{F}$-quasi-bipartite (since $G$ is $f$-quasi-bipartite and since $I \subseteq E$ ).
(A) For every $C \in \mathcal{C}_{\mathcal{F}}$ fix some inclusion minimal $p$-cover $I_{C} \subseteq I$ of $\mathcal{F}(C)$. In Lemma 4 we show the following:
(i) Each $I_{C}$ partitions into $p$ inclusion minimal 1-covers $I_{C}^{1}, \ldots, I_{C}^{p}$ of $\mathcal{F}(C)$.
(ii) Each $\mathcal{F}(C)$ has a unique inclusion maximal set $M_{C}$ and each $I_{C}^{j}$ has a unique edge $e_{C}^{j}$ that covers $M_{C}$, which we call the prime edge of $I_{C}^{j}$.
(B) In Lemma 5 we show that for distinct $C, C^{\prime} \in \mathcal{C}_{\mathcal{F}}$ and any $1 \leq j, j^{\prime} \leq p$, if $I_{C}^{j} \cap I_{C^{\prime}}^{j^{\prime}} \neq \emptyset$ then $I_{C}^{j} \cap I_{C^{\prime}}^{j^{\prime}}=\left\{e_{C}^{j}\right\}$ or $I_{C}^{j} \cap I_{C^{\prime}}^{j^{\prime}}=\left\{e_{C^{\prime}}^{j^{\prime}}\right\}$, see Fig. 1(a); this property is since $I$ is $\mathcal{F}$-quasi-bipartite. Consequently, for every $e \in I$ there is at most one set $I_{C}^{j}$ such that $e \in I_{C}^{j}$ and $e \neq e_{C}^{j}$.
(C) Construct an auxiliary bipartite graph $\mathcal{H}$ with node- and edge-costs as follows, see Fig. 1 (b) The node parts of $\mathcal{H}$ are the prime edges and $\mathcal{C}_{\mathcal{F}}$. Each node $e$ of $\mathcal{H}$ that is a prime edge inherits its cost $c(e)$ in $G$, and is connected to each $C \in \mathcal{C}_{\mathcal{F}}$ such that $e \in I_{C}^{j}$ for some $j$ by an edge of cost $c\left(I_{C}^{j}\right)-c(e)$ (this edge represents the set $I_{C}^{j}$ ). Since for every $e \in I$ at most one set $I_{C}^{j}$ contains $e$ as a non-prime edge, and since the sets $I_{C}^{j}$ are pairwise disjoint, the total cost of $\mathcal{H}$ is at most 2 times the cost of $I$.
(D) Every node $C \in \mathcal{C}_{\mathcal{F}}$ of $\mathcal{H}$ has at least $p$ neighbors in $\mathcal{H}$ (the prime edges of the sets $I_{C}^{1}, \ldots, I_{C}^{p}$ ). In Lemma 6 we show that $\mathcal{H}$ contains a star $S_{\mathcal{H}}$ with leaf set $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ such that $\frac{c\left(S_{\mathcal{H}}\right)}{|\mathcal{C}|} \leq \frac{1}{p} \cdot \frac{c(\mathcal{H})}{\left|\mathcal{C}_{\mathcal{F}}\right|} \leq \frac{2}{p} \cdot \frac{\tau}{\left|\mathcal{C}_{\mathcal{F}}\right|}$. Then the edge subset $S \subseteq I$ that corresponds to $S_{\mathcal{H}}$ covers $\mathcal{F}(\mathcal{C})$, and $S, \mathcal{C}$ satisfy inequality (1).
(E) To find $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}_{\mathcal{F}}$ and a cover $S$ of $\mathcal{F}(\mathcal{C})$ that satisfies (11), we make a similar construction: now $\mathcal{H}$ has node set $E \cup \mathcal{C}$, every node $e \in E$ of $\mathcal{H}$ has cost equal to the cost of $e$ in $G$, and in $\mathcal{H}$ each node $C \in \mathcal{C}_{\mathcal{F}}$ is connected to each node $e \in E$ by an edge of cost being the minimum cost of an edge set $S$ such that $S \cup\{e\}$ covers $\mathcal{F}(C)$. In such a graph $\mathcal{H}$ we can find a star $S_{\mathcal{H}}$ with leaf set $\mathcal{C}$ that minimizes $\frac{c\left(S_{\mathcal{H}}\right)}{|\mathcal{C}|}$ using the method of Klein \& Ravi [21]; see also step 3 of the implementation discussed in the next section.

### 3.2 Implementation

Here we briefly discuss a simple implementation of the entire algorithm. We start with the particular case of the Min-Cost Rooted $\operatorname{Subset}\left(k_{0}, k\right)$ -Edge-Connection Augmentation problem. In what follows let $n=|V|$ and $m=|E|$. As a pre-processing step, we assign unit capacities to edges in $E$ and compute a $k_{0}$-flow from the root $r$ to each $t \in T$. This can be done in $O(k m|T|)$ time using the Ford-Fulkerson algorithm. Let us consider iteration $\ell$ of Algorithm 1. when $\max (f)=k-\ell$. We will assume that we already have a flow on zero cost edges of value $k-\ell-1$ to each $t \in T$, and perform the following steps.

1. We increase the flow by 1 to each $t \in T$, and discard terminals for which the flow can be further increased by 2 . This can be done in $O(m|T|)$ time.
2. To compute the cost of an edge of $\mathcal{H}$ between nodes $C$ and $e$, we add a "dummy" edge of cost 0 from $r$ to some terminal in every core distinct from $C$, set the cost of $e$ to 0 , and compute a minimum cost edge set that increases the $r C$-flow by 1 ; the later problem admits a linear time reduction to the shortest path problem and thus can be implemented in $O\left(n^{2}\right)$ time. The number of edges in $\mathcal{H}$ is $O(m|T|)$, hence $\mathcal{H}$ can be constructed in $O\left(n^{2} m|T|\right)$ time.
3. We can sort the edges of $\mathcal{H}$ by increasing cost in $O(m|T| \log n)$ time. Then finding a (nontrivial) star $S^{e}$ in $\mathcal{H}$ with a specific center $e$ that minimizes $\frac{c\left(S^{e}\right)}{|\mathcal{C}|}$ can be done in time linear in the degree of $e$ in $\mathcal{H}$ as follows. We take the lowest cost edge incident to $e$ into $S^{e}$ and then add edges incident to $e$ one by one in increasing cost order until reaching a local minimum of $\frac{c\left(S^{e}\right)}{|\mathcal{C}|}$; see [21]. The overall time for computing all stars $S^{e}$ is $O(m n \log n)$, which is dominated by the time $O\left(n^{2} m|T|\right)$ of the construction of $\mathcal{H}$.
4. At iteration $\ell$ we need to construct the graph $\mathcal{H}$ at most $|T|$ times, hence the overall time per iteration $\ell$ is $O\left(n^{2} m|T|^{2}\right)$. And since we have $k-k_{0}$ iterations, the overall running time is $\left(k-k_{0}\right) \cdot O\left(n^{2} m|T|^{2}\right)=O\left(k n^{6}\right)$.
We note that while the running time of the described implementation is somewhat high, it is still much lower than that of Chan et al. [2].

The implementation of steps 1,3,4 for the Min-Cost Set Function Edge Cover problem under Assumption 1 is similar. For step 2, for any $C \in \mathcal{C}_{\mathcal{F}}$ and $e \in E \backslash I$ we need to find in polynomial time a min-cost edge set $S=S(e, C)$ such that $S \cup\{e\}$ covers $\mathcal{F}(C)$. For this, it is sufficient to find a min-cost cover of $\mathcal{F}(C)$ after resetting the cost of $e$ to zero. The family $\mathcal{F}(C)$ is a $T$-intersecting family that has a unique core; such a family is called a ring. It is known that a min-cost edge-cover of a ring can be found in polynomial time under Assumption 1 (c.f. [9,28), by a standard primal dual algorithm.

### 3.3 Proofs of Lemmas

Now we turn to formal proofs of Lemmas 45 and 6 mentioned in our roadmap. At each step we will specify the part of our roadmap that is proved.

A $T$-intersecting family $\mathcal{R}$ that has a unique core $C$ is called a ring. Then $C$ is the intersection of all sets in $\mathcal{R}$, and $\mathcal{R}$ also has a unique inclusion maximal set $M$ which is the union of all sets in $\mathcal{R}$. The following lemma is a folklore.

Lemma 3 If $\mathcal{F}$ is a $T$-intersecting family then $\mathcal{F}(C)$ is a ring family for any $C \in \mathcal{C}_{\mathcal{F}}$; thus $\mathcal{F}(C)$ also has a unique inclusion maximal set $M_{C}$. Furthermore, $M_{C} \cap M_{C^{\prime}} \cap T=\emptyset$ for any distinct $C, C^{\prime} \in \mathcal{C}_{\mathcal{F}}$.

The next lemma gives two additional known properties of rings; c.f. [8] for the first property and [28, Lemma 2.6 and Corollary 2.7] for the second. These two properties imply part (A).

Lemma 4 Let $\mathcal{R}$ be a ring with minimal member $C$ and maximal member $M$.
(i) Any p-cover of $\mathcal{R}$ is a union of $p$ edge disjoint covers of $\mathcal{R}$.
(ii) Let $I$ be an inclusion minimal cover of $\mathcal{R}$. Then there is an ordering $e_{1}, e_{2}, \ldots, e_{q}$ of $I$ and a nested family $C=C_{1} \subset C_{2} \cdots \subset C_{q}=M$ of sets in $\mathcal{R}$ such that for every $j=1, \ldots, q, e_{j}$ is the unique edge in $I$ that enters $C_{j}$ (namely, $e_{j}$ has head in $C_{j}$ and tail not in $C_{j}$ ).

Lemmas 3 and 4 (i) imply the following lemma that implies parts (B,C).
Lemma 5 Let I be an $\mathcal{F}$-quasi-bipartite cover of a $T$-intersecting family $\mathcal{F}$. For $C \in \mathcal{C}_{\mathcal{F}}$ let $I_{C} \subseteq I$ be an inclusion minimal cover of $\mathcal{F}(C)$, and let $e_{C}$ be the unique (by Lemma 4(i)) edge in $I_{C}$ that covers $M_{C}$. Let $C, C^{\prime} \in \mathcal{C}_{\mathcal{F}}$ be distinct and let $e \in I_{C} \cap I_{C^{\prime}}$. Then $e=\left\{e_{C}\right\}$ or $e=\left\{e_{C^{\prime}}\right\}$.

Proof Suppose that $e \neq e_{C}$ and we will show that then $e=e_{C^{\prime}}$. Note that $e$ does not cover $M_{C}$, hence $e$ has both ends in $M_{C}$, by the minimality of $I_{C}$ and Lemma 4 (ii). Since $I$ is $\mathcal{F}$-quasi-bipartite, $e$ has an end $t$ in $M_{C} \cap T$. By Lemma 3, $t \notin M_{C^{\prime}}$, hence by the minimality of $I_{C^{\prime}}$ we must have $e=e_{C^{\prime}}$.

The next lemma implies part (D).

Lemma 6 Let $H=(A \cup B, E)$ be a bipartite graph with edge- and node- costs $\{c(e): e \in E\} \cup\{c(a): a \in A\}$ and let $\mathcal{S}$ be the set of stars in $H$ with center in $A$ and leaves in $B$. If the degree of every $b \in B$ is at least $p$ then there is $S^{*} \in \mathcal{S}$ such that $\frac{c\left(S^{*}\right)}{\left|L\left(S^{*}\right)\right|} \leq \frac{1}{p} \cdot \frac{c(G)}{|B|}$, where $L\left(S^{*}\right)$ is the set of leaves of $S^{*}$.

Proof For $S \in \mathcal{S}$ let $c_{S}$ denote the cost of $S$ and let $\mathbf{c}=\left\{c_{S}: S \in \mathcal{S}\right\}$ be a vector of costs of the stars. For an integer $q$ let $\mathcal{L}(q)$ be the following set of linear constraints:

$$
\begin{array}{ll}
\sum_{L(S) \ni b} x_{S} \geq q & \forall b \in B \\
0 \leq x_{S} \leq 1 & \forall S \in \mathcal{S}
\end{array}
$$

Note that the characteristic vector $\mathbf{x}$ of the inclusion maximal stars in $\mathcal{S}$ satisfies the set of constraints $\mathcal{L}(p)$ and that $\mathbf{c} \cdot \mathbf{x}=c(H)$. Thus the vector $\mathbf{y}=\mathbf{x} / p$ satisfies $\mathcal{L}(1)$ and $\mathbf{c} \cdot \mathbf{y}=c(H) / p$. Let $S^{*}=\arg \max _{S \in \mathcal{S}} \frac{|L(S)|}{c(S)}$. Then
$\frac{\left|L\left(S^{*}\right)\right|}{c\left(S^{*}\right)}(\mathbf{c} \cdot \mathbf{y}) \geq \sum_{S \in \mathcal{S}} \frac{|L(S)|}{c_{S}} c_{S} y_{S}=\sum_{S \in \mathcal{S}}|L(S)| y_{S}=\sum_{b \in B} \sum_{L(S) \ni b} y_{S} \geq \sum_{b \in B} 1=|B|$.
The first inequality is by the choice of $S^{*}$ and the second inequality is since $\mathbf{y}$ satisfies $\mathcal{L}(1)$.

From this we get that $\frac{\left|L\left(S^{*}\right)\right|}{c\left(S^{\prime}\right)} \geq \frac{|B|}{\mathbf{c} \cdot \mathbf{y}}$, so $\frac{c\left(S^{*}\right)}{\left|L\left(S^{*}\right)\right|} \leq \frac{\mathbf{c} \cdot \mathbf{y}}{|B|}=\frac{\mathbf{c} \cdot \mathbf{x} / p}{|B|}=\frac{1}{p} \cdot \frac{c(H)}{|B|}$.
This concludes the proof of Lemma 2, and thus also the proofs Theorem 1 and Corollary 1 are complete.

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