

UPPER BOUNDS FOR EDGE-ANTIPODAL AND SUBEQUILATERAL POLYTOPES

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ABSTRACT. A polytope in a finite-dimensional normed space is *subequilateral* if the length in the norm of each of its edges equals its diameter. Subequilateral polytopes occur in the study of two unrelated subjects: surface energy minimizing cones and edge-antipodal polytopes. We show that the number of vertices of a subequilateral polytope in any d -dimensional normed space is bounded above by $(\frac{d}{2} + 1)^d$ for any $d \geq 2$. The same upper bound then follows for the number of vertices of the edge-antipodal polytopes introduced by I. Talata (Period. Math. Hungar. **38** (1999), 231–246). This is a constructive improvement to the result of A. Pór (to appear) that for each dimension d there exists an upper bound $f(d)$ for the number of vertices of an edge-antipodal d -polytopes. We also show that in d -dimensional Euclidean space the only subequilateral polytopes are equilateral simplices.

1. NOTATION

Denote the d -dimensional real linear space by \mathbb{R}^d , a norm on \mathbb{R}^d by $\|\cdot\|$, its unit ball by B , and the ball with centre x and radius r by $B(x, r)$. Denote the diameter of a set $C \subseteq \mathbb{R}^d$ by $\text{diam}(C)$, and (if it is measurable) its volume (or d -dimensional Lebesgue measure) by $\text{vol}(C)$. The *dual norm* $\|\cdot\|^*$ is defined by $\|x\|^* := \sup\{\langle x, y \rangle : \|y\| \leq 1\}$, where $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^d . Denote the number of elements of a finite set S by $|S|$. The *difference body* of a set $S \subseteq \mathbb{R}^d$ is $S - S := \{x - y : x, y \in S\}$. A *polytope* is the convex hull of finitely many points in some \mathbb{R}^d . A *d -polytope* is a polytope of dimension d . A *convex body* C is a compact convex subset of \mathbb{R}^d with nonempty interior. The boundary of C is denoted by $\text{bd } C$. Given any convex body C we define the *relative norm* $\|\cdot\|_C$ *determined by* C to be the norm with unit ball $C - C$, or equivalently,

$$\|x\|_C := \sup\{\lambda > 0 : a + \lambda x \in C \text{ for some } a \in C\}.$$

See [9, 1, 17] for background on polytopes, convexity, and finite-dimensional normed spaces.

This material is based upon work supported by the South African National Research Foundation under Grant number 2053752.

2. INTRODUCTION

2.1. Antipodal and edge-antipodal polytopes. A d -polytope P is *antipodal* if for any two vertices x and y of P there exist two parallel hyperplanes, one through x and one through y , such that P is contained in the closed slab bounded by the two hyperplanes. Klee [10] posed the problem of finding an upper bound for the number of vertices of an antipodal d -polytope in terms of d . Danzer and Grünbaum [7] proved the sharp upper bound of 2^d . See [12] for a recent survey.

A d -polytope P is *edge-antipodal* if for any two vertices x and y joined by an edge there exist two parallel hyperplanes, one through x and one through y , such that P is contained in the closed slab bounded by the two hyperplanes. This notion was introduced by Talata [18], who conjectured that the number of vertices of an edge-antipodal 3-polytope is bounded above by a constant. Csikós [6] proved an upper bound of 12, and K. Bezdek, Bisztriczky and Böröczky [2] gave the sharp upper bound of 8. Pór [15] proved that the number of vertices of an edge-antipodal d -polytope is bounded above by a function of d . However, his proof is existential, with no information on the size of the upper bound. Our main result is an explicit bound.

Theorem 1. *Let $d \geq 2$. Then the number of vertices of an edge-antipodal d -polytope is bounded above by $(\frac{d}{2} + 1)^d$.*

In the plane, an edge-antipodal polytope is clearly antipodal, and in this case the above theorem is sharp. The bound given is not sharp for $d \geq 3$ (since the bound in Theorem 2 below is not sharp). In [2] it is stated without proof that all edge-antipodal 3-polytopes are antipodal. On the other hand, Talata has an example of an edge-antipodal d -polytope that is not antipodal for each $d \geq 4$ (see [6] and Section 4 below). Most likely the largest number of vertices of an edge-antipodal d -polytope has an upper bound exponential in d , perhaps even 2^d . We also mention the paper by Bisztriczky and Böröczky [3] discussing edge-antipodal 3-polytopes.

Theorem 1 is proved by considering a metric relative of edge-antipodal polytopes, discussed next.

2.2. Equilateral and subequilateral polytopes. A polytope P is *equilateral* with respect to a norm $\|\cdot\|$ on \mathbb{R}^d if its vertex set is an *equidistant set*, i.e., the distance between any two vertices is a constant. This notion was first considered by Petty [14], who showed that equilateral polytopes are antipodal, hence have at most 2^d vertices. We now introduce the following natural weakening of this notion, analogous to the weakening from antipodal to edge-antipodal. We say that a d -polytope P is *subequilateral* with respect to a norm $\|\cdot\|$ on \mathbb{R}^d if the length of each of its edges equals its diameter.

Although not explicitly given a name, the vertex sets of subequilateral polytopes appear in the study of surface energy minimizing cones by Lawlor and Morgan [11]; see Section 4 for a discussion.

It is well-known and easy to prove that an edge-antipodal polytope P is subequilateral with diameter 1 in the relative norm $\|\cdot\|_P$ determined by P [18, 6]. It is also easy to see that any subequilateral polytope is edge-antipodal. In order to prove Theorem 1 it is therefore sufficient to bound the number of vertices of a subequilateral d -polytope.

Theorem 2. *Let $d \geq 2$. Then the number of vertices of a subequilateral d -polytope with respect to some norm $\|\cdot\|$ is bounded above by $(\frac{d}{2} + 1)^d$.*

The proof is in Section 3. In two-dimensional normed spaces subequilateral polytopes are always equilateral. Therefore, the above theorem is sharp for $d = 2$. By analyzing equality in the proof of Theorem 2, it can be seen that the bound is not sharp for $d \geq 3$. Since edge-antipodal 3-polytopes have at most 8 vertices, with equality only for parallelepipeds [2], it follows that subequilateral 3-polytopes with respect to any norm has size at most 8, with equality only if the unit ball of the norm is a parallelepiped homothetic to the polytope.

We finally mention that in Euclidean d -space \mathbb{E}^d the only subequilateral polytopes are equilateral simplices, and give a proof. In the proof we have to consider subequilateral polytopes in spherical spaces, making it possible to formulate a more general theorem for spaces of constant curvature. Note that if we restrict ourselves to a hemisphere of the d -sphere \mathbb{S}^d in \mathbb{E}^{d+1} , the notion of a polytope can be defined without ambiguity. The definition of a subequilateral polytope then still makes sense in a hemisphere of \mathbb{S}^d , as well as in hyperbolic d -space \mathbb{H}^d .

Theorem 3. *Let P be a subequilateral d -polytope in either \mathbb{E}^d , \mathbb{H}^d , or a hemisphere of \mathbb{S}^d . Then P is an equilateral d -simplex.*

Proof. The proof is by induction on $d \geq 1$, with $d = 1$ trivial and $d = 2$ easy. Suppose now $d \geq 3$. Let P be a subequilateral d -polytope in any of the three spaces. By induction all facets of P are equilateral simplices. In particular, P is simplicial. Since $d \geq 3$, it is sufficient to show that P is simple (see section 4.5 and exercise 4.8.11 of [9]).

Consider any vertex v with neighbours v_1, \dots, v_k , $k \geq d$. Then v_1, \dots, v_k are contained in an open hemisphere S of the $(d-1)$ -sphere of radius $\text{diam}(P)$ and centre v . (This sphere will be isometric to some sphere in \mathbb{E}^d , not necessarily of radius $\text{diam}(P)$.)

Consider the $(d-1)$ -polytope P' in S generated by v_1, \dots, v_k and any facet of P' with vertex set $F \subset \{v_1, \dots, v_k\}$. There exists a great sphere C of S passing through F with P' in one of the closed hemispheres determined by C . It follows that the hyperplane H generated by C and v passes through $F \cup \{v\}$, and P is contained in one of the closed half spaces bounded by H . Therefore, $F \cup \{v\}$ is the vertex set of a facet of P .

Similarly, it follows that for any vertex set F of a facet of P containing v , $F \setminus \{v\}$ is the vertex set of a facet of P' . Therefore, any edge $v_i v_j$ of P' is an edge of P , hence of length the diameter of P . It follows that the distance between v_i and v_j in H is the diameter of P' as measured in H . This shows that P' is subequilateral in H , and so by induction is an equilateral $(d-1)$ -simplex. Therefore, $k = d$, giving that P is a simple polytope, which finishes the proof. \square

3. A MEASURE OF NON-EQUIDISTANCE

The key to the proof of Theorem 2 is a lower bound for the distance between two nonadjacent vertices of a subequilateral polytope. For any finite set of points V we define

$$\lambda(V; \|\cdot\|) = \text{diam}(V) / \min_{x,y \in V, x \neq y} \|x - y\|.$$

Since $\lambda(V; \|\cdot\|) \geq 1$, with equality if and only if V is equidistant in the norm $\|\cdot\|$, this functional measures how far V is from being equidistant. The next lemma generalizes the theorem of Petty [14] and Soltan [16] that the number of points in an equidistant set is bounded above by 2^d . In [8] a proof of the 2^d -upper bound was given using the isodiametric inequality for finite-dimensional normed spaces due to Busemann (equation (2.2) on p. 241 of [4]; see also Mel'nikov [13]). However, since the isodiametric inequality has a quick proof using the Brunn-Minkowski inequality [5], it is not surprising that the latter inequality occurs in the following proof.

Lemma 1. *Let V be a finite set in a d -dimensional normed space. Then $|V| \leq (\lambda(V; \|\cdot\|) + 1)^d$.*

Proof. Let $\lambda = \lambda(V; \|\cdot\|)$. By scaling we may assume that $\text{diam}(V) = \lambda$. Then $\|x - y\| \geq 1$ for all $x, y \in V$, $x \neq y$, hence the balls $B(v, 1/2)$, $v \in V$, have disjoint interiors. Define $C = \bigcup_{v \in V} B(v, 1/2)$. Then $\text{vol}(C) = |V|(1/2)^d \text{vol}(B)$ and $\text{diam}(C) \leq 1 + \lambda$. By the Brunn-Minkowski inequality [5] we obtain $\text{vol}(C - C)^{1/d} \geq \text{vol}(C)^{1/d} + \text{vol}(-C)^{1/d}$. Noting that $C - C \subseteq (1 + \lambda)B$, the result follows. \square

In order to find an upper bound on the number of vertices of a subequilateral polytope with vertex set V , it remains to bound $\lambda(V; \|\cdot\|)$ from above.

Lemma 2. *Let $d \geq 2$ and let V be the vertex set of a subequilateral d -polytope. Then $\lambda(V; \|\cdot\|) \leq d/2$.*

Proof. Let P be a subequilateral d -polytope of diameter 1, and let V be its vertex set. We have to show that $\|x - y\| \geq 2/d$ for any distinct $x, y \in V$. Since this follows from the definition if xy is an edge of P , we assume without loss that xy is not an edge of P . Then xy intersects the convex hull P' of $V \setminus \{x, y\}$ in a (possibly degenerate) segment, say $x'y'$, with x, x', y', y

in this order on xy . Let F_x and F_y be facets of P' containing x' and y' , respectively.

We show that $\|x - x'\| \geq 1/d$. For each vertex z of F_x , xz is an edge of P , hence $\|x - z\| = 1$. By Carathéodory's theorem [1, (2.2)], there exist d vertices z_1, \dots, z_d of the $(d-1)$ -polytope F_x and real numbers $\lambda_1, \dots, \lambda_d$ such that

$$x' = \sum_{i=1}^d \lambda_i z_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^d \lambda_i = 1.$$

Suppose without loss that $\lambda_d = \max_i \lambda_i$. Then $\lambda_d \geq 1/d$. By the triangle inequality we obtain

$$\begin{aligned} \|x' - z_d\| &= \left\| \sum_{i=1}^{d-1} \lambda_i (z_i - z_d) \right\| \leq \sum_{i=1}^{d-1} \lambda_i \|z_i - z_d\| \\ &\leq \sum_{i=1}^{d-1} \lambda_i = 1 - \lambda_d \leq 1 - \frac{1}{d}, \end{aligned}$$

and

$$\begin{aligned} \|x - x'\| &\geq \|x - z_d\| - \|x' - z_d\| \\ &\geq 1 - \left(1 - \frac{1}{d}\right) = \frac{1}{d}. \end{aligned}$$

Similarly, $\|y - y'\| \geq 1/d$, and we obtain $\|x - y\| \geq 2/d$. \square

Lemmas 1 and 2 now imply Theorem 2. \square

4. CONCLUDING REMARKS

4.1. Sharpness of Lemma 2. The following example shows that Lemma 2 cannot be improved in general. Consider the subspace $X = \{(x_1, \dots, x_{d+1}) : \sum_{i=1}^d x_i = 0\}$ of \mathbb{R}^{d+1} with the ℓ_1 norm $\|(x_1, \dots, x_{d+1})\|_1 := \sum_{i=1}^{d+1} |x_i|$. Let the standard unit vector basis of \mathbb{R}^{d+1} be e_1, \dots, e_{d+1} . Let $c = \sum_{i=1}^d e_i$. Then $V = \{de_i - c : i = 1, \dots, d\} \cup \{\pm 2e_{d+1}\}$ is the vertex set of a d -polytope P in X , with all intervertex distances equal to $2d$, except for the distance between $\pm 2e_{d+1}$, which is 4. It follows that P is subequilateral and $\lambda(V; \|\cdot\|) = d/2$.

However, the above polytope P is in fact antipodal, and so it is equilateral in $\|\cdot\|_P$, which gives $\lambda(V; \|\cdot\|_P) = 1$. It is easy to see that for any polytope P subequilateral with respect to some norm $\|\cdot\|$, and with vertex set V , we have $\lambda(V, \|\cdot\|) \leq \lambda(V, \|\cdot\|_P)$. One may therefore hope that for the norm $\|\cdot\|_P$ the upper bound in Lemma 2 may be improved, thus giving a better bound in Theorem 1. The following example shows that any such improved upper bound will still have to be at least $(d-1)/2$, indicating that essentially new ideas will be needed to improve the upper bounds in Theorems 1 and 2.

We consider Talata's example [6] of an edge-antipodal polytope that is not antipodal. Let $d \geq 4$, e_1, \dots, e_d be the standard basis of \mathbb{R}^d , $p =$

$\frac{2}{d-1} \sum_{i=1}^{d-1} e_i$, and $\lambda = (d-1)/2 - \varepsilon > 1$ for some small $\varepsilon > 0$. Then the polytope P with vertex set $V = \{o, e_1, \dots, e_d, p, e_d + \lambda p\}$ is edge-antipodal but not antipodal. In fact, $\text{diam}(V) \leq 1$ by definition of $\|\cdot\|_P$, and since $\|e_d - o\|_P = 1$ and $\|p - o\|_P = 1/\lambda$, we obtain $\lambda(V, \|\cdot\|_P) \geq \lambda$, which is arbitrarily close to $(d-1)/2$.

4.2. Subequilateral polytopes in the work of Lawlor and Morgan.

Define the $\|\cdot\|$ -energy of a hypersurface S in \mathbb{R}^d to be $\|S\| := \int_S \|n(x)\| dx$, where $n(x)$ is the Euclidean unit normal at $x \in S$. In [11] a sufficient condition is given to obtain an energy minimizing hypersurface partitioning a convex body. We restate a special case of the ‘‘General Norms Theorem I’’ in [11, pp. 66–67] in terms of subequilateral polytopes. (In the notation of [11] we take all the norms Φ_{ij} to be the same. Then the points p_1, \dots, p_m in the hypothesis form an equidistant set with respect to the dual norm. The weakening of the hypothesis in the last sentence of the General Norms Theorem I is easily seen to be equivalent to the requirement that p_1, \dots, p_m is the vertex set of a subequilateral polytope.) We refer to [11] for the simple and enlightening proof using the divergence theorem.

Lawlor-Morgan Theorem. *Let $\|\cdot\|$ be a norm on \mathbb{R}^n , and let $p_1, \dots, p_m \in \mathbb{R}^n$ be the vertex set of a subequilateral polytope of $\|\cdot\|$ -diameter 1. Let $\Sigma = \bigcup H_{ij} \subset C$ be a hypersurface which partitions some convex body C into regions R_1, \dots, R_m with R_i and R_j separated by a piece H_{ij} of a hyperplane such that the parallel hyperplane passing through $p_i - p_j$ supports the unit ball B at $p_i - p_j$.*

Then for any hypersurface $M = \bigcup M_{ij}$ which also separates the $R_i \cap \text{bd } C$ from each other in C , with the regions touching $R_i \cap \text{bd } C$ and $R_j \cap \text{bd } C$ facing each other across M_{ij} , we have $\|\Sigma\|^ \leq \|M\|^*$, i.e. Σ minimizes $\|\cdot\|^*$ -energy, where $\|\cdot\|^*$ is the norm dual to $\|\cdot\|$.*

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