On almost precipitous ideals.

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Abstract

We answer questions concerning an existence of almost precipitous ideals raised in [5]. It is shown that every successor of a regular cardinal can carry an almost precipitous ideal in a generic extension of L. In $L[\mu]$ every regular cardinal which is less than the measurable carries an almost precipitous non-precipitous ideal. Also, results of [4] are generalized- thus assumptions on precipitousness are replaced by those on ∞-semi precipitousness.

1 On semi precipitous and almost precipitous ideals

Definition 1.1 Let κ be a regular uncountable cardinal, τ a ordinal and I a κ-complete ideal over κ. We call I τ-*almost precipitous* iff every generic ultrapower of I is wellfounded up to the image of τ.

Clearly, any such I is τ-almost precipitous for each $\tau < \kappa$. Also, note if $\tau \geq (2^\kappa)^+$ and I is τ-almost precipitous, then I is precipitous.

Definition 1.2 Let κ be a regular uncountable cardinal. We call κ *almost precipitous* iff for each $\tau < (2^\kappa)^+$ there is τ-almost precipitous ideal over κ.

It was shown in [5] that $\aleph_1$ is almost precipitous once there is an $\aleph_1$-Erdős cardinal. The following questions were raised in [5]:
1. Is $\aleph_1$-Erdős cardinal needed?
2. Can cardinals above $\aleph_1$ be almost precipitous without a measurable cardinal in an inner model?

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We will construct two generic extensions of L such $\aleph_1$ will be almost precipitous in the first and $\aleph_2$ in the second.

Some of the ideas of Donder and Leviski [1] will be crucial here.

Definition 1.3 (Donder- Levinski [1]) Let κ be a cardinal and τ be a limit ordinal of cofinality above κ or $\tau = On$. κ is called *τ-semi-precipitous* iff there exists a forcing notion P such the following is forced by the weakest condition:

there exists an elementary embedding $j : V_\tau \rightarrow M$ such that

1. $crit(j) = \kappa$
2. M is transitive.

κ is called $< \lambda$- semi-precipitous iff it is τ-semi-precipitous for every limit ordinal $\tau < \lambda$ of cofinality above κ.
κ is called a semi-precipitous iff it is τ-semi-precipitous for every limit ordinal τ of cofinality above κ.
κ is called ∞-semi-precipitous iff it is On-semi-precipitous.
Note if κ is a semi-precipitous, then it is not necessarily ∞-semi-precipitous, since by Donder and Levinski [1] semi-precipitous cardinals are compatible with $V = L$, and ∞-semi-precipitous cardinals imply an inner model with a measurable.
Let us call

$$F = \{X \subseteq \kappa \mid 0_P \Vdash \kappa \in j(X)\}$$

a τ-semi-precipitous filter. Note that such F is a normal filter over κ.

Lemma 1.4 *Let F be a τ-almost precipitous normal filter over κ for some ordinal τ above κ. Then F is τ-semi-precipitous.*

Proof. Force with F^+. Let $i : V \rightarrow N = V \cap {}^\kappa V/G$ be the corresponding generic embedding. Set $j = i \restriction \tau$. Then $j : V_\tau \rightarrow (V_{i(\tau)})^N$. Set $M = (V_{i(\tau)})^N$. We claim that M is well founded. Suppose otherwise. Then there is a sequence $\langle g_n \mid n < \omega \rangle$ of functions such that

1. $g_n \in V$
2. $g_n : \kappa \rightarrow V_\tau$
3. $\{\alpha < \kappa \mid g_{n+1}(\alpha) \in g_n(\alpha)\} \in G$

Replace each g_n by a function $f_n : \kappa \to \tau$. Thus, set $f_n(\alpha) = rank(g_n(\alpha))$. Clearly, still we have

$$\{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in G.$$

But this means that N is not well-founded below the image of τ. Contradiction.
$\square$

Note that the opposite direction does not necessary hold. Thus for $\tau \geq (2^\kappa)^+$, τ-almost precipitousness implies precipitousness and hence a measurable cardinal in an inner model. By Donder and Levinski [1], it is possible to have semi-precipitous cardinals in L.

The following is an analog of a game that was used in [5] with connection to almost precipitous ideals.

Definition 1.5 (The game $\mathcal{G}_\tau(F)$)

Let F be a normal filter on κ and let $\tau > \kappa$ be an ordinal.
The game $\mathcal{G}_\tau(F)$ is defined as follows:
Player **I** starts by picking a set A_0 in F^+. Player **II** chooses a function $f_1 : A_0 \to \tau$ and either a partition $\langle B_i | i < \xi < \kappa\rangle$ of A_0 into less than κ many pieces or a sequence $\langle B_\alpha | \alpha < \kappa\rangle$ of disjoint subsets of κ so that

$$\triangledown_{\alpha<\kappa} B_\alpha \supseteq A_0.$$

The first player then supposed to respond by picking an ordinal α_2 and a set $A_2 \in F^+$ which is a subset of A_0 and of one of B_i's or B_α's.
At the next stage the second player supplies again a function $f_3 : A_2 \to \tau$ and either a partition $\langle B_i | i < \xi < \kappa\rangle$ of A_2 into less than κ many pieces or a sequence $\langle B_\alpha | \alpha < \kappa\rangle$ of disjoint subsets of κ so that

$$\triangledown_{\alpha<\kappa} B_\alpha \supseteq A_2.$$

The first player then supposed to respond by picking a stationary set A_4 which is a subset of A_2 and of one of B_i's or B_α's on which everywhere f_1 is either above f_3 or equal f_3 or below f_3. In addition he picks an ordinal α_4 such that

$$\alpha_2, \alpha_4 \text{ respect the order of } f_1 \restriction A_4, f_3 \restriction A_4,$$

i.e.

$$\alpha_2 < \alpha_4 \text{ iff } f_1 \restriction A_4 < f_3 \restriction A_4,$$

$$\alpha_2 > \alpha_4 \text{ iff } f_1 \restriction A_4 > f_3 \restriction A_4$$

and

$$\alpha_2 = \alpha_4 \text{ iff } f_1 \restriction A_4 = f_3 \restriction A_4$$

. Intuitively, α_{2n} pretends to represent f_{2n-1} in a generic ultrapower.
Continue further in the same fashion.
Player **I** wins if the game continues infinitely many moves. Otherwise Player **II** wins.

Clearly it is a determined game.

The following lemma is analogous to [5] (Lemma 3).

Lemma 1.6 *Suppose that λ is a κ-Erdős cardinal. Then for each ordinal $\tau < \lambda$ Player **II** does not have a winning strategy in the game $\mathcal{G}_\tau(Cub_\kappa)$.*

Proof. Suppose otherwise. Let σ be a strategy of two. Find a set $X \subset \lambda$ of cardinality κ such that σ does not depend on ordinals picked by Player **I** from X. In order to get such X let us consider a structure

$$\mathfrak{A} = \langle H(\lambda), \in, \lambda, \kappa, \mathcal{P}(\kappa), F, \mathcal{G}_\tau(F), \sigma\rangle.$$

Let X be a set of κ indiscernibles for $\mathfrak{A}$.

Pick now an elementary submodel M of $H(\chi)$ for $\chi > \lambda$ big enough of cardinality less than κ, with $\sigma, X \in M$ and such that $M \cap \kappa \in On$. Let $\alpha = M \cap \kappa$. Let us produce an infinite play in which the second player uses σ. This will give us the desired contradiction.
Consider the set $S = \{f(\alpha) | f \in M, f$ is a partial function from κ to $\tau\}$. Obviously, S is countable. Hence we can fix an order preserving function $\pi : S \to X$.
Let one start with $A_0 = \kappa$. Consider $\sigma(A_0)$. Clearly, $\sigma(A_0) \in M$. It consists of a function $f_1 : A_0 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \kappa\rangle$ of disjoint subsets of κ so that

$$\bigtriangledown_{\xi<\kappa} B_\xi \supseteq A_0.$$

Now, $\alpha \in A_0$, hence there is $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_0 \cap B_{\xi^*} \in M$ and $\alpha \in A_0 \cap B_{\xi^*}$. Let $A_2 = A_0 \cap B_{\xi^*}$. Note that $A_2 \cap C \neq \emptyset$, for every closed unbounded subset C of κ which belongs to M, since α is in both A_2 and C.
Pick $\alpha_2 = \pi(f_1(\alpha))$.
Consider now the answer of two which plays according to σ. It does not depend on α_2, hence it is in M. Let it be a function $f_3 : A_2 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \kappa\rangle$ of disjoint subsets of κ so that

$$\bigtriangledown_{\xi<\kappa} B_\xi \supseteq A_2.$$

As above find $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_2 \cap B_{\xi^*} \in M$ and $\alpha \in A_2 \cap B_{\xi^*}$. Let $A_2' = A_2 \cap B_{\xi^*}$. Split it into three sets $C_<, C_=, C_>$ such that

$$C_< = \{\nu \in A_2' | f_3(\nu) < f_1(\nu)\},$$

$$C_= = \{\nu \in A_2' | f_3(\nu) = f_1(\nu)\},$$

$$C_> = \{\nu \in A_2' | f_3(\nu) > f_1(\nu)\}.$$

Clearly, α belongs to only one of them, say to $C_<$. Set then $A_4 = C_<$. Then, clearly, $A_4 \in M$, it is stationary and $f_3(\alpha) < f_1(\alpha)$. Set $\alpha_4 = \pi(f_3(\alpha))$.

Continue further in the same fashion.

□

It follows that the first player has a winning strategy.

The next game was introduced by Donder and Levinski in [1].

Definition 1.7 A set R is called κ-*plain* iff

1. $R \neq \emptyset$,
2. R consists of normal filters over κ,
3. for all $F \in R$ and $A \in F^+$, $F + A \in R$.

Definition 1.8 (The game $H_R(F, \tau)$)

Let R be a κ-plain, $F \in R$ be a normal filter on κ and let $\tau > \kappa$ be an ordinal.
The game $H_R(F\tau)$ is defined as follows. Set $F_0 = F$. Let $1 \leq i < \omega$. Player **I** plays at stage i a pair (A_i, f_i), where $A_i \subseteq \kappa$ and $f_i : \kappa \to \tau$. Player **II** answers by a pair (F_i, γ_i), where $F_i \in R$ and γ_i is an ordinal. The rules are as follows:

1. For $0 \leq i < \omega, A_{i+1} \in (F_i)^+$
2. For $0 \leq i < \omega, F_{i+1} \supseteq F_i[A_{i+1}]$

Player **II** wins iff for all $1 \leq i, k \leq n < \omega : (f_i <_{F_n} f_k) \to (\gamma_i < \gamma_k)$

Donder and Levinski [1] showed that an existence of a winning strategy for Player **II** in the game $H_R(F, \lambda)$ for some R, F is equivalent to κ being τ- semi precipitous.

Next two lemmas deal with connections between winning strategies for the games $\mathcal{G}_\tau(F)$ and $H_R(F, \tau)$.

Lemma 1.9 *Suppose that Player **II** has a winning strategy in the game $H_R(F,\tau)$, for some κ-plain R, a normal filter $F \in R$ over κ and an ordinal τ. Then Player **I** has a winning strategy in the game $\mathcal{G}_\tau(F)$.*

Proof. Let σ be a winning strategy of Player **II** in $H_R(F,\tau)$. Define a winning strategy δ for Player **I** in the game $\mathcal{G}_\tau(F)$. Let the first move according to δ be κ. Suppose that Player **II** responds by a function $f_1 : \kappa \to \tau$ and a partition $\mathcal{B}_1$ of κ to less then κ many subsets or a sequence $\mathcal{B}_1 = \langle B_\alpha \mid \alpha < \kappa\rangle$ of κ many subsets such that $\nabla_{\alpha<\kappa} B_\alpha \supseteq \kappa$. Turn to the strategy σ. Let $\sigma(\kappa, f_1) = (F_1, \gamma_1)$, for some $F_1 \supseteq F, F_1 \in R$ and an ordinal γ_1. Now we let Player **I** pick $A_1 \in (F_1)^+$ such that there is a set $B \in \mathcal{B}_1$ with $A_1 \subseteq B$ (he can always choose such an A_1 because F_1 is normal and $\bigtriangledown_{\alpha<\kappa} B_\alpha \in (F_1)^+$) and let the respond according to δ be (A_1, γ_1). Player **II** will now choose a function $f_2 : A_1 \to \lambda$ and a partition $\mathcal{B}_2$ of A_1 or a sequence $\mathcal{B}_2 = \langle B_\alpha \mid \alpha < \kappa\rangle$, $\bigtriangledown_{\alpha<\kappa} B_\alpha \supseteq A_1$. Back in $H_R(F,\tau)$, we consider the answer according σ of Player **II** to (A_1, f_2), i.e. $\sigma((\kappa, f_1), (A_1, f_2)) = (F_2, \gamma_2)$. Choose $A_2 \in (F_2)^+$ such that there is a set $B \in \mathcal{B}_2$ with $A_2 \subseteq B$ (it is always possible to find such A_2 because F_2 is normal and $\bigtriangledown_{\alpha<\kappa} B_\alpha \in (F_2)^+$) on which either $f_1 < f_2$ or $f_1 > f_2$ or $f_1 = f_2$. Let the respond according to δ be (A_2, γ_2).
Continue in a similar fashion. The play will continue infinitely many moves. Hence Player **I** will always win once using the strategy δ.
□

Lemma 1.10 *Suppose that Player **I** has a winning strategy in the game $\mathcal{G}_\tau(F)$, for a normal filter F over κ and an ordinal τ. Then Player **II** has a winning strategy in the game $H_R(D,\tau)$ for some κ-plain R and $D \in R$.*

Proof. Let σ be a winning strategy of Player **I** in $\mathcal{G}_\tau(F)$. Set

$$J = \{X \subseteq \kappa \mid X \text{ and any of its subsets are never used by } \sigma\},$$

and for every finite play $t = \langle t_1, ..., t_{2n}\rangle$

$$J_t = \{X \subseteq \kappa \mid X \text{ and any of its subsets are never used by } \sigma \text{ in the continuation of } t\}.$$

It is not hard to see that such J and J_t's are normal ideals over κ. Denote by D and D_t the corresponding dual filters.
Pick R to be a κ-plain which includes D and all D_t's.

Define a winning strategy δ for Player **II** in the game $H_R(D, \tau)$. Let (A_1, g_1) be the first move in $H_R(D, \tau)$. Then $A_1 \in D^+$. Hence σ picks A_1 in a certain play t as a move of Player **I** in the game $\mathcal{G}_\tau(F)$. Continue this play, and let Player **II** responde by a trivial partition of A_1 consisting of A_1 itself and by function g_1 restricted to A_1. Let (B_1, γ_1) be the respond of Player **I** according to σ. Set $t_1 = t^\frown(\{A_1\}, g_1)$. Then $B_1 \in D_{t_1}$. Now we set the respond of Player **II** according to δ to be (D_{t_1}, γ_1).
Continue in similar fashion.

□

Theorem 1.11 *Suppose that λ is a κ-Erdős cardinal, then κ is τ-semi precipitous for every $\tau < \lambda$.*

Proof. It follows by Lemmas 1.6,1.10.

□

Combining the above with Theorem 17 of [5], we obtain the following:

Theorem 1.12 *Assume that $2^{\aleph_1} = \aleph_2$ and $\|f\| = \omega_2$, for some $f : \omega_1 \to \omega_1$. Let $\tau < \aleph_3$. If there is a τ-semi-precipitous filter over $\aleph_1$, then there is a normal τ-almost precipitous filter over $\aleph_1$ as well.*

By Donder and Levinski [1], $0^\#$ implies that the first indiscernible c_0 for L is in L τ-semi-precipitous for each τ. They showed [1](Theorem 7) that the property "κ is τ-semi-precipitous " relativizes down to L. Also it is preserved under κ-c.c. forcings of cardinality $\leq \kappa$ ([1](Theorem 8)).

Now combine this with 1.12. We obtain the following:

Theorem 1.13 *Suppose that κ is $< \kappa^{++}$-semi-precipitous cardinal in L. Let G be a generic subset of the Levy Collapse $Col(\omega, < \kappa)$. Then for each $\tau < \kappa^{++}$, κ carries a τ-almost precipitous normal ideal in $L[G]$.*

Proof. In order to apply 1.12, we need to check that there is $f : \omega_1 \to \omega_1$ with $\|f\| = \omega_2$. Suppose otherwise. Then by Donder and Koepke [2] (Theorem 5.1) we will have wCC(ω_1) (the weak Chang Conjecture for ω_1). Again by Donder and Koepke [2] (Theorem D), then $(\aleph_2)^{L[G]}$ will be almost $< (\aleph_1)^L$-Erdös in L. But note that $(\aleph_2)^{L[G]} = (\kappa^+)^L$ and in L, $2^\kappa = \kappa^+$. Hence, in L, we must have $2^\kappa \to (\omega)^2_\kappa$, as a particular case of 2^κ being almost $< \aleph_1$-Erdös. But $2^\kappa \not\to (3)^2_\kappa$. Contradiction.

□

Corollary 1.14 *The following are equivalent:*

1. *Con(there exists an almost precipitous cardinal),*
2. *Con(there exists an almost precipitous cardinal with normal ideals witnessing its almost precipitousness),*
3. *Con(there exists $< \kappa^{++}$-semi-precipitous cardinal κ).*

In particular the strength of existence of an almost precipitous cardinal is below $0^{\#}$.

2 An almost precipitous ideal on ω_2

In this section we will construct a model with $\aleph_2$ being almost precipitous.

The initial assumption will be an existence of a Mahlo cardinal κ which carries a $(2^\kappa)^+$-semi precipitous normal filter F with $\{\tau < \kappa \mid \tau \text{ is a regular cardinal }\} \in F$.
Again by Donder and Levinski [1] this assumption is compatible with L. Thus, under $0^\sharp$, the first indiscernible will be like this in L.

Assume $V = L$.

Let $\langle P_i, \underset{\sim}{Q}_j \mid i \leq \kappa, j < \kappa\rangle$ be Revised Countable Support iteration (see [9]) so that for each $\alpha < \kappa$, if α is an inaccessible cardinal (in V),then $\underset{\sim}{Q}_\alpha$ is $Col(\omega_1, \alpha)$ which turns it to $\aleph_1$ and $\underset{\sim}{Q}_{\alpha+1}$ will be the Namba forcing which changes the cofinality of α^+ (which is now $\aleph_2$) to ω. In all other cases let $\underset{\sim}{Q}_\alpha$ be the trivial forcing.

By [9](Chapter 9), the forcing P_κ turns κ into $\aleph_2$, preserves $\aleph_1$, does not add reals and satisfies the κ -c.c. Let G be a generic subset of P_κ.

By Donder and Levinsky ([1]), a κ-c.c. forcing preserves semi precipitousness of F. Hence F is $\kappa^{++} = \aleph_4$-semi precipitous in $L[G]$. In addition,

$$\{\tau < \kappa \mid \mathrm{cof}(\tau) = \omega_1\} \in F$$

and

$$\{\tau < \kappa \mid \mathrm{cof}((\tau^+)^V) = \omega\} \in F.$$

Now, there is a forcing Q in $L[G]$ so that in $L[G]^Q$ we have a generic embedding $j : L_{\kappa^{++}}[G] \to M$ such that M is a transitive and $\kappa \in j(A)$ for every $A \in F$. By elementarity, then M is of the form $L_\lambda[G^*]$, for some $\lambda > \kappa^{++}$, and $G^* \subseteq j(P_\kappa)$ which is L_λ-generic.
Note that Q_κ collapses κ to $(\aleph_1)^M$ because it was an inaccessible cardinal, and at the very next stage its successor changes the cofinality to ω. That means that there is a function

$H \in L_{\kappa^{++}}[G]$ such that $j(H)(\kappa) : \omega \to (\aleph_3)^{L[G]}$ is an increasing and unbounded in $(\kappa^+)^L = (\aleph_3)^{L[G]}$ function.
We will use such H as a replacement of the corresponding function of [4]. Together with the fact that in the model $L[G]$ we have a filter on $\aleph_2$ which is $\aleph_4$ semi precipitous this will allow us to construct τ- almost precipitous filter on $\aleph_2$, for every $\tau < \aleph_4$.

2.1 The construction

Fix $\tau < \kappa^{++}$.

By [1], we can assume that $Q = Col(\omega, \tau^\kappa)$ Denote by $\mathcal{B}$ the complete Boolean algebra RO(Q). Further by $\leq$ we will mean the order of $\mathcal{B}$.

For each $p \in \mathcal{B}$ set

$$F_p = \{X \subseteq \kappa | p \Vdash \kappa \in \underset{\sim}{j}(X)\}$$

We will use the following easy lemma:

Lemma 2.1 *1. $p \leq q \to F_p \supseteq F_q$*

2. $X \in (F_p)^+$ iff there is a $q \leq p, q \Vdash \kappa \in j(X)$

3. Let $X \in (F_p)^+$, then for some $q \leq p$, $F_q = F_p + X$

Proof. (1) and (2) are trivial. Let us prove (3).
Suppose that $X \in (F_p)^+$. Set $q = ||\kappa \in \underset{\sim}{j}(X)||^{\mathcal{B}} \wedge p$. We claim that $F_q = F_p + X$. The inclusion $F_q \supseteq F_p + X$ is trivial. Let us show that $F_p + X \supseteq F_q$. Suppose not, then there are $Y \in (F_p)^+, Y \subseteq X$ and $Z \in F_q$ such that $Y \cap Z = \emptyset$. But $Y \in (F_p)^+$, so we can find $s \leq p$ such that $s \Vdash \kappa \in \underset{\sim}{j}(Y)$. Now, $s \leq p$ and $s \Vdash \kappa \in \underset{\sim}{j}(X)$, since $Y \subseteq X$. Hence, $s \leq q$. But then

$$s \Vdash \kappa \in \underset{\sim}{j}(Y), \kappa \in \underset{\sim}{j}(Z), \underset{\sim}{j}(Z \cap Y) = \emptyset.$$

Contradiction.
□

Define $\{A_{n\alpha} \mid \alpha < \kappa^+, n < \omega\}$ as in [4]:

$$A_{n\alpha} = \{\eta > \kappa \mid \exists p \in \mathcal{B} \quad p \Vdash \underset{\sim}{H}(\eta)(n) = \underset{\sim}{h}_\alpha(\eta)\},$$

where $\langle h_\alpha \mid \alpha < \kappa^+\rangle$ is a sequence of κ^+ canonical functions from κ to κ (in $V^{\mathcal{B}}$). Note that here H is only cofinal and not onto, as in [4].

The following lemmas were proved in [4] and hold without changes in the present context:

Lemma 2.2 *For every $n < \omega$ there is an ordinal $\alpha < \kappa^+$ so that $A_{n\alpha} \in (F_{1_{\mathcal{B}}})^+$*

Lemma 2.3 *For every $\alpha < \kappa^+$ and $p \in \mathcal{B}$ there is $n < \omega$ and $\alpha < \beta < \kappa^+$ so that $A_{n\beta} \in (F_p)^+$*

Lemma 2.4 *Let $n < \omega$ and $p \in \mathcal{B}$. Then the set:*

$$\{A_{n\alpha} \mid \alpha < \kappa^+ \text{ and } A_{n\alpha} \in (F_p)^+\}$$

is a maximal antichain in $(F_p)^+$.

The following is an analog of a lemma due Assaf Rinot in [4], 3.5.

Lemma 2.5 *Let $\mathcal{D}$ be a family of κ^+ dense subsets of $\mathcal{B}$, there exists a sequence $\langle p_\alpha | \alpha < \kappa^+\rangle$ such that for all $Z \in (F_{1_{\mathcal{B}}})^+$,$p' \in Q$ and $n < \omega$ if*

$$Z_{n,p'} = \{\alpha < \kappa^+ | A_{n\alpha} \cap Z \in (F_{p'})^+\}$$

has cardinality κ^+ then :

1. *For any $p \in \mathcal{B}$ there exists $\alpha \in Z_{n,p'}$ with $p \geq p_\alpha$.*
2. *For any $D \in \mathcal{D}$ there exists $\alpha \in Z_{n,p'}$ with $p_\alpha \Vdash \kappa \in j(A_{n\alpha} \cap Z)$,$p_\alpha \leq p'$ and $p_\alpha \in D$.*

Proof. Let $\{S_i \mid i < \kappa^+\} \subseteq [\kappa^+]^{\kappa^+}$ be some partition of κ^+, $\{D_\alpha \mid \alpha < \kappa^+\}$ an enumeration of $\mathcal{D}$,$\{q_\alpha \mid \alpha < \kappa^+\}$ an enumeration of Q and let $\lhd$ be a well ordering of $\kappa^+ \cup \kappa^+ \times \kappa^+$ of order type κ^+. Now, fix a surjective function $\varphi : \kappa^+ \to \{(Z, n, p) \in ((F_{1_{\mathcal{B}}})^+, \omega, Q) \mid |Z_{n,p}| = \kappa^+\}$. We would like to define a function $\psi : \kappa^+ \to \kappa^+ \cup \kappa^+ \times \kappa^+$ and the sequence $\langle p_\alpha | \alpha < \kappa^+\rangle$. For that, we now define two sequences of ordinals $\{L_\alpha \mid \alpha < \kappa^+\}, \{R_\alpha \mid \alpha < \kappa^+\}$ and the values of ψ and the sequence on the intervals $[L_\alpha, R_\alpha]$ by recursion on $\alpha < \kappa^+$. For $\alpha = 0$ we set $L_0 = R_0 = 0$,$\psi(0) = 0$ and $p_0 = q_0$.
Now, suppose that $\{L_\beta, R_\beta \mid \beta < \alpha\}$ and $\psi \restriction \bigcup_{\beta<\alpha} [L_\beta, R_\beta]$ were defined.Take i to be the unique index such that $\alpha \in S_i$.Let $(Z, n, p) = \varphi(i)$ and set $L_\alpha = \min(\kappa^+ \setminus \cup_{\beta<\alpha} [L_\beta, R_\beta])$, $R_\alpha = \min(Z_{n,p} \setminus L_\alpha)$.
Now, for each $\beta \in [L_\alpha, R_\alpha]$ we set $\psi(\beta) = t$,where:

$$t = min_\lhd(\kappa^+ \cup \{i\} \times \kappa^+) \setminus \psi''(Z_{n,p} \cap L_\alpha).$$

If $t \in \kappa^+$ then we set $p_\beta = q_t$ for each $\beta \in [L_\alpha, R_\alpha]$.Otherwise , $t = (i, \delta)$ for some $\delta < \kappa^+$ and because $A_{nR_\alpha} \cap Z \in F_p^+$ and D_δ is dense we can find some $q \in D_\delta$, $q \leq p$, $q \Vdash \kappa \in \underset{\sim}{j}(A_{nR_\alpha} \cap Z)$

and set $p_\beta = q$ for each $\beta \in [L_\alpha, R_\alpha]$.This completes the construction.
Now, we would like to check that the construction works. Fix $Z \in F^+_{1_\mathcal{B}}$ $p \in Q$ and $n < \omega$ so that $|Z_{n,p}| = \kappa^+$.Let $i < \kappa^+$ be such that $\varphi(i) = Z_{n,p}$ and notice that the construction insures that $\psi'' Z_n = \kappa^+ \cup \{i\} \times \kappa^+$.
(1) Let $p' \in \mathcal{B}$:There exists a $t < \kappa^+$ so that $q_t \leq p'$.Let $\alpha \in Z_n$ be such that $\psi(\alpha) = t$, so $p_\alpha = q_t \leq p'$.
(2) Let $D \in \mathcal{D}$. There exist $\delta < \kappa^+$ and $\alpha \in Z_{n,p}$ such that $D_\delta = D$ and $\psi(\alpha) = (i, \delta)$.Then, by the construction we have that $p_\alpha \in D_\delta$, $p_\alpha \Vdash \kappa \in j(A_{n\alpha} \cap Z)$ and $p_\alpha \leq p$.□

Define $\mathcal{D} = \{D_f \mid f \in (\tau^\kappa)^V\}$, where

$$D_f = \{p \in \mathcal{B} \mid \exists \gamma \in On \quad p \Vdash j(\check{f})(\kappa) = \check{\gamma}\}$$

and let $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ be as in lemma 2.5.

We turn now to the construction of filters which will be similar to those of [4]).

Start with $n = 0$. Let $\alpha < \kappa^+$. Consider three cases:

Case I: If $|\{\xi < \kappa^+ \mid A_{0\xi} \in (F_{1_\mathcal{B}})^+\}| = \kappa^+$ and $p_\alpha \Vdash \kappa \in j(A_{0\alpha})$ then we define $q_{<\alpha>} = p_\alpha$ and extend $F_{1_\mathcal{B}}$ to $F_{q_{<\alpha>}}$.

Case II: If I fails but $A_{0\alpha} \in (F_{1_\mathcal{B}})^+$ then we define $q_{<\alpha>} = \|\kappa \in j(A_{0\alpha})\|_\mathcal{B}$ and extend our filter to $F_{q_{<\alpha>}}$.

Case III: If $A_{0\alpha} \in \check{F}_\emptyset$ (the dual ideal of $F_{1_\mathcal{B}}$) then $q_{<\alpha>}$ is not defined.

Notice that by Lemma 2.2, there exists some $\alpha < \kappa^+$ with $A_{0\alpha} \in (F_{1_\mathcal{B}})^+$, thus $\{\alpha < \kappa^+ \mid F_{q_{\langle\alpha\rangle}}$ is defined $\}$ is non-empty.

Definition 2.6 Set $F_0 = \bigcap \{F_{q_{\langle\alpha\rangle}} \mid \alpha < \kappa^+, F_{q_{\langle\alpha\rangle}}$ is defined $\}$, and denote the corresponding dual ideals by $I_{q_{\langle\alpha\rangle}}$ and I_0.

Clearly, $I_0 = \bigcap \{I_{q_{\langle\alpha\rangle}} \mid \alpha < \kappa^+, I_{q_{\langle\alpha\rangle}}$ is defined $\}$. Also, $F_0 \supseteq F_{1_\mathcal{B}}$ and $I_0 \supseteq \check{F}_\emptyset$, since each $F_{q_{\langle\alpha\rangle}} \supseteq F_{1_\mathcal{B}}$ and $I_{q_{\langle\alpha\rangle}} \supseteq \check{F}_\emptyset$. Note that F_0 is a κ complete, normal and proper filter since it is an intersection of such filters and also I_0 is .

We now describe the successor step of the construction, i.e., $m = n + 1$.

Let $\sigma : m \to \kappa^+$ be a function with F_{p_σ} defined and $\alpha < \kappa^+$. There are three cases:

Case I: If $|\{\xi < \kappa^+ \mid A_{m\xi} \in F^+_{p_\sigma}\}| = \kappa^+$, $p_\alpha \leq p_\sigma$ and $p_\alpha \Vdash \kappa \in j(A_{m\alpha})$, then we define $q_{\sigma^\frown\alpha} = p_\alpha$ and extend F_{p_σ} to $F_{q_{\sigma^\frown\alpha}}$.

Case II: If Case I fails, but $A_{m\alpha} \in (F_\sigma)^+$, then let $q_{\sigma^\frown\alpha} = \|\kappa \in j(A_{m\alpha})\|_{\mathcal{B}} \wedge q_\sigma$, and extend F_{q_σ} to $F_{q_{\sigma^\frown\alpha}}$.

Case III: If $A_{m\alpha} \in I_{p_\sigma}$, then $q_{\sigma^\frown\alpha}$ and $F_{q_{\sigma^\frown\alpha}}$ would not be defined.

This completes the construction.

Definition 2.7 Let $F_{n+1} = \bigcap\{F_{p_\sigma} \mid \sigma : n+2 \to \kappa^+, F_{p_\sigma} \text{ is defined }\}$, and define the corresponding dual ideals I_{n+1}, I_{p_σ} .

Notice that all F_ns and I_ns are κ complete, proper and normal as an intersection of such filters and ideals respectively.

Definition 2.8 Let F_ω be the closure under ω intersections of $\bigcup_{n<\omega} F_n$.

Let $I_\omega =$ the closure under ω unions of $\bigcup_{n<\omega} I_n$.

Lemma 2.9 *$F \subseteq F_0 \subseteq ... \subseteq F_n \subseteq ... \subseteq F_\omega$ and $I \subseteq I_0 \subseteq ... \subseteq I_n \subseteq ... \subseteq I_\omega$, and I_ω is the dual ideal to F_ω.*

Lemma 2.10 *Let $s : m \to \kappa^+$ with F_{p_s} defined; then:*

1. *$\{\alpha < \kappa^+ \mid F_{s^\frown\alpha}$ is defined $\} = \{\xi < \kappa^+ \mid A_{m\xi} \in F^+_{p_s}\}$;*
2. *There exists an extension $\sigma \supseteq s$ such that F_{p_σ} is defined and:*

$$|\{\xi < \kappa^+ \mid A_{\mathrm{dom}(\sigma)\xi} \in F^+_\sigma\}| = \kappa^+.$$

Proof. 1) is clear from the construction above. For 2), let us assume that for every extension $\sigma \supseteq s$ such that F_{p_σ} is defined :

$$|\{\xi < \kappa^+ \mid A_{\mathrm{dom}(\sigma)\xi} \in F^+_\sigma\}| \leq \kappa.$$

That means that $\Sigma = \{\sigma : n \to \kappa^+ | n \geq m \text{ and } \sigma \supseteq s\}$ is of cardinality less or equal κ, so $\nu = \bigcup_{\sigma\in\Sigma} \mathrm{ran}(\sigma)$ is less then κ^+ and p_s or some extension of it will force that $j(H)(\kappa)$ is bounded, contradiction.

□

From now on the proof will be the same as in [4](Theorem 2.5) and we get that F_ω is the desired filter.

3 Constructing of almost precipitous ideals from semi-precipitous

Suppose κ is a λ semi-precipitous cardinal for some ordinal λ which is a successor ordinal $> \kappa$ or a limit one with $\mathrm{cof}(\lambda) > \kappa$. Let P be a forcing notion witnessing this. Then, for each generic $G \subseteq P$, in $V[G]$ we have an elementary embedding $j : V_\lambda \to M$ with $cp(j) = \kappa$ and M is transitive. Consider

$$U = \{X \subseteq \kappa \mid X \in V, \kappa \in j(X)\}.$$

Then U is a $V-$normal ultrafilter over κ. Let $i_U : V \to V \cap {}^\kappa V/U$ be the corresponding elementary embedding. Note that $V \cap {}^\kappa V/U$ need not be well founded, but it is well founded up to the image of λ. Thus, denote $V \cap {}^\kappa V/U$ by N. Define $k : (V_{i(\lambda)})^N \to M$ in a standard fashion by setting

$$k([f]_U) = j(f)(\kappa),$$

for each $f : \kappa \to V_\lambda, f \in V$. Then k will be elementary embedding, and so $(V_{i(\lambda)})^N$ is well founded.

For every $p \in P$ set

$$F_p = \{X \subset \kappa \mid p \Vdash \kappa \in \underset{\sim}{j}(X)\}.$$

Clearly, if G is a generic subset of P with $p \in G$ and U_G is the corresponding V-ultrafilter, then $F_p \subseteq U_G$.

Note that, if for some $p \in P$ the filter F_p is κ^+-saturated, then each U_G with $p \in G$ will be generic over V for the forcing with F_p-positive sets. Thus, every maximal antichain in F_p^+ consists of at most κ many sets. Let $\langle A_\nu \mid \nu < \kappa \rangle \in V$ be such maximal antichain. Without loss of generality we can assume that $\min(A_\nu) > \nu$, for each $\nu < \kappa$. Then there is $\nu^* < \kappa$ with $\kappa \in j(A_{\nu^*})$. Hence $A_{\nu^*} \in U_G$ and we are done.

It follows that in such a case N which is the ultrapower by U_G is fully well founded.

Note that in general if some forcing P produces a well founded N, then κ is ∞-semi precipitous. Just i and N will witness this.

Our aim will be to prove the following:

Theorem 3.1 *Assume that $2^\kappa = \kappa^+$ and κ carries a λ-semi-precipitous filter for some limit ordinal λ with $\mathrm{cof}(\lambda) > \kappa$. Suppose in addition that there is a forcing notion P witnessing λ-semi-precipitous with corresponding N ill founded. Then*

1. *if $\lambda < \kappa^{++}$, then κ is λ-almost precipitous witnessed by a normal filter,*

2. *if $\lambda \geq \kappa^{++}$, then κ is an almost precipitous witnessed by a normal filters.*

Proof. The proof will be based on an extension of the method of constructing normal filters of [4] which replaces restrictions to positive sets by restrictions to filters. An additional idea will be to use a witness of a non-well-foundedness in the construction in order to limit it to ω many steps.

Let κ, τ, P be as in the statement of the theorem. Preserve the notation that we introduced above. Then

$$0_P \Vdash (V_{i(\lambda)})^N \text{ is well founded and } N \text{ is ill founded.}$$

Fix a sequence $\langle \underset{\sim}{g}_n \mid n < \omega \rangle$ of names of functions witnessing an ill foundedness of N, i.e.

$$0_P \Vdash [\underset{\sim}{g}_n] > [\underset{\sim}{g}_{n+1}],$$

for every $n < \omega$. Note that, as was observed above, for every $p \in P$, the filter F_p is not κ^+-saturated.

Fix some $\tau < \kappa^{++}, \tau \leq \lambda$. We should construct a normal τ-almost precipitous filter over κ.

For each $p \in P$ choose a maximal antichain $\{A_{p\beta} \mid \beta < \kappa^+\}$ in F_p^+.

Let $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ enumerate all the functions from κ to τ. Fix an enumeration $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ of $F_{0_P}^+$.

Start now an inductive process of extending of F_{1_P}.

Let $n = 0$. Assume for simplicity that there is a function $g_0 : \kappa \to On \in V$ so that $1_P \Vdash \check{g}_0 = \underset{\sim}{g}_0$.

We construct inductively a sequence of ordinals $\langle \xi_{0\beta} \mid \beta < \kappa^+ \rangle$ and a sequence of conditions $\langle p_{0\beta} \mid \beta < \kappa^+ \rangle$. Let $\alpha < \kappa^+$.

Case I. There is a $\xi < \kappa^+$ so that $\xi \neq \xi_{0\beta}$, for every $\beta < \alpha$ and $X_\alpha \cap A_{1\xi} \in F_{0_P}^+$.

Then let $\xi_{0\alpha}$ be the least such ξ. We would like to attach an ordinal to $f_{\xi_{0\alpha}}$. Let us pick $p \in P$, such that $p \Vdash \kappa \in j(X_\alpha \cap A_{1\xi})$ and for some γ such that $p \Vdash j(\underset{\sim}{f}_{\xi_{0\alpha}})(\kappa) = \gamma$. Now, set $p_{0\alpha} = p$ and extend F_{0_P} to $F_{p_{0\alpha}}$.

Case II. Not Case I.

Then we will not define $F_{p_{0\alpha}}$. Set $\xi_{0\alpha} = 0$ and $p_{0\alpha} = 0_P$.

Note that if Case I fails then we have $X_\alpha \subseteq \nabla_{\beta<\kappa} A_{1\xi_{\tau(\beta)}}$ mod F_{0_P} for a surjective $\tau : \kappa \to \alpha$.

Set $F_0 = \bigcap\{F_{p_{0\alpha}} \mid \alpha < \kappa^+$ and $F_{p_{0\alpha}}$ is defined $\}$, and denote the corresponding dual ideals by $I_{p_{0\alpha}}$ and I_0.

Clearly, $I_0 = \bigcap\{I_{p_{0\alpha}} \mid \alpha < \kappa^+\}$. Also, $F_0 \supseteq F_{0_P}$ and $I_0 \supseteq \check{F}_{0_P}$, since each $F_{p_{0\alpha}} \supseteq F_{0_P}$ and $I_{p_{0\alpha}} \supseteq \check{F}_{0_P}$. Note that F_0 is a κ complete, normal and proper filter since it is an intersection of such filters and also I_0 is.

We now describe the successor step of the construction, i.e., $n = m + 1$.

Let $\sigma : m \to \kappa^+$. Find some $p \in P, p \geq p_\sigma$ and a function $g_m : \kappa \to On \in V$ such that $p \Vdash \check{g_m} = \underset{\sim}{g}_m$, $p \Vdash \underset{\sim}{g}_m < \underset{\sim}{g}_{m-1}$. Denote $S_\sigma = \{\nu \mid g_m(\nu) < g_{m-1}(\nu)\}$. We extend F_{p_σ} to $F_p + S_\sigma$. By 2.1, there is $q_\sigma \in P$, $q_\sigma \geq p$ and $F_{q_\sigma} \supseteq F_p + S_\sigma$.

We construct now by induction a sequence of ordinals $\langle \xi_{\sigma\beta} \mid \beta < \kappa^+\rangle$ and a sequence of conditions $\langle p_{\sigma\beta} \mid \beta < \kappa^+\rangle$. Let $\alpha < \kappa^+$:

Case I. There is $\xi < \kappa^+$ so that $\xi \neq \xi_{\sigma\beta}$ for every $\beta < \alpha$ and $X_\alpha \cap A_{q_\sigma \xi} \in F^+_{q_\sigma}$.

Then let $\xi_{\sigma\alpha}$ be the least such ξ. We would like to attach an ordinal to $f_{\xi_{\sigma\alpha}}$. Let us pick $p \in P$ so that $p \leq q_\sigma$, $p \Vdash \kappa \in j(X_\alpha \cap A_{q_\sigma \xi_{\sigma\alpha}})$ and there is an ordinal γ such that $p \Vdash j(\underset{\sim}{f}_{\xi_{\sigma\alpha}})(\kappa) = \gamma$. Now, set $p_{\sigma\alpha} = p$ and extend F_{q_σ} to $F_{p_{\sigma\alpha}}$.

Case II. Case I fails.

Then we will not define $F_{p_{\sigma\alpha}}$. Set $\xi_{\sigma\alpha} = 0$ and $p_{\sigma\alpha} = 0_P$.

This completes the construction.

Set $F_n = \bigcap\{F_{p_{\sigma\alpha}} \mid \sigma : m \to \kappa^+, \alpha < \kappa^+$ and $F_{p_{\sigma\alpha}}$ is defined $\}$, and denote the corresponding dual ideals by $I_{p_{\sigma\alpha}}$ and I_n.

will use the following:

Definition 3.2 Let F_ω be the closure under ω intersections of $\bigcup_{n<\omega} F_n$.

Let $I_\omega =$ the closure under ω unions of $\bigcup_{n<\omega} I_n$.

Lemma 3.3 *$F_0 \subseteq ... \subseteq F_n \subseteq ... \subseteq F_\omega$ and $I_0 \subseteq ... \subseteq I_n \subseteq ... \subseteq I_\omega$, and I_ω is the dual ideal to F_ω.*

Our purpose now will be to show that we cannot continue the construction further beyond ω and then we would be able to show that F_ω is a τ -almost precipitous filter.

Lemma 3.4 $F^+_\omega \subseteq \cup\{F_{p_\sigma} \mid \sigma \in {}^{<\omega}\kappa^+\}$.

Proof. Let $X \in (F_\omega)^+$ and assume that $X \notin F_{p_\sigma}$ for each $\sigma \in [\kappa^+]^{<\omega}$ so that F_{p_σ} is defined. Let us show that then there are at most κ many σ's so that $X \in F^+_{p_\sigma}$. Thus, for n=0, $\{\alpha < \kappa^+ \mid X \cap A_{1\alpha} \in F^+_{1_P}\}$ is of cardinality less or equal κ. Suppose otherwise. Let $\nu < \kappa^+$ be such that $X = X_\nu$. Then F_{p_ν} is defined according to Case I and $X \in F_{p_\nu}$. Contradiction. For every $\nu < \kappa^+$ with $X \in F^+_\nu$, the set $\{\alpha < \kappa^+ \mid X \cap A_{q_{\langle 0,\nu\rangle}\alpha} \in F^+_{q_{\langle 0,\nu\rangle}}\}$ is of cardinality less or equal κ. Otherwise, we must have that for $\xi < \kappa^+$ with $X = X_\xi$ the filter $F_{p_{\langle 0,\nu\rangle\xi}}$ is defined according to Case I and $X \in F_{p_{\langle 0,\nu\rangle\xi}}$. We continue in a similar fashion and obtain that the set $T = \{\sigma \in [\kappa^+]^{<\omega} \mid F_{p_\sigma}$ is defined , $X \in F^+_{p_\sigma}\}$ is of cardinality at most κ. Also note, that for every $\sigma \in T$ the set

$$B_\sigma = \{\beta < \kappa^+ \mid A_{q_\sigma\beta} \cap X \in F^+_{q_\sigma}\}$$

is of cardinality at most κ. Otherwise, we can always find $\xi, \alpha < \kappa^+$ so that $X = X_\alpha$, $X_\alpha \cap A_{q_\sigma\xi} \in F^+_{q_\sigma}$ and $\xi \neq \xi_{\sigma\beta}$, for every $\beta < \alpha$. Then, according to Case 1, $X_\alpha \in F_{q_{\sigma\xi_{\sigma\alpha}}}$.

For every $\sigma \in T$, fix $\psi_\sigma : \kappa \longleftrightarrow B_\sigma$. Note that

$$X \setminus \nabla^{\psi_\sigma}_{\beta<\kappa} A_{q_\sigma\psi_\sigma(\beta)}$$

is in the ideal I_{q_σ}.

Now, let $n = 0$. Turn the family $\{A_{0_P\psi_0(\gamma)} \mid \gamma < \kappa\}$ into a family of disjoint sets as follows:

$$A'_{0_P\psi(0)} := A_{0_P\psi(0)} - \{0\}$$

and for each $\gamma < \kappa$ let

$$A'_{0_P\psi(\gamma)} := A_{0_P\psi(\gamma)} - (\bigcup_{\beta<\gamma} A_{0_P\psi(\beta)} \cup (\gamma + 1)).$$

Note that

$$\nabla^{\psi_0}_{\beta<\kappa} A'_{0_P\psi_0(\beta)} = \{\nu < \kappa \mid \exists\ \beta < \nu \text{ so that } \nu \in A'_{0_P\psi_0(\beta)}\}$$

and, because $\nu \in A'_{1\psi_0(\beta)} \to \nu > \beta$, we get that the right hand side is equal to

$$\bigcup\{A'_{0_P\psi_0(\gamma)} \mid \gamma < \kappa\}.$$

Also note that

$$\nabla^{\psi_0}_{\beta<\kappa} A'_{0_P\psi_0(\beta)} = \nabla^{\psi_0}_{\beta<\kappa} A_{0_P\psi_0(\beta)}.$$

So $\{X \cap A'_{\alpha_0\psi(\gamma)} \mid \gamma < \kappa\}$ is still a maximal antichain in $F^+_{0_P}$ below X and $X \subseteq \nabla^{\psi_0}_{\beta<\kappa} A'_{0_P\psi_0(\beta)}$ mod F_{0_P}. Set $R_0 := X \setminus \cup_{\beta<\kappa} A'_{0_P\psi(\beta)}$. Then $R_0 \in I_{0_P}$.

Now, for each $\beta < \kappa$ with $F_{p_{\sigma_\beta}} = F_{p_{\langle\psi(\beta)\rangle}}$ defined, let us turn the family $\{A_{q_{\sigma_\beta}\psi_{\sigma_\beta}(\gamma)} \mid \gamma < \kappa\}$ into a disjoint one $\{A'_{q_{\sigma_\beta}\psi_{\sigma_\beta}(\gamma)} \mid \gamma < \kappa\}$ as described above. Then

$$R_{\sigma_\beta} := X \cap A'_{0_P\psi(\beta)} \setminus \bigcup_{\gamma<\kappa} (A'_{q_{\sigma_\beta}\psi_{\sigma_\beta}(\gamma)} \cap S_{\sigma_\beta}) \in I_{\sigma_\beta},$$

where S_{σ_β} was defined during the construction above. Set $R_1 = \cup\{R_{\sigma_\beta} \mid \sigma_\beta \in T\}$.

Claim 1 $R_1 \in I_0$.

Proof. Suppose otherwise. Then $R_1 \in (F_0)^+$. Note that $R_1 \subseteq \cup\{X \cap A'_{0_P\psi(\beta)} \mid \langle\psi(\beta)\rangle \in T\}$ and that the right hand side is a disjoint union. Maximality of $\{X \cap A'_{0_P\psi(\beta)} \mid \beta < \kappa\}$ implies that $R_1 \cap A'_{0_P\psi(\alpha)} \in F^+_{p_{\sigma_\alpha}}$, for some $\alpha < \kappa$. But $R_1 \cap A'_{0_P\psi(\alpha)} = R_{\sigma_\alpha}$ and $R_{\sigma_\alpha} \in I_{\sigma_\alpha}$, contradiction.
$\square$ of the claim.

Continue similar for each $n < \omega$. We will have $R_n \in I_{n-1}$. Set

$$R_\omega := \bigcup_{n<\omega} R_n.$$

Then $R_\omega \in I_\omega$ and $X - R_\omega \in (F_\omega)^+$. Now, let $\alpha \in X - R_\omega$. We can find a non decreasing sequence $\langle p_n \mid n < \omega\rangle$ and $\langle \beta_n \mid n < \omega\rangle$ so that

$$\alpha \in \bigcap_{n<\omega} (A'_{p_n\beta_n} \cap S_{p_n}).$$

Recall that $g_{n+1}(\nu) < g_n(\nu)$, for each $n < \omega$ and $\nu \in \bigcap_{k\leq\ n+1}(A'_{p_k\beta_k} \cap S_{p_k})$. So the intersection $\bigcap_{n<\omega}(A'_{p_n\beta_n} \cap S_{p_n})$ must be empty, but on the other hand, α is a member of this intersection. Contradiction.
$\square$

Lemma 3.5 *Generic ultapower by F_ω is well founded up to the image of τ*

Proof. Suppose that $\langle \underset{\sim}{h}_n \mid n < \omega\rangle$ is a sequence of $(F_\omega)^+$-names of old (in V) functions from κ to τ. Let $G \subseteq (F_\omega)^+$ be a generic ultrafilter. Choose $X_0 \in G$ and a function $h_0 : \kappa \to \tau, h_0 \in V$ so that $X_0 \Vdash_{F^+_\omega} \check{h}_0 = \underset{\sim}{h}_0$. Let $\alpha_0 < \kappa^+$ be so that $f_{\alpha_0} = h_0$. By Lemma 3.4, we can find $\sigma_0 \in [\kappa^+]^{<\omega}$ such that $F_{p_{\sigma_0}}$ is defined and $X_0 \in F_{p_{\sigma_0}}$. Note that at the next stage of the construction there will be β with $A_{p_{\sigma_0}\alpha_0} \in F_{p_{\sigma_0}\beta}$, and so the value of $j(\underset{\sim}{f}_{\alpha_0})(\kappa)$ will be decided. Denote this value by γ_0. Assume for simplicity that $A_{p_{\sigma_0}\alpha_0} \cap X_0$ is

in G (otherwise we could replace X_0 by another positive set using density). Continue below $A_{p_{\sigma_0}\alpha_0} \cap X_0$ and pick $X_1 \in G$ and a function $h_1 : \kappa \to \tau, h_1 \in V$ so that $X_1 \Vdash_{F^+_\omega} \check{h}_1 = \underset{\sim}{h}_1$. Let $\alpha_1 < \kappa^+$ be so that $f_{\alpha_1} = h_1$. By Lemma 3.4, we can find $\sigma_1 \in [\kappa^+]^{<\omega}$ such that $F_{p_{\sigma_1}}$ is defined, $\sigma_1 \supseteq \sigma_0$ and $X_1 \in F_{p_{\sigma_1}}$. Again, note that at the next stage of the construction there will be β with $A_{p_{\sigma_1}\alpha_1} \in F_{p_{\sigma_1}\beta}$, and so the value of $j(\underset{\sim}{f}_{\alpha_1})(\kappa)$ will be decided. Denote this value by γ_1. Continue the process for every $n < \omega$. There must be $k < m < \omega$ such that $\gamma_k \leq \gamma_m$ and $X_m \cap A_{\sigma_m\alpha_m} \in G$. So the sequence $\langle [h_n]_G \mid n < \omega \rangle$ is not strictly decreasing.
$\square$

Let us deduce now some conclusions concerning an existence of almost precipitous filters. The following answers a question raised in [5].

Corollary 3.6 *Assume $0^\sharp$. Then every cardinal can be an almost precipitous witnessed by normal filters in a generic extension of L .*

Proof. By Donder, Levinski [1], every cardinal can be semi-precipitous in a generic extension of L. Now apply 3.1. Clearly, there is no saturated ideals in $L[0^\sharp]$.
$\square$

Corollary 3.7 *Assume there are class many Ramsey cardinals. Then every cardinal is an almost precipitous witnessed by normal filters.*

Proof. It follows from 1.6 and 3.1.
$\square$

Corollary 3.8 *Assume $V = L[U]$ with U a normal ultrafilter over κ. Then*

1. *every regular cardinal less than κ is an almost precipitous witnessed by normal filters and non precipitous,*

2. *for each $\tau \leq \kappa^+$, κ carries a normal τ-almost precipitous non precipitous filter.*

Proof. Let η be a regular cardinal less than κ. By 1.11, η is $< \kappa$-semi-precipitous. Note that no cardinal less than κ can be ∞-semi precipitous. Hence, η is almost an precipitous witnessed by a normal filter, by 3.1. This proves (1).
Now,

$$A = \{\eta < \kappa \mid \eta \text{ is an almost precipitous witnessed by a normal filter and non precipitous } \}$$

is in U. Hence, in $M \simeq {}^\kappa V/U$, for each $\tau < (\kappa^{++})^M$ there is a normal τ-almost precipitous non precipitous filter F_τ over κ. Then F_τ remains such also in V, since ${}^\kappa M \subseteq M$.
$\square$

We do not know if (2) remains valid once we replace $\tau \leq \kappa^+$ by $\tau < \kappa^{++}$.

Let us turn to the case of ∞-semi precipitous cardinals which was not covered by Theorem 3.1
Combining constructions of [4] with the present ones (mainly, replacing restrictions to sets by restrictions to filters) we obtain the following.

Theorem 3.9 *Assume that $\aleph_1$ is ∞-semi precipitous and $2^{\aleph_1} = \aleph_2$. Suppose that for some witnessing this forcing P*

$$0_P \Vdash_P \underset{\sim}{i}(\aleph_1) > (\aleph_1^+)^V.$$

Then $\aleph_1$ is almost precipitous witnessed by normal filters.

Theorem 3.10 *Assume that κ is ∞-semi precipitous, $2^\kappa = \kappa^+$ and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ. Suppose that for some witnessing this forcing P*

1. $0_P \Vdash_P \underset{\sim}{i}(\kappa) > (\kappa^+)^V$
2. $0_P \Vdash_P \kappa \in \{\nu < \underset{\sim}{i}(\kappa) \mid \mathrm{cof}(\nu) = \kappa^-\}$.

Then κ is almost precipitous witnessed by normal filters.

Theorem 3.11 *Suppose that there is no inner model satisfying $(\exists\alpha \quad o(\alpha) = \alpha^{++})$. Assume that $\aleph_1$ is ∞-semi precipitous and $2^{\aleph_1} = \aleph_2$. If $\aleph_3$ is not a limit of measurable cardinals of the core model, then there exists a normal precipitous ideal on $\aleph_1$.*

Theorem 3.12 *Suppose that there is no inner model satisfying $(\exists\alpha \quad o(\alpha) = \alpha^{++})$. Assume that κ is ∞-semi precipitous, $2^\kappa = \kappa^+$ and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ. Suppose that for some witnessing this forcing P*

$$0_P \Vdash_P \kappa \in \{\nu < \underset{\sim}{i}(\kappa) \mid \mathrm{cof}(\nu) = \kappa^-\}.$$

If κ^{++} is not a limit of measurable cardinals of the core model, then there exists a normal precipitous ideal on κ.

Theorem 3.13 *Assume that $\aleph_1$ is ∞-semi precipitous. Let P be a witnessing this forcing such that*

$$0_P \Vdash_P \underset{\sim}{i}(\aleph_1) > (\aleph_1^+)^V.$$

Then, after forcing with $Col(\aleph_2, |P|)$, there will be a normal precipitous filter on $\aleph_1$.

Theorem 3.14 *Assume that κ is ∞-semi precipitous and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ. Let P be a witnessing this forcing such that*

1. $0_P \Vdash_P \underset{\sim}{i}(\kappa) > (\kappa^+)^V$
2. $0_P \Vdash_P \kappa \in \{\nu < \underset{\sim}{i}(\kappa) \mid \mathrm{cof}(\nu) = \kappa^-\}$.

Then, after forcing with $Col(\kappa^+, |P|)$, there will be a normal precipitous filter on κ.

Sketch of the proof of 3.13. Let P be a forcing notion witnessing ∞-semi precipitousness such that

$$0_P \Vdash_P \underset{\sim}{i}(\aleph_1) > (\aleph_1^+)^V.$$

Fix a function H such that for some $p \in P$

$$p \Vdash_P \underset{\sim}{i}(H)(\kappa) : \omega \to^{\text{onto}} (\kappa^+)^V,$$

where here and further κ will stand for $\aleph_1$. Assume for simplicity that $p = 0_P$. Let $\langle h_\alpha \mid \alpha < \kappa^+\rangle$ be a sequence of the canonical functions from κ to κ. For every $\alpha < \kappa^+$ and $n < \omega$ set

$$A_{n\alpha} = \{\nu \mid H(\nu)(n) = h_\alpha(\nu)\}.$$

Then, the following hold:

Lemma 3.15 *For every $\alpha < \kappa^+$ and $p \in P$ there is $n < \omega$ so that $A_{n\alpha} \in F_p^+$.*

Lemma 3.16 *Let $n < \omega$ and $p \in P$. Then the set*

$$\{A_{n\alpha} \mid \alpha < \kappa^+ \textit{ and } A_{n\alpha} \in F_p^+\}$$

is a maximal antichain in F_p^+.

Denote by

$$Col(\aleph_2, P) = \{t \mid t \text{ is a partial function of cardinality at most } \aleph_1 \text{ from } \aleph_2 \text{ to } P\}.$$

Let $G \subseteq Col(\aleph_2, P)$ be a generic and $C = \bigcup G$.

We extend F_{0_P} now as follows.
Start with $n = 0$. If $|\{\alpha \mid A_{0\alpha} \in F_{0_P}^+| < \kappa^+$, then set $F_0 = F_{0_P}$.
Suppose otherwise. Let $\alpha < \kappa^+$. If $A_{0\alpha}$ in the ideal dual to F_{0_P}, then set $F_{0\alpha} = F_{0_P}$. If

$A_{0\alpha} \in F^+_{0_P}$, then we consider $F_{C(\alpha)}$. If $A_{0\alpha} \notin F^+_{C(\alpha)}$, then pick some $p(0\alpha) \in P$ forcing $\kappa \in \underset{\sim}{i}(A_{0\alpha})$ and set $F_{0\alpha} = F_{p(0\alpha)}$. If $A_{0\alpha} \in F^+_{C(\alpha)}$, then pick some $p(0\alpha) \in P, p(0\alpha) \geq C(\alpha)$ forcing $\kappa \in \underset{\sim}{i}(A_{0\alpha})$ and set $F_{0\alpha} = F_{p(0\alpha)}$.
Set $F_0 = \bigcap\{F_{0\alpha} \mid \alpha < \kappa^+\}$.

Let now $n = 1$. Fix some $\gamma < \kappa^+$ with $F_{0\gamma}$ defined. If $|\{\alpha \mid A_{1\alpha} \in F^+_{0\gamma}| < \kappa^+$, then we do nothing. Suppose that it is not the case. Let $\alpha < \kappa^+$. We define $F_{\langle 0\gamma, 1\alpha\rangle}$ as follows:

- if $A_{1\alpha} \notin F^+_{0\gamma}$, then set $F_{\langle 0\gamma,1\alpha\rangle} = F_{0\gamma}$,
- if $A_{1\alpha} \in F^+_{0\gamma}$, then consider $F_{C(\alpha)}$. If there is no p stronger than both $C(\alpha), p(0\gamma)$ and forcing $\kappa \in \underset{\sim}{i}(A_{1\alpha})$, then pick some $p(\langle 0\gamma, 1\alpha\rangle) \geq p(0\alpha)$ which forces $\kappa \in \underset{\sim}{i}(A_{1\alpha})$ and set $F_{\langle 0\gamma,1\alpha\rangle} = F_{p(\langle 0\gamma,1\alpha\rangle)}$. Otherwise, pick some $p(\langle 0\gamma, 1\alpha\rangle) \geq C(\alpha), p(0\alpha)$ which forces $\kappa \in \underset{\sim}{i}(A_{1\alpha})$ and set $F_{\langle 0\gamma,1\alpha\rangle} = F_{p(\langle 0\gamma,1\alpha\rangle)}$.

Set $F_1 = \bigcap\{F_{\langle 0\gamma,1\alpha\rangle} \mid \alpha, \gamma < \kappa^+\}$.
Continue by induction and define similar filters F_s, F_n and conditions $p(s)$ for each $n < \omega, s \in [\omega \times \kappa^+]^{<\omega}$.

Finally set

$$F_\omega = \text{ the closure under } \omega \text{ intersections of } \bigcup_{n<\omega} F_n.$$

The arguments like those of 3.1 transfer directly to the present context. We refer to [4] which contains more details.
Let us prove the following crucial lemma.

Lemma 3.17 *F_ω is a precipitous filter.*

Proof. Suppose that $\langle \underset{\sim}{g}_n | n < \omega\rangle$ is a sequence of F^+_ω-names of old (in V) functions from $\kappa \to On$.
Let $G \subseteq F^+_\omega$ be a generic ultrafilter. Pick a set $X_0 \in G$ and a function

$$g_0 : \kappa \to On$$

in V such that

$$X_0 \Vdash_{F^+_\omega} \dot{g}_0 = \check{g}_0.$$

Pick some $t_0 \in Col(\aleph_2, P)$, $t \subseteq C$ such that

$$\langle t_0, X_0\rangle \Vdash_{Col(\aleph_2,P)*F^+_\omega} \underset{\sim}{g}_0 = \check{g}_0$$

and for some $s_0 = \langle \xi_0, ..., \xi_n \rangle \in [\omega \times \kappa^+]^{<\omega}$

$$t_0 \Vdash X_0 \in \underset{\sim}{F}_{s_0},$$

moreover, for each $i \leq n$, $\xi_i \in \mathrm{dom}(t_0)$ and $t_0(\xi_n) = p(s_0)$.

Claim 2 For each $\langle t, Y \rangle \in Col(\aleph_2, P) * \underset{\sim}{F}^+_\omega$ with $\langle t, Y \rangle \geq \langle t_0, X_0 \rangle$ there are $\langle q_0, Z_0 \rangle \geq \langle t, Y \rangle, \rho_0 \in On$ and s'_0 extending s_0 such that

1. $q(s'_0(|s'_0|)) \leq p(s'_0)$,
2. $q \Vdash_{Col(\aleph_2, P)} \check{Z}_0 \in \underset{\sim}{F}_{s'_0}$,
3. $p(s'_0) \Vdash_P \underset{\sim}{i}(g_0)(\kappa) = \check{\rho}_0$.

Proof. Suppose for simplicity that $\langle t, Y \rangle = \langle t_0, X_0 \rangle$. We know that t_0 decides F_{s_0}, $t_0(s_0(|s_0|)) = p(s_0)$ and $X_0 \in F_{s_0}$. Find s extending s_0 of the smallest possible length such that the set $B = \{\alpha \mid A_{|s|\alpha} \in F^+_{s_0}\}$ has cardinality κ^+. Remember that we do not split F_{s_0} before getting to such s. Pick some $\alpha \in B \backslash \mathrm{dom}(t_0)$. $A_{|s|\alpha} \in F^+_{s_0}$, hence there is some $p' \in P, p' \geq p(s_0)$ which forces $\kappa \in \underset{\sim}{i}(A_{|s|\alpha})$. Find some $p \in P, p \geq p'$ and ρ_0 such that

$$p \Vdash_P \underset{\sim}{i}(g_0)(\kappa) = \rho_0.$$

Extend now t_0 to t by adding to it $\langle \alpha, p \rangle$. Let $s'_0 = s^\frown \alpha$ and $Z_0 = X_0 \cap A_{|s|\alpha}$.
$\square$ of the claim.

By the genericity we can find $\langle q_0, Z_0 \rangle$ as above in $C * G$. Back in $V[C, G]$, find $X_1 \subseteq Z_0$ in G and a function

$$g_1 : \kappa \to On$$

in V such that

$$X_1 \Vdash_{F^+_\omega} \dot{g}_1 = \check{g}_1.$$

Proceed as above only replacing X_0 by X_1. This will define q_1, Z_1 and ρ_1 for g_1 as in the claim.

Continue the process for each $n < \omega$. The ordinals ρ_n will witness the well foundness of the sequence $\langle [g_n]_G | n < \omega \rangle$
$\square$
$\square$

Note that if there is a precipitous ideal (not a normal one) over κ, then we can use its positive sets as P of Theorems 3.13, 3.14. The cardinality of this forcing is 2^κ. So adding a Cohen subset to κ will suffice.

Embeddings witnessing ∞-semi precipitousness may have a various sources. Thus for example they may come from strong, supercompact, huge cardinals etc or their generic relatives. An additional source of examples is Woodin Stationary Tower forcings, see Larson [6].

Corollary 3.18 *Suppose that δ is a Woodin cardinal and there is $f : \omega_1 \to \omega_1$ with $\|f\| \geq \omega_2$. Then in $V^{Col(\aleph_2,\delta)}$ there is a normal precipitous ideal over $\aleph_1$.*

Remark. Woodin following Foreman, Magidor and Shelah [3] showed that $Col(\aleph_1, \delta)$ turns $NS_{\aleph_1}$ into a presaturated ideal. On the other hand Schimmerling and Velickovic [8] showed that there is no precipitous ideals on $\aleph_1$ in $L[E]$ up to at least a Woodin limit of Woodins. Also by [8], there is $f : \omega_1 \to \omega_1$ with $\|f\| \geq \omega_2$ in $L[E]$ up to at least a Woodin limit of Woodins.

Proof. Let δ be a Woodin cardinal. Force with $\mathbf{P}_{<\delta}$, (refer to the Larson book [6] for the definitions) above a stationary subset of ω_1. This will produce a generic embedding $i : V \to N$ with a critical point ω_1, N is transitive and $i(\omega_1) > (\omega_2)^V$. The cardinality of $\mathbf{P}_{<\delta}$ is δ. So 3.13 applies.
□

Similar, using 3.14, one can obtain the following:

Corollary 3.19 *Suppose that δ is a Woodin cardinal, $\kappa < \delta$ is the immediate successor of κ^-, $(\kappa^-)^{<\kappa^-} = \kappa^-$ and there is $f : \kappa \to \kappa$ with $\|f\| \geq \kappa^+$. Then in $V^{Col(\kappa^+,\delta)}$ there is a normal precipitous ideal over κ.*

4 Extension of an elementary embedding

Donder and Levinsky [1] showed that κ-c.c. forcings preserve semi-precipitousness of a cardinal κ. Let us show that κ^+-distributive forcings preserve semi-precipitousness of a cardinal κ, as well.

Lemma 4.1 *Let κ be a semi-precipitous cardinal and let $\bar{P}$ be a κ^+-distributive forcing. Then, $V^{\bar{P}} \models$ " κ is semi-precipitous ".*

Proof. Fix a cardinal λ so that $\bar{P} \in V_\lambda$. Let as show that κ remains a λ-semi-precipitous in $V^{\bar{P}}$. It is enough for every $p \in \bar{P}$ to find a generic subset G of $\bar{P}$ with $p \in G$, such that κ is a λ-semi-precipitous in $V[G]$. Fix some $p_0 \in \bar{P}$.
In V, κ is λ-semi-precipitous so the forcing $Q = Col(\omega, \mu)$, with $\mu \geq \lambda$ big enough, produces an elementary embedding $j : V_\lambda \to M \backsimeq (V_\lambda)^\kappa/U$, with M transitive and U a normal V-ultrafilter over κ (in V^Q).
Note that $|\bar{P}| = \aleph_0$ in V^Q. So there is a set $G \in V^Q$ which is a V-generic subset of $\bar{P}$ with $p_0 \in G$. Set

$$G^* = \{p \in \bar{P} \mid \text{ there is a } q \in \bar{P}, p \geq j(q)\}.$$

Clearly, G^* is directed and we would like to show that it meets every open dense subset of $j(\bar{P})$ which belongs to M. Let D be such a subset. There is a function $f \in V_\lambda, f : \kappa \to V_\lambda$ so that $[f]_U = D$. We can assume that for each $\alpha < \kappa$ $f(\alpha)$ is an open dense subset of $\bar{P}$. $\bar{P}$ is κ^+-distributive, hence $\bigcap\{f(\alpha) \mid \alpha < \kappa\} = D'$ is a dense subset of $\bar{P}$. So $G \cap D' \neq \emptyset$. Let $q \in G \cap D'$. Then $j(q) \in G^*$ which implies that $G^* \cap D \neq \emptyset$.
Now it is easy to extend j to $j^* : V_\lambda[G] \to M[G^*]$.

So, in V^Q, we found a V-generic subset G of $\bar{P}$ with $p_0 \in G$ and an elementary embedding of $V_\lambda[G]$ into a transitive model. Note that this actually implies λ-semi-precipitousness of κ in $V[G]$. Thus, force with Q/G over $V[G]$. Clearly, $V[G]^{Q/G} = V^Q$. Hence the forcing Q/G produces the desired elementary embedding.
□

We can use the previous lemma in order to show the following:

Theorem 4.2 *Suppose that κ is a λ-semi-precipitous, for some $\lambda > (2^\kappa)^+$. Then κ will be an almost precipitous after adding of a Cohen subset to κ^+.*

Proof. First note that if κ caries a precipitous filter, then this filter will remain precipitous in the extension. By Lemma 4.1, κ caries a λ-semi-precipitous filter in $V^{Cohen(\kappa^+)}$. If there is a precipitous filter over κ, then we are done. Suppose that it is not the case. Note that in the generic extension we have $2^\kappa = \kappa^+$, so the results of Section 3 apply and give the desired conclusion.
□

5 A remark on pseudo-precipitous ideals

Pseudo-precipitous ideals were introduced by T. Jech in [7]. The original definition was based on a game. We will use an equivalent definition, also due to T. Jech [7].
Let I be a normal ideal over κ. Consider the forcing notion Q_I which consists of normal ideals J extending I. We say that J_1 is stronger than J_2, if $J_1 \supseteq J_2$.
Let G be a generic subset of Q_I. Then $\bigcup G$ is a prime ideal with respect to V. Let F_G denotes its dual V-ultrafilter.

Definition 5.1 (Jech [7]) An ideal I is called a pseudo-precipitous iff I forces in Q_I that ${}^{\kappa}V \cap V/\underset{\sim}{F}_G$ is well founded.

T. Jech [7] asked how strong is the consistency of "there is a pseudo-precipitous ideal on $\aleph_1$"?

Note that if U is a normal ultrafilter over κ then the corresponding forcing is trivial and F_G is always U. In particular, U is pseudo-precipitous.

Let us address the consistency strength of existence of a pseudo-precipitous ideal over a successor cardinal.

Theorem 5.2 *If there is a pseudo-precipitous ideal over a successor cardinal then there is an inner model with a strong cardinal. In particular, an existence of precipitous ideal does not necessary imply an existence of a pseudo-precipitous one.*

Remark 5.3 By Jech [7], any normal saturated ideal is pseudo-saturated. S. Shelah showed that starting with a Woodin cardinal it is possible to construct a model with a saturated ideal on $\aleph_1$. So the strength of existence of a pseudo-precipitous ideal requires at least a strong but not more than a Woodin cardinal.

Proof. Suppose that I is a pseudo-precipitous ideal over $\lambda = \kappa^+$. Assume

$$I \Vdash_{Q_I} j(\lambda) > (\lambda^+)^V,$$

just otherwise we will have large cardinals. This is basically due to Mitchell, see Lemmas 2.31, 2.32 of [4].

Find $J \geq I$ and a function H such that

$$J \Vdash_{Q_I} j(H)(\lambda) : \kappa \to^{onto} (\lambda^+)^V.$$

Fix $\langle h_\nu \mid \nu < \lambda^+\rangle$ canonical functions. Now there is $\xi < \kappa$ such that for λ^+ ordinals $\nu < \lambda^+$, we have

$$A_\nu := \{\alpha < \lambda \mid H(\alpha)(\xi) > h_\nu(\alpha)\} \in J^+.$$

Extend J to J' by adding to it all the compliments of A_ν's and their subsets. Then J' will be a normal ideal extending J. Now extend J' to J'' deciding $j(H)(\lambda)(\xi)$. Let η be the decided value. Then for each $\nu < \lambda^+$ we have $\eta > \nu$. But

$$J \Vdash_{Q_I} \mathrm{ran}(j(H)(\lambda)) = (\lambda^+)^V.$$

Contradiction.

$\square$

The following natural question remain open:

Question: Suppose that I is a pseudo-precipitous. Is I a precipitous?

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