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An $L(1/3)$ Discrete Logarithm Algorithm for Low Degree Curves[*]

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Abstract. We present an algorithm for solving the discrete logarithm problem in Jacobians of families of plane curves whose degrees in X and Y are low with respect to their genera. The finite base fields $\mathbb{F}_q$ are arbitrary, but their sizes should not grow too fast compared to the genus. For such families, the group structure and discrete logarithms can be computed in subexponential time of $L_{q^g}(1/3, O(1))$. The runtime bounds rely on heuristics similar to the ones used in the number field sieve or the function field sieve.

Key words. Algebraic curve, Discrete logarithm, Subexponentiality.

1. Introduction

The discrete logarithm problem (DLP) is the keystone for the security of cryptosystems based on elliptic curves and on Jacobian groups of more general algebraic curves. While to date, elliptic curves provide a very broad range of groups for which no algorithm improves over the generic ones for attacking the DLP, the same does not hold for higher genus curves. A variety of algorithms exists to tackle the DLP on Jacobians of curves, depending on whether the problem is being considered with the field size or the genus growing to infinity, or possibly both. For a general overview on algorithms for the DLP, see the survey [12]. The outcome is that for implementing cryptographic primitives, curves of genus 3 and higher have clear practical disadvantages over curves of genus 2 and elliptic curves. Yet, studying the DLP on these curves is important in particular because of the Weil descent strategy, which reduces the DLP on elliptic curves over extension fields to the DLP in the Jacobian of a curve of higher genus. Therefore, besides

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the better understanding of the general picture that one may obtain by studying large genus curves, an algorithm for solving the DLP in the large genus case may eventually become a threat for some elliptic curve cryptosystems.

The following is a general strategy for solving the DLP in groups enjoying in particular a suitable notion of size (for more details on an appropriate model, see [13]). A first phase consists in collecting relations involving elements of a chosen factor base, which is a subset of the group under consideration formed by elements of relatively small size. Thereafter, the logarithms of these elements are deduced by linear algebra. Depending on the exact algorithm employed, the output of this computation either gives the logarithm of a chosen set of group elements, or in more advanced algorithms, the ability to compute the logarithms of arbitrary elements at a relatively low cost. The resulting complexity is usually of subexponential nature, namely of the form

$$L_N(\alpha, c) = e^{c(\log N)^\alpha (\log\log N)^{1-\alpha}}$$

for $\alpha \in (0, 1)$ and $c > 0$, where N is the group size.

Quite early on, it appeared that this approach could be adapted to a family of hyperelliptic curves over a fixed base field $\mathbb{F}_q$ and of genus g growing to infinity. In this case the algorithm from [2] solves the DLP in subexponential time $L_{q^g}(1/2, O(1))$. This complexity is heuristic. It is established under the assumption that a given family of polynomials behaves similarly to random polynomials of the same degree. Later on, rigorous results for smoothness of divisors have led to proofs of the subexponential running time, and the algorithm has been generalised to further classes of curves [7,11,13,15,19,24]. These results imply that given a family of algebraic curves of growing genus g over a base field $\mathbb{F}_q$ with $\log q$ bounded by some polynomial in g, solving the DLP is possible in proven subexponential time $L_{q^g}(1/2, O(1))$.

We briefly mention, at the opposite end of the spectrum, the DLP on a family of curves of fixed genus over a base field $\mathbb{F}_q$ with q growing to infinity. In this case, analogous algorithms have a complexity which is exponential in $\log q$ [9,10,16]. This case is not studied here.

Subexponential algorithms are known in other common contexts, namely integer factorisation and computation of discrete logarithms in finite fields. Proven algorithms of complexity $L(1/2)$ exist; however, the most efficient algorithms for these problems are the number field sieve [4,17] and the function field sieve [1] and their derivatives, which achieve a heuristic complexity of $L(1/3)$. For a long time, it has been an open problem to decide whether such a complexity can be achieved for solving the discrete logarithm problem in Jacobian groups of algebraic curves.

We answer this question positively for a relatively large class of curves and present a probabilistic algorithm of heuristic subexponential complexity $L_{q^g}(1/3, O(1))$ for solving the discrete logarithm problem in Jacobians of curves of genus g over finite fields $\mathbb{F}_q$. Here, we consider families of curves $\mathcal{C}_i(X, Y)$ of genus g_i over finite fields $\mathbb{F}_{q_i}$. We require $g_i \geq (\log q_i)^2$, and the degrees in X and Y must stay within the non-empty interval with end points $\approx g_i^\alpha$ and $\approx g_i^{1-\alpha}$, where $1/3 \leq \alpha \leq 2/3$. Our constraint on the curve equation is the key for producing principal divisors of small degree, in a manner analogous to the function field sieve. The computation of individual logarithms, once the relation collection and linear algebra steps have been completed, is performed using a special-Q descent strategy.

A previous related result appeared in [14]; however, this earlier version has been considerably improved. First, the class of curves to which our algorithm applies has been expanded. Furthermore, the computation of discrete logarithms no longer has complexity $L(1/3+\varepsilon, o(1))$, but rather $L(1/3, O(1))$. This raises the question of determining explicitly the constant represented by $O(1)$. Assuming the family of curves satisfies $\deg_X \mathcal{C}_i \cdot \deg_Y \mathcal{C}_i \leq \kappa g_i$, the exact complexity of our algorithm is $L(1/3, (64\kappa/9)^{1/3})$, which is a familiar complexity in the context of the number field sieve. We mention that subsequently to [14], Diem has presented at the 10th Workshop on Elliptic Curve Cryptography (ECC 2006) an algorithm based on similar ideas [8]; he argued that computing discrete logarithms for non-singular plane curves can be solved in $L(1/3, (64/9)^{1/3}+\varepsilon)$ for any $\varepsilon > 0$. We show that the same complexity is also achieved using a slight modification of our algorithm and that it is valid for a class of curves strictly including those handled by Diem's algorithm.

The article is organised as follows. Section 2 gives an informal presentation of the algorithm. Section 3 provides the necessary tools for the precise statement and analysis of the algorithm, which is given in Sects. 4 and 5. Some corner and special cases are studied in Sect. 6.

2. Main Idea

2.1. *Relation Collection*

Before describing our algorithm with all its technical details on the most general class of curves, we sketch in this section the main idea yielding a complexity of $L_{q^g}(1/3, O(1))$ for a restricted class of curves. We provide a simplified analysis by hand waving; Sect. 3 is devoted to a more precise description of the heuristics used and of the smoothness properties needed for the analysis.

Let $\mathbb{F}_q$ be a fixed finite field. We consider a family of C_{nd} curves over $\mathbb{F}_q$, that is, curves of the form

$$\mathcal{C} : Y^n + X^d + f(X, Y)$$

without affine singularities such that $\gcd(n, d) = 1$ and any monomial $X^i Y^j$ occurring in f satisfies $ni + dj < nd$ (see [23]). Such a curve has genus $g = \frac{(n-1)(d-1)}{2}$; we assume that g tends to infinity, and that $n \approx g^\alpha$ and $d \approx g^{1-\alpha}$ for some $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ (we use the symbol $\approx$, meaning "about the same size" with no precise definition). The non-singular model of a C_{nd} curve has a unique point at infinity, which is $\mathbb{F}_q$-rational; so there is a natural bijection between degree zero divisors and affine divisors, and in the following, we shall only be concerned with effective affine divisors. Choose as factor base $\mathcal{F}$ the about $L_{q^g}(1/3, O(1))$ prime divisors of smallest degree, that is, of degree bounded by some $B \in \mathbb{N}$ with $B \approx \log_q L_{q^g}(1/3, O(1))$.

To obtain relations, consider functions $\varphi(X, Y) \in \mathbb{F}_q[X, Y]$ such that

$$k = \deg_Y \varphi \approx g^{\alpha-1/3} \quad \text{and} \quad \delta = \deg_X \varphi \approx g^{2/3-\alpha}.$$

Whenever the affine part $\operatorname{div}(\varphi)$ of the divisor of φ is smooth with respect to the factor base, it yields a relation, and we have to estimate the probability of this event.

Let N be the norm of the function field extension $\mathbb{F}_q(\mathcal{C}) = \mathbb{F}_q(X)[Y]/(Y^n + X^d + f(X,Y))$ relative to $\mathbb{F}_q(X)$. For a given function φ on the curve, if $\operatorname{div}\varphi$ contains only places of inertia degree 1, then $\operatorname{div}\varphi$ is B-smooth if and only if the norm of φ is. We have

$$\begin{aligned}\deg_X \mathrm{N}(\varphi) &= \deg \operatorname{Res}_Y\bigl(\varphi(X,Y), Y^n + X^d + f(X,Y)\bigr),\\ &\le \deg_X \varphi \deg_Y \mathcal{C} + \deg_Y \varphi \deg_X \mathcal{C} = n\delta + kd \approx g^{2/3}.\end{aligned}$$

Heuristically, we assume that the norm behaves like a random polynomial of degree about $g^{2/3}$. Then it is B-smooth with probability $1/L_{q^g}(1/3, O(1))$. (This is the same theorem as the one stating that a random polynomial of degree g is $\log_q L_{q^g}(1/2, O(1))$-smooth with probability $1/L_{q^g}(1/2, O(1))$, cf., for instance, Theorem 2.1 of [3].) Equivalently, we may assume heuristically that $\operatorname{div}(\varphi)$ behaves like a random effective divisor of the same degree $\deg_X \mathrm{N}(\varphi)$. Then the standard results on arithmetic semigroups (cf. Sect. 3) yield again that $\operatorname{div}(\varphi)$ is smooth with probability $1/L_{q^g}(1/3, O(1))$.

So the expected time for obtaining $|\mathcal{F}| = L_{q^g}(1/3, O(1))$ relations is $L_{q^g}(1/3, O(1))$. With the same complexity, one can solve a linear system and obtain the discrete logarithms of the elements of $\mathcal{F}$. If the group structure was not known in advance, it is also possible to deduce it from a Smith normal form computation, which lies again in the same complexity class.

It remains to show that the search space is sufficiently large to yield the required $L_{q^g}(1/3, O(1))$ relations, or otherwise said, that the number of candidates for φ is at least $L_{q^g}(1/3, O(1))$. The number of φ is about

$$q^{k\delta} \approx q^{g^{1/3}} < e^{(g^{1/3}(\log q)^{1/3})(\log(g \log q))^{2/3}} = L_{q^g}\bigl(1/3, O(1)\bigr).$$

The previous inequality in the place of the desired equality shows that a more rigorous analysis requires a careful handling of the $\log q$ factors in the exponent; in particular, k or δ has to be slightly increased. Moreover, the constant exponent in the subexponential function needs to be taken into account.

2.2. *Individual Logarithms*

After Sect. 2.1, the discrete logarithms of the elements of the factor base $\mathcal{F}$ are known. Now, to solve a general discrete logarithm problem, we need to be able to rewrite any element in terms of elements of $\mathcal{F}$. The classical tool for doing so is the special-Q descent strategy as introduced by Coppersmith [6].

The input is a place $Q = \operatorname{div}(u(X), Y - v(X))$, for which the discrete logarithm is sought. While not all elements can be written in that form, most of them can; so without loss of generality, by randomising the input, we may assume the special form. The degree of Q is $\deg u \le g$, and $\deg v < \deg u$.

One step of the special-Q descent rewrites a place of degree $\approx g^{1/3+\tau}$ for some $\tau \in [0, 2/3]$ as a sum of places of degrees bounded by $g^{1/3+\tau/2}$. Thus, the place Q of degree at most g is first rewritten as a sum of places of degrees bounded by $g^{2/3}$. Each of them is then rewritten as a sum of places of degrees bounded by $g^{1/2}$, and so on. We end up with a tree of places, whose leaves have a degree as close to $g^{1/3}$ as we wish. Therefore, pushing the special-Q descent far enough, we can hope to obtain leaves that are in $\mathcal{F}$, so that the discrete logarithms of all the elements of the tree, including that of Q, can be deduced.

Let us now sketch how one step of the special-Q descent works in our case: We consider a place $Q = \operatorname{div}(u(X), Y - v(X))$, with $\deg v < \deg u \approx g^{1/3+\tau}$ for some $\tau \in [0, 2/3]$. The polynomial functions on the curve having a zero at Q and of degree in Y bounded by $k \approx g^{\alpha-1/3+\tau/2}$ form an $\mathbb{F}_q[X]$-lattice generated by

$$\big(u(X), Y - v(X), Y^2 - \big(v(X)^2 \bmod u(X)\big), \ldots, Y^k - \big(v(X)^k \bmod u(X)\big)\big).$$

We consider $\mathbb{F}_q[X]$-linear combinations of these basis elements that have a small degree in X: Allowing coefficients in the combination to have a degree up to $\approx g^{2/3-\alpha+\tau/2}$, the corresponding functions have a degree in X bounded by $\approx g^{2/3-\alpha+\tau/2}$. Among the $\approx q^{g^{1/3+\tau}}$ such functions, we limit ourselves to a sieving space of size about $q^{g^{1/3}}$.

The degree of the affine part of the divisor of each function φ in the sieving space is bounded by $n \deg_X \varphi + d \deg_Y \varphi \approx g^{2/3+\tau/2}$. Since there are about $q^{g^{1/3}}$ of them, one can expect to find one whose divisor is $\approx g^{1/3+\tau/2}$-smooth (apart from the place Q that is present in the divisor by construction). We have then rewritten Q as a sum of divisors of degree at most $\approx g^{1/3+\tau/2}$ in time $L(1/3)$.

In this description, we have been vague with respect to the degree bounds, and it is necessary to be more accurate, especially when τ is getting close to 0. This motivates the following section, in which we examine in more detail the smoothness results and heuristics that are needed for the algorithm.

3. Smoothness

The algorithm presented in this article relies on finding relations as smooth divisors of random polynomial functions of low degree. As with other algorithms of this kind, for instance [2], its running time analysis will be heuristic. The main heuristic assumption is that certain principal divisors are as likely to be smooth as random divisors of the same degree, for which the desired smoothness probabilities can be proved. In this section, we collect the needed smoothness results before discussing our heuristics in more detail.

We suppose that all curves are given by absolutely irreducible plane affine models

$$\mathcal{C} : F(X, Y)$$

with $F \in \mathbb{F}_q[X, Y]$, where $\mathbb{F}_q$ is the exact constant field of the function field of $\mathcal{C}$. Arithmetic of elements of the Jacobian group of such curves is detailed in [18]. In particular, operations such as splitting a divisor into a sum of places can be performed in polynomial time.

Essentially, we are interested in a factor base $\mathcal{F}$ consisting of the places of degree bounded by some parameter μ (a few technical modifications are necessary and will be discussed later in this section). Then an effective divisor of degree ν is called $\mathcal{F}$-smooth or μ-smooth if it is composed only of places in $\mathcal{F}$. The probability of μ-smoothness is ruled by the usual results on smoothness probabilities in arithmetic semigroups such as the integers or polynomials over a finite field, cf. [21].

Unfortunately, most results in the literature are stated for a fixed semigroup and give asymptotics for μ and ν tending to infinity, whereas we need information that is uniform over an infinite family of curves. Notice, however, the purely combinatorial nature of the question: How many objects of size up to ν can be built from irreducible blocks of size up to μ? The answer depends only on the number of building blocks of any given size, and it turns out that its main term is the same uniformly over all semigroups under consideration. This can be exploited to prove combinatorially, in the same spirit as for hyperelliptic curves in [15], the following general result, which is Theorem 13 of [19]:

Theorem 1 (Heß). *Let* $0 < \varepsilon < 1$, $\gamma = \frac{3}{1-\varepsilon}$ *and* ν, μ *and* $u = \frac{\nu}{\mu}$ *such that* $3\log_q(14g+4) \le \mu \le \nu^{\varepsilon}$ *and* $u \ge 2\log(g+1)$. *Denote by* $\psi(\nu,\mu)$ *the number of* μ*-smooth effective divisors of degree* ν. *Then for* μ *and* ν *sufficiently large* (*with an explicit bound depending only on* ε, *but not on* q *or* g),

$$\frac{\psi(\nu,\mu)}{q^{\nu}} \ge e^{-u\log u\left(1+\frac{\log\log u+\gamma}{\log u}\right)} = e^{-u\log u(1+o(1))}.$$

Denote by

$$L(\alpha,c) = L_{q^g}(\alpha,c) = e^{c(g\log q)^{\alpha}(\log(g\log q))^{1-\alpha}}$$

for $0 \le \alpha \le 1$ and $c > 0$ the subexponential function with respect to $g\log q$, and let

$$\mathcal{M} = \mathcal{M}_{q^g} = \log_q(g\log q) = \frac{\log(g\log q)}{\log q}.$$

The parameter $g\log q$ will be the input size for the class of curves we consider; more intrinsically, this is the logarithmic size of the group in which the discrete logarithm problem is defined.

Proposition 2. *For some* $0 < \beta < \alpha \le 1$ *and* $c, d > 0$, *let*

$$\nu = \left\lfloor \log_q L(\alpha,c)\right\rfloor = \left\lfloor cg^{\alpha}\mathcal{M}^{1-\alpha}\right\rfloor \quad \text{and} \quad \mu = \left\lceil \log_q L(\beta,d)\right\rceil = \left\lceil dg^{\beta}\mathcal{M}^{1-\beta}\right\rceil.$$

Assume that there is a constant $\rho > \frac{1-\alpha}{\alpha-\beta}$ *such that* $g \ge (\log q)^{\rho}$. *Then for* g *sufficiently large,*

$$\frac{\psi(\nu,\mu)}{q^{\nu}} \ge L\left(\alpha-\beta, -\frac{c}{d}(\alpha-\beta)+o(1)\right),$$

where $o(1)$ *is a function that is bounded in absolute value by a constant* (*depending on* α, β, c, d *and* ρ) *times* $\frac{\log\log(g\log q)}{\log(g\log q)}$.

Proof. One computes

$$u = \frac{\nu}{\mu} \le \frac{c}{d}\left(\frac{g\log q}{\log(g\log q)}\right)^{\alpha-\beta}$$

(the inequality being due only to the rounding of ν and μ),

$$\log u = (\alpha - \beta)\log(g \log q)\big(1 + o(1)\big)$$

and

$$\frac{\log\log u}{\log u} = o(1),$$

with both $o(1)$ terms being of the form stipulated in the proposition. Applying Theorem 1 yields the desired result. Its prerequisites are satisfied since

$$\begin{aligned}\limsup \frac{\log \mu}{\log \nu} &= \limsup \frac{\beta \log g - (1-\beta)\log\log q}{\alpha \log g - (1-\alpha)\log\log q}\\ &\leq \limsup \frac{\beta \log g}{\alpha \log g - \frac{1-\alpha}{\rho}\log g}\\ &= \frac{\beta}{\alpha - \frac{1-\alpha}{\rho}} =: \varepsilon' < 1\end{aligned}$$

because of the definition of ρ; then ε is taken to be any value strictly larger than ε' and less than 1. □

The choice of μ shall insure that the factor base size, that is about q^{μ}, becomes subexponential. But the necessary rounding of μ, which may increase q^{μ} by a factor of almost q, may result in more than subexponentially many elements in the factor base when q grows too fast compared to g.

Proposition 3. *Let $0 < \beta < 1$ and $\rho \geq \frac{1-\beta}{\beta}$. If $g \geq (\log q)^{\rho}$, then $q = L(\beta, o(1))$ for $g \to \infty$.*

Proof. One computes

$$q = e^{\log q} = e^{(\log q)^{1-\beta}(\log q)^{\beta}}.$$

Since $g \geq (\log q)^{\rho}$ with $\rho \geq \frac{1-\beta}{\beta}$, one gets $(\log q)^{1-\beta} \leq g^{\beta}$, so that $q \leq e^{(g \log q)^{\beta}}$. Compared to $L(\beta, 1)$, the term $(\log(g \log q))^{1-\beta}$ is missing in the exponent; since this term tends to infinity, the result follows. □

Corollary 4. *Let $0 < \beta < 1$, $\rho > \frac{1-\alpha}{\alpha-\beta}$ and $\rho \geq \frac{1-\beta}{\beta}$, and $g \geq (\log q)^{\rho}$. Then Proposition 2 remains valid for an arbitrary rounding of μ and ν, and $q^{\mu} = L(\beta, d + o(1))$.*

Proof. Let k be any integer. By Proposition 3,

$$\nu + k = \big\lfloor \log_q\big(q^k L(\alpha, c)\big)\big\rfloor = \big\lfloor \log_q L\big(\alpha, c + o(1)\big)\big\rfloor,$$

which shows that ν may be replaced by $\nu + k$ in Proposition 2. The same argumentation holds for μ. □

We need to deal with a few technicalities related to the potential singularities and the places at infinity of our curves. To this purpose, we augment the factor base as follows; this addition of a polynomial number of divisors is negligible compared to the subexponential factor base size. Furthermore, the computational expense incurred by these additions is also negligible, since the algorithms in [18] have polynomial complexity.

- Add to $\mathcal{F}$ all the places corresponding to the resolution of singularities, regardless of their degrees, whose number is bounded by $\frac{(d-1)(d-2)}{2}$ with $d = \deg F$. The algorithm can then be described as if the curves were non-singular.
- Add to $\mathcal{F}$ the infinite places corresponding to non-singularities, regardless of their degrees, whose number is bounded by d by Bézout's theorem. Then a divisor is $\mathcal{F}$-smooth if and only if its affine part is.

The correctness and the running time analysis of our algorithm depend on two heuristics, that are classical in the context of discrete logarithm computations by collecting smooth relations. First of all, the smoothness probabilities of Proposition 2 should also apply to the special way in which we create the relations.

Heuristic 5. *Let D of degree ν be the affine part of the divisor of a uniformly randomly chosen polynomial φ with imposed bounds on the degrees in X and Y. Then the probability of D to be $\mathcal{F}$-smooth is asymptotically the same as that of a random effective affine divisor of degree ν to be μ-smooth. If φ is additionally constrained to have a zero in a special place Q, the same holds for* $\operatorname{div} \varphi - Q$.

The first part of the heuristic covers the initial relation collection phase as described in Sect. 2.1, the second part is needed for the special Q-descent of Sect. 2.2 for computing individual logarithms. They ensure that relations are found sufficiently quickly. Next, one needs to make sure that the found relations are sufficiently varied to capture the complete Jacobian group.

Heuristic 6. *The probability that the relations found by the algorithm span the full relation lattice is the same as for random relations.*

Here, the *full relation lattice* designates the lattice L such that the Jacobian group of $\mathcal{C}$ over $\mathbb{F}_q$ is isomorphic to the quotient by L of the free Abelian group over the factor base. *Randomness of relations* is to be understood as the uniform distribution on the set of relations with coefficients between 0 and the order of the Jacobian group.

Depending on the choice of $\mathcal{F}$, it is not immediately clear why Heuristic 6 should hold. For instance, assume that $\mathcal{F}$ contains places of inertia degree larger than 1 with respect to the function field extension $\mathbb{F}_q(X)[Y]/(\mathcal{C})$ over $\mathbb{F}_q(X)$, that is, places corresponding to ideals $(u, v(X,Y))$ with $u \in \mathbb{F}_q[X]$ and $\deg_Y v > 1$. If φ is limited to being linear in Y, then no such place may occur in a relation, so that the relation lattice cannot have full rank.

In practice, however, inert places should be very rare. This is justified by the observation that these places have a Dirichlet density of 0: A place of degree μ and inertia degree f dividing μ corresponds to a closed point on $\mathcal{C}$ with X-coordinate in $\mathbb{F}_{q^{\mu/f}}$ and Y-coordinate in $\mathbb{F}_{q^\mu}$, of which there are on the order of $q^{\mu/f}$. Clearly, places with

$f \geq 2$ are completely negligible. Notice now that the proof of Theorem 1 is entirely combinatorial and relies on the fact that there are essentially q^μ / μ places of degree μ. As this is still the case when restricting to non-inert places, the proof of the theorem should carry over. This motivates an a priori artificial restriction of the factor base to non-inert places.

To summarise, we rely on the validity of Heuristics 5 and 6 for the factor base $\mathcal{F}$ of smoothness parameter μ containing the following places:

- All places corresponding to the resolution of singularities.
- All places at infinity (i.e. places where the function X has a negative valuation).
- The affine non-inert places of degree bounded by μ, or otherwise said, the places corresponding to prime ideals of the form $(u, Y - v)$ with $u \in \mathbb{F}_q[X]$ irreducible of degree at most μ and $v \in \mathbb{F}_q[X]$ of degree less than $\deg u$.

4. Relation Search

For the time being, we assume that all groups we are dealing with are cyclic, of known order and with a known generator which is part of the factor base. Discrete logarithms are taken with respect to this generator. We discuss the complications arising when one of these conditions is not satisfied at the end of Sect. 5.

We are now ready to formulate precisely the algorithm outlined in Sect. 2, together with its complexity analysis. We start by the relation collection and linear algebra phases as sketched in Sect. 2.1.

Theorem 7. *Let $(\mathcal{C}_i(X,Y))_{i \in \mathbb{N}}$ be a family of plane curves of genus g_i over $\mathbb{F}_{q_i}$ of degrees n_i in Y and d_i in X. Assume that there are constants $\kappa > 0$ and $\rho \geq 2$ such that*

$$\frac{n_i d_i}{g_i} \leq \kappa, \tag{1}$$

$$\frac{n_i}{(g_i/\mathcal{M}_i)^{1/3}} \to \infty, \qquad \frac{d_i}{(g_i/\mathcal{M}_i)^{1/3}} \to \infty \quad \text{with } \mathcal{M}_i = \frac{\log(g_i \log q_i)}{\log q_i}, \tag{2}$$

$$g_i \geq (\log q_i)^\rho. \tag{3}$$

Let b be defined by

$$b = \sqrt[3]{\frac{8\kappa}{9}}.$$

There exists an algorithm that computes a factor base with $L_{q_i^{g_i}}(1/3, b)$ elements, together with the discrete logarithms of all the factor base elements, in an expected running time of

$$L_{q_i^{g_i}}\bigl(1/3, c + o(1)\bigr) \quad \text{with } c = \sqrt[3]{\frac{64\kappa}{9}}$$

under Heuristics 5 and 6.

Proof. For the sake of notational clarity, we drop all indices i in the following.

Let ν, $\delta > 0$ be constants to be optimised later. Consider polynomials $\varphi(X,Y) \in \mathbb{F}_q[X,Y]$, seen as functions on $\mathcal{C}$, of degrees bounded by $\lceil \nu \frac{n}{(g/\mathcal{M})^{1/3}} \rceil$ in Y and $\lceil \delta \frac{\kappa g/n}{(g/\mathcal{M})^{1/3}} \rceil$ in X. Then (2) implies that

$$\deg_Y \varphi \le \nu \frac{n}{(g/\mathcal{M})^{1/3}} \big(1+o(1)\big) \quad \text{and} \quad \deg_X \varphi \le \delta \frac{\kappa g/n}{(g/\mathcal{M})^{1/3}} \big(1+o(1)\big). \tag{4}$$

The affine part of the divisor of φ has a degree bounded by

$$\begin{aligned}\deg_X \mathrm{Res}_Y(\varphi, \mathcal{C}) &\le \deg_X \varphi \deg_Y \mathcal{C} + \deg_Y \varphi \deg_X \mathcal{C} \\ &\le \big(\delta \kappa g^{2/3} \mathcal{M}^{1/3} + \nu n d g^{-1/3} \mathcal{M}^{1/3}\big) \cdot \big(1+o(1)\big) \\ &\le \kappa \big(\delta + \nu + o(1)\big) g^{2/3} \mathcal{M}^{1/3} \quad \text{by (1)} \\ &= \log_q L\big(2/3, \kappa\big(\delta+\nu+o(1)\big)\big).\end{aligned}$$

Let $b > 0$ be a constant to be optimised later, and choose a smoothness bound of $\lceil \log_q(L(1/3,b)) \rceil$. Then by (3) and Corollary 4, the factor base size is in $L(1/3, b+o(1))$, and by Corollary 4 and Heuristic 5, the smoothness probability of the divisor of φ is at least

$$L\bigg(1/3, -\frac{\kappa(\nu+\delta)}{3b} + o(1)\bigg).$$

The number of different φ that satisfy the chosen degree bounds is at least

$$q^{\kappa \nu \delta g^{1/3} \mathcal{M}^{2/3}} = L(1/3, \kappa \nu \delta).$$

So the expected number of relations obtained from all these φ is bounded below by $L(1/3, \kappa(\nu\delta - \frac{\nu+\delta}{3b}) + o(1))$. For the linear algebra to succeed, according to Heuristic 6, we need the number of relations to exceed the factor base size. To minimise the relation collection effort, we choose ν and δ such that equality holds, that is,

$$\kappa \nu \delta - \frac{\kappa(\nu+\delta)}{3b} = b. \tag{5}$$

On the other hand, we wish to choose the parameters such that the time taken by the (sparse) linear algebra phase, which is $L(1/3, 2b+o(1))$, is comparable with the time taken by the relation collection:

$$\kappa \nu \delta = 2b. \tag{6}$$

Substituting $\kappa \nu \delta$ from (6) into (5), we obtain

$$\nu + \delta = \frac{3b^2}{\kappa}.$$

So the sum and product of ν and δ are known, and ν and δ are the roots of the quadratic polynomial

$$X^2 - \frac{3b^2}{\kappa}X + \frac{2b}{\kappa}.$$

For the roots to exist as real numbers, the discriminant of the quadratic polynomial must be non-negative, which is equivalent to

$$b \geq \sqrt[3]{\frac{8\kappa}{9}}.$$

Since we want to minimise the effort, we choose b minimal and reach equality above. Then

$$\nu = \delta = \sqrt{\frac{2b}{\kappa}} = \sqrt[3]{\frac{8}{3\kappa}}.$$

The total running time becomes $L(1/3, c + o(1))$ with

$$c = 2b = \sqrt[3]{\frac{64\kappa}{9}}. \qquad \square$$

5. Computing Discrete Logarithms

We now turn to the precise description and analysis of the special-Q descent strategy outlined in Sect. 2.2.

Theorem 8. *Under the assumptions of Theorem 7, once the relation collection and linear algebra steps have been completed, the logarithm of any divisor in the Jacobian group of $\mathcal{C}_i$ over $\mathbb{F}_{q_i}$ can be computed in time*

$$L_{q_i^{g_i}}(1/3, b + \varepsilon) \quad \textit{with } b = \sqrt[3]{\frac{8\kappa}{9}} \textit{ and any } \varepsilon > 0.$$

Notice that this complexity is well below that of Theorem 7 for the relation collection and linear algebra phases.

Proof. Without loss of generality, one may assume that the element whose logarithm is sought is a place of degree bounded by g and of inertia degree 1; cf. the discussion at the end of Sect. 3.

More precisely, let $Q = \operatorname{div}(u(X), Y - v(X))$ be a place with $\deg v < \deg u \leq \log_q L(1/3 + \tau, c)$ for some $c > 0$ and $0 \leq \tau \leq 2/3$. The place we start with has $\tau = \frac{2}{3}$ and $c = 1$.

We consider the polynomial functions on the curve having a zero at Q, and in particular the lattice of polynomials φ of degree in Y bounded by k with

$$k = \left\lfloor \sigma \frac{n}{(g/\mathcal{M})^{1/3 - \tau/2}} \right\rfloor,$$

where $\sigma > 0$ is a constant to be determined later. These φ form an $\mathbb{F}_q[X]$-lattice generated by $(v_0(X), Y - v_1(X), Y^2 - v_2(X), \dots, Y^k - v_k(X))$ with $v_0 = u$ and $v_i = v^i \bmod u$ for $i \geq 1$.

Let $L(1/3, e + o(1))$ be the effort we are willing to expend for one smoothing step, where $e > 0$ is a parameter to be optimised later. Then we need a sieving space of the same size, and are thus looking for $L(1/3, e + o(1))$ distinct $(k+1)$-tuples of polynomials $(\alpha_0(X), \alpha_1(X), \dots, \alpha_k(X))$ and corresponding functions

$$\varphi = -\alpha_0(X)v_0(x) + \sum_{i=1}^{k} \alpha_i(X)\big(Y^i - v_i(X)\big) = \sum_{i=1}^{k} \alpha_i(X)Y^i - \sum_{i=0}^{k} \alpha_i(X)v_i(X).$$

At the same time, we wish to minimise the degree of φ in X. Recall that the degree of v_i is bounded by $D := \log_q L(1/3 + \tau, c)$. Then for any integer z, linear algebra on the lattice yields q^{kz} different tuples such that the degrees of the α_i and that of $\sum_i \alpha_i v_i$ are at most $\frac{D}{k} + z$. Choose z so as to obtain a sieving space of size $L(1/3, e + o(1))$, that is, solve $q^{kz} = L(1/3, e + o(1))$, or

$$z = \frac{1}{n} \log_q L\big(2/3 - \tau/2, e/\sigma + o(1)\big).$$

Now the degree of φ in X is bounded from above by $\frac{D}{k} + z$ with $\frac{D}{k} = \frac{1}{2} \log_q L(2/3 + \tau/2, c/\sigma)$. Whenever τ is bounded away from zero, the value of z is thus negligible compared to that of D/k. However, to encompass in a unified treatment the case where τ approaches zero, we crudely bound $-\tau/2$ by $+\tau/2$ in the expression for z to obtain

$$\deg_X \varphi \leq \frac{1}{n} \log_q L\big(2/3 + \tau/2, (c + e)/\sigma + o(1)\big).$$

The degree of the affine part of the divisor of φ is again, as in the proof of Theorem 7, bounded by

$$\begin{aligned} \deg_X \varphi \deg_Y \mathcal{C} + \deg_Y \varphi \deg_X \mathcal{C} &\leq n \deg_X \varphi + kd, \\ &\leq \log_q L\big(2/3 + \tau/2, (c + e)/\sigma + \sigma\kappa + o(1)\big) \end{aligned}$$

since

$$kd \leq \sigma \frac{nd}{(g/\mathcal{M})^{1/3 - \tau/2}} \overset{(1)}{\leq} \sigma \frac{\kappa g}{(g/\mathcal{M})^{1/3 - \tau/2}} = \log_q L(2/3 + \tau/2, \sigma\kappa).$$

So out of the $L(1/3, e + o(1))$ possible φ, we expect by Corollary 4 and Heuristic 5 that one is $\log_q L(1/3 + \tau/2, c')$-smooth for

$$c' = \frac{1}{3e}\big((c + e)/\sigma + \sigma\kappa\big).$$

To minimise this quantity, we let $\sigma = \sqrt{(c + e)/\kappa}$, so that

$$c' = \frac{2\sqrt{\kappa}}{3e}\sqrt{c + e}. \tag{7}$$

Let us summarise the procedure: Starting with Q of degree $g = \log_q L(1/3+2/3, 1)$, we use the technique above (with $\tau_0 = 2/3$, $c_0 = 1$) to smooth it into places of degree at most $\log_q L(1/3 + \tau_1, c_1)$ with $\tau_1 = 1/3$ and $c_1 = 2\sqrt{\kappa(c_0 + e)}/3e$. Each of these is then smoothed again into places of degree at most $\log_q L(1/3 + \tau_2, c_2)$, and so on, following the formulae

$$\tau_i = \frac{1}{3 \cdot 2^{i-1}}, \qquad c_i = \frac{2\sqrt{\kappa}}{3e}\sqrt{c_{i-1} + e}.$$

After i steps, we get places of degree at most

$$\log_q L_{q^g}\left(\frac{1}{3} + \frac{1}{3 \cdot 2^{i-1}}, c_i\right) = \log_q L_{q^g}\left(\frac{1}{3}, c_i\, \mathcal{M}^{\frac{1}{3 \cdot 2^{i-1}}}\right).$$

We need to bound the c_i. Studying the function $f(x) = \alpha\sqrt{x + \beta}$ yields that the sequence (c_i) converges to a finite limit c_∞, obtained by solving $c' = c$ in (7), so that

$$c_\infty = \chi/2\big(\chi + \sqrt{\chi^2 + 4e}\big), \quad \text{where } \chi = \frac{2\sqrt{\kappa}}{3e}.$$

Fix an arbitrary constant $\xi > 0$. After a certain number of steps, depending only on e, κ and ξ, we have $c_i < c_\infty \cdot (1 + \xi)$. Furthermore, after $O(\log\log g)$ steps, we can also bound the expression $\mathcal{M}^{\frac{1}{3 \cdot 2^{i-1}}}$ by $(1 + \xi)$.

It follows that for any positive constant ξ, by building a special-Q descent tree of depth $O(\log\log g)$, we can smooth elements down to a degree

$$\log_q L_{q^g}\left(\frac{1}{3}, c_\infty(1 + \xi)\right).$$

Each node in the tree has arity bounded by g, so the number of nodes in the tree is in $g^{O(\log\log g)} = L_{q^g}(1/3, o(1))$ and has no influence on the overall complexity. We finally compute the effort needed to reach $c_\infty = b$. We have $9b^3 = 8\kappa$, and we write $9e^3 = E\kappa$, with E to be determined. The equation $b = c_\infty$ simplifies as:

$$\left(\frac{8}{E}\right)^{1/3} = \frac{2}{E}\big(1 + \sqrt{1 + E}\big).$$

The latter holds for $E = 8$, which gives $e = b$. We therefore conclude that the special-Q descent finishes within time $L_{q^g}(1/3, b + \varepsilon)$ for any fixed $\varepsilon > 0$.

So far, we have remained silent about the exact nature of the $o(1)$ terms. As long as a fixed number of them is involved, this does not pose any problem. But the number of smoothing steps and thus ultimately the number of applications of Theorem 1 is not constant. So at first sight, it is not clear whether the sum of all the $o(1)$ terms is still in $o(1)$. However, since the depth of the tree is in $O(\log\log g)$, and since according to Proposition 2 the $o(1)$ is actually a constant times $\frac{\log\log(g\log q)}{\log(g\log q)}$, the overall function still tends to 0 and is a $o(1)$. □

The Non-cyclic Case In general, the Jacobian group need not be cyclic, but may have up to $2g$ invariant factors. In this case, we call "discrete logarithm" of an element its coefficient vector with respect to a basis of the invariant factor decomposition. In other words, we need to compute a tuple of scalars instead of a single one.

We assume that the group order is still known and start by considering the comparatively easy case that we are given two elements P and Q, where Q is a multiple of P, and we wish to compute the unknown multiplier, the discrete logarithm of Q to the base P. Write down the relation matrix exactly as in Theorem 7, and perform two descents as in Theorem 8 for decomposing P and Q as sums of factor base elements. The right hand sides of the two decompositions are appended to the relation matrix. An element of the kernel of this matrix modulo the group order gives the sought relationship between P and Q. The discrete logarithm can be deduced from it if the coefficient corresponding to Q is coprime to the group order; using techniques as in [13], this can be guaranteed to happen with probability approaching 1. The final complexity is then the same as in Theorem 7.

This approach generalises immediately to the non-cyclic case if an explicit basis $\{P_i\}$ of the invariant factors is known together with the exact orders of the basis elements. Then the discrete logarithm of an element Q as a tuple with respect to the P_i may be obtained as follows. After decomposing the P_i and Q over the factor base as in Theorem 8, the matrix may be augmented by the right hand sides of all these decompositions. An element of the kernel yields the sought expression of Q in terms of the P_i as long as the coefficient corresponding to Q is coprime with the group order. Again, the total complexity is as in Theorem 7.

We finally show how to obtain the group structure if only the group order is known. The classical approach is to compute a Smith Normal Form (SNF) of the relation matrix obtained in Theorem 7, but this is more costly than a sparse kernel computation. Using the knowledge of the group order and the fact that for divisor class groups of curves there is a known set of generators of polynomial size, Heß shows in [19, Lemma 50] how to tweak the SNF computation to keep the same low complexity as before. In our context, after having computed the relation matrix as in Theorem 7 and a set of generators of polynomial cardinality r, we apply r times Theorem 8 to obtain a decomposition of each generator in terms of the factor base elements. The right hand sides of these decompositions are appended to the matrix. Then some r kernel elements are computed by sparse linear algebra modulo the group order, yielding relations between the generators. Using the randomisation techniques of [13], one may ensure that these relations are uniformly distributed over all kernel elements. It is then easy to compute a Smith Normal Form (SNF) of this matrix of polynomial size, thus giving an explicit basis for the group structure. The overall complexity is then again the same as for Theorem 7.

Group Order If the group order is unknown, it may be obtained alongside the invariant factors from the SNF of the relation matrix of Theorem 7; but computing the SNF, while still being of complexity $L(1/3)$, would needlessly increase the constant of the subexponential function.

Instead, one may use the point counting algorithm due to Lauder and Wan [20], which has a complexity that is polynomial in p, the degree of the finite field extension and the degree of the curve equation. Notice that by (1), the latter is in $O(g)$. If p is very small compared to g, for instance, in the extreme case that p is fixed, then Lauder and Wan's algorithm has an overall polynomial time complexity. But even in the most general setting in which Theorem 7 applies, we have $q = L(1/3, o(1))$ by Corollary 4, so that computing the group order takes only time $L(1/3, o(1))$.

In practice, SNF computations may still be faster than Lauder and Wan's algorithm in corner cases. It may then be worthwhile to switch to the algorithm of [5] for $\mathcal{C}_{ab}$ curves, which has a quasi-linear complexity in p; or to that of [22] for superelliptic curves, which has a square-root complexity in p.

6. Limit Cases and Special Classes of Curves

6.1. *n Close to* $(g/\mathcal{M})^{1/3}$

In this and the following section, we examine what happens when the hypothesis (2) of Theorem 7 is not satisfied. First, we consider the case $0 < \liminf \frac{n_i}{(g_i/\mathcal{M}_i)^{1/3}} =: \lambda < \infty$ (the symmetric condition for d_i is handled analogously). To simplify the presentation, we assume that we have switched to a subsequence that approaches the limit, and drop again all indices i.

Following the proof of Theorem 7, we see that the degree in Y of φ poses problem: It tends to $\lceil \nu\lambda \rceil$, which is a constant, so that (4) is not valid any more. Define $\nu^* = \frac{\lceil \nu\lambda \rceil}{\lambda} < \nu + \frac{1}{\lambda}$; then (4) holds with ν^* in the place of ν.

We now have to optimise the constant in the subexponential function giving the total complexity, $2b$, subject to (5) and (6), in which all occurrences of ν have been replaced by ν^*. As with ν we loose one degree of freedom, the solution to the optimisation problem becomes worse, and we will end up with a higher total complexity. In fact, (5) and (6) in the two variables b and δ admit a unique solution b, $\delta > 0$, which is easily computed. The analysis of the individual logarithm computation step is modified along the same lines, with an increased effort value.

It is interesting to study what happens when $\lambda \to 0$. This entails $\nu^* \sim \frac{1}{\lambda} \to \infty$ (here, $\sim$ denotes equivalence in the sense that the quotient of the left and the right hand side tends to 1). The solution to (5) and (6) is uniquely determined by ν^* and yields in particular

$$b \sim \sqrt{\frac{\kappa}{3}} \cdot \frac{1}{\sqrt{\lambda}}.$$

Similarly, in the special-Q descent step, we have

$$\deg_Y \varphi = k = \sigma \frac{n}{(g/\mathcal{M})^{1/3-\tau/2}} = \sigma\lambda (g/\mathcal{M})^{\tau/2}.$$

Assuming the worst case scenario, which is τ very close to 0 (corresponding to the end of the descent), we must ensure that $\sigma\lambda \geq 1$. We thus have to replace the optimal σ by $\sigma^* \sim \frac{1}{\lambda}$. This changes the equation giving c' as a function of c. For the limit of the sequence c_i to match b, we thus have to adapt the effort value e. We obtain:

$$e \sim \sqrt{\frac{\kappa}{3}} \cdot \frac{1}{\sqrt{\lambda}}.$$

Given that b and e tend to infinity when $\lambda \to 0$, we expect that a complexity of $L(1/3)$ will no longer be achievable using the presented algorithm when n grows more slowly than $(g/\mathcal{M})^{1/3}$; this is confirmed by the following analysis.

6.2. *n Below* $(g/\mathcal{M})^{1/3}$

When the lower bound for n_i has the form $\lambda(g/\mathcal{M})^{\alpha}$ with $\alpha < 1/3$, then we have $d = \log_q L(1-\alpha, O(1))$ at best. This implies that in the algorithm depicted in this article, both in the relation collection and individual logarithm steps, the best possible upper bound for the norm of the functions φ is $\deg_X \mathrm{N}(\varphi) \le \log_q L(1-\alpha, O(1))$. We then obtain an algorithm of complexity

$$L\left(\frac{1-\alpha}{2}, c\right) \quad \text{for some } c > 0.$$

Following exactly the lines of the proofs of Theorems 7 and 8, it is also possible to make the constant c in the expression above completely explicit.

6.3. *Curves with a Low Weighted Degree*

Theorem 9. *Assume that the family of curves of Theorem 7 satisfies the following additional constraint*: $\kappa = 2$, *and each monomial* $X^j Y^k$ *occurring in the equation of* $\mathcal{C}$ *has* $nj + dk \le nd$. *For instance*, *the curves may be* $\mathcal{C}_{nd}$ *curves.*

Then the relation collection and the linear algebra phases are performed in time $L_{q^g}(1/3, c + o(1))$ *with* $c = \sqrt[3]{\frac{64}{9}}$.

Remark. The case of plane non-singular curves of total degree $\approx \sqrt{g}$, which has been studied by Diem in [8], is included in the theorem. In this case, one has additionally $n \approx d \approx \sqrt{g}$ and $\alpha = 1/2$.

Proof. We use the notation of the proof of Theorem 7. Instead of bounding the degrees of X and Y in φ separately ("taking φ from a rectangle"), we take φ of bounded weighted degree ("from a triangle"). The monomials $X^j Y^k$ occurring in φ are required to satisfy $nj + dk \le \lambda g^{2/3}\mathcal{M}^{1/3}$ for some parameter λ replacing ν and δ and to be optimised later.

Then

$$\deg_X \mathrm{Res}_Y(\varphi, \mathcal{C}) \le \lambda g^{2/3}\mathcal{M}^{1/3} = \log_q L_{q^g}(2/3, \lambda),$$

which yields a smoothness probability of

$$L\left(1/3, -\frac{\lambda}{3b} + o(1)\right).$$

The biggest power of X in φ is $\frac{\lambda g^{2/3}\mathcal{M}^{1/3}}{n}$, the biggest power of Y is $\frac{\lambda g^{2/3}\mathcal{M}^{1/3}}{d}$. The number of allowed monomials is given by the product of these two quantities divided by 2, so that the search space has size about

$$q^{\frac{\lambda^2 g^{4/3}\mathcal{M}^{2/3}}{2nd}} \ge q^{\lambda^2 g^{1/3}\mathcal{M}^{2/3}/(2\kappa)} = L\big(1/3, \lambda^2/4\big).$$

So the expected number of relations becomes $L(1/3, \lambda(3b\lambda - 4)/12b)$, which should be the same as the factor base size. Thus, $b = \lambda(3b\lambda - 4)/(12b)$. Equating the time spent in the relation collection and in the linear algebra phase, we get $\lambda^2/4 = 2b$. These two equations are solved by

$$b = \sqrt[3]{\frac{8}{9}} \quad \text{and} \quad \lambda = \sqrt[3]{\frac{64}{3}}$$

and yield a total complexity of $L(1/3, c)$ with

$$c = 2b = \sqrt[3]{\frac{64}{9}}. \qquad \square$$

To conclude, we note that the runtime for computing individual logarithms by special-Q descent derived in Sect. 5 is still dominated by the improved runtime for relation collection and linear algebra in this special case. Therefore, while an analogously improved approach to individual logarithms using functions "from a triangle" would work, it would not have any effect on the total complexity, and we omit its analysis.

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