A Note on Extensions of Infinitary Logic^{*}

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November 3, 2018

Abstract

We show that a strong form of the so called Lindström's Theorem [4] fails to generalize to extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ) compactness property and the Löwenheim-Skolem theorem down to κ . With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to $< \kappa$.

By a well-known theorem of Lindström [4], first order logic $L_{\omega\omega}$ is the strongest logic which satisifies the compactness theorem and the downward Löwenheim-Skolem theorem. For weakly compact κ , the infinitary logic $L_{\kappa\omega}$ satisfies both the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to κ . In [1] Jon Barwise pointed out that $L_{\kappa\omega}$ is not maximal with respect to these properties, and asked what is the strongest logic based on a weakly compact cardinal κ which still satisfies the (κ, κ) -compactness

^{*}We are indebted to Lauri Hella, Tapani Hyttinen and Kerkko Luosto for useful suggestions.

[†]Research partially supported by the United States-Israel Binational Science Foundation. Publication number [ShVa:726]

[‡]Research partially supported by grant 40734 of the Academy of Finland.

property and some other natural conditions suggested by κ . We prove (Corollary 5) that for weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to κ . This shows that there is no extension of $L_{\kappa\omega}$ which would satisify the most obvious generalization of Lindström's Theorem. A stronger result (Theorem 11) is proved under an additional assumption.

We use the notation and terminology of [2, Chapter II] as much as possible. We will work with concrete logics such as first order logic $L_{\omega\omega}$, infinitary logic $L_{\kappa\lambda}$ and their extensions $L_{\omega\omega}(\{Q_i : i \in I\})$ and $L_{\kappa\lambda}(\{Q_i : i \in I\})$ by generalized quantifiers. Therefore it is not at all critical which definition of a logic one uses as long as these logics are included and some basic closure properties are respected. We use $\mathcal{L} \leq \mathcal{L}'$ to denote the sublogic relation. Let \mathcal{P} be a property of logics. A logic \mathcal{L}^* is strongest extension of \mathcal{L} with \mathcal{P} , if

1. $\mathcal{L} \leq \mathcal{L}^*$,

2. \mathcal{L}^* has property \mathcal{P} ,

and whenever a logic \mathcal{L}' has the properties 1 and 2, then $\mathcal{L}' \leq \mathcal{L}^*$.

Let \mathcal{L} be a logic. For infinite cardinals κ and λ , \mathcal{L} is (κ, λ) -compact if for all $\Phi \subseteq \mathcal{L}$ of power κ , if each subset of Φ of cardinality $< \lambda$ has a model, then Φ has a model. \mathcal{L} is κ -compact if it is (κ, ω) -compact. κ is weakly compact for \mathcal{L} if \mathcal{L} is (κ, κ) -compact. \mathcal{L} is fully compact if it is κ -compact for all κ . \mathcal{L} has the Löwenheim-Skolem property down to κ , denoted by $\mathrm{LS}(\kappa)$ if every $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $\leq \kappa$. If every sentence $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $< \kappa$, we say that \mathcal{L} satisfies $\mathrm{LS}(<\kappa)$. $\mathrm{otp}(R)$ denotes the order-type of the well-ordering R.

Theorem 1 [4] The logic $L_{\omega\omega}$ is the strongest extension of $L_{\omega\omega}$ with \aleph_0 compactness and $LS(\aleph_0)$.

Let C be a non-trivial class of regular cardinals. Let

 $Q_C^{cf}xy\phi(x,y,\vec{z}) \iff \phi(\cdot,\cdot,\vec{z})$ is a linear order with cofinality in C.

By [9], $L_{\omega\omega}(\mathbf{Q}_C^{\mathrm{cf}})$ is always fully compact. For C an interval we use the notation $\mathbf{Q}_{[\kappa,\lambda)}^{\mathrm{cf}}$ and $\mathbf{Q}_{[\kappa,\lambda]}^{\mathrm{cf}}$.

Proposition 2 There is no strongest κ -compact extension of $L_{\omega\omega}$. In fact:

- 1. there are fully compact logics \mathcal{L}_n , $n < \omega$, such that $\mathcal{L}_n \leq \mathcal{L}_{n+1}$ for all $n < \omega$, but no \aleph_0 -compact logic can extend each \mathcal{L}_n .
- There is an ℵ₀-compact logic L₁ and a fully compact logic L₂ such that no ℵ₀-compact logic can extend both L₁ and L₂.

Proof. Let $\mathcal{L}_n = L_{\omega\omega}(\{\mathbf{Q}_{[\aleph_{\omega},\infty]}^{\text{cf}}\} \cup \{\mathbf{Q}_{\aleph_l}^{\text{cf}}: l < n\})$. By [9], each \mathcal{L}_n is fully compact. Clearly, no \aleph_0 -compact logic can extend each \mathcal{L}_n .

For the second claim, let \mathcal{L}_1 be the logic $L_{\omega\omega}(Q_1)$, where Q_1 is the quantifier "there exists uncountable many" introduced by Mostowski [8]. This logic is \aleph_0 -compact [3], see [2, Chapter IV] for more recent results. Let \mathcal{L}_2 be the logic $L_{\omega\omega}(Q_B)$, where Q_B is the quantifier "there is a branch" introduced by Shelah [10]. More exactly,

$$Q_B xytuM(x)T(y)(t \le u)$$

if and only if \leq_T is a partial order of $T \subseteq M$ and there are D, \leq_D, f and B such that:

- 1. \leq_D is a total order of $D \subseteq M$
- 2. $f: \langle T, \leq_T \rangle \to \langle D, \leq_D \rangle$ is strictly increasing
- 3. $\forall s \in D \exists p \in T(f(p) = s)$
- 4. $B \subseteq T$ is totally ordered by \leq_T
- 5. $\forall b \in B((p \in T\& p \leq_T b) \to p \in B)$
- 6. $\forall s \in D \exists b \in B(s \leq_D f(b)).$

The reader is referred to [10] for a proof of the full compactness of \mathcal{L}_2 .

Suppose there were an \aleph_0 -compact logic \mathcal{L} containing both \mathcal{L}_1 and \mathcal{L}_2 as a sublogic. It is easy to see that the class of countable well-orders can be expressed as a relativized pseudoelementary class in \mathcal{L} . This contradicts \aleph_0 -compactness of \mathcal{L} . \Box

Lauri Hella pointed out that by elaborating the proof of claim (2) of the above proposition, we can make \mathcal{L}_1 fully compact. It was proved in [11] that, assuming GCH, there is no strongest extension of $\mathcal{L}_{\omega\omega}$ which is \aleph_0 -compact. Our proof of (2) of the above proposition is essentially the same as a note, based on a suggestion of Paolo Lipparini, added after Theorem 8 of [11].

Proposition 3 Suppose $\kappa > \aleph_0$. There is no strongest extension of $L_{\kappa^+\omega}$ with $LS(\kappa)$

Proof. Let $\mathcal{L}_1 = L_{\kappa^+\omega}(Q_{\aleph_0}^{\text{ef}})$ and $\mathcal{L}_2 = L_{\kappa^+\omega}(Q_{[\aleph_1,\kappa]}^{\text{ef}})$. By using standard arguments with elementary chains of submodels, it is easy to see that both \mathcal{L}_1 and \mathcal{L}_2 have $\mathrm{LS}(\kappa)$, but the consistent sentence

R is a linear order with no last element \wedge

$$\neg \mathbf{Q}_{\aleph_0}^{\mathrm{cf}} xy R(x, y) \land \neg \mathbf{Q}_{[\aleph_1, \kappa]}^{\mathrm{cf}} xy R(x, y)$$

has no models of size $\leq \kappa$. \Box

It was proved in [11] that there is no strongest extension of $\mathcal{L}_{\omega\omega}$ with $\mathrm{LS}(\omega)$.

Lemma 4 Suppose κ is weakly compact. Then κ is weakly compact for $L_{\kappa\omega}(Q_{\{\aleph_0\}}^{c_f})$ and for $\mathcal{L}_{\kappa\omega}(Q_{[\aleph_1,\kappa]}^{c_f})$. Moreover, if $\kappa > \omega$, these logics satisfy $LS(\kappa)$.

Proof. The claim concerning $\mathrm{LS}(\kappa)$ is proved with a standard elementary chain argument. We prove the weak compactness of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. The case of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{\{\aleph_0\}})$ is similar, but easier. For this end, suppose T is a set of sentences of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ and $|T| = \kappa$. We may assume $T \subseteq \kappa$. If $\alpha < \kappa$, then we assume that there is a model $\mathfrak{M}_{\alpha} \models T \cap \alpha$. In view of $\mathrm{LS}(\kappa)$, it is not a loss of generality to assume that $\mathfrak{M}_{\alpha} = \langle H(\kappa), R_{\alpha} \rangle$, where $R_{\alpha} \subseteq \kappa \times \kappa$. Let $R(\alpha, \beta, \gamma) \iff R_{\alpha}(\beta, \gamma)$. By weak compactness there is a transitive M of cardinality κ such that

$$\langle H(\kappa), \epsilon, T, R \rangle \prec_{L_{\kappa\kappa}} \langle M, \epsilon, T^*, R^* \rangle$$

and $\kappa \in M$. Let $\mathfrak{M} = \langle M, S \rangle$, where $S(x, y) \iff R^*(\kappa, x, y)$. We claim that $\mathfrak{M} \models T$. We need only worry about the cofinality-quantifier. Cofinalities $< \kappa$ can be expressed in $L_{\kappa\kappa}$, so they are preserved both ways. Therefore also cofinality κ is preserved, and no other cofinalities can occur as the models have cardinality κ . \Box

Since the logics $L_{\kappa\omega}(\mathbf{Q}_{\aleph_0}^{\mathrm{cf}})$ and $\mathcal{L}_{\kappa\omega}(\mathbf{Q}_{[\aleph_1,\kappa]}^{\mathrm{cf}})$ cannot both be a sublogic of a logic with $\mathrm{LS}(\kappa)$, we get from the above lemma:

Corollary 5 Suppose $\kappa > \omega$ is weakly compact. Then there is no strongest extension of $L_{\kappa\omega}$ for which κ is weakly compact and which has $LS(\kappa)$.

The logic $L_{\kappa\omega}$ actually satisfies the property $\mathrm{LS}(<\kappa)$ which is stronger than $\mathrm{LS}(\kappa)$. To prove a result like the above corollary for the property $\mathrm{LS}(<\kappa)$ we have to work a little harder. At the same time we extend the proof to extensions of $L_{\kappa\kappa}$. Here the cofinality quantifiers $\mathrm{Q}_C^{\mathrm{cf}}$ will not help as $\mathrm{Q}_{\{\lambda\}}^{\mathrm{cf}}$ is definable in $L_{\kappa\kappa}$ for $\lambda < \kappa$. Therefore we use more refined order-type quantifiers.

Definition 6 Let $L_{\kappa\lambda}(Q)$ denote the formal extension of $L_{\kappa\lambda}$ by the generalized quantifier symbol $Qxy\phi(x, y, \vec{z})$. If \mathcal{Y} is a class of ordinals, we get a logic $L_{\kappa\lambda}(Q, \mathcal{Y})$ from $L_{\kappa\lambda}(Q)$ by defining the semantics by

$$\mathfrak{A} \models Qxy\phi(x, y, \vec{c}) \iff otp(\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in \mathcal{Y}.$$

If $\phi \in L_{\kappa\lambda}(Q, \mathcal{Y})$ and $\mathfrak{A} \models \phi$, we say that $\mathfrak{A} \models \phi$ holds in the \mathcal{Y} -interpretation.

If \mathfrak{A} is a model, then

$$p(\mathfrak{A},\mathcal{Y},\kappa,\lambda)$$

(

is the supremum of all $\operatorname{otp}(\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\})$ where $\phi \in L_{\kappa\lambda}(\mathcal{Y})$, $\vec{c} \in A^{<\lambda}$ and $\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}$ is well-ordered.

Lemma 7 Suppose $\kappa \geq \lambda$, $\phi \in L_{\kappa\lambda}(Q)$, \mathfrak{A} is a model, $\vec{a} \in A^{<\lambda}$, and $\mathcal{Y}' \cap o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda) = \mathcal{Y}$. Then $\mathfrak{A} \models \phi(\vec{a})$ in the \mathcal{Y} -interpretation if and only if $\mathfrak{A} \models \phi(\vec{a})$ in the \mathcal{Y}' -interpretation.

Proof. This is a straightforward induction of the length of the formula ϕ .

- **Lemma 8** 1. Suppose $\kappa > \omega$, $\phi \in L_{\kappa\kappa}(Q)$, and ϕ has a model \mathfrak{A} in the \mathcal{Y} interpretation. Then there is a submodel \mathfrak{B} of \mathfrak{A} of cardinality $\leq 2^{\kappa}$ and $\mathcal{Y}' \subseteq (2^{\kappa})^+$ such that $\mathcal{Y}' \cap \kappa = \mathcal{Y}$ and $\mathfrak{B} \models \phi$ in the \mathcal{Y}' -interpretation.
 - 2. Suppose $\kappa = \kappa^{<\kappa}$, $T \subseteq L_{\kappa\kappa}(Q)$, $|T| \leq \kappa$ and T has a model \mathfrak{A} in the \mathcal{Y} -interpretation. Then for all $\xi < \kappa^+$ there is a submodel \mathfrak{B} of \mathfrak{A} of cardinality $\leq \kappa$ and $\mathcal{Y}' \subseteq \kappa^+$ such that $\mathcal{Y} \cap \xi = \mathcal{Y}' \cap \xi$ and $\mathfrak{B} \models T$ in the \mathcal{Y}' -interpretation.

Proof. We may assume $|A| \geq 2^{\kappa}$. Let us expand \mathfrak{A} by

1. A well-ordering \prec the order-type of which exceed all the order-types of well-orderings definable by subformulas of ϕ with parameters in A.

- 2. A new predicate P which contains those elements d of A for which $otp(\{\langle a, b \rangle : a \prec b \prec d\}) \in \mathcal{Y}$
- 3. A prediacte F which codes an isomorphism from each well-ordering, definable by a subformula of ϕ with parameters in A, onto an initial segment of \prec .

Let $\langle \mathfrak{A}, \prec, P, F \rangle$ be the expanded structure and $\langle \mathfrak{B}, \prec^*, P^*, F^* \rangle$ an $L_{\kappa\kappa}$ -elementary substructure of it of cardinality $\leq 2^{\kappa}$. Let

$$\mathcal{Y}' = \{ \operatorname{otp}(\{ \langle a, b \rangle \in B^2 : a \prec^* b \prec^* d\} : d \in P^* \}$$

It is easy to see that $\mathfrak{B} \models \phi$ in the \mathcal{Y}' -interpretation. \Box

Let π be the canonical well-ordering of ordered triples of ordinals. We say that a pair (δ_1, Z_1) , where $Z_1 \subseteq \delta_1$ codes a pair (δ_2, Z_2) , where $Z_2 \subseteq \delta_2$, if there is a bijection $f : \delta_2 \to \delta_1$ such that

- 1. δ_1 is closed under π
- 2. $\pi(0, \alpha, \beta) \in Z_1 \iff f(\alpha) < f(\beta)$
- 3. $\pi(1,0,\alpha) \in Z_1 \iff f(\alpha) \in Z_2.$

Definition 9 A cardinal κ satisfies $\Diamond(WC)$ if it is weakly compact and there is a sequence $\langle A_{\alpha} : \alpha < \kappa \rangle$ such that

- 1. $A_{\alpha} \subseteq \alpha$ for $\alpha < \kappa$.
- 2. $(\forall A \subseteq \kappa)(\{\lambda < \kappa : A_{\lambda} = A \cap \lambda\} \in \mathcal{I}^+)$, where \mathcal{I} is the weakly compact ideal on κ .

Proposition 10 1. If κ is measurable $> \omega$, then κ satisfies $\Diamond(WC)$.

- 2. If κ is weakly compact $> \omega$, then there is a generic extension which preserves all cardinals and in which κ satisfies $\Diamond(WC)$.
- 3. If V=L, then every weakly compact cardinal $> \omega$ satisfies $\Diamond(WC)$.

Theorem 11 Suppose $\kappa > \omega$ satisfies $\Diamond(WC)$ and $2^{\kappa} = \kappa^+$. Then there is no strongest extension of $L_{\kappa\kappa}$ for which κ is weakly compact and which has $LS(<\kappa)$.

Proof. We shall construct two sets $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \kappa^+$ such that κ is weakly compact for the logics $L_{\kappa\kappa}(Q, \mathcal{Y}^i)$ and these logics satisfy $\mathrm{LS}(<\kappa)$, but no logic containing both $L_{\kappa\kappa}(Q, \mathcal{Y}^1)$ and $L_{\kappa\kappa}(Q, \mathcal{Y}^2)$ satisfies $\mathrm{LS}(<\kappa)$. The sets \mathcal{Y}^i are constructed by induction together with ordinals $\xi^i_{\alpha} < \kappa^+$ such that:

$$\begin{array}{ll} \mathcal{Y}^{i} = \bigcup_{\alpha < \kappa^{+}} \mathcal{Y}^{i}_{\alpha} \\ \mathcal{Y}^{i}_{0} = \emptyset & \xi^{i}_{0} = 0 \\ \mathcal{Y}^{i}_{\alpha} = \mathcal{Y}^{i}_{\beta} \cap \xi^{i}_{\alpha} & \text{for } \alpha < \beta \\ \xi^{i}_{\alpha} \leq \xi^{i}_{\beta} & \text{for } \alpha < \beta \\ \mathcal{Y}^{i}_{\nu} = \bigcup_{\alpha < \nu} \mathcal{Y}^{i}_{\alpha}, \quad \xi^{i}_{\nu} = \bigcup_{\alpha < \nu} \xi^{i}_{\alpha}, \text{ for } \nu = \cup \nu \\ \mathcal{Y}^{1}_{\alpha} \cap \mathcal{Y}^{2}_{\alpha} = \emptyset & \text{for } \alpha < \kappa \\ \mathcal{Y}^{i}_{\alpha} \subseteq \xi^{i}_{\alpha} & \text{for } \alpha < \kappa^{+} \end{array}$$

First we define \mathcal{Y}^i_{α} for $\alpha < \kappa$ in such a way that $L_{\kappa\kappa}(\mathcal{Y}^i)$ will in the end have the property $\mathrm{LS}(<\kappa)$.

Let S_1, S_2 be a partition of cardinals $< \kappa$ into two stationary sets. Let $\{\phi_{\nu}^i : \nu \in S_i\}$ list all $L_{\kappa\kappa}(Q)$ -sentences so that each sentence is listed as ϕ_{ν}^i for stationary many $\nu \in S_i$.

Suppose $\alpha = \lambda + 1$ and $\xi_{\lambda}^{i} = \lambda$. Suppose $\lambda \in S_{i}$.

Case 1. Suppose that (λ, A_{λ}) codes some pair (ξ, Z) . In this case we let

$$egin{aligned} \mathcal{Y}^i_lpha &= \mathcal{Y}^i_\lambda \cup (Z\setminus\lambda), \xi^i_lpha &= \xi \ \mathcal{Y}^{3-i}_lpha &= \mathcal{Y}^{3-i}_\lambda. \end{aligned}$$

Case 2. Otherwise we let $\xi_{\alpha}^{i} = \lambda$, $\mathcal{Y}_{\alpha}^{i} = \mathcal{Y}_{\lambda}^{i}$, $\mathcal{Y}_{\alpha}^{3-1} = \mathcal{Y}_{\lambda}^{3-i}$.

Suppose then $\alpha = \lambda + 2$, $\xi_{\lambda}^i = \lambda \in S_i$ and we have defined $\xi_{\lambda+1}^i$ and $\mathcal{Y}_{\lambda+1}^i$.

Case 3. The sentence ϕ_{λ}^{i} has a model in the \mathcal{Y} -interpretation for some $\mathcal{Y} \subseteq \kappa^{+}$ with $\mathcal{Y} \cap \xi_{\lambda+1}^{i} = \mathcal{Y}_{\lambda+1}^{i}$. By Lemma 8 part 2, ϕ_{λ}^{i} has a model \mathfrak{A} of cardinality $< \kappa$ in the \mathcal{Y} -interpretation for some $\mathcal{Y} \subseteq \kappa$ of cardinality $< \kappa$ with $\mathcal{Y} \cap \xi_{\lambda+1}^{i} = \mathcal{Y}_{\lambda+1}^{i}$. Let μ be minimal such that $\phi_{\lambda}^{i} \in \mathcal{L}_{\mu\mu}(\mathcal{Y})$. Let $\xi_{\lambda+2}^{i} = o(\mathfrak{A}, \mathcal{Y}, \mu, \mu)$ and $\mathcal{Y}_{\lambda+2}^{i} = \mathcal{Y}$. Let $\mathcal{Y}_{\lambda+2}^{3-i} = \mathcal{Y}_{\lambda+1}^{3-i}$.

Case 4. Otherwise $\xi_{\lambda+1}^i = \xi_{\lambda}^i$, $\mathcal{Y}_{\alpha}^i = \mathcal{Y}_{\lambda+1}^i$, $\mathcal{Y}_{\alpha}^{3-1} = \mathcal{Y}_{\lambda+1}^{3-i}$.

Finally for all other $\alpha \leq \kappa$ we let ξ^i_{α} and \mathcal{Y}^i_{α} be defined canonically.

This ends the construction of \mathcal{Y}^i_{α} for $\alpha \leq \kappa$. Note that $\mathcal{Y}^1_{\kappa} \cap \mathcal{Y}^2_{\kappa} = \emptyset$. Moreover, if ϕ^i_{ν} has a model in the \mathcal{Y} -interpretation for some $\mathcal{Y} \supseteq \mathcal{Y}^i_{\kappa}$, then, by construction, ϕ^i_{ν} has a model of cardinality $< \kappa$ in the \mathcal{Y}^i_{κ} -interpretation. Let $\mathcal{Y}^i_{\kappa+1} = \mathcal{Y}^i_{\kappa} \cup \{\kappa\}$ and $\xi^i_{\kappa+1} = \kappa + 2$. Next we shall define \mathcal{Y}^i_{α} and ξ^i_{α} for $\kappa + 1 < \alpha < \kappa^+$. For this, let $\langle T_{\alpha} : \kappa < \alpha < \kappa^+ \rangle$ enumerate all $L_{\kappa\kappa}(Q)$ -theories of cardinality $\leq \kappa$ in a language of cardinality $\leq \kappa$ which satisfy the condition that every subset of cardinality $< \kappa$ has a model in the \mathcal{Y}^i_{κ} -interpretation. Here we use the assumption $2^{\kappa} = \kappa^+$. We may assume $T_{\alpha} \subseteq H(\kappa)$ for all α .

Suppose \mathcal{Y}^i_{β} and ξ^i_{β} have been defined for $\beta < \alpha$. If $\alpha = \bigcup \alpha$, \mathcal{Y}^i_{α} and ξ^i_{α} are defined canonically. So assume $\alpha = \beta + 1$. Let $T : H(\kappa) \to H(\kappa)$ be the function $T(a) = T_{\beta} \cap a$. If $a \in H(\kappa)$, then T(a) has a model \mathfrak{B}_a in the \mathcal{Y}^i_{κ} -interpretation. By construction, we may assume $\mathfrak{B}_a \in H(\kappa)$. Let $B : H(\kappa) \to H(\kappa)$ be the function $B(a) = \mathfrak{B}_a$. Let $Z \subseteq \kappa$ code $(\xi^i_{\beta}, \mathcal{Y}^i_{\beta})$. By $\diamondsuit(WC), W = \{\lambda < \kappa : A_{\lambda} = Z \cap \lambda\} \in \mathcal{I}^+$, where \mathcal{I} is the weakly compact ideal on κ . Let $A : \kappa \to H(\kappa)$ be the function $A(\alpha) = A_{\alpha}$. By the definition of \mathcal{I} , there are a transitive set M and A^*, W^*, Y^*, R^* such that

$$\langle H(\kappa), \epsilon, A, W, \mathcal{Y}^i_{\kappa}, B, T \rangle \prec_{\kappa\kappa} \langle M, \epsilon, A^*, W^*, Y^*, B^*, T^* \rangle$$

and $\kappa \in W^*$. Now $A^*(\kappa) = Z$ and, by construction, $Y^* \cap \xi^i_\beta = \mathcal{Y}^i_\beta$

It is clear now that $B(\kappa)$ is a model of T_{α} in the Y*-interpretation. By Lemma 8 there is a model \mathfrak{B} of cardinality $\leq \kappa$ of T_{β} in the Y**-interpretation for some Y** with $Y^{**} \cap \xi^{i}_{\beta} = \mathcal{Y}^{i}_{\beta}$. Let $\xi^{i}_{\alpha} = o(\mathfrak{B}, Y^{**}, \kappa, \kappa)$ and $\mathcal{Y}^{i}_{\alpha} = Y^{**} \cap \xi^{i}_{\alpha}$. Finally, let $\mathcal{Y}^{i} = \bigcup_{\alpha < \kappa^{+}} \mathcal{Y}^{i}_{\alpha}$.

Claim 1. $L_{\kappa\kappa}(\mathcal{Y}^i)$ satisfies the LS($< \kappa$)-property.

Suppose ϕ is a sentence of $L_{\kappa\kappa}(\mathcal{Y}^i)$ with a model. Let $\lambda \in S_i$ such that $\xi^i_{\lambda} = \lambda$ and $\phi^i_{\lambda} = \phi$. By the construction of $\mathcal{Y}^i_{\lambda+2}$ there is a model of ϕ of cardinality $< \kappa$.

Claim 2. $L_{\kappa\kappa}(\mathcal{Y}^i)$ is weakly κ -compact.

Suppose $T \subseteq L_{\kappa\kappa}(\mathcal{Y}^i)$ is given and every subset of T of cardinality $< \kappa$ has a model in the \mathcal{Y}^i -interpretation. Then $T = T_\alpha$ for some α . By construction, every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i \cap \kappa$ -interpretation. Thus the definition of \mathcal{Y}^i_α is made so that T_α has a model \mathfrak{B} in the \mathcal{Y} interpretation for some \mathcal{Y} such that $\mathcal{Y} \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa) = \mathcal{Y}^i \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa)$. Thus by Lemma 7, $\mathfrak{B} \models T_\alpha$ in the \mathcal{Y}^i -interpretation. The Claim is proved. We can now finish the proof of the theorem. In a logic in which both the quantifier $Q_{\mathcal{Y}^1}$ and $Q_{\mathcal{Y}^2}$ are definable, we can say that the order-type of a well-ordering is in $\mathcal{Y}^1 \cap \mathcal{Y}^2$. Thus such a logic cannot satisfy $\mathrm{LS}(<\kappa)$. \Box

It is interesting to note that a proof like above would not be possible for the following stronger Löwenheim-Skolem property: A filter-family is a family $\mathcal{F} = (\mathcal{F}(A))_{A \neq \emptyset}$, where $\mathcal{F}(A)$ is always a filter on the set A. Luosto [6] defines the concept of a (κ^+, ω) -neat filter family. We will not repeat the definition here, its elements are invariance under bijections, fineness, κ^+ -completeness, normality and upward relativizability (all defined in [6]). Suppose \mathcal{L} is a logic of the form $L_{\kappa\lambda}(\vec{Q})$ for some sequence \vec{Q} of generalized quantifiers. We say that \mathcal{L} has the \mathcal{F}, κ -persistency property, if for all models \mathfrak{A} and $B \in \mathcal{F}(A)$, we have $\mathfrak{A} \upharpoonright B \prec \mathfrak{A}$. Luosto proves that if \mathcal{L}_1 and \mathcal{L}_2 both satisfy the \mathcal{F}, κ -persistency property, then there is \mathcal{L}_3 such that $\mathcal{L}_1 \leq \mathcal{L}_3$, $\mathcal{L}_2 \leq \mathcal{L}_3$ and \mathcal{L}_3 satisfies the \mathcal{F}, κ -persistency property. Lipparini [5] proves a similar result for families of limit ultrafilters related closely to compactness.

Tapani Hyttinen pointed out that the assumption $2^{\kappa} = \kappa^+$ is not needed in Theorem 11, if κ is assumed to be measurable.

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