MODEL THEORY OF THE REGULARITY AND REFLECTION SCHEMES

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ABSTRACT. This paper develops the model theory of ordered structures that satisfy Keisler's *regularity scheme,* and its strengthening $\mathsf{REF}(\mathcal{L})$ (the *reflection scheme*) which is an analogue of the reflection principle of Zermelo-Fraenkel set theory. Here $\mathcal{L}$ is a language with a distinguished linear order $<$, and $\mathsf{REF}(\mathcal{L})$ consists of the universal closure of formulas of the form

$$\exists x \forall y_1 < x \cdots \forall y_n < x \ \varphi(y_1, \cdots, y_n) \leftrightarrow \varphi^{<x}(y_1, \cdots, y_n),$$

where $\varphi(y_1, \cdots, y_n)$ is an $\mathcal{L}$-formula, $\varphi^{<x}$ is the $\mathcal{L}$-formula obtained by restricting all the quantifiers of φ to the initial segment determined by x, and x is a variable that does not appear in φ. Our results include:

Theorem. *The following five conditions are equivalent for a complete first order theory T in a countable language $\mathcal{L}$ with a distinguished linear order*:

(1) *Some model of T has an elementary end extension with a first new element.*
(2) $T \vdash \mathsf{REF}(\mathcal{L})$.
(3) *T has an ω_1-like model that continuously embeds ω_1.*
(4) *For some regular uncountable cardinal κ, T has a κ-like model that continuously embeds a stationary subset of κ.*
(5) *For some regular uncountable cardinal κ, T has a κ-like model $\mathfrak{M}$ that has an elementary extension in which the supremum of M exists.*

Moreover, if κ is a regular cardinal satisfying $\kappa = \kappa^{<\kappa}$, then each of the above conditions is equivalent to:

(6) *T has a κ^+-like model that continuously embeds a stationary subset of κ.*

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1. INTRODUCTION AND PRELIMINARIES

The reflection principle[1] for Zermelo-Fraenkel set theory implies that for any set theoretical formula $\varphi(x)$ (possibly with parameters) there is a rank initial segment V_α of the universe that is φ-reflective, i.e., for any $s \in V_\alpha$, $\varphi(s)$ holds in the universe iff $\varphi(s)$ holds in V_α. This paper develops the basic model theory of ordered structures that satisfy an analogue - dubbed the *reflection scheme* - of the reflection principle in set theory. The reflection scheme was first explicitly formulated in Schmerl's work [Schm-2, Sec.1] on the model theory of ordered structures that continuously embed stationary subsets of uncountable cardinals, a topic that is intimately related to the study of extensions of first order logic with stationary quantifiers on power-like models, first introduced by Shelah [Sh-1], and recently revisited in his joint work with Väänänen [SV].

Given a language $\mathcal{L}$ with a distinguished symbol $<$ for a linear order, the reflection scheme over $\mathcal{L}$, denoted $\mathsf{REF}(\mathcal{L})$, consists of the sentence "$<$ is a linear order without a last element" plus the universal closure of formulas of the form

$$\exists x \forall y_1 < x \cdots \forall y_1 < x\, \varphi(y_1, \cdots, y_n, w_1, \cdots, w_r) \leftrightarrow \varphi^{<x}(y_1, \cdots, y_n, w_1, \cdots, w_r),$$

where $\varphi(y_1, \cdots, y_n, w_1, \cdots, w_r)$ is an $\mathcal{L}$-formula with the displayed free variables, $\varphi^{<x}$ is the $\mathcal{L}$-formula obtained by restricting all the quantifiers of φ to the initial segment determined by x, and x is a variable that does not appear in φ. In other words, models of $\mathsf{REF}(\mathcal{L})$ are precisely those $\mathcal{L}$-structures

$$\mathfrak{M} = (M, <_M, \cdots)$$

such that $(M, <_M)$ is a linear order without a last element such that for any $\mathcal{L}$-formula $\varphi(y_1, \cdots, y_n, w_1, \cdots, w_r)$ and any choice of parameters $c_1, \cdots, c_r$ in $\mathfrak{M}$, there is some $m \in M$ such that for every sequence of elements $a_1, \cdots, a_n$ below m, the sentence $\varphi(a_1, \cdots, a_n, c_1, \cdots, c_r)$ holds in $\mathfrak{M}$ iff it holds in the submodel of $\mathfrak{M}$ whose universe is the initial segment $\{x \in M : x <_M m\}$.

It is easy to see, using a Löwenheim-Skolem argument, that if κ is a regular uncountable cardinal, and $<$ is the natural order on κ, then every expansion of the structure $(\kappa, <)$ satisfies the reflection scheme.

[1] This particular formulation is one of the three variants of the reflection principle stated in [Jec, Theorem 2.14], and is due to Lévy who refined an earlier version by Montague. See [Kan, Sec. 2] for more detail on the history of the reflection principle.

Another rich source of examples of models of the reflection scheme come from set theory. By a well-known forcing construction [F], every countable[2] model $\mathfrak{M} := (M, E)$ of ZFC (where $E = \in^{\mathfrak{M}}$) has an expansion to a model $(\mathfrak{M}, <)$ that satisfies the statement GW expressing "$<$ is global well-ordering all of whose proper initial segments form a set", and which also satisfies $\mathsf{ZF}(\{<, \in\})$, i.e., all instances of the replacement scheme for formulas of the extended language $\{\in, <\}$. By a slight modification of the proof of the reflection principle one can show that all instances of the reflection scheme in the language $\{\in, <\}$ are provable in the theory $\mathsf{ZF}(\{<, \in\}) + \mathsf{GW}$. In particular, this shows that $\mathsf{ZFC} + \mathsf{REF}(\{<, \in\})$ is a conservative extension of ZFC.

In order to motivate and situate the results of this paper, we first discuss a related first order scheme known as the *regularity scheme*. The regularity scheme was introduced implicitly by Keisler in [Ke-1] and [Ke-2], and further investigated in [Sh-2]. The model theory of the regularity scheme is closely tied to the study of κ-like models and generalized quantifiers of the form "there exist κ-many" (where κ is an infinite cardinal).

- Throughout the paper, $\mathcal{L}$ is a countable language with a distinguished linear order $<$.

The *regularity scheme* $\mathsf{REG}(\mathcal{L})$ consists of the sentence "$<$ is a linear order with no last element" plus the universal closure of axioms of the form

$$[\forall v\ \exists x > v\ \exists y < z\ \varphi(x, y)] \to [\exists y < z\ \forall v\ \exists x > v\ \varphi(x, y)],$$

where φ is an $\mathcal{L}$-formula. Note that every model of $\mathsf{REF}(\mathcal{L})$ is also a model of $\mathsf{REG}(\mathcal{L})$ (but not vice versa). It is well-known that the regularity scheme is equivalent to the *collection scheme* $\mathsf{COLL}(\mathcal{L})$, consisting of the sentence "$<$ is a linear order with no last element" plus the universal closure of axioms of the following form

$$(\forall x < z\ \exists y_1\ \cdots \exists y_k\ \varphi(x, \overline{y})) \to (\exists v\ \forall x < z\ \exists y_1 < v \cdots \exists y_k < v\ \varphi(x, \overline{y}))\,,$$

[2]If $\mathfrak{M}$ satisfies the statement "the universe is ordinal definable from some set" then the countability assumption of $\mathfrak{M}$ (and the forcing argument) can be bypassed. In particular, all models of $\mathsf{ZF} + \exists x(\mathbf{V} = \mathbf{L}[\mathbf{x}])$ expand to models of the reflection scheme. However, uncountable models of ZFC that have no expansion to $\mathsf{REF}(\{<, \in\})$ exist. To see this, recall that by a classical theorem of Easton [Ea], there is a (countable) model $\mathfrak{M}_0$ of ZFC with a proper class of pairs that has no definable choice function. If $\mathfrak{M}$ is a "rather classless" elementary extension of $\mathfrak{M}_0$, then $\mathfrak{M}$ has no expansion to $\mathsf{REF}(\{<, \in\})$. See [En-1] for more information on rather classless models of set theory.

where $\varphi(x, \overline{y}) := \varphi(x, y_1 \cdots y_k)$. See [Ho, Lemma 6.1.6] for a level-by-level refinement of the equivalence of $\mathsf{REG}(\mathcal{L})$ and $\mathsf{COLL}(\mathcal{L})$.

Example 1.1.

(1) If κ is a regular infinite cardinal, then every expansion of a κ-like linear order[3] satisfies the regularity scheme. As observed earlier, if κ is an *uncountable* regular cardinal and $<$ is the natural order on κ, then every expansion of $(\kappa, <)$ satisfies the reflection scheme. More generally, if $(X, \lhd)$ is a κ-like linear order that continuously embeds[4] a stationary subset of κ, then any expansion of $(X, \lhd)$ satisfies the reflection scheme.

(2) All instances of $\mathsf{REG}(\mathcal{L}_{\mathsf{PA}})$ are provable in PA, where $\mathcal{L}_{\mathsf{PA}}$ is the language of PA (Peano arithmetic). In this context $\mathsf{REG}(\mathcal{L}_{\mathsf{PA}})$ plus the scheme $I\Delta_0$ of bounded induction is known to be equivalent to PA. See [Kay, Theorem 7.3] and [MP] for more detail. It is easy to see that PA disproves many instances of $\mathsf{REF}(\mathcal{L}_{\mathsf{PA}})$, e.g., the sentence "there is no last element" is never reflected.

(3) Zermelo-Fraenkel set theory ZF plus the axiom $\mathbf{V} = \mathbf{OD}$ (expressing "all sets are ordinal definable") proves all instances of the reflection scheme in the language of $\{<_{\mathbf{OD}}, \in\}$, where $<_{\mathbf{OD}}$ is the canonical well-ordering of the ordinal-definable sets. This follows from the reflection principle in ZF.

(4) $\mathsf{ZF}\backslash\{$Power Set Axiom$\}$ plus the axiom $\mathbf{V} = \mathbf{L}$ (expressing "all sets are constructible") proves all instances of the reflection scheme in the language $\mathcal{L} = \{<_{\mathbf{L}}, \in\}$, where $<_{\mathbf{L}}$ is the canonical well-ordering of the constructible universe. This is a nontrivial result that follows from coupling the provability of the so-called "β_k-model reflection scheme" within $\mathsf{Z}_2 + \mathsf{DC}$ (second order arithmetic plus the dependent choice scheme), with the canonical one-to-one correspondence between models of $\mathsf{ZF}\backslash\{$Power Set Axiom$\}+$ "all sets are finite or countable" and models of $\mathsf{Z}_2 + \mathsf{DC}$. See [Si, Sec. VII.7] for the former, and [Si, Sec. VII.3] for the latter.

(5) The theory T of pure linear orders with no maximum element proves every instance of $\mathsf{REG}(\{<\})$. This follows from coupling Theorem 1.2 below with a result of Rosenstein [R, Theorem 13.58], which states that every countable linear order without

[3] Recall that a linear order $(X, \lhd)$ is κ-like if $|X| = \kappa$, but for every $x \in X$, $|\{y \in X : y \lhd x\}| < \kappa$.

[4] A linear order $\mathbb{L}$ continuously embeds a subset S of κ if there is an order preserving injection f from κ to $\mathbb{L}$ such that for all limit ordinals $\alpha \in S$, $f(\alpha)$ is the supremum of $\{f(\beta) : \beta < \alpha\}$.

a maximum element has an e.e.e. It is easy to see that (a) T is consistent with $\mathsf{REF}(\{<\})$, but (b) T does not prove some instance of $\mathsf{REF}(\{<\})$.

Before stating the next result, let us recall a key definition:

- Given $\mathcal{L}$-structures $\mathfrak{M} = (M, <_M, \cdots)$ and $\mathfrak{N} = (N, <_N, \cdots)$, $\mathfrak{M}$ is *end extended* by $\mathfrak{N}$, written $\mathfrak{M} \subseteq_{\mathsf{e}} \mathfrak{N}$, if $\mathfrak{M}$ is a submodel of $\mathfrak{N}$ such that $x <_N y$ whenever $x \in M$ and $y \in N \backslash M$. We abbreviate "elementary end extension" by "e.e.e"; and write $\mathfrak{M} \prec_{\mathsf{e}} \mathfrak{N}$ when $\mathfrak{N}$ is a proper e.e.e. of $\mathfrak{M}$.

The following fundamental theorem establishes various model theoretic characterizations of the regularity scheme. It is fairly straightforward to use Theorem 1.2 to derive several important theorems of model theory, including: Vaught's two cardinal theorem, the countable compactness of the logic $L(Q)$ with the extra quantifier "there exist uncountably many", and the recursive enumerability of the set of valid sentences of $L(Q)$.

Theorem 1.2. (Keisler) *The following are equivalent for a complete first order theory T formulated in the language $\mathcal{L}$.*
(1) *Some model of T has an e.e.e.*
(2) *T proves $\mathsf{REG}(\mathcal{L})$.*
(3) *Every countable model of T has an e.e.e.*
(4) *Every countable model of T has an ω_1-like e.e.e.*
(5) *T has a κ-like model for some regular cardinal κ.*

Proof (outline):

(1) $\Rightarrow$ (2) : Routine.
(2) $\Rightarrow$ (3) : This uses a standard application of the Henkin-Orey omitting types theorem, see [CK, Theorem 2.2.18] for a similar proof.
(3) $\Rightarrow$ (4) : Start with any countable model of T and use part (3) ω_1-times (while taking unions at limits).
(4) $\Rightarrow$ (5) : ω_1 is a regular cardinal, assuming ZFC in the metatheory (see Example 1.1.1).
(5) $\Rightarrow$ (1) : Suppose $\mathfrak{M}$ is a κ-like model of T. If $\kappa = \omega$, then $(M, <_M)$ has order-type ω and any elementary extension of $\mathfrak{M}$ is automatically an end extension. On the other hand, if $\kappa > \omega$, then a Löwenheim-Skolem argument reveals that $\mathfrak{M}$ has a proper initial elementary submodel.

$\square$

Corollary 1.3. *Every consistent theory T extending* $\mathsf{REG}(\mathcal{L})$ *has a completion T^* in a language $\mathcal{L}^*$ extending $\mathcal{L}$ by one new unary relation symbol such that T^* is Skolemized and contains* $\mathsf{REG}(\mathcal{L}^*)$.

Proof: Given T, use Theorem 1.2 to build an ω_1-like model $\mathfrak{M}$ of T, and let $\lhd$ well-order M of order-type ω_1. Since the expanded structure $(\mathfrak{M}, \lhd)$ has definable Skolem functions, we can let $T^* = Th(\mathfrak{M}, \lhd)$.

□

Remark 1.4.

(1) In part (4) of Theorem 1.2, ω_1 cannot be replaced by ω_2. To see this, let $\mathfrak{M} := (H(\omega_1)^{\mathbf{L}}, <_{\mathbf{L}}, \in)$, where $H(\omega_1)^{\mathbf{L}}$ is the collection of hereditarily countable sets in the sense of the constructible universe $\mathbf{L}$. It is well-known that $\mathfrak{M}$ satisfies the theory specified in Example 1.1.4. Let $\mathfrak{M}_0$ be a countable model that is elementarily equivalent to $\mathfrak{M}$. Note that an e.e.e. of $\mathfrak{M}_0$ cannot be of cardinality more than ω_1 since the set of natural numbers in the sense of $\mathfrak{M}$ is countable, and $\mathfrak{M}$ satisfies the sentence "every set is finite or countable".

(2) In part (5) of Theorem 1.2, κ cannot in general be chosen as ω_2. To establish this claim, let $\mathfrak{M}$ be as in (1) above, and note that within $\mathfrak{M}$, one can define a *special Aronszajn tree in a first order manner* by repeating the classical construction internally in $\mathfrak{M}$. This shows that if ω_2 has the tree property (i.e., there is no ω_2-Aronszajn tree), then $Th(\mathfrak{M})$ cannot have an ω_2-like model[5].

(3) Certain theories T containing the regularity scheme have the property that *every* model of T has an e.e.e. The celebrated MacDowell-Specker Theorem[6] shows that PA is such a theory (see [Kay] or [KS] for an exposition). In contrast, it is known that every completion of ZFC has an ω_1-like model that does not have an e.e.e., see [Kau, Theorem 4.2] or [En-1, Theorem 1.5]. Moreover, as shown in [En-2], there is a scheme Φ in the usual language of set theory such that: (a) every completion of $\mathsf{ZFC} + \Phi$ has a θ-like model for any uncountable $\theta \geq \omega_1$, and (b) it is consistent (relative to ZFC + "there is an ω-Mahlo cardinal") that the only completions of ZFC that have an ω_2-like model are those that satisfy Φ.

[5]The work of Mitchell and Silver shows that over ZFC, the statement "$\aleph_2$ has the tree property" is equiconsistent with "there is uncountable weakly compact", see [Jec, Theorems 28.23 and 28.24].

[6]Shelah [Sh-2, Theorem 2.5] extended the MacDowell-Specker Theorem to a wider class of models. See [Schm, Sec.6] for an exposition.

(4) Rubin [Sh-2, Theorem 2.1.2] refined (2) $\Rightarrow$ (3) of Theorem 1.2 by showing that for any countable linear order $\mathbb{L}$, and any countable model $\mathfrak{M}_0$ of $\mathsf{REG}(\mathcal{L})$ with definable Skolem functions, there is an elementary extension $\mathfrak{M}_{\mathbb{L}}$ of $\mathfrak{M}_0$ such that the lattice of intermediate submodels $\{\mathfrak{M} : \mathfrak{M}_0 \preceq \mathfrak{M} \preceq \mathfrak{M}_{\mathbb{L}}\}$ (ordered under $\prec$) is isomorphic to the Dedekind completion of $\mathbb{L}$. Since there are continuum many nonisomorphic countable Dedekind complete linear orders, this shows that every countable complete Skolemized extension of $\mathsf{REG}(\mathcal{L})$ has continuum many countable nonisomorphic models.

The following is a reformulation of the classical two-cardinal theorems of Chang [CK, Theorem 7.2.7] and Jensen [Jen]. An analogue of Theorem 1.5.1 will be established in Theorem 2.10.

Theorem 1.5. *Suppose T is a consistent theory formulated in the language $\mathcal{L}$ such that T proves $\mathsf{REG}(\mathcal{L})$.*

(1) (Chang) *If κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, then T has a κ^+-like model.*
(2) (Jensen) *If κ is a singular strong limit cardinal and $\Box_\kappa$ holds, then T has a κ^+-like model.*

Remark 1.6.

(1) The converse of part (1) of Theorem 1.5 is false (this answers a question posed by Chang in his original paper [C]). To see this, suppose the universe of set theory $\mathbf{V}$ is obtained by forcing over the constructible universe $\mathbf{L}$ such that (a) $2^\omega = \omega_2$, and (b) the cardinals of $\mathbf{L}$ are the cardinals of $\mathbf{V}$ (this can be easily arranged, e.g., by adding ω_2-many Cohen reals to $\mathbf{L}$). Let T be as in Theorem 1.5, and recall that the submodel $\mathbf{L}(T)$ satisfies the continuum hypothesis. Therefore by Theorem 1.5 (a), there is a model $\mathfrak{M}$ of T such that $\mathbf{L}(T)$ thinks "$\mathfrak{M}$ is ω_2-like". But since cardinals are preserved in the passage between $\mathbf{L}$ and $\mathbf{V}$, $\mathfrak{M}$ is ω_2-like in $\mathbf{V}$. Chang's Theorem has been recently revisited in the work of Villegas-Silva [V], which employs the existence of a coarse $(\kappa, 1)$-morass (instead of $\kappa^{<\kappa} = \kappa$) to establish the conclusion of Theorem 1.5.1 for theories T formulated in languages of cardinality κ.
(2) Shelah [Sh-3] has isolated a square principle (denoted $\Box^{b^*}_\kappa$) that is *equivalent* to the two-cardinal transfer principle $(\omega_1, \omega) \rightarrow (\kappa^+, \kappa)$. See [KV] for a detailed discussion of the role of $\Box^{b^*}_\kappa$ and related square principles in model theory.

2. PRINCIPAL RESULTS

It is well-known[7] that every elementary extension of a model of PA or ZF can be "split" into a cofinal elementary extension (denoted $\preceq_{\mathsf{cof}}$) followed by an elementary end extension. The first result of this section shows that what is at work here is the regularity scheme. Note that by (1) $\Rightarrow$ (2) of Theorem 1.2, the converse of Theorem 2.1 is also true.

Theorem 2.1. (Splitting Theorem). *Suppose $\mathfrak{M} \models \mathsf{REG}(\mathcal{L})$ with $\mathfrak{M} \prec \mathfrak{N}$. Let $\mathfrak{M}^*$ be the submodel of $\mathfrak{M}$ whose universe M^* is the convex hull of M in $\mathfrak{N}$, i.e.,*

$$M^* := \{x \in N : \exists y \in M \ (x <_N y)\}.$$

Then

$$\mathfrak{M} \preceq_{\mathsf{cof}} \mathfrak{M}^* \preceq_{\mathsf{e}} \mathfrak{N}.$$

Proof: It suffices to show that $\mathfrak{M}^* \preceq \mathfrak{N}$. We use the Tarski-test by supposing that $\mathfrak{N} \models \exists x \varphi(a_1, \cdots, a_n, x)$, where each $a_i \in M^*$. Let $c \in M$ such that each $a_i < c$. Then, by invoking $\mathsf{REG}(\mathcal{L})$ in $\mathfrak{M}$ (in the guise of $\mathsf{COLL}(\mathcal{L})$), there must be some $b \in M$ such that $\mathfrak{M}$ satisfies the sentence

$$\forall z_1 < c \ \cdots \forall z_n < c \ (\exists x \varphi(z_1, \cdots, z_n, x) \to \exists x < b \ \varphi(z_1, \cdots, z_n, x)).$$

Since $\mathfrak{N}$ satisfies the same sentence, this shows that we can find $a_{n+1} \in M^*$ such that $\mathfrak{N} \models \varphi(a_1, \cdots, a_n, a_{n+1})$. $\square$

Another notion that can be fruitfully generalized from the model theory of arithmetic is the important notion of *tallness*.

- A model is *tall* iff it can be written as an e.e.e. chain with no last element.

The following theorem is well-known in the context of models of PA. Recall that $\mathfrak{M}$ is *recursively saturated*[8] if for every finite sequence $\mathbf{m}$ of elements of M, every finitely realizable type over the expanded model $(\mathfrak{M}, \mathbf{m})$ that is recursive (computable) in $\mathcal{L}$ is realized in $\mathfrak{M}$.

Theorem 2.2. *The following three conditions are equivalent for a model $\mathfrak{M}$ of $REG(\mathcal{L})$ with definable Skolem functions.*
(1) *For every c in M there is some d in M such that $\mathfrak{M} \models \tau(c) < d$ for every definable $\mathcal{L}$-term τ.*
(2) *$\mathfrak{M}$ is tall.*
(3) *$\mathfrak{M}$ has a cofinal recursively saturated elementary extension.*

[7] See [Kay] for PA and [Ke-2, Lemma C, p. 138] for ZF.

[8] Usually recursive saturation is defined for structures whose vocabulary $\mathcal{L}$ is effectively presented. Our definition here is more general and applies to structures in all countable languages.

Proof (outline):
(1) $\Rightarrow$ (2): The Splitting Theorem, coupled with (1) shows that the set of elementary initial segments of $\mathfrak{M}$ are unbounded in $\mathfrak{M}$.
(2) $\Rightarrow$ (3): The key observation here is that (2) can be used to show that if $\Sigma(x_1, \cdots, x_n)$ is any finitely satisfiable type over some expansion $(\mathfrak{M}, \mathbf{m})$ of $\mathfrak{M}$, then there is some $c \in M$ such that the "bounded" type

$$\overline{\Sigma}(x_1, \cdots, x_n) := \Sigma(x_1, \cdots, x_n) \cup \{x_i < c : 1 \leq i \leq n\}$$

is also finitely satisfiable. Since $\overline{\Sigma}$ is $\mathcal{L}$-recursive iff Σ is $\mathcal{L}$-recursive, the Splitting Theorem and a routine modification of the usual elementary chains proof of the existence of recursively saturated elementary extensions of prescribed structures together show that $\mathfrak{M}$ has a cofinal recursively saturated elementary extension.

(3) $\Rightarrow$ (1) : Suppose $\mathfrak{N}$ is a cofinal recursively saturated elementary extension of $\mathfrak{M}$. Given any $c \in M$, by recursive saturation there is a bound $d_1 \in N$ for the elements generated by c in $\mathfrak{N}$ via definable terms. Choose $d \in M$ with $d_1 < d$ and observe that by elementarity, for all definable terms $\tau(x)$,

$$\mathfrak{M} \models \tau(c) < d.$$

□

By putting Theorem 2.2 together with the fact that the notions of recursive saturation and resplendence coincide for countable models [BS, 2.3(ii)], we may conclude that *if $\mathfrak{M}$ is a countable $\mathcal{L}$-model of* $\mathsf{REG}(\mathcal{L})$*, then $\mathfrak{M}$ is tall iff $\mathfrak{M}$ has a resplendent cofinal elementary extension.* The next result (Theorem 2.3) removes the countability hypothesis from the aforementioned equivalence. The proof uses the notion of 'total resplendence', defined as follows: an $\mathcal{L}$-structure $\mathfrak{M}$ is *totally resplendent* if the following condition is satisfied:

> For any formula $\varphi(R)$ in the language $\mathcal{L} \cup \{R\}$, where R is a new n-ary predicate, whenever some elementary extension of $\mathfrak{M}$ expands to a model of $\varphi(R)$, then there is some relation symbol $S \in \mathcal{L}$ such that $\mathfrak{M}$ satisfies $\varphi(S)$.

It is well-known that the usual existence proof of resplendent models can be modified to yield a totally resplendent elementary extension $\mathfrak{M}$ of any prescribed structure $\mathfrak{M}_0$ (note that in general the language of $\mathfrak{M}$ extends the language of $\mathfrak{M}_0$).

Theorem 2.3. *Every tall model of* $\mathsf{REG}(\mathcal{L})$ *has a cofinal resplendent elementary extension.*

Proof: Let $\mathfrak{M}$ be a tall model of $\mathsf{REG}(\mathcal{L})$. Then $\mathfrak{M}$ can be written as the union of an e.e.e. chain $\langle \mathfrak{M}_\alpha : \alpha < \kappa \rangle$, where κ is some infinite cardinal. We wish to build, by simultaneous recursion on α, a chain of models $\langle \mathfrak{N}_\alpha : \alpha < \kappa \rangle$ and a chain of languages $\langle \mathcal{L}_\alpha : \alpha < \kappa \rangle$ with satisfying the following conditions for each $\alpha < \kappa$:

(1) $\mathfrak{N}_\alpha$ is an $\mathcal{L}_\alpha$-structure and $\mathfrak{M}_\alpha \prec \mathfrak{N}_\alpha \upharpoonright \mathcal{L}$;
(2) $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_\delta \subseteq \mathcal{L}_\alpha$ and $\mathfrak{N}_\delta \prec \mathfrak{N}_\alpha \upharpoonright \mathcal{L}_\delta$ whenever $\delta < \alpha$;
(3) $\mathfrak{N}_{\alpha+1} \models c < d$ for all $c \in N_\alpha$ and $d \in M_{\alpha+1} \backslash M_\alpha$;
(4) $\mathfrak{N}_\alpha$ is totally resplendent; and
(5) $N_\alpha \cap M = M_\alpha$.

Let us first verify that the existence of a chain of models satisfying the above properties establishes Theorem 2.3. Let

$$\mathfrak{N} := \bigcup_{\alpha<\kappa} (\mathfrak{N}_\alpha \upharpoonright \mathcal{L}),$$

and note that by (1) and (2), $\mathfrak{M} \prec \mathfrak{N}$, and by (3) $\mathfrak{M}$ is cofinal in $\mathfrak{N}$. Also, by coupling (2) and (4) with the Robinson consistency theorem [CK, Theorem 2.2.23], $\mathfrak{N}$ is resplendent.

$\langle \mathfrak{N}_\alpha : \alpha < \kappa \rangle$ is built by recursion on α as follows. Our effort will be focused on dealing with successor ordinals α since $\mathfrak{N}_0$ can be chosen to be a totally resplendent elementary extension of $\mathfrak{M}_0$ such that $N_0 \cap M = M_0$, and $\mathfrak{N}_\alpha$ can be defined as the union of $\langle \mathfrak{N}_\gamma : \gamma < \alpha \rangle$ for limit α. If α is a successor ordinal of the form $\theta + 1$, then we first build a model $\mathfrak{N}^*_{\theta+1}$ that satisfies the following two properties:

(i) $\mathfrak{N}^*_{\theta+1} \models c < d$ for all $c \in N_\theta$ and $d \in M_{\theta+1} \backslash M_\theta$; and
(ii) $\mathfrak{N}^*_{\theta+1}$ realizes the commuting diagram below. In the diagram, $\rightarrow$ denotes elementary embedding, and $\rightarrow_{\mathsf{e}}$ denotes elementary *end* embedding.

$$\begin{tikzcd}
 & \mathfrak{N}^*_{\theta+1} & \\
\mathfrak{N}_\theta \arrow[ur] & & \mathfrak{M}_{\theta+1} \arrow[ul] \\
 & \mathfrak{M}_\theta \arrow[ul] \arrow[ur, "\mathsf{e}"'] &
\end{tikzcd}$$

Note that $N_\theta \cap M_{\theta+1} = M_\theta$ by inductive hypothesis. Thanks to the compactness theorem, the construction of $\mathfrak{N}^*_{\theta+1}$ is reduced to verifying the consistency of the theory T_θ obtained by augmenting the union of the elementary diagrams of $\mathfrak{N}_\theta$ and $\mathfrak{M}_{\theta+1}$ with sentences of the form $(c < d)$ where $c \in N_\theta$ and $d \in M_{\theta+1} \backslash M_\theta$.

Let T_0 be a finite subset of T_θ. By using conjunctions, we may assume that T_0 is of the form

$$\{\varphi(c_1, \cdots, c_r), \psi(d_1, \cdots, d_s), c_r < d_1\},$$

where $c_1, \cdots, c_r$ is an increasing finite sequence from N_θ, $d_1, \cdots, d_s$ is an increasing finite sequence from $M_{\theta+1}$, the sentence $\varphi(c_1, \cdots, c_r)$ holds in $\mathfrak{N}_\theta$, and the sentence $\psi(d_1, \cdots, d_s)$ holds in $\mathfrak{M}_{\theta+1}$. Note that φ and ψ may contain (suppressed) parameters from M_θ. We shall verify that T_0 is interpretable in $\mathfrak{N}_\theta$. By coupling the assumption that $\psi(d_1, \cdots, d_s)$ holds in $\mathfrak{M}_{\theta+1}$ with the assumption that $\mathfrak{M}_{\theta+1}$ is an e.e.e. of $\mathfrak{M}_\theta$, it is easy to see that $\mathfrak{M}_\theta$ satisfies the sentence

$$\forall x \exists y_1 > x \cdots \exists y_s > x \ \psi(y_1, \cdots, y_s).$$

Since $\mathfrak{M}_\theta \prec \mathfrak{N}_\theta$, the above sentence is also true in $\mathfrak{N}_\theta$, and therefore there is an increasing sequence $e_1, \cdots, e_s$ in N_θ with $c_r < e_1$ such that $\mathfrak{N}_\theta$ satisfies

$$\varphi(c_1, \cdots, c_r) \wedge \psi(e_1, \cdots, e_s) \wedge c_r < e_1.$$

This concludes the verification of the consistency of T_θ. Let $\mathfrak{N}^*_{\theta+1}$ be a model of T_θ and choose $\mathfrak{N}_{\theta+1}$ to be a totally resplendent model whose reduct to $\mathcal{L}_\theta$ is an elementary extension of $\mathfrak{N}^*_\theta$. Note that it is easy to ensure that $N_{\theta+1} \cap M_{\theta+2} = M_{\theta+1}$.

□

Schlipf [Schl, Sec.3] showed that every resplendent model of PA or ZF is isomorphic to a proper initial elementary submodel of itself. Our next theorem generalizes Schlipf's result. Before stating it, we need a new definition.

- Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures with a distinguished linear order $<$, and $\mathfrak{M}$ is a submodel of $\mathfrak{N}$. $\mathfrak{N}$ is said to be a *blunt*[9] *extension* of $\mathfrak{M}$ if the supremum of M in $(N, <_N)$ exists, i.e., if $\{x \in N : \forall m \in M (m <_N x)\}$ has a first element.

Theorem 2.4. *Suppose $\mathfrak{M}$ is a resplendent model of* $\mathsf{REG}(\mathcal{L})$*. Then there is some $\mathfrak{M}_0 \prec_{\mathsf{e}} \mathfrak{M}$ such that $\mathfrak{M}_0 \cong \mathfrak{M}$. Moreover, if $\mathfrak{M}$ is a model of* $\mathsf{REF}(\mathcal{L})$*, then we can further require that $\mathfrak{M}_0 \prec_{\mathsf{e}}^{\mathsf{blunt}} \mathfrak{M}$.*

Proof: Add a new unary predicate $U(x)$ to $\mathcal{L}$, and consider the theory $T(U)$ in the expanded language obtained by augmenting $Th(\mathfrak{M}, m)_{m \in M}$ with a scheme S that expresses "the submodel determined by U is a proper initial elementary submodel". Note that by Theorem 1.2, $T(U)$ is consistent. Now augment the language of $T(U)$ with a new unary function symbol f and let $\overline{T}(U, f)$ be $T(U)$ plus a scheme that expresses "f is an isomorphism between the submodel determined by U and the whole model". We claim that $\overline{T}(U, f)$ is also consistent. To verify

[9]Note that a blunt extension need not be an end extension. We already commented (Remark 1.4.3) that no $\aleph_1$-like model of ZF has an e.e.e., but in contrast, by Corollary 2.6 of this paper, every $\aleph_1$-like model of ZF with a definable global well-ordering has a blunt elementary extension.

this, it suffices to show that every countable subtheory T_0 of $\overline{T}(U)$ that includes S has a model that is isomorphic to a proper initial segment of itself. To this end, choose a countable recursively saturated model $(\mathfrak{M}, U_M)$ of T_0, and let $\mathfrak{M}_U$ be the submodel of $\mathfrak{M}$ whose universe is U_M. Notice that

$$\mathfrak{M}_U \text{ is recursively saturated, and } \mathfrak{M}_U \prec_{\mathsf{e}} \mathfrak{M}.$$

This allows us to invoke the *pseudo-uniqueness* of countable recursively saturated models [BS, 1.4(iii)] to conclude that $\mathfrak{M}_U \cong \mathfrak{M}$. Therefore $\overline{T}(U, f)$ is consistent and $\mathfrak{M}$ has an elementary extension that is isomorphic to a proper initial segment of itself. We may now employ resplendence to conclude that $\mathfrak{M}$ is also isomorphic to a proper initial segment of itself.

Next, we verify the 'moreover' clause. If $\mathfrak{M}$ is a resplendent model of $\mathsf{REF}(\mathcal{L})$, then repeat the same argument as above, but (1) replace the scheme S with $\mathsf{REF}(\mathcal{L})$, and (2) replace $T(U)$ with $T_{\mathsf{blunt}}(U)$ obtained by adding the sentence "U is the set of predecessors of some element" to $T(U)$. Then use the fact that any countable recursively saturated model $\mathfrak{M}$ of $\mathsf{REF}(\mathcal{L})$ has an element $c \in M$ such that the submodel of $\mathfrak{M}$ whose universe is the set of predecessors of c is an elementary submodel of $\mathfrak{M}$.

□

Corollary 2.5. *Every resplendent model of* $\mathsf{REF}(\mathcal{L})$ *with definable Skolem functions has a blunt minimal*[10] *elementary end extension.*

Proof: Let $\mathfrak{M}$ be a resplendent model of $\mathsf{REF}(\mathcal{L})$ with definable Skolem functions. By the 'moreover' clause of Theorem 2.4 there is some blunt e.e.e. $\mathfrak{N}$ of $\mathfrak{M}$, with $c = \min(N \backslash M)$. Let $\mathfrak{M}(c)$ be the (elementary) submodel of $\mathfrak{N}$ generated by $M \cup \{c\}$ via the definable terms of $\mathfrak{N}$. To see that $\mathfrak{M}(c)$ is a minimal e.e.e. of $\mathfrak{M}$, suppose $\mathfrak{N}^* \preceq \mathfrak{M}(c)$, with $\mathfrak{N}^* \supsetneq M$. It suffices to show that for any $d \in N^* \backslash M$, c belongs to the submodel generated by $M \cup \{d\}$. To verify this, choose a definable term $\tau(x)$ (with suppressed parameters from M) such that $d = \tau(x)$ and then observe that for all $m \in M$, $\mathfrak{N} \models \tau(m) \neq d$ since $\mathfrak{M} \prec \mathfrak{N}$. Therefore $\mathfrak{N}$ satisfies the sentence S expressing "there is a least x such that $\tau(x) = d$". So S is also true in $\mathfrak{N}^*$, and moreover, any element witnessing the existential claim of S in $\mathfrak{N}^*$ should be equal to c by the assumption that $\mathfrak{N}^* \preceq \mathfrak{M}(c)$.

□

[10] Recall that $\mathfrak{B}$ is a *minimal* elementary extension of $\mathfrak{A}$, if $\mathfrak{A} \prec \mathfrak{B}$ and there is no $\mathfrak{C}$ with $\mathfrak{A} \preceq \mathfrak{C} \preceq \mathfrak{B}$.

Corollary 2.6. *Every tall model of* $\mathsf{REF}(\mathcal{L})$ *has a blunt elementary extension. In particular, every model of* $\mathsf{REF}(\mathcal{L})$ *of uncountable cofinality has a blunt elementary extension.*

Proof: This is an immediate consequence of putting Theorems 2.3 and 2.4 together.

□

The next result is the analogue of Theorem 1.2 for models of the reflection scheme. We should point out that the equivalence of conditions (1), (3), (5), and (6) of Theorem 2.7 was stated (but not proved) in [Schm-2, Sec.1].

Theorem 2.7. *The following are equivalent for a complete first order theory* T *formulated in the language* $\mathcal{L}$.
(1) *Some model of* T *has a blunt e.e.e.*
(2) *There are models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *of* T *such that* $\mathfrak{M}_1$ *has an e.e.e. and* $\mathfrak{M}_2$ *has a blunt elementary extension.*
(3) $T \vdash \mathsf{REF}(\mathcal{L})$.
(4) *Every countable recursively saturated model of* T *has a countable blunt recursively saturated e.e.e.*
(5) T *has an* ω_1*-like e.e.e. that continuously embeds* ω_1.
(6) T *has a* κ*-like model for some regular uncountable cardinal* κ *that continuously embeds a stationary subset of* κ.
(7) T *has a* κ*-like model for some regular uncountable cardinal* κ *that has a blunt elementary extension.*

Proof (outline):

(1) $\Rightarrow$ (2) : Trivial.
(2) $\Rightarrow$ (3) : Since $\mathfrak{M}_1$ has an e.e.e., T satisfies $\mathsf{REG}(\mathcal{L})$ by Theorem 1.2. This allows us to invoke the Splitting Theorem to show that $\mathfrak{M}_2$ has a cofinal elementary extension that has a blunt e.e.e. The rest is easy.
(3) $\Rightarrow$ (4) : This is an immediate consequence of the 'moreover' clause of Theorem 2.4, and the resplendence property of countable recursively saturated models.
(4) $\Rightarrow$ (5) : Start with a countable recursively saturated model of T, and build an elementary chain of length ω_1 by invoking the 'moreover' clause of Theorem 2.4 ω_1-times, while taking unions at limits. Note that recursive saturation is preserved at limit stages.
(5) $\Rightarrow$ (6) : Our metatheory is ZFC, one of whose theorems is that ω_1 is a regular cardinal (see Example 1.1.1).
(6) $\Rightarrow$ (7) : A Löwenheim-Skolem argument reveals that if T has a κ-like model $\mathfrak{M}$ that continuously embeds a stationary subset of κ for

some regular uncountable cardinal κ, then $\mathfrak{M}$ satisfies $\mathsf{REF}(\mathcal{L})$. Therefore Corollary 2.6 can be invoked to obtain a blunt elementary extension of $\mathfrak{M}$.
(7) $\Rightarrow$ (1) : Suppose κ is a regular uncountable cardinal, and $\mathfrak{M}$ is a κ-like model of T such that $\mathfrak{M} \prec^{\mathsf{blunt}} \mathfrak{N}$ with $c = \sup M$ in $(N, <_N)$. Note that $\mathfrak{M}$ must satisfy the regularity scheme, so by the Splitting Theorem there is a (unique) model $\mathfrak{M}^*$ such that

$$\mathfrak{M} \preceq_{\mathsf{cof}} \mathfrak{M}^* \prec_{\mathsf{e}} \mathfrak{N}.$$

This shows that $\mathfrak{M}^* \prec_{\mathsf{e}}^{\mathsf{blunt}} \mathfrak{N}$ since $c = \min(N \backslash M^*)$.

$\square$

Remark 2.8.

(1) In contrast with part (3) of Theorem 1.2, not all countable models of the reflection scheme have a blunt e.e.e. For example, no e.e.e. of the Shepherdson-Cohen minimal model of set theory can be blunt. This follows from [En-3, Theorem 3.11 and Corollary 3.12] which also shows that every consistent extension of ZF has a countable model that has no blunt e.e.e. With a little more work, one can even show that each consistent extension of ZF has a countable model that has no blunt elementary extension.

(2) A number of central results about stationary logic $L(aa)$ can be derived, via the reduction method[11], as corollaries of Theorem 2.7. In particular, the countable compactness of $L(aa)$, as well as the recursive enumerability of the set of valid sentences of $L(aa)$ can be directly derived from Theorem 2.7.

Using the strategy of the proof of Corollary 1.3 from Theorem 1.2, we can derive the following corollary from Theorem 2.7.

Corollary 2.9. *Every consistent theory T extending $\mathsf{REF}(\mathcal{L})$ has a completion T^* in a language $\mathcal{L}^* \supseteq \mathcal{L}$ extending $\mathcal{L}$ by a new binary relation symbol such that T^* is Skolemized and contains $\mathsf{REF}(\mathcal{L}^*)$.*

Our last theorem is the analogue of Chang's Theorem 1.5.1, whose proof is based on an adaptation of Chang's original proof. This result was stated without proof in [Schm-2, Sec.1].

[11]See [Eb, Sec.3.2] or [Schm-3, Sec.1] for more on the reduction method. [Eb, Theorem 3.2.2] couples the reduction method with a theorem of Hutchinson [Hu] concerning blunt elementary end extensions of models of set theory to establish the $\aleph_0$-compactness of $L(aa)$ and recursive enumerability of the set of valid sentences of $L(aa)$.

Theorem 2.10. *Suppose T is a consistent theory containing $\mathsf{REF}(\mathcal{L})$, and κ is a regular cardinal with $\kappa = \kappa^{<\kappa}$. Then T has a κ^+-like model that continuously embeds the stationary subset $\{\alpha < \kappa^+ : cf(\alpha) = \kappa\}$ of κ^+.*

Proof: By Theorem 2.7, T has an ω_1-like model $\mathfrak{M}$ that continuously embeds ω_1. Without loss of generality $\mathfrak{M}$ is of the form $(\omega_1, <_M, \cdots)$. Expand $\mathfrak{M}$ by adjoining a binary relation E_0 such that (ω_1, E_0) satisfies a weak fragment of set theory, known in the literature as VS (Vaught set theory), consisting of sentences of the following form for each positive $n \in \omega$ (where E_0 interprets E):

$$\forall x_1 \cdots \forall x_n \exists y \forall z (zEy \leftrightarrow \bigvee_{i=1}^{n} z = x_i).$$

The importance of adjoining E_0 will become clear later in the proof, but notice the important fact that the expansion $(\mathfrak{M}, E_0)$ satisfies $REF(\overline{\mathcal{L}})$, where $\overline{\mathcal{L}} = \mathcal{L} \cup \{E\}$. Since $\kappa = \kappa^{<\kappa}$, by a classical theorem of model theory [CK, Prop. 5.1.5] there is a saturated model

$$\mathfrak{A} := (A, <_A, \cdots, E_A)$$

of power κ of $Th(\mathfrak{M}, E_0)$.

The general plan of the proof is to build a chain of models $\langle \mathfrak{A}_\alpha : \alpha < \kappa^+ \rangle$ satisfying the following two conditions:

(1) $\mathfrak{A}_\alpha \cong \mathfrak{A}$ and $\mathfrak{A}_\alpha \prec_{\mathsf{e}} \mathfrak{A}$ for each $\alpha < \kappa^+$.
(2) $\mathfrak{A}_\alpha \prec_{\mathsf{e}}^{\mathsf{blunt}} \mathfrak{A}_{\alpha+1}$ for each $\alpha < \kappa^+$ of cofinality κ.

Note that the existence of such a chain immediately establishes the theorem, since the model obtained by taking the union of the chain would then be a κ^+-like model of T that continuously embeds $\{\alpha < \kappa^+ : cf(\alpha) = \kappa\}$.

Putting the resplendence property of saturated models [CK, Theorem 5.3.1 and Exercise 5.3.5] with the 'moreover' clause of Theorem 2.4 shows that $\mathfrak{A}$ has a blunt e.e.e. that is isomorphic to $\mathfrak{A}$. It is easy to see that this fact can be used ω-times to obtain a sequence of models $\langle \mathfrak{A}_n : n \in \omega \rangle$ such that $\mathfrak{A}_n \cong \mathfrak{A}$ and $\mathfrak{A}_n \prec_{\mathsf{e}}^{\mathsf{blunt}} \mathfrak{A}_{n+1}$ for each $n \in \omega$. However, the union of $\langle \mathfrak{A}_n : n \in \omega \rangle$ is a model of cofinality ω and therefore is not isomorphic to $\mathfrak{A}$. The following central claim, however, will allow us to construct the desired chain of models:

Claim ($\clubsuit$). *Suppose $\theta < \kappa^+$ and $\langle \mathfrak{A}_\alpha : \alpha < \theta \rangle$ is an e.e.e. chain such that $\mathfrak{A}_\alpha \cong \mathfrak{A}$ for each $\alpha < \theta$, and let*

$$\mathfrak{B}_\theta := \bigcup_{\alpha<\theta} \mathfrak{A}_\alpha.$$

$\mathfrak{B}_\theta$ has an e.e.e. $\mathfrak{C} \cong \mathfrak{A}$ and moreover, if $cf(\theta) = \kappa$, then $\mathfrak{B}_\theta \prec_{\mathsf{e}}^{\mathsf{blunt}} \mathfrak{C}$.

The rest of the proof will be devoted to the verification of the above claim. Notice that if $cf(\theta) = \kappa$, and $\mathfrak{B}_\theta$ is as in the statement of Claim ($\clubsuit$), then $\mathfrak{B}_\theta \equiv \mathfrak{A}$, and $\mathfrak{B}_\theta$ is a saturated model of power κ . Since elementary equivalent saturated models of the same cardinality are isomorphic [CK, Theorem 5.1.13], this shows that

$$\mathfrak{B}_\theta \cong \mathfrak{A}.$$

Coupled with the earlier observation that $\mathfrak{A}$ has a blunt e.e.e., this shows that the verification of Claim ($\clubsuit$) would be complete once we verify that $\mathfrak{B}_\theta$ *can be elementarily embedded as an initial segment of* $\mathfrak{A}$ *whenever* $cf(\theta) < \kappa$. Of course the κ^+-universality of saturated models of power κ [CK, Theorem 5.1.12] implies that there is an elementary embedding j mapping $\mathfrak{B}_\theta$ into $\mathfrak{A}$. We shall take advantage of the availability of the $\in$-like relation E to show that we can arrange the range of j to be an *initial segment* of A. It is easy to see that such an embedding j can be constructed by a back-and-forth construction of length κ, once we establish the following nontrivial sub-claim:

Claim ($\spadesuit$). *Suppose $\lambda = cf(\theta) < \kappa$ and $(\mathfrak{B}_\theta, b_\alpha)_{\alpha<\lambda} \equiv (\mathfrak{A}, a_\alpha)_{\alpha<\lambda}$. Then*

(a) $\forall c \in A\ \exists \alpha (c < a_\alpha) \Rightarrow \exists d \in B_\theta\ (\mathfrak{B}_\theta, d, b_\alpha)_{\alpha<\lambda} \equiv (\mathfrak{A}, c, a_\alpha)_{\alpha<\lambda}$.
(b) $\forall d \in B_\theta\ \exists c \in A\ (\mathfrak{B}_\theta, d, b_\alpha)_{\alpha<\lambda} \equiv (\mathfrak{A}, c, a_\alpha)_{\alpha<\lambda}$.

The proof of part (b) of Claim ($\spadesuit$) is routine and uses κ^+-universality of κ-saturated models, therefore we shall concentrate on the proof of part (a), whose proof is tricky. Suppose that we are given some $c <_A a_{\alpha_0}$ for some $\alpha_0 < \lambda$. We are looking for some $d \in \mathfrak{B}_\theta$ such that

$$(\mathfrak{B}_\theta, d, b_\alpha)_{\alpha<\lambda} \equiv (\mathfrak{A}, c, a_\alpha)_{\alpha<\lambda}.$$

Let $\Sigma(x)$ be the 1-type of c over $(\mathfrak{A}, c, a_\alpha)_{\alpha<\lambda}$. It is easy to see, using the assumption of Claim ($\spadesuit$), that $\Sigma(x)$ is finitely satisfiable in $(\mathfrak{B}_\theta, b_\alpha)_{\alpha<\lambda}$. We wish to show that indeed $\Sigma(x)$ is realized in $(\mathfrak{B}_\theta, b_\alpha)_{\alpha<\lambda}$. Let S be the set of all of finite subsets of $\Sigma(x)$, and for each $s \in \mathcal{S}$, choose a realization $r_s \in B_\theta$ of the formulas in s. Note that for every $s \in S$

$$\mathfrak{B}_\theta \models r_s < b_{\alpha_0}.$$

For each $s \in S$, we wish to find an element $m_s \in \mathfrak{B}_\theta$ that satisfies the following two conditions:

$(*)$ If $s \subseteq s' \in S$, then $\mathfrak{B}_\theta \models r_{s'}\ E\ m_s$, and

$(**)$ $\mathfrak{B}_\theta \models \forall x(xEm_s \rightarrow \bigwedge_{\varphi \in s} \varphi(x))$.

Consider the 1-type $\Gamma_s(v)$ defined as follows

$$\Gamma_s(v) := \{r_{s'}\ E\ v : s \subseteq s' \in S\} \cup \{\forall x(xEv \rightarrow \bigwedge_{\varphi \in s} \varphi(x))\}.$$

The crucial observation is that s is a finite set of formulas, and therefore there are only a finite number of parameters that are used in the formulas in s. This shows that there is an ordinal $\alpha_s < \theta$ such that all the parameters used in s come from A_{α_s} and $b_{\alpha_0} \in A_{\alpha_a}$ (the latter condition ensures that all the parameters mentioned in $\Gamma_s(v)$ come from A_{α_s}). It is easy to see, using $\mathfrak{A}_{\alpha_s} \prec \mathfrak{B}_\theta$, that $\Gamma_s(v)$ is finitely satisfiable in $\mathfrak{A}_{\alpha_s}$. Coupled with the assumption that $\mathfrak{A}_{\alpha_s}$ is a saturated model, this shows that $\Gamma_s(v)$ is realized in $\mathfrak{A}_{\alpha_s}$, and therefore also realized in $\mathfrak{B}_\theta$ by some element m_s, as desired. Our final task is to find an element $d \in \mathfrak{B}_\theta$ such that for all $s \in S$, $\mathfrak{B}_\theta \models dEm_s$. Consider the following 1-type

$$\Pi(v) := \{vEm_s : s \in S\}.$$

Recall that $m_s \in A_{\alpha_a}$ for each $s \in S$. Note that $\Pi(v)$ is finitely satisfiable in $\mathfrak{A}_{\alpha_s}$ since S is closed under finite unions. Therefore $\Pi(v)$ is realized in $\mathfrak{A}_{\alpha_s}$ by some element d. It is now easy to check that d is the desired element satisfying condition (a) of Claim (♠).

□

Remark 2.11. In light of Theorem 2.7, one might wonder whether Theorem 2.10 be strengthened by arranging a κ^+-like model in which κ^+ itself can be continuously embedded. The following example shows that such a strengthening is impossible. Consider the linear order $\mathbb{L}$ obtained by inserting a copy of the rationals $\mathbb{Q}$ between any two consecutive ordinals in ω_1. Note that $\mathbb{L}$ is a dense linear order that continuously embeds ω_1, and all the initial segments determined by elements of $\mathbb{L}$ have the order-type of the non-negative rationals $\mathbb{Q}^{\geq 0}$. This allows us to define a relation $R(x, y, z, w)$ such that for any fixed choice of x and y, $R(x,\ y, z, w)$ codes the graph of an order-preserving bijection between the initial segments of $\mathbb{L}$ determined by x and y. It is easy to see that $Th(\mathbb{L}, R)$ has no model that continuously embeds any stationary subset S of a cardinal such that at least two members of S have different cofinalities. In particular, $Th(\mathbb{L}, R)$ has no model that continuously embeds any cardinal $\kappa \geq \omega_2$.

3. OPEN QUESTIONS

Question 3.1. *Is there a countable theory containing* $\mathsf{REF}(\mathcal{L})$ *that has no* ω_2*-like model that continuously embeds* $\{\alpha < \omega_2 : cf(\alpha) = \omega\}$?

- This question is motivated by Theorem 2.10 and Remark 2.11.

Question 3.2. *Let* $\kappa \rightarrow_{\mathsf{c.u.b.}} \theta$ *abbreviate the transfer relation "every sentence with a* κ*-like model that continuously embeds a stationary subset of* κ *also has a* θ*-like model that continuously embeds a c.u.b. subset of* θ*". Is there a model of* ZFC *in which the only inaccessible cardinals* κ *such that the transfer relation* $\kappa \rightarrow_{\mathsf{c.u.b.}} \omega_2$ *holds are those cardinals* κ *that are* n*-subtle for each* $n \in \omega$*?*

- Notice that Theorem 2.7 implies that $\kappa \rightarrow_{\mathsf{c.u.b.}} \omega_1$ for every regular uncountable cardinal κ. To motivate this question, first let $\kappa \rightarrow \theta$ abbreviate "every sentence with a κ-like model also has a θ-like model". The following three results suggest that Question 3.2 might have a positive answer: (1) Schmerl and Shelah [SS] showed that $\kappa \rightarrow \theta$ holds for $\theta \geq \omega_1$, if κ is n-Mahlo for each $n \in \omega$; (2) Schmerl [Schm-1] proved that (relative to the consistency of an ω-Mahlo cardinal) there is a model of ZFC in which the only inaccessible cardinals κ such that $\kappa \rightarrow \omega_2$ holds are precisely those inaccessible cardinals κ that are n-Mahlo for each $n \in \omega$; and (3) Schmerl [Schm-2] established that $\kappa \rightarrow_{\mathsf{c.u.b.}} \theta$ holds for all $\theta \geq \omega_1$ if κ is n-subtle for each $n \in \omega$.

Question 3.3. *Can Theorem 2.10 be strengthened by (1) weakening the hypothesis* $\kappa = \kappa^{<\kappa}$ *to Shelah's square principle* $\square_\kappa^{b^*}$ *(mentioned in Remark 1.6.1), or (2) by using coarse* $(\kappa, 1)$ *morasses so as to allow* T *to have cardinality* κ*?*

- Schmerl [Schm-2] states that Jensen's proof of Theorem 1.5.2 can be modified to establish the conclusion of Theorem 2.7 for singular limit κ using $\square_\kappa$. Coarse $(\kappa, 1)$ morasses were mentioned in Remark 1.6.1.

Question 3.4. (Schmerl) *Given a language* $\mathcal{L}$ *with a distinguished linear order, is there a scheme of* $\mathcal{L}$*-formulas that axiomatizes the theory of the class of* $\mathcal{L}$*-structures that continuously embed some regular uncountable cardinal* κ*?*

- As observed by Schmerl (private communication), one can use Theorem 2.7 to show that the answer to the above question is in the positive if "scheme" is replaced by "recursively enumerable set".

Question 3.5. *Does every pure linear order (of any cardinality) with no last element have an elementary end extension?*

- This is motivated by Rosenstein's theorem mentioned in Example 1.1.5.

Question 3.6. *Let* $<$ *be the natural order on* ω^ω *and suppose* $(A, \lhd)$ *and* $(\omega^\omega, <)$ *are elementarily equivalent. Does* $(A, \lhd)$ *have a blunt e.e.e.?*

- Recall that by a classical theorem of Ehrenfeucht [Eh], $(\omega^\omega, <) \prec (\mathbf{Ord}, <)$. We do not know the answer to Question 3.6 even when A is countable.

Question 3.7. *Is it true that a cofinal elementary extension of a recursively saturated model of* $\mathsf{REF}(\mathcal{L})$ *is recursively saturated?*

- It is known that every cofinal extension of a recursively saturated model of PA or ZF is recursively saturated. Schmerl has pointed out that the answer to Question 3.7 is negative if $\mathsf{REF}(\mathcal{L})$ is replaced by $\mathsf{REG}(\mathcal{L})$: consider the saturated model $\mathfrak{M}_0 = (\omega + \mathbb{Q}\mathbb{Z}, <)$ of $Th(\omega, <)$, and the non-recursively saturated cofinal elementary extension $\mathfrak{M}_1 := (\omega + (\mathbb{Q} + 2 + \mathbb{Q})\mathbb{Z}, <)$ of $\mathfrak{M}_0$.

Question 3.8. *Does every resplendent model of* $\mathsf{REG}(\mathcal{L})$ *with definable Skolem functions have a minimal e.e.e. ?*

- This is motivated by Corollary 2.5. By a theorem of Rubin [Sh-2, Theorem 2.1.1], every countable model of $\mathsf{REG}(\mathcal{L})$ with definable Skolem functions has a minimal e.e.e. We have been able to modify Rubin's proof to show that every *saturated* model of $\mathsf{REG}(\mathcal{L})$ with definable Skolem functions has a minimal e.e.e., but we suspect that Question 3.8 has a negative answer.

Question 3.9. *Does every countable model of* $\mathsf{REG}(\mathcal{L})$ (*or* $\mathsf{REF}(\mathcal{L})$) *expand to a Skolemized model of* $\mathsf{REG}(\mathcal{L}^*)$ (*or* $\mathsf{REF}(\mathcal{L}^*)$)?

- This question is motivated by Corollaries 1.3 and 2.9.

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References

[BS] J. Barwise and J. Schlipf, *An introduction to recursively saturated and resplendent models*, **J. Sym. Logic** vol. 41 (1976), no. 2, pp. 531–536.

[C] C.C. Chang, *A note on the two cardinal problem*, **Proc. Amer. Math. Soc.** vol. 16 (1965), pp. 1148-1155.

[CK] C.C. Chang and H.J. Keisler, **Model Theory** (third edition), Studies in Logic and the Foundations of Mathematics, 73. North-Holland Publishing Co., Amsterdam, 1990.

[Ea] W. Easton, *Powers of regular cardinals*, **Ann. Math Logic** vol. 1, pp. 139-178. (1970).

[Eb] H.-D Ebbinghaus, *Extended logics: the general framework*, in **Model Theoretic Logics** (J. Barwise and S. Feferman, eds.), Springer-Verlag, New York 1985, pp. 25-76.

[Eh] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, **Fund. Math.** vol. 49 (1960/1961) pp. 129–141.

[En-1] A. Enayat, *On certain elementary extensions of models of set theory*, **Trans. Amer. Math. Soc.** vol. 283 (1984), pp. 705-715.

[En-2] ———, *Power-like models of set theory*, **J. Sym. Logic** vol. 66, pp. 1766-1782 (2001).

[En-3] ———, *Models of set theory with definable ordinals*, **Arch. Math. Logic** vol. 44, pp. 363-385 (2005).

[F] U. Felgner, *Comparisons of the axioms of local and universal choice*, **Fund. Math.** vol. 71, pp. 43-62 (1971).

[Ho] W. Hodges, **Building Models by Games**, London Mathematical Society Student Texts, 2. Cambridge University Press, Cambridge, 1985.

[Hu] J. Hutchinson, *Elementary extensions of countable models of set theory*, **J. Sym. Logic** vol. 41 (1976), no. 1, 139–145.

[Jec] T. Jech, **Set Theory**, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[Jen] R. Jensen, *The fine structure of the constructible hierarchy*, **Ann. Math. Logic** vol. 4 (1972), pp. 229–308.

[Kan] A. Kanamori, *Levy and set theory*, **Ann. Pure Appl. Logic** vol. 140 (2006), 233–252.

[Kau] M. Kaufmann, *Blunt and topless end extensions of models of set theory*, **J. Sym. Logic** vol. 48 (1984), pp. 1053-1073..

[Kay] R. Kaye, **Models of Peano Arithmetic**, Oxford University Press, Oxford, 1991.

[Ke-1] H.J. Keisler, *Some model theoretic results on ω-logic*, **Israel J. Math.** vol. 4 (1966), pp.249-261.

[Ke-2] ———, **Model theory for infinitary logic**, Studies in Logic and the Foundations of Mathematics, vol. 62. North-Holland Publishing Co., Amsterdam-London, 1971.

[KV] J. Kennedy and J. Väänänen, *Applications of regular filters and square principles in model theory*, **Quaderni di Matematica** vol. 17, (A. Andretta, ed.), Department of Mathematics, Seconda Università di Napoli, Caserta, 2005.

[KS] R. Kossak and J. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford University Press, Oxford 2007.

[MP] G. Mills and J. Paris, *Regularity in models of arithmetic*, **J. Sym. Logic** vol. 49 (1984), pp. 272-280.

[M] S. Mohsenipour, **Elementary End Extensions in Model Theory and Set Theory**, Doctoral Dissertation (2005), IPM, Tehran, Iran.

[R] J.G. Rosenstein, **Linear Orderings,** Academic Press, New York-London, 1982.

[Schl] J. Schlipf, *Toward model theory through recursive saturation,* **J. Sym. Logic** vol. 43 (1978), pp. 183–206.

[Schm-1] J. Schmerl, *Generalizing special Aronszajn trees,* **J. Sym. Logic** vol. 39 (1974), pp. 732–740.

[Schm-2] ——, *On κ-like structures which embed stationary and closed unbounded subsets,* **Ann. Math. Logic** vol. 10 (1976), pp. 289–314.

[Schm-3] ——,*Transfer theorems and their applications to logics*, in **Model Theoretic Logics** (J. Barwise and S. Feferman, eds.), Springer-Verlag, New York 1985, pp. 177-209.

[SS] J. Schmerl and S. Shelah, *On power-like models for hyperinaccessible cardinals,* **J. Sym. Logic** vol. 37 (1972), pp. 531–537.

[Sh-1] S. Shelah, *Generalized quantifiers and compact logic,* **Trans. Amer. Math. Soc.** vol. 204 (1975), pp. 342–364.

[Sh-2] ——, *End extensions and the number of countable models,* **J. Sym. Logic,** vol. 43 (1978), pp. 550-562.

[Sh-3] ——, *"Gap 1" two-cardinal principles and the omitting types theorem* for $\mathcal{L}(Q)$, **Israel J. Math.** vol. 65 (1989), pp. 133–152.

[SV] S. Shelah and J. Väanänen, *Recursive logic frames*, **MLQ Math. Log. Q.** vol. 52 (2006), pp. 151–164.

[Si] S.G. Simpson, **Subsystems of Second Order Arithmetic,** Springer-Verlag, Berlin, 1999.

[V] L. M. Villegas-Silva, *A gap 1 cardinal transfer theorem*, **MLQ Math. Log. Q**. vol. 2 (2006), pp. 340–350.

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