MORE ABOUT λ–SUPPORT ITERATIONS OF $({<}\lambda)$–COMPLETE FORCING NOTIONS

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Abstract. This article continues Rosłanowski and Shelah [8, 9, 10, 11, 12] and we introduce here a new property of $({<}\lambda)$–strategically complete forcing notions which implies that their λ–support iterations do not collapse λ^+ (for a strongly inaccessible cardinal λ).

1. Introduction

The systematic studies of iterations with uncountable supports which do not collapse cardinals were intensified with articles Shelah [13, 14]. Those works started the development of a theory parallel to that of "proper forcing in CS iterations", but the drawback there was that the corresponding properties were more like those in the case of "not adding new reals in CS iterations of proper forcings". If we want to investigate cardinal characteristics associated with ${}^\lambda\lambda$ (in a manner it was done for cardinal characteristics of the continuum), we naturally are interested in iterating forcing notions which do add new elements of ${}^\lambda\lambda$. The study of λ–support iterations of such forcing notions (for an uncountable cardinal λ) has a quite long history already. For instance, Kanamori [6] considered iterations of λ–Sacks forcing notion (similar to the forcing $\mathbb{Q}^{2,\bar{E}}$; see Definition 3.7 and Remark 3.8) and he proved that under some circumstances these iterations preserve λ^+. Fusion properties of iterations of other tree–like forcing notions were used in Friedman and Zdomskyy [4] and Friedman, Honzik and Zdomskyy [3]. In particular, they showed that λ–support iterations of a close relative of $\mathbb{Q}^2_\lambda$ from Definition 3.1 do not collapse λ^+. Several conditions ensuring that λ^+ is not collapsed in λ–support iterations were introduced in a series of previous works Rosłanowski and Shelah [8, 9, 10, 11, 12]. Also Eisworth [2] introduced a condition of this type. Each of those conditions was meant to be applicable to some natural forcing notions adding a new member of ${}^\lambda\lambda$ without adding new elements of ${}^{<\lambda}\lambda$. In some sense, they explained why the relevant forcings can be iterated (without collapsing cardinals).

In the present paper we introduce *semi–pure properness* (Definition 2.3) and we show that for an inaccessible cardinal λ, λ–support iterations of semi–purely proper forcing notions are proper in the standard sense (Theorem 2.7). The cases of successor λ and/or weakly inaccessible λ will be treated in a subsequent paper [7].

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The semi–pure properness is designed to cover the forcing notion $\mathbb{Q}^2_\lambda$ mentioned above (and its relatives given in 3.1, 3.7), but we hope it is much more general. This property has a flavor of *fuzzy properness over quasi–diamonds* of [10, Definition A.3.6] and even more so of *being reasonably merry* of [11, Definition 6.3]. There is also some similarity with *pure B^*–boundedness* of [11, Definition 2.2]. However, the exact relationships between these and other properness conditions are not clear.

While there are some similarities between conditions studied so far, we are far from the state that was achieved for CS iterations and the concept of properness. The considered properties are (unfortunatelly) tailored to fit particular forcing notions and they do not provide any satisfactory general framework covering all examples. The search for the "right" notion of λ–propernes is still far from being completed.

Basic definitions concerning strategically complete forcing notions, their iterations and trees of conditions are reminded in the further part of the Introduction. In the second section of the paper we prove our Iteration Theorem 2.7 and in the following section we present the forcing notions to which this theorem applies. Some special properties of and relationships between the forcings from the third section are investigated in the fourth section.

1.1. **Notation.** Our notation is rather standard and compatible with that of classical textbooks (like Jech [5]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

(1) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta \ldots$) and also by i, j (with possible sub- and superscripts).

    Cardinal numbers will be called κ, λ; λ **will be always assumed to be a regular uncountable cardinal such that** $\lambda^{<\lambda} = \lambda$; in most instances λ is even assumed to be **strongly inaccessible**.

    Also, χ will denote a *sufficiently large* regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ. Moreover, we fix a well ordering $<^*_\chi$ of $\mathcal{H}(\chi)$.

(2) We will consider several games of two players. One player will be called *Generic* or *Complete* or just *COM*, and we will refer to this player as "she". Her opponent will be called *Antigeneric* or *Incomplete* or just *INC* and will be referred to as "he".

(3) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it).

    By "λ–support iterations" we mean iterations in which domains of conditions are of size $\leq \lambda$. However, on some occasions we will pretend that conditions in a λ–support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ are total functions on ζ^* and for $p \in \lim(\bar{\mathbb{Q}})$ and $\alpha \in \zeta^* \setminus \mathrm{dom}(p)$ we will let $p(\alpha) = \underset{\sim}{\emptyset}_{\mathbb{Q}_\alpha}$.

(4) A filter on λ is a non-empty family of subsets of λ closed under supersets and intersections and do not containing $\emptyset$. A filter is $(<\lambda)$–complete if it is closed under intersections of $<\!\lambda$ members. (Note: we do allow principal filters or even $\{\lambda\}$.)

For a filter D on λ, the family of all D–positive subsets of λ is called D^+. (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.) By a normal filter on λ we mean *proper uniform* filter closed under diagonal intersections.

(5) By a *sequence* we mean a function whose domain is a set of ordinals. For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is the order type of its domain and it is denoted by $\mathrm{lh}(\eta)$.

(6) A tree is a $\vartriangleleft$–downward closed set of sequences. A complete λ–tree is a tree $T \subseteq {}^{<\lambda}\lambda$ such that every $\vartriangleleft$-chain of size less than λ has an $\vartriangleleft$-bound in T and for each $\eta \in T$ there is $\nu \in T$ such that $\eta \vartriangleleft \nu$.

Let $T \subseteq {}^{<\lambda}\lambda$ be a tree. For $\eta \in T$ we let

$$\mathrm{succ}_T(\eta) = \{\alpha < \lambda : \eta^\frown\langle\alpha\rangle \in T\} \quad \text{and} \quad (T)_\eta = \{\nu \in T : \nu \vartriangleleft \eta \text{ or } \eta \trianglelefteq \nu\}.$$

We also let $\mathrm{root}(T)$ be the shortest $\eta \in T$ such that $|\mathrm{succ}_T(\eta)| > 1$ and $\lim_\lambda(T) = \{\eta \in {}^\lambda\lambda : (\forall \alpha < \lambda)(\eta{\restriction}\alpha \in T)\}$.

1.2. **Background on trees of conditions.**

Definition 1.1. Let $\mathbb{P}$ be a forcing notion.

(1) For an ordinal γ and a condition $r \in \mathbb{P}$, let $\Game_0^\gamma(\mathbb{P}, r)$ be the following game of two players, *Complete* and *Incomplete*:

> the game lasts at most γ moves and during a play the players construct a sequence $\langle (p_i, q_i) : i < \gamma\rangle$ of pairs of conditions from $\mathbb{P}$ in such a way that
> $$(\forall j < i < \gamma)(r \leq p_j \leq q_j \leq p_i)$$
> and at the stage $i < \gamma$ of the game, first Incomplete chooses p_i and then Complete chooses q_i.

Complete wins if and only if for every $i < \gamma$ there are legal moves for both players.

(2) We say that the forcing notion $\mathbb{P}$ is *strategically* $(<\gamma)$*–complete* (*strategically* $(\leq\gamma)$*–complete*, respectively) if Complete has a winning strategy in the game $\Game_0^\gamma(\mathbb{P}, r)$ (in the game $\Game_0^{\gamma+1}(\mathbb{P}, r)$, respectively) for each condition $r \in \mathbb{P}$.

(3) Let a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be such that ${}^{<\lambda}N \subseteq N$, $|N| = \lambda$ and $\mathbb{P} \in N$. We say that a condition $p \in \mathbb{P}$ is $(N,\mathbb{P})$*–generic in the standard sense* (or just: $(N,\mathbb{P})$*–generic*) if for every $\mathbb{P}$–name $\underset{\sim}{\tau} \in N$ for an ordinal we have $p \Vdash$“ $\underset{\sim}{\tau} \in N$ ”.

(4) $\mathbb{P}$ is λ*–proper in the standard sense* (or just: λ*–proper*) if there is $x \in \mathcal{H}(\chi)$ such that for every model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfying

$${}^{<\lambda}N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad \mathbb{P}, x \in N,$$

and every condition $q \in N \cap \mathbb{P}$ there is an $(N,\mathbb{P})$–generic condition $p \in \mathbb{P}$ stronger than q.

Definition 1.2 (Compare [10, Def. A.1.7], see also [9, Def. 2.2])**.** (1) Let γ be an ordinal, $\emptyset \neq w \subseteq \gamma$. *A* $(w,1)^\gamma$*–tree* is a pair $\mathcal{T} = (T, \mathrm{rk})$ such that

- $\mathrm{rk} : T \longrightarrow w \cup \{\gamma\}$,
- if $t \in T$ and $\mathrm{rk}(t) = \varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon\rangle$,

- $(T, \vartriangleleft)$ is a tree with root $\langle\rangle$ and
- if $t \in T$, then there is $t' \in T$ such that $t \trianglelefteq t'$ and $\mathrm{rk}(t') = \gamma$.

(2) If, additionally, $\mathcal{T} = (T, \mathrm{rk})$ is such that every chain in T has a $\vartriangleleft$–upper bound it T, we will call it *a standard* $(w, 1)^\gamma$*–tree*
We will keep the convention that $\mathcal{T}^x_y$ is (T^x_y, rk^x_y).

(3) Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma\rangle$ be a λ–support iteration. *A tree of conditions in* $\bar{\mathbb{Q}}$ is a system $\bar{p} = \langle p_t : t \in T\rangle$ such that
- (T, rk) is a $(w, 1)^\gamma$–tree for some $w \subseteq \gamma$,
- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
- if $s, t \in T$, $s \vartriangleleft t$, then $p_s = p_t{\restriction}\mathrm{rk}(s)$.

If, additionally, (T, rk) is a standard tree, then $\bar{p}$ is called *a standard tree of conditions*.

(4) Let $\bar{p}^0, \bar{p}^1$ be trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T\rangle$. We write $\bar{p}^0 \le \bar{p}^1$ whenever for each $t \in T$ we have $p^0_t \le p^1_t$.

Note that our standard trees and trees of conditions are a special case of that [10, Def. A.1.7] when $\alpha = 1$.

2. Semi-purity and iterations

In this section we introduce a new property of $({<}\lambda)$–complete forcing notions: *semi–pure properness*. Then we prove that if λ is strongly inaccessible, then λ–support iterations of semi–pure proper forcing notions are proper in the standard sense (so they preserve stationarity of relevant sets and do not collapse λ^+).

Definition 2.1. Let $f : \lambda \longrightarrow \lambda+1$. *A forcing notion with f–complete semi-purity* is a triple $(\mathbb{Q}, \le, \bar{\le}_{\mathrm{pr}})$ such that $\bar{\le}_{\mathrm{pr}} = \langle \le^\alpha_{\mathrm{pr}} : \alpha < \lambda\rangle$ and $\le, \le^\alpha_{\mathrm{pr}}$ are transitive and reflexive (binary) relations on $\mathbb{Q}$ satisfying for each $\alpha < \lambda$:

(a) $\le^\alpha_{\mathrm{pr}} \subseteq \le$,
(b) $(\mathbb{Q}, \le)$ is strategically $({<}\lambda)$–complete and $(\mathbb{Q}, \le^\alpha_{\mathrm{pr}})$ is strategically $({\le}\kappa)$–complete for all infinite cardinals $\kappa < f(\alpha)$.

If $(\mathbb{Q}, \le, \bar{\le}_{\mathrm{pr}})$ is a forcing notion with semi-purity, then all our forcing terms (like "forces", "name", etc) refer to $(\mathbb{Q}, \le)$. The relations $\le^\alpha_{\mathrm{pr}}$ have an auxiliary character only and if we want to refer to them we add "α–purely" (so "stronger" refers to $\le$ and "α–purely stronger" refers to $\le^\alpha_{\mathrm{pr}}$).

Remark 2.2. Note that unlike in [11, Definition 2.1], in semi-purity we do not require any kind of pure decidability.

Definition 2.3. Let $f : \lambda \longrightarrow \lambda + 1$ and let $(\mathbb{Q}, \le, \bar{\le}_{\mathrm{pr}})$ be a forcing notion with f–complete semi-purity. Suppose that D is a normal filter on λ (e.g., the club filter).

(1) A sequence $\bar{Y} = \langle Y_\alpha : \alpha < \lambda\rangle$ is called *an indexing sequence* whenever $\emptyset \ne Y_\alpha \subseteq {}^\alpha\lambda$ and $|Y_\alpha| < \lambda$ for each $\alpha < \lambda$.

(2) For an indexing sequence $\bar{Y}$, a system $\bar{q} = \langle q_{\alpha,\eta} : \alpha < \lambda \ \&\ \eta \in Y_\alpha\rangle \subseteq \mathbb{Q}$ and a condition $p \in \mathbb{Q}$ we define a game $\Game^{\mathrm{aux}}_{\bar{Y}}(p, \bar{q}, \mathbb{Q}, \le, \bar{\le}_{\mathrm{pr}}, D)$ between two players, COM and INC as follows. A play of $\Game^{\mathrm{aux}}_{\bar{Y}}(p, \bar{q}, \mathbb{Q}, \le, \bar{\le}_{\mathrm{pr}}, D)$ lasts λ steps during which the players choose successive terms of a sequence $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda\rangle$. These terms are chosen so that

(a) $r_\alpha, r'_\alpha \in \mathbb{Q}$, $A_\alpha \in D$, $\eta_\alpha \in {}^\alpha\lambda$ and for $\alpha < \beta < \lambda$:

$$p = r_0 \leq r_\alpha \leq r'_\alpha \leq r_\beta \quad \text{and} \quad A_\beta \subseteq A_\alpha \quad \text{and} \quad \eta_\alpha \lhd \eta_\beta,$$

(b) at a stage α of the play, first COM chooses $(r_\alpha, A_\alpha, \eta_\alpha)$ and then INC picks $r'_\alpha \geq r_\alpha$.

At the end, COM wins the play $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda\rangle$ if and only if both players had always legal moves (so the play really lasted λ steps) and

($\odot$) if $\gamma \in \mathop{\triangle}\limits_{\alpha<\lambda} A_\alpha$ is limit, then $\eta_\gamma \in Y_\gamma$ and $q_{\gamma,\eta_\gamma} \leq^\gamma_{\rm pr} r_\gamma$.

(3) If COM has a winning strategy in $\Game^{\rm aux}_{\bar{Y}}(p, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ then we say that *the condition p is aux-generic over $\bar{q}, D$.*

(4) Let $\bar{Y}$ be an indexing sequence and $p \in \mathbb{Q}$. A game $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ between two players, Generic and Antigeneric, is defined as follows. A play of the game lasts λ steps during which the players construct a sequence $\langle \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$. At stage $\alpha < \lambda$ of the play, first Generic chooses a system $\bar{p}^\alpha = \langle p_{\alpha,\eta} : \eta \in Y_\alpha\rangle$ of pairwise incompatible conditions from $\mathbb{Q}$. Then Antigeneric answers by picking a system $\bar{q}^\alpha = \langle q_{\alpha,\eta} : \eta \in Y_\alpha\rangle$ of conditions from $\mathbb{Q}$ satisfying

$$p_{\alpha,\eta} \leq^\alpha_{\rm pr} q_{\alpha,\eta} \qquad \text{for all } \eta \in Y_\alpha.$$

At the end, Generic wins the play $\langle \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$ if and only if, letting $\bar{q} = \langle q_{\alpha,\eta} : \alpha < \lambda \ \&\ \eta \in Y_\alpha\rangle$,

($\boxdot$) there is an aux-generic condition $p^* \geq p$ over $\bar{q}, D$.

(5) A forcing notion $\mathbb{Q}$ is *f–semi-purely proper over an indexing sequence $\bar{Y}$ and a filter D* if for some sequence $\bar{\leq}_{\rm pr}$ of binary relations on $\mathbb{Q}$, $(\mathbb{Q}, \leq, \bar{\leq}_{\rm pr})$ is a forcing with the f–complete semi-purity and for every $p \in \mathbb{Q}$ Generic has a winning strategy in $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$. We then say that *the sequence $\bar{\leq}_{\rm pr}$ witnesses the semi-pure properness of $\mathbb{Q}$.*

(6) If D is the club filter on λ, then we omit it and we write $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr})$ etc. If $\leq^\alpha_{\rm pr} = \leq_{\rm pr}$ for all $\alpha < \lambda$, then we write $\leq_{\rm pr}$ instead of $\bar{\leq}_{\rm pr}$, like in $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \leq_{\rm pr})$. If $f(\alpha) = \lambda$ for all α, then we write λ instead of f (in phrases like λ–complete semi–purity etc).

Observation 2.4. *If $f, g : \lambda \longrightarrow \lambda + 1$ and $f \leq g$, then "g–semi-purely proper" implies "f–semi-purely proper".*

The proof of the following proposition may be considered as an introduction to the more complicated and general proof of Theorem 2.7 dealing with the iterations.

Proposition 2.5. *Assume that $f : \lambda \longrightarrow \lambda + 1$, $\omega + \alpha < f(\alpha)$ for $\alpha < \lambda$ and D is a normal filter on λ. Let $\bar{Y} = \langle Y_\alpha : \alpha < \lambda\rangle$ be an indexing sequence. If a forcing notion $\mathbb{Q}$ is f–semi-purely proper over $\bar{Y}, D$, then it is λ–proper in the standard sense.*

Proof. Let $\bar{\leq}_{\rm pr}$ be a sequence witnessing the semi-pure properness of $\mathbb{Q}$. Assume $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfies

$${}^{<\lambda}N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad (\mathbb{Q}, \leq, \bar{\leq}_{\rm pr}), \bar{Y}, D \ldots \in N.$$

Let $p \in N \cap \mathbb{Q}$. Fix a winning strategy $\mathbf{st} \in N$ of Generic in $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ and pick a list $\langle \tau_\alpha : \alpha < \lambda\rangle$ of all $\mathbb{Q}$–names for ordinals from N.

Consider a play of $\Game^{\rm main}_{\bar{Y}}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ in which Generic uses $\mathbf{st}$ and Antigeneric chooses his answers as follows. At stage $\alpha < \lambda$ of the play, after Generic played $\bar{p}^\alpha = \langle p_{\alpha,\eta} : \eta \in Y_\alpha \rangle$, Antigeneric picks the $<^*_\chi$–first sequence $\bar{q}^\alpha = \langle q_{\alpha,\eta} : \eta \in Y_\alpha \rangle$ such that for each $\eta \in Y_\alpha$:

$(*)_\eta$ $p_{\alpha,\eta} \leq^\alpha_{\rm pr} q_{\alpha,\eta}$,

$(**)_\eta$ if $\beta < \alpha$ and there is a condition q α–purely stronger than $q_{\alpha,\eta}$ and forcing a value to $\underset{\sim}{\tau}_\beta$, then $q_{\alpha,\eta}$ already forces a value to $\underset{\sim}{\tau}_\beta$.

Note that since $(\mathbb{Q}, \leq^\alpha_{\rm pr})$ is strategically $(\leq|\alpha|)$–complete, there are conditions $q \in \mathbb{Q}$ satisfying $(*)_\eta + (**)_\eta$. One checks inductively that $\bar{p}^\alpha, \bar{q}^\alpha \in N$ for all $\alpha < \lambda$ (remember $\mathbf{st} \in N$ and the choice of "the $<^*_\chi$–first"). The play $\langle \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda \rangle$ is won by Generic, so there is a condition $p^* \geq p$ which is aux-generic over $\bar{q} = \langle q_{\alpha,\eta} : \alpha < \lambda \ \&\ \eta \in Y_\alpha \rangle$ and D. We claim that p^* is $(N, \mathbb{Q})$–generic. So suppose towards contradiction that $p^+ \geq p^*$, $p^+ \Vdash \underset{\sim}{\tau}_\beta = \zeta$, $\beta < \lambda$ but $\zeta \notin N$. Consider a play $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda \rangle$ of $\Game^{\rm aux}_{\bar{Y}}(p^*, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ in which COM follows her winning strategy and INC plays:

- $r'_0 = p^+$, and for $\alpha > 0$ he lets $r'_\alpha = r_\alpha$.

Let $\gamma \in \underset{\alpha<\lambda}{\triangle} A_\alpha$ be a limit ordinal greater than β. Since the play was won by COM, we have $\eta_\gamma \in Y_\gamma$ and $q_{\gamma,\eta_\gamma} \leq^\gamma_{\rm pr} r_\gamma$. Since $p^+ \leq r_\gamma$, we know that $r_\gamma \Vdash \underset{\sim}{\tau}_\beta = \zeta$ and hence (by $(**)_{\eta_\gamma}$) $q_{\gamma,\eta_\gamma} \Vdash \underset{\sim}{\tau}_\beta = \zeta$. However, $q_{\gamma,\eta_\gamma} \in N$, contradicting $\zeta \notin N$. $\square$

Lemma 2.6. *Assume that λ is a regular uncountable cardinal, $f : \lambda \longrightarrow \lambda + 1$ and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ is a λ–support iteration such that for every $\xi < \lambda$:*

$$\Vdash_{\mathbb{P}_\xi} \text{" } (\underset{\sim}{\mathbb{Q}}_\xi, \leq, \bar{\leq}_{\rm pr}) \text{ is a forcing notion with } f\text{–complete semi-purity "}.$$

Let $\mathcal{T} = (T, {\rm rk})$ be a standard $(w,1)^\gamma$–tree, $w \in [\gamma]^{<\lambda}$, and let $\bar{p} = \langle p_t : t \in T \rangle$ be a tree of conditions in $\mathbb{P}_\gamma$. Suppose that $\alpha < \lambda$ and Υ is a set of $\mathbb{P}_\gamma$–names for ordinals such that $|T| \cdot |\Upsilon| < f(\alpha)$. Then there exists a tree of conditions $\bar{q} = \langle q_t : t \in T \rangle$ such that

$(\circledast)_1$ *$\bar{p} \leq \bar{q}$ and if $t \in T$, $\xi \in w \cap {\rm rk}(t)$, then $q_t{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p_t(\xi) \leq^\alpha_{\rm pr} q_t(\xi)$, and*

$(\circledast)_2$ *if $\underset{\sim}{\tau} \in \Upsilon$, $t \in T$, ${\rm rk}(t) = \gamma$ and there is a condition $q \in \mathbb{P}_\gamma$ such that*
- *$q_t \leq q$, and $q{\restriction}\xi \Vdash_{\mathbb{P}_\xi} q_t(\xi) \leq^\alpha_{\rm pr} q(\xi)$ for all $\xi \in w$, and*
- *q forces a value to $\underset{\sim}{\tau}$,*

then q_t forces a value to $\underset{\sim}{\tau}$.

Proof. Let $\kappa = |T| \cdot |\Upsilon| < f(\alpha)$ (and we may assume κ is infinite as otherwise arguments are trivial). Let $\leq^{\rm pr}_w$ be a binary relation on $\mathbb{P}_\gamma$ defined by
$p \leq^{\rm pr}_w q$ if and only if
$p \leq_{\mathbb{P}_\gamma} q$ and for each $\xi \in w$, $q{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p(\xi) \leq^\alpha_{\rm pr} q(\xi)$.
The relation $\leq^{\rm pr}_w$ is extended to trees of conditions in the natural way.

For $\xi \in \gamma \setminus w$ let $\underset{\sim}{\mathbf{st}}^0_\xi$ be a $\mathbb{P}_\xi$–name for a wining strategy of Complete in $\Game^{\kappa+1}_0\big((\underset{\sim}{\mathbb{Q}}_\xi, \leq), \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}\big)$ such that it instructs her to play $\emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}$ as long as Incomplete plays $\emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}$. For $\xi \in w$ let $\underset{\sim}{\mathbf{st}}^1_\xi$ be a name for a similar strategy for the game $\Game^{\kappa+1}_0\big((\underset{\sim}{\mathbb{Q}}_\xi, \leq^\alpha_{\rm pr}), \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}\big)$.

Let $\langle (t_i, \underset{\sim}{\tau}_i) : i < \kappa \rangle$ list all members of $\{t \in T : {\rm rk}(t) = \gamma\} \times \Upsilon$ (with possible repetitions). By induction on $i \leq \kappa$ we choose trees of conditions $\bar{q}^i = \langle q^i_t : t \in T \rangle$ and $\bar{r}^i = \langle r^i_t : t \in T \rangle$ such that

(α) $\bar{p} \leq^{\rm pr}_w \bar{q}^0$, $\bar{q}^i \leq^{\rm pr}_w \bar{r}^i \leq^{\rm pr}_w \bar{q}^j \leq^{\rm pr}_w \bar{r}^j$ for $i < j \leq \kappa$,

(β) for each $t\in T$, $j\leq\kappa$ and $\xi\in{\rm rk}(t)\setminus w$,

$$q^j_t{\restriction}\xi\Vdash_{\mathbb{P}_\xi} \text{“ the sequence } \langle (q^i_t(\xi),r^i_t(\xi):i\leq j\rangle \text{ is a legal partial play of } \Game^{\kappa+1}_0\big((\mathunderaccent\tilde-3{\mathbb{Q}}_\xi,\leq),\mathunderaccent\tilde-3{\emptyset}_{\mathbb{Q}_\xi}\big) \text{ in which Complete follows } \mathunderaccent\tilde-3{\mathbf{st}}^0_\xi \text{ ”,}$$

(γ) for each $t\in T$, $j\leq\kappa$ and $\xi\in{\rm rk}(t)\cap w$,

$$q^j_t{\restriction}\xi\Vdash_{\mathbb{P}_\xi} \text{“ the sequence } \langle (q^i_t(\xi),r^i_t(\xi):i\leq j\rangle \text{ is a legal partial play of } \Game^{\kappa+1}_0\big((\mathunderaccent\tilde-3{\mathbb{Q}}_\xi,\leq^\alpha_{\rm pr}),\mathunderaccent\tilde-3{\emptyset}_{\mathbb{Q}_\xi}\big) \text{ in which Complete follows } \mathunderaccent\tilde-3{\mathbf{st}}^1_\xi \text{ ”,}$$

(δ) for each $i<\kappa$, if there is a condition $q\in\mathbb{P}_\gamma$ such that
 (a) $q^i_{t_i}\leq^{\rm pr}_w q$, and
 (b) q forces a value to $\mathunderaccent\tilde-3{\tau}_i$,
 then already $q^i_{t_i}$ forces the value to $\mathunderaccent\tilde-3{\tau}_i$.

So suppose we have defined $\bar{q}^j,\bar{r}^j$ for $j<i$. Stipulating $\bar{r}^{-1}=\bar{p}$, $t_\kappa=t_0$, and $\mathunderaccent\tilde-3{\tau}_\kappa=\mathunderaccent\tilde-3{\tau}_0$ we ask if there is a condition $q\in\mathbb{P}_\gamma$ such that $r^j_{t_i}\leq^{\rm pr}_w q$ for all $j<i$ which forces a value to $\mathunderaccent\tilde-3{\tau}_i$. If there are such conditions, let $q^i_{t_i}$ be one of them. Otherwise let $q^i_{t_i}$ be any $\leq^{\rm pr}_w$–bound to $\{r^j_{t_i}:j<i\}$ (there is such a bound by $(\beta)+(\gamma)$). Then for $t\in T\setminus\{t_i\}$ define q^i_t so that letting $s=t\cap t_i$:

- if $\xi<{\rm rk}(s)$, then $q^i_t(\xi)=q^i_{t_i}(\xi)$,
- if ${\rm rk}(s)\leq\xi<{\rm rk}(t)$, $\xi\notin w$, then $q^i_t(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name such that
$$q^i_t{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{“ } q^i_t(\xi) \text{ is a } \leq\text{–upper bound to } \{r^j_t(\xi):j<i\} \text{ ”,}$$
- if ${\rm rk}(s)\leq\xi<{\rm rk}(t)$, $\xi\in w$, then $q^i_t(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name such that
$$q^i_t{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{“ } q^i_t(\xi) \text{ is a } \leq^\alpha_{\rm pr}\text{–upper bound to } \{r^j_t(\xi):j<i\} \text{ ”.}$$

It should be clear that the above demands correctly define a tree of conditions $\bar{q}^i=\langle q^i_t:t\in T\rangle$ (note the choice of “the $<^*_\chi$–first names”). Finally, we choose $\bar{r}^i$ so that (the respective instances of) conditions $(\beta)+(\gamma)$ are satisfied. To ensure we end up with a tree of conditions, at each coordinate we choose “the $<^*_\chi$–first names for the answers given by the respective strategies”.

After the inductive process is completed, put $\bar{q}=\bar{q}^\kappa$. □

Theorem 2.7. *Assume that λ is a strongly inaccessible cardinal, $f:\lambda\longrightarrow\lambda+1$ and $\bar{\kappa}=\langle\kappa_\alpha:\alpha<\lambda\rangle$ is a sequence of infinite cardinals such that $(\kappa_\alpha)^{|\alpha|}<f(\alpha)$ for all $\alpha<\lambda$, and suppose also that D is a normal filter on λ. For $\xi<\gamma$ let $\bar{Y}^\xi=\langle Y^\xi_\alpha:\alpha<\lambda\rangle$ be an indexing sequence such that $|Y^\xi_\alpha|\leq\kappa_\alpha$. Let $\bar{\mathbb{Q}}=\langle\mathbb{P}_\xi,\mathunderaccent\tilde-3{\mathbb{Q}}_\xi:\xi<\gamma\rangle$ be a λ–support iteration such that*

$$\Vdash_{\mathbb{P}_\xi}\text{“ } \mathunderaccent\tilde-3{\mathbb{Q}}_\xi \text{ is } f\text{–semi-purely proper over } \bar{Y}^\xi, D^{\mathbf{V}^{\mathbb{P}_\xi}} \text{ ”}$$

for every $\xi<\gamma$ (where $D^{\mathbf{V}^{\mathbb{P}_\xi}}$ is the normal filter on λ generated in $\mathbf{V}^{\mathbb{P}_\xi}$ by D). Then $\mathbb{P}_\gamma=\lim(\bar{\mathbb{Q}})$ is λ–proper in the standard sense.

Proof. The proof is very similar to that of [11, Theorem 2.7].

Abusing our notation, the names for the forcing relation and a witness for the semi-pure properness of $\mathunderaccent\tilde-3{\mathbb{Q}}_\xi$ will be denoted $\leq$ and $\bar{\leq}_{\rm pr}=\langle\leq^\alpha_{\rm pr}:\alpha<\lambda\rangle$, respectively. For each $\xi<\gamma$ let $\mathunderaccent\tilde-3{\mathbf{st}}^0_\xi$ be the $<^*_\chi$–first $\mathbb{P}_\xi$–name for a winning strategy of Complete in $\Game^\lambda_0(\mathunderaccent\tilde-3{\mathbb{Q}}_\xi,\mathunderaccent\tilde-3{\emptyset}_{\mathbb{Q}_\xi})$ such that it instructs Complete to play $\mathunderaccent\tilde-3{\emptyset}_{\mathbb{Q}_\xi}$ as long as her opponent plays $\mathunderaccent\tilde-3{\emptyset}_{\mathbb{Q}_\xi}$.

Let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be such that ${}^{<\lambda}N \subseteq N$, $|N| = \lambda$ and $\bar{\mathbb{Q}}, D, \langle \bar{Y}^\xi, (\mathbb{Q}_\xi, \leq, \bar{\leq}_{\rm pr}) : \xi < \gamma\rangle, \ldots \in N$. Let $p \in N \cap \mathbb{P}_\gamma$ and let $\langle \tau_\alpha : \alpha < \lambda\rangle$ list all $\mathbb{P}_\gamma$–names for ordinals from N. Note that if $\xi \in \gamma \cap N$, then $\mathbf{st}^0_\xi \in N$.

By induction on $\delta < \lambda$ we will choose

$(\otimes)_\delta$ $\mathcal{T}_\delta, w_\delta, r^-_\delta, r_\delta, \bar{p}^\delta_*, \bar{q}^\delta_*$ and $\bar{p}_{\delta,\xi}, \bar{q}_{\delta,\xi}, \mathbf{st}_\xi$ for $\xi \in N \cap \gamma$

so that the following demands are satisfied.

$(*)_0$ All objects listed in $(\otimes)_\delta$ belong to N. After stage $\delta < \lambda$ of the construction, these objects are known for δ and $\xi \in w_\delta$.

$(*)_1$ $r^-_\delta, r_\delta \in \mathbb{P}_\gamma$, $r^-_0(0) = r_0(0) = p(0)$, $w_\delta \subseteq \gamma$, $|w_\delta| = |\delta + 1|$, $w_0 = \{0\}$, $w_\delta \subseteq w_{\delta+1}$, and if δ is limit then $w_\delta = \bigcup\limits_{\alpha<\delta} w_\alpha$, and

$$\bigcup_{\alpha<\lambda} \mathrm{dom}(r_\alpha) = \bigcup_{\alpha<\lambda} w_\alpha = N \cap \gamma.$$

$(*)_2$ For each $\alpha < \delta < \lambda$ we have $\big(\forall \xi \in w_{\alpha+1}\big)\big(r_\alpha(\xi) = r_\delta(\xi)\big)$ and $p \leq r^-_\alpha \leq r_\alpha \leq r^-_\delta \leq r_\delta$.

$(*)_3$ If $\xi \in (\gamma \setminus w_\delta) \cap N$, then

$$r_\delta{\restriction}\xi \Vdash \text{“ the sequence } \langle r^-_\alpha(\xi), r_\alpha(\xi) : \alpha \leq \delta\rangle \text{ is a legal partial play of } \partial^\lambda_0\big(\mathbb{Q}_\xi, \emptyset_{\mathbb{Q}_\xi}\big) \text{ in which Complete follows } \mathbf{st}^0_\xi \text{ ”}$$

and if $\xi \in w_{\delta+1} \setminus w_\delta$, then $\mathbf{st}_\xi \in N$ is a $\mathbb{P}_\xi$–name for a winning strategy of Generic in $\partial^{\rm main}_{\bar{Y}^\xi}(r_\delta(\xi), \mathbb{Q}_\xi, \leq, \bar{\leq}_{\rm pr}, D^{\mathbf{V}^{\mathbb{P}_\xi}})$. (And $\mathbf{st}_0 \in N$ is a winning strategy of Generic in $\partial^{\rm main}_{\bar{Y}^0}(p(0), \mathbb{Q}_0, \leq, \bar{\leq}_{\rm pr}, D)$.)

$(*)_4$ $\mathcal{T}_\delta = (T_\delta, \mathrm{rk}_\delta)$ is a standard $(w_\delta, 1)^\gamma$–tree, $T_\delta = \bigcup\limits_{\alpha\leq\gamma} \prod\limits_{\xi \in w_\delta\cap\alpha} Y^\xi_\delta$ (so T_δ consists of all sequences $\bar{t} = \langle t_\xi : \xi \in w_\delta \cap \alpha\rangle$ where $\alpha \leq \gamma$ and $t_\xi \in Y^\xi_\delta$).

$(*)_5$ $\bar{p}^\delta_* = \langle p^\delta_{*,t} : t \in T_\delta\rangle$ and $\bar{q}^\delta_* = \langle q^\delta_{*,t} : t \in T_\delta\rangle$ are standard trees of conditions, $\bar{p}^\delta_* \leq \bar{q}^\delta_*$.

$(*)_6$ For $t \in T_\delta$ we have that $\mathrm{dom}(p^\delta_{*,t}) = \big(\mathrm{dom}(p) \cup \bigcup\limits_{\alpha<\delta} \mathrm{dom}(r_\alpha) \cup w_\delta\big) \cap \mathrm{rk}_\delta(t)$ and for each $\xi \in \mathrm{dom}(p^\delta_{*,t}) \setminus w_\delta$:

$$p^\delta_{*,t}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \text{“ if the set } \{r_\alpha(\xi) : \alpha < \delta\} \cup \{p(\xi)\} \text{ has an upper bound in } \mathbb{Q}_\xi, \text{ then } p^\delta_{*,t}(\xi) \text{ is such an upper bound ”}.$$

$(*)_7$ For $\xi \in N \cap \gamma$, $\bar{p}_{\delta,\xi} = \langle p^\xi_{\delta,\eta} : \eta \in Y^\xi_\delta\rangle$ and $\bar{q}_{\delta,\xi} = \langle q^\xi_{\delta,\eta} : \eta \in Y^\xi_\delta\rangle$ are $\mathbb{P}_\xi$–names for systems of conditions in $\mathbb{Q}_\xi$ indexed by Y^ξ_δ.

$(*)_8$ If $\xi \in w_{\beta+1} \setminus w_\beta$, $\beta < \lambda$, then

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \langle \bar{p}_{\alpha,\xi}, \bar{q}_{\alpha,\xi} : \alpha < \lambda\rangle \text{ is a play of } \partial^{\rm main}_{\bar{Y}^\xi}(r_\beta(\xi), \mathbb{Q}_\xi, \leq, \bar{\leq}_{\rm pr}, D^{\mathbf{V}^{\mathbb{P}_\xi}}) \text{ in which Generic uses } \mathbf{st}_\xi \text{ ”}.$$

$(*)_9$ If $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \xi < \gamma$, then for each $\eta \in Y^\xi_\delta$

$$q^\delta_{*,t} \Vdash_{\mathbb{P}_\xi} \text{“ } p^\xi_{\delta,\eta} = p^\delta_{*,t\cup\{\langle\xi,\eta\rangle\}}(\xi) \text{ and } q^\xi_{\delta,\eta} = q^\delta_{*,t\cup\{\langle\xi,\eta\rangle\}}(\xi) \text{ ”}.$$

$(*)_{10}$ If $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \gamma$ and $\alpha < \delta$ and there is a condition $q \in \mathbb{P}_\gamma$ such that
(a) $q^\delta_{*,t} \leq q$, and
(b) $q{\restriction}\xi \Vdash_{\mathbb{P}_\xi} q^\delta_{*,t}(\xi) \leq^\delta_{\rm pr} q(\xi)$ for all $\xi \in w_\delta$ and
(c) q forces a value to τ_α,
then already the condition $q^\delta_{*,t}$ forces the value to τ_α.

$(*)_{11}$ $\mathrm{dom}(r^-_\delta)=\mathrm{dom}(r_\delta)=\bigcup\limits_{t\in T_\delta}\mathrm{dom}(q^\delta_{*,t})$ and if $t\in T_\delta$, $\xi\in\mathrm{dom}(r_\delta)\cap\mathrm{rk}_\delta(t)\setminus w_\delta$, and $q^\delta_{*,t}{\restriction}\xi\le q\in\mathbb{P}_\xi$, $r_\delta{\restriction}\xi\le q$, then

$$q\Vdash_{\mathbb{P}_\xi} \text{“ if the set } \{r_\alpha(\xi):\alpha<\delta\}\cup\{q^\delta_{*,t}(\xi),p(\xi)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\xi, \text{ then } r^-_\delta(\xi) \text{ is such an upper bound ”}.$$

We start with fixing an increasing continuous sequence $\langle w_\alpha:\alpha<\lambda\rangle$ of subsets of $N\cap\gamma$ such that the demands of $(*)_1$ are satisfied. Now, by induction on $\delta<\lambda$ we choose the other objects. So assume that we have defined all objects listed in $(\otimes)_\alpha$ for $\alpha<\delta$.

To ensure $(*)_0$, whenever we say “choose an X such that…” we mean “choose the $<^*_\chi$–first X such that…”. This convention will guarantee that our choices are from N.

If δ is a successor ordinal and $\xi\in w_\delta\setminus w_{\delta-1}$, then let $\underset{\sim}{\mathbf{st}}_\xi\in N$ be a $\mathbb{P}_\xi$–name for a winning strategy of Generic in $\Game^{\mathrm{main}}_{\bar{Y}^\xi}(r_{\delta-1}(\xi),\underset{\sim}{\mathbb{Q}}_\xi,\le,\bar{\le}_{\mathrm{pr}},D^{\mathbf{V}^{\mathbb{P}_\xi}})$. We also pick $\underset{\sim}{\bar{p}}_{\alpha,\xi},\underset{\sim}{\bar{q}}_{\alpha,\xi}$ for $\alpha<\delta$ so that $(*)_7+(*)_8$ hold (note that we already know $r_{\delta-1}(\xi)$ and by $(*)_2$ it is going to be equal to $r_\delta(\xi)$).

Clause $(*)_4$ fully describes $\mathcal{T}_\delta$. Note that, by the assumptions on $\bar{Y},\bar{\kappa}$,

$(*)_{12}$ $|T_\delta|\le(\kappa_\delta)^{|\delta|}<f(\delta)$ so also $|T_\delta|\cdot|\delta|<f(\delta)$.

For each $\xi\in w_\delta$ we choose a $\mathbb{P}_\xi$–name $\underset{\sim}{\bar{p}}_{\delta,\xi}$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \underset{\sim}{\bar{p}}_{\delta,\xi}=\langle \underset{\sim}{p}^\xi_{\delta,\eta}:\eta\in Y^\xi_\delta\rangle \text{ is given to Generic by } \underset{\sim}{\mathbf{st}}_\xi \text{ as an answer to } \langle \underset{\sim}{\bar{p}}_{\alpha,\xi},\underset{\sim}{\bar{q}}_{\alpha,\xi}:\alpha<\delta\rangle \text{ in the game } \Game^{\mathrm{main}}_{\bar{Y}^\xi}(r_\beta(\xi),\underset{\sim}{\mathbb{Q}}_\xi,\le,\bar{\le}_{\mathrm{pr}},D^{\mathbf{V}^{\mathbb{P}_\xi}}),\text{ ”}$$

where $\beta<\delta$ is such that $\xi\in w_{\beta+1}\setminus w_\beta$. (Note that for each $\xi\in w_\delta$ and distinct $\eta_0,\eta_1\in Y^\xi_\delta$ we have $\Vdash_{\mathbb{P}_\xi}$“ the conditions $\underset{\sim}{p}^\xi_{\delta,\eta_0},\underset{\sim}{p}^\xi_{\delta,\eta_1}$ are incompatible”.) Next we choose a tree of conditions $\bar{p}^\delta_*=\langle p^\delta_{*,t}:t\in T_\delta\rangle$ such that for each $t\in T_\delta$:

- $\mathrm{dom}(p^\delta_{*,t})=\big(\mathrm{dom}(p)\cup\bigcup\limits_{\alpha<\delta}\mathrm{dom}(r_\alpha)\cup w_\delta\big)\cap\mathrm{rk}_\delta(t)$ and
- for $\xi\in\mathrm{dom}(p^\delta_{*,t})\setminus w_\delta$, $p^\delta_{*,t}(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for a condition in $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$p^\delta_{*,t}{\restriction}\xi\Vdash_{\mathbb{P}_\xi} \text{“ if the set } \{r_\alpha(\xi):\alpha<\delta\}\cup\{p(\xi)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\xi, \text{ then } p^\delta_{*,t}(\xi) \text{ is such an upper bound ”},$$

- $p^\delta_{*,t}(\xi)=\underset{\sim}{\bar{p}}^\xi_{\delta,(t)_\xi}$ for $\xi\in\mathrm{dom}(p^\delta_{*,t})\cap w_\delta$.

Because of $(*)_{12}$ we may use Lemma 2.6 to pick a tree of conditions $\bar{q}^\delta_*=\langle q^\delta_{*,t}:t\in T_\delta\rangle$ such that

- $\bar{p}^\delta_*\le\bar{q}^\delta_*$,
- if $t\in T_\delta$, $\xi\in w_\delta\cap\mathrm{rk}_\delta(t)$, then $q^\delta_{*,t}{\restriction}\xi\Vdash_{\mathbb{P}_\xi} p^\delta_{*,t}(\xi)\le^\delta_{\mathrm{pr}} q^\delta_{*,t}(\xi)$,
- if $t\in T_\delta$, $\mathrm{rk}_\delta(t)=\gamma$ and $\alpha<\delta$ and there is a condition $q\in\mathbb{P}_\gamma$ such that
  (a) $q^\delta_{*,t}\le q$, and
  (b) $q{\restriction}\xi\Vdash_{\mathbb{P}_\xi} q^\delta_{*,t}(\xi)\le^\delta_{\mathrm{pr}} q(\xi)$ for all $\xi\in w_\delta$ and
  (c) q forces a value to $\underset{\sim}{\tau}_\alpha$,
  then $q^\delta_{*,t}$ forces a value to $\underset{\sim}{\tau}_\alpha$.

Note that if $\xi\in w_\delta$, $t\in T_\delta$, $\mathrm{rk}_\delta(t)=\xi$ and $\eta_0,\eta_1\in Y^\xi_\delta$ are distinct, then

$$q^\delta_{*,t}\Vdash_{\mathbb{P}_\xi} \text{“ the conditions } q^\delta_{*,t\cup\{\langle\xi,\eta_0\rangle\}}(\xi),q^\delta_{*,t\cup\{\langle\xi,\eta_1\rangle\}}(\xi) \text{ are incompatible ”}.$$

Therefore we may choose $\mathbb{P}_\xi$–names $\underset{\sim}{q}^\xi_{\delta,\eta}$ (for $\xi \in w_\delta$) such that

- $\Vdash_{\mathbb{P}_\xi}$ "$\underset{\sim}{\bar{q}}_{\delta,\xi} = \langle \underset{\sim}{q}^\xi_{\delta,\eta} : \eta \in Y^\xi_\delta\rangle$ is a system of conditions in $\underset{\sim}{\mathbb{Q}}_\xi$ indexed by Y^ξ_δ",
- $\Vdash_{\mathbb{P}_\xi}$ " $(\forall \eta \in Y^\xi_\delta)(\underset{\sim}{p}^\xi_{\delta,\eta} \leq^\delta_{\rm pr} \underset{\sim}{q}^\xi_{\delta,\eta})$ ",
- if $t \in T_\delta$, $\mathrm{rk}_\delta(t) > \xi$, then $q^\delta_{*,t\restriction\xi} \Vdash_{\mathbb{P}_\xi} q^\delta_{*,t}(\xi) = \underset{\sim}{q}^\xi_{\delta,(t)_\xi}$.

Finally, we define $r^-_\delta, r_\delta \in \mathbb{P}_\gamma$ so that

$$\mathrm{dom}(r^-_\delta) = \mathrm{dom}(r_\delta) = \bigcup_{t\in T_\delta} \mathrm{dom}(q^\delta_{*,t})$$

and

- $r^-_0(0) = r_0(0) = p(0)$,
- if $\xi \in w_{\alpha+1}$, $\alpha < \delta$, then $r^-_\delta(\xi) = r_\delta(\xi) = r_\alpha(\xi)$,
- if $\xi \in \mathrm{dom}(r^-_\delta) \setminus w_\delta$, then $r^-_\delta(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for an element of $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$r^-_\delta\restriction\xi \Vdash_{\mathbb{P}_\xi} \begin{array}{l} \text{" } r^-_\delta(\xi) \text{ is an upper bound of } \{r_\alpha(\xi) : \alpha < \delta\} \cup \{p(\xi)\} \text{ and} \\ \text{if } t \in T_\delta,\ \ \mathrm{rk}_\delta(t) > \xi, \text{ and } q^\delta_{*,t}\restriction\xi \in \underset{\sim}{G}_{\mathbb{P}_\xi} \text{ and the set} \\ \{r_\alpha(\xi) : \alpha < \delta\} \cup \{q^\delta_{*,t}(\xi), p(\xi)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\xi, \\ \text{then } r^-_\delta(\xi) \text{ is such an upper bound ",} \end{array}$$

  and $r_\delta(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for an element of $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$r_\delta\restriction\xi \Vdash_{\mathbb{P}_\xi} \begin{array}{l} \text{" } r_\delta(\xi) \text{ is given to Complete by } \underset{\sim}{\mathbf{st}}^0_\xi \text{ as the answer to} \\ \langle r^-_\alpha(\xi), r_\alpha(\xi) : \alpha < \delta\rangle^\frown\langle r^-_\delta(\xi)\rangle \text{ in the game } \Game^\lambda_0(\underset{\sim}{\mathbb{Q}}_\xi, \underset{\sim}{\emptyset}_{\mathbb{Q}_\xi}) \text{ ".} \end{array}$$

It follows from $(*)_2 + (*)_3$ from the previous stages that r^-_δ, r_δ are well defined and $p, r_\alpha \leq r^-_\delta \leq r_\delta$ for $\alpha < \delta$ (using induction on $\xi \in \mathrm{dom}(r_\delta)$).

This completes the description of the inductive definition of the objects listed in $(\otimes)_\delta$; it should be clear from the construction that demands $(*)_0$–$(*)_{11}$ are satisfied. For each $\xi \in w_{\beta+1} \setminus w_\beta$, $\beta < \lambda$, look at the sequence $\langle \underset{\sim}{\bar{p}}_{\delta,\xi}, \underset{\sim}{\bar{q}}_{\delta,\xi} : \delta < \lambda\rangle$ and use $(*)_8$ to choose a $\mathbb{P}_\xi$–name $q(\xi)$ for a condition in $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{" } q(\xi) \geq r_\beta(\xi) \text{ is aux-generic over } \langle \underset{\sim}{q}^\xi_{\delta,\eta} : \delta < \lambda \ \&\ \eta \in Y^\xi_\delta\rangle \text{ and } D^{\mathbf{V}^{\mathbb{P}_\xi}} \text{ "}$$

(if $\xi = 0$ then $q(0) \geq r_0(0)$ is aux-generic over $\langle q^0_{\delta,\eta} : \delta < \lambda \ \&\ \eta \in Y^0_\delta\rangle$, D). This determines a condition $q \in \mathbb{P}_\gamma$ with $\mathrm{dom}(q) = N \cap \gamma$. It follows from $(*)_2$ that $p \leq r_\beta \leq q$ for all $\beta < \lambda$.

Let us argue that q is $(N, \mathbb{P}_\gamma)$–generic. Let $\underset{\sim}{\tau} \in N$ be a $\mathbb{P}_\gamma$–name for an ordinal, say $\underset{\sim}{\tau} = \underset{\sim}{\tau}_{\alpha^*}$, $\alpha^* < \lambda$, and let us show that $q \Vdash \underset{\sim}{\tau} \in N$. So suppose towards contradiction that $q' \geq q$, $q' \Vdash \underset{\sim}{\tau} = \zeta$, $\zeta \notin N$. For each $\xi \in N \cap \gamma$ fix a $\mathbb{P}_\xi$–name $\underset{\sim}{\mathbf{st}}^+_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \begin{array}{l} \text{" } \underset{\sim}{\mathbf{st}}^+_\xi \text{ is a winning strategy of COM in} \\ \Game^{\rm aux}_{\bar{Y}^\xi}\big(q(\xi), \langle \underset{\sim}{q}^\xi_{\delta,\eta} : \delta < \lambda \ \&\ \eta \in Y^\xi_\delta\rangle, \underset{\sim}{\mathbb{Q}}_\xi, \leq, \bar{\leq}_{\rm pr}, D^{\mathbf{V}^{\mathbb{P}_\xi}}\big) \text{ ".} \end{array}$$

Construct inductively a sequence

$$\langle r^+_\alpha, r'_\alpha, \eta_\alpha(\xi), \underset{\sim}{\eta}_\alpha(\xi), \langle A^i_\alpha(\xi), \underset{\sim}{A}^i_\alpha(\xi) : i < \lambda\rangle, \underset{\sim}{A}_\alpha(\xi) : \alpha < \lambda \ \&\ \xi \in N \cap \gamma\rangle$$

such that the following demands $(*)_{13}$–$(*)_{15}$ are satisfied.

$(*)_{13}$ $r^+_\alpha, r'_\alpha \in \mathbb{P}_\gamma$, $r^+_0 = q$, $r'_0 \geq q'$ and $r^+_\beta \leq r'_\beta \leq r^+_\alpha$ for $\beta < \alpha < \lambda$.

$(*)_{14}$ For each $\xi\in N\cap\gamma$ and $\alpha<\lambda$ we have that $\eta_\alpha(\xi)\in{}^\alpha\lambda$, $A^i_\alpha(\xi)\in D$, $\underset{\sim}{\eta}_\alpha(\xi)$ is a $\mathbb{P}_\xi$–name for a member of ${}^\alpha\lambda$, $\underset{\sim}{A}^i_\alpha(\xi)$ is a $\mathbb{P}_\xi$–name for a member of D and $\underset{\sim}{A}_\alpha(\xi)$ is a $\mathbb{P}_\xi$–name for a member of $D^{\mathbf{V}^{\mathbb{P}_\xi}}$, and

$\Vdash_{\mathbb{P}_\xi}$ " $\langle(r^+_\alpha(\xi),\underset{\sim}{A}_\alpha(\xi),\underset{\sim}{\eta}_\alpha(\xi),r'_\alpha(\xi)):\alpha<\lambda\rangle$ is a result of a play of $\eth^{\rm aux}_{\bar{Y}^\xi}\big(q(\xi),\langle\underset{\sim}{q}^\xi_{\delta,\eta}:\delta<\lambda\ \&\ \eta\in Y^\xi_\delta\rangle,\underset{\sim}{\mathbb{Q}}_\xi,\leq,\bar{\leq}_{\rm pr},D^{\mathbf{V}^{\mathbb{P}_\xi}}\big)$ in which COM follows the strategy $\underset{\sim}{\mathbf{st}}^+_\xi$ ".

$(*)_{15}$ For $j,\beta\leq\alpha<\lambda$ and $\xi\in w_\alpha$ we have

$$r'_\alpha{\restriction}\xi\Vdash\text{" }\underset{\sim}{\eta}_\alpha(\xi)=\eta_\alpha(\xi)\ \ \&\ \ \mathop{\triangle}_{i<\lambda}\underset{\sim}{A}^i_\alpha(\xi)\subseteq\underset{\sim}{A}_\alpha(\xi)\ \ \&\ \ \underset{\sim}{A}^j_\beta(\xi)=A^j_\beta(\xi)\text{ ".}$$

(It should be clear how to carry out the construction; remember $\mathbb{P}_\gamma$ is $({<}\lambda)$–strategically complete, so in particular it does not add new members of ${}^\alpha\lambda$ for $\alpha<\lambda$.) Take a limit ordinal $\varepsilon>\alpha^*$ such that $\varepsilon\in\bigcap\limits_{\xi\in w_\varepsilon}\bigcap\limits_{i,j<\varepsilon}A^i_j(\xi)$. Then, by $(*)_{13}$–$(*)_{15}$, for each $\xi\in w_\varepsilon$ we have

$$r^+_\varepsilon{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{" }\varepsilon\in\mathop{\triangle}_{\alpha<\lambda}\underset{\sim}{A}_\alpha(\xi)\ \text{ and }\ \underset{\sim}{\eta}_\varepsilon(\xi)=\bigcup_{\alpha<\varepsilon}\eta_\alpha(\xi)=\eta_\varepsilon(\xi)\in Y^\xi_\varepsilon\text{ "}$$

and consequently, by $(*)_{14}$,

$(*)_{16}$ $r^+_\varepsilon{\restriction}\xi\Vdash_{\mathbb{P}_\xi}$ " $\underset{\sim}{q}^\xi_{\varepsilon,\eta_\varepsilon(\xi)}\leq^\varepsilon_{\rm pr}r^+_\varepsilon(\xi)$ " for each $\xi\in w_\varepsilon$.

Also note that

$(*)_{17}$ $p\leq r_\delta\leq q\leq r^+_\varepsilon$ for all $\delta<\lambda$.

Let $t\in T_\varepsilon$ be such that ${\rm rk}_\varepsilon(t)=\gamma$ and $(t)_\xi=\eta_\varepsilon(\xi)$ for $\xi\in w_\varepsilon$. By induction on $\xi\leq\gamma$, $\xi\in N$, we show that $q^\varepsilon_{*,t}{\restriction}\xi\leq_{\mathbb{P}_\xi}r^+_\varepsilon{\restriction}\xi$. So let us assume that $\xi<\gamma$ and we have shown that $q^\varepsilon_{*,t}{\restriction}\xi\leq_{\mathbb{P}_\xi}r^+_\varepsilon{\restriction}\xi$. If $\xi\in w_\varepsilon$ then by $(*)_9+(*)_{16}$ we have $q^\varepsilon_{*,t}{\restriction}(\xi+1)\leq_{\mathbb{P}_{\xi+1}}r^+_\varepsilon{\restriction}(\xi+1)$. So assume $\xi\notin w_\varepsilon$. Now, by $(*)_5$, $p^\varepsilon_{*,t}{\restriction}\xi\leq r^+_\varepsilon{\restriction}\xi$, so

$$r^+_\varepsilon{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{" }r_\alpha(\xi)\leq p^\varepsilon_{*,t}(\xi)\text{ for all }\alpha<\varepsilon\text{ "}$$

(remember $(*)_{17}+(*)_6$), and hence

$$r^+_\varepsilon{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{" }r_\alpha(\xi)\leq q^\varepsilon_{*,t}(\xi)\text{ for all }\alpha<\varepsilon\text{ ".}$$

Consequently, it follows from $(*)_{11}$ that

$$r^+_\varepsilon{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{" }q^\varepsilon_{*,t}(\xi)\leq r^-_\varepsilon(\xi)\leq r_\varepsilon(\xi)\leq r^+_\varepsilon(\xi)\text{ "}$$

and thus $q^\varepsilon_{*,t}{\restriction}(\xi+1)\leq_{\mathbb{P}_{\xi+1}}r^+_\varepsilon{\restriction}(\xi+1)$.

Now, since $q^\varepsilon_{*,t}\leq r^+_\varepsilon$ and $(*)_{16}$ holds, we may use the condition $(*)_{10}$ to conclude that $q^\varepsilon_{*,t}\Vdash_{\mathbb{P}_\gamma}\underset{\sim}{\tau}=\zeta$ (remember $q'\leq r^+_\varepsilon$, $\alpha^*<\varepsilon$) and consequently $\zeta\in N$, a contradiction. □

Remark 2.8. Semi–pure properness is very similar to being reasonably merry of [11, Section 6]. Despite of some differences in the parameters involved, one may suspect that the games are essentially the same if $\leq^\delta_{\rm pr}=\leq$. This would suggest that semi–pure properness is a weaker condition than being reasonably merry. However, the index sets Y_δ here are known *before* the master game starts, while in [11] the index sets I_δ are decided at the stage δ of the game. This makes our present notion somewhat stronger. Note that in our proof of the Iteration Theorem 2.7 we really have to know Y_δ's in advance – we cannot decide names for them and take care of

$(*)_8 + (*)_9$ at the same time. (This obstacle was not present in the proof of [11, Theorem 6.4] as there we did not deal with the auxilary relations $\leq^\delta_{\rm pr}$.)

It should be noted that some of the λ–semi-purely proper forcing notions discussed in the next section (see Proposition 3.6) are not reasonably merry as they do not have the bounding property of [11, Theorem 6.4(b)].

Problem 2.9. Are there any relationships between semi–pure properness and the properties introduced in [10, Definition A.3.6], [11, Defnitions 2.2, 6.3] ?

3. The Forcings

In this section we will show that our "last forcing standing" $\mathbb{Q}^2_\lambda$ and some of its relatives fit the framework of semi–pure properness (so their λ–support iterations preserve λ^+). A slight modification of $\mathbb{Q}^2_\lambda$ was used in iterations in Friedman and Zdomskyy [4] and Friedman, Honzik and Zdomskyy [3]. It was called Miller(λ) there and the main difference between the two forcings is in condition [4, Definition 2.1(vi)].

The filter D from the previous section will be the club filter, so it is not mentioned; also until Proposition 3.9 the auxiliary relations $\leq^\alpha_{\rm pr}$ do not depend on α, so instead of $\bar{\leq}_{\rm pr}$ we have just $\leq_{\rm pr}$ and $f(\alpha) = \lambda$ so we write λ instead of f (see Definition 2.3(6)).

For our results we have to assume that λ is strongly inaccessible; the case of successor λ remains untreated here (we will deal with it in a subsequent paper).

Definition 3.1. (1) Let $\mathbb{T}^{\rm club}$ be the family of all complete λ–trees $T \subseteq {}^{<\lambda}\lambda$ such that
- if $t \in T$, then $|{\rm succ}_T(t)| = 1$ or ${\rm succ}_T(t)$ is a club of λ, and
- $(\forall t \in T)(\exists s \in T)(t \vartriangleleft s \ \&\ |{\rm succ}_T(s)| > 1)$.

(2) We define a forcing notion $\mathbb{Q}^2_\lambda$ as follows:
a condition in $\mathbb{Q}^2_\lambda$ is a tree $T \in \mathbb{T}^{\rm club}$ such that
- if $\langle t_i : i < j\rangle \subseteq T$ is $\vartriangleleft$–increasing, $|{\rm succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup\limits_{i<j} t_i$, then ($t \in T$ and) $|{\rm succ}_T(t)| > 1$,

the order $\leq$ of $\mathbb{Q}^2_\lambda$ is the inverse inclusion, i.e., $T_1 \leq T_2$ if and only if $T_2 \subseteq T_1$.

(3) Forcings notions $\mathbb{Q}^1_\lambda, \mathbb{Q}^3_\lambda, \mathbb{Q}^4_\lambda$ are defined analogously, but
a condition in $\mathbb{Q}^1_\lambda$ is a tree $T \in \mathbb{T}^{\rm club}$ such that for every λ–branch $\eta \in \lim_\lambda(T)$ the set $\{\alpha \in \lambda : |{\rm succ}_T(\eta{\restriction}\alpha)| > 1\}$ contains a club of λ,
a condition in $\mathbb{Q}^3_\lambda$ is a tree $T \in \mathbb{T}^{\rm club}$ such that for some club $C \subseteq \lambda$ we have
$$(\forall t \in T)({\rm lh}(t) \in C \ \Rightarrow\ |{\rm succ}_T(t)| > 1),$$
a condition in $\mathbb{Q}^4_\lambda$ is a tree $T \in \mathbb{T}^{\rm club}$ such that
$$(\forall t \in T)({\rm root}(T) \vartriangleleft t \ \Rightarrow\ |{\rm succ}_T(t)| > 1).$$

(4) For $\ell = 1, 2, 3, 4$ we define a binary relation $\leq_{\rm pr}$ on $\mathbb{Q}^\ell_\lambda$ by
$T_1 \leq_{\rm pr} T_2$ if and only if $T_1 \leq T_2$ and ${\rm root}(T_1) = {\rm root}(T_2)$.

(5) Let $\mathbb{Q}^{1,*}_\lambda$ consists of all conditions $T \in \mathbb{Q}^1_\lambda$ such that for each λ–branch $\eta \in \lim_\lambda(T)$ the set $\{\alpha \in \lambda : |{\rm succ}_T(\eta{\restriction}\alpha)| > 1\}$ **is** a club of λ.

(6) Let $\mathbb{Q}^{3,*}_\lambda$ consists of all conditions $T \in \mathbb{Q}^3_\lambda$ such that for some club $C \subseteq \lambda$ we have

- if $t \in T$ and $\mathrm{lh}(t) \in C$, then $|\mathrm{succ}_T(t)| > 1$, and
- if $t \in T$ and $\mathrm{lh}(t) \notin C$, then $|\mathrm{succ}_T(t)| = 1$.

Observation 3.2. *Let $T \in \mathbb{T}^{\mathrm{club}}$. Then $T \in \mathbb{Q}^{1,*}_\lambda$ if and only if there exists a sequence $\langle F_\alpha : \alpha < \lambda\rangle$ of fronts of T such that*

- *if $\alpha < \beta < \lambda$, $t \in F_\beta$, then there is $s \in F_\alpha$ such that $s \vartriangleleft t$,*
- *if $\delta < \lambda$ is limit, $t_\alpha \in F_\alpha$ (for $\alpha < \delta$) are such that $t_\alpha \vartriangleleft t_\beta$ whenever $\alpha < \beta < \delta$, then $\bigcup\limits_{\alpha<\delta} t_\alpha \in F_\delta$,*
- *for each $t \in T$, $|\mathrm{succ}_T(t)| > 1$ if and only if $t \in \bigcup\limits_{\alpha<\lambda} F_\alpha$.*

Observation 3.3. $\mathbb{Q}^4_\lambda \subseteq \mathbb{Q}^{3,*}_\lambda \subseteq \mathbb{Q}^2_\lambda = \mathbb{Q}^{1,*}_\lambda \subseteq \mathbb{Q}^1_\lambda$ *and* $\mathbb{Q}^{3,*}_\lambda \subseteq \mathbb{Q}^3_\lambda \subseteq \mathbb{Q}^1_\lambda$, *and* $\mathbb{Q}^{3,*}_\lambda$ *is a dense subforcing of* $\mathbb{Q}^3_\lambda$.

Observation 3.4. *Let $\ell \in \{1,2,3,4\}$.*

(1) *$(\mathbb{Q}^\ell_\lambda, \le, \le_{\mathrm{pr}})$ is a forcing notion with λ–complete semi-purity.*
(2) *Moreover, the relations $(\mathbb{Q}^\ell_\lambda, \le)$ and $(\mathbb{Q}^\ell_\lambda, \le_{\mathrm{pr}})$ are $(<\lambda)$–complete.*

Lemma 3.5. *Let $0 < \ell \le 4$. Assume that $T^\delta \in \mathbb{Q}^\ell_\lambda$ and $F_\delta \subseteq T^\delta$ (for $\delta < \lambda$) are such that*

(i) *F_δ is a front of T^δ, $T^{\delta+1} \subseteq T^\delta$, and $F_\delta \subseteq T^{\delta+1}$,*
(ii) *if δ is limit, then $T^\delta = \bigcap\limits_{i<\delta} T^i$ and $F_\delta = \big\{t \in T^\delta : (\forall \xi < \delta)(\exists i < \mathrm{lh}(t))(t{\restriction}i \in F_\xi)$ and $(\forall i < \mathrm{lh}(t))(\exists \xi < \delta)(\exists j < \mathrm{lh}(\nu))(i < j \ \&\ \nu{\restriction}j \in F_\xi)\big\}$,*
(iii) *$(\forall t \in F_{\delta+1})(\exists s \in F_\delta)(s \vartriangleleft t)$,*
(iv) *if $t \in F_\delta$ and $|\mathrm{succ}_{T^\delta}(t)| > 1$, then $|\mathrm{succ}_{T^{\delta+1}}(t)| > 1$.*

Then $S \stackrel{\mathrm{def}}{=} \bigcap\limits_{\delta<\lambda} T^\delta \in \mathbb{Q}^\ell_\lambda$.

Proof. Plainly, S is a tree closed under unions of $\vartriangleleft$–chains shorter than λ, and by (i)–(iii) we see that for each $t \in S$ there is $s \in S$ such that $t \vartriangleleft s$. Hence S is a complete λ–tree.

Also, for each $\alpha < \lambda$ we have

(v) F_α is a front of S and for all $\beta \ge \alpha$
$$\{t \in S : (\exists s \in F_\alpha)(t \trianglelefteq s)\} = \{t \in T_\beta : (\exists s \in F_\alpha)(t \trianglelefteq s)\}.$$

Hence every splitting node in S splits into a club. Suppose now that $s \in S$ and let $\eta \in \lim_\lambda(S)$ be such that $s \vartriangleleft \eta$. Since $T^i \in \mathbb{Q}^1_\lambda$ (remember 3.3), the set $\{\alpha < \lambda : |\mathrm{succ}_{T^i}(\eta{\restriction}\alpha)| > 1\}$ contains a club (for each $i < \lambda$). Also the set $\{\alpha < \lambda : \eta{\restriction}\alpha \in F_\alpha\}$ is a club (remember (iii)+(ii)). So we may pick a limit ordinal $\delta < \lambda$ such that $\mathrm{lh}(s) < \delta$, $\eta{\restriction}\delta \in F_\delta$ and $|\mathrm{succ}_{T^i}(\eta{\restriction}\delta)| > 1$ for all $i < \delta$. Then (by (ii)) also $|\mathrm{succ}_{T^\delta}(\eta{\restriction}\delta)| > 1$ and hence (by (iv)+(iii)+(v)) $|\mathrm{succ}_S(\eta{\restriction}\delta)| > 1$ (and $s \vartriangleleft \eta{\restriction}\delta$). So we may conclude that $S \in \mathbb{T}^{\mathrm{club}}$. We will argue that $S \in \mathbb{Q}^\ell_\lambda$ considering the four possible values of ℓ separately.

Case $\ell = 1$

Suppose $\eta \in \lim_\lambda(S)$. Then for each $\delta < \lambda$ the set $\{\alpha < \lambda : |\mathrm{succ}_{T^\delta}(\eta{\restriction}\alpha)| > 1\}$ contains a club and thus the set
$$A \stackrel{\mathrm{def}}{=} \big\{\alpha < \lambda : \alpha \text{ is limit and } (\forall \delta < \alpha)(|\mathrm{succ}_{T^\delta}(\eta{\restriction}\alpha)| > 1) \text{ and } \eta{\restriction}\alpha \in F_\alpha\big\}$$
contains a club. But if $\alpha \in A$, then also $|\mathrm{succ}_{T^\alpha}(\eta{\restriction}\alpha)| > 1$ and hence $|\mathrm{succ}_S(\eta{\restriction}\alpha)| > 1$ (remember (ii)+(iv)).

Case $\ell = 2$
Suppose that a sequence $\langle s_i : i < j\rangle \subseteq S$ is $\lhd$–increasing and $|\mathrm{succ}_S(s_i)| > 1$ for all $i < j$. Let $s = \bigcup\limits_{i<j} s_i$ and $\delta = \mathrm{lh}(s)$. Then also $|\mathrm{succ}_{T^\delta}(s_i)| > 1$ (for all $i < j$) and hence $|\mathrm{succ}_{T^\delta}(s)| > 1$. By (v)+(iv)+(iii)+(i) we easily conclude $|\mathrm{succ}_S(s)| > 1$ (note that $s \trianglelefteq t$ for some $t \in F_\delta$).
Case $\ell = 3$
Let $C_\delta \subseteq \lambda$ be a club such that

$$\alpha \in C_\delta \ \&\ t \in T^\delta \cap {}^\alpha\lambda \quad \Rightarrow \quad |\mathrm{succ}_{T^\delta}(t)| > 1.$$

Set $C = \mathop{\triangle}\limits_{\delta<\lambda} C_\delta$. Then for each limit $\alpha \in C$ and $t \in S \cap {}^\alpha\lambda$ we have that $|\mathrm{succ}_{T^\delta}(t)| > 1$ for all $\delta < \alpha$, and hence also $|\mathrm{succ}_{T^\alpha}(t)| > 1$ (by (ii)). Invoking (v)+(iv) we see that $|\mathrm{succ}_S(t)| > 1$ whenever $t \in S$, $\mathrm{lh}(t) \in C$ is limit.
Case $\ell = 4$
If $\mathrm{root}(S) \lhd s \in S$, then $|\mathrm{succ}_{T^\delta}(s)| > 1$ for all $\delta < \lambda$ and hence $|\mathrm{succ}_S(s)| > 1$ (remember (v)). $\square$

Proposition 3.6. *Let λ be a strongly inaccessible cardinal, $Y_\delta = {}^\delta\delta$ for $\delta < \lambda$ and $\bar{Y} = \langle Y_\delta : \delta < \lambda\rangle$. Then the forcing notions $(\mathbb{Q}^\ell_\lambda, \leq, \leq_{\mathrm{pr}})$ for $\ell \in \{2, 3, 4\}$ are λ-semi–purely proper over $\bar{Y}$.*

Proof. Let $1 < \ell \leq 4$, $T \in \mathbb{Q}^\ell_\lambda$. Consider the following strategy **st** of Generic in the game $\Game^{\mathrm{main}}_{\bar{Y}}(T, \mathbb{Q}^\ell_\lambda, \leq, \leq_{\mathrm{pr}})$.

In the course of the play, in addition to her innings $\langle T_{\delta,\eta} : \eta \in Y_\delta\rangle$, Generic chooses also sets $A_\delta \subseteq Y_\delta$ and conditions $T^\delta \in \mathbb{Q}^\ell_\lambda$ so that T^δ is decided before the stage δ of the game. Suppose that the two players arrived to a stage $\delta < \lambda$. If $\delta = 0$ then Generic lets $T^0 = T$ and if δ is limit, then she puts $T^\delta = \bigcap\limits_{i<\delta} T^i$ (in both cases $T^\delta \in \mathbb{Q}^\ell_\lambda$). Now Generic determines A_δ and $\langle T_{\delta,\eta} : \eta \in Y_\delta\rangle$ as follows. She sets $A_\delta = T^\delta \cap Y_\delta$ and then she lets $\langle T_{\delta,\eta} : \eta \in Y_\delta\rangle \subseteq \mathbb{Q}^\ell_\lambda$ be a system of pairwise incompatible conditions chosen so that

- if $\eta \in A_\delta$ then $T_{\delta,\eta} = (T^\delta)_\eta$.

Generic's inning at this stage is $\langle T_{\delta,\eta} : \eta \in Y_\delta\rangle$. After this Antigeneric answers with a system $\langle S_{\delta,\eta} : \eta \in Y_\delta\rangle \subseteq \mathbb{Q}^\ell_\lambda$ such that $T_{\delta,\eta} \leq_{\mathrm{pr}} S_{\delta,\eta}$, and then Generic writes aside

$$T^{\delta+1} \stackrel{\mathrm{def}}{=} \big\{t \in T^\delta : \big(\exists\eta \in A_\delta\big)\big(\eta \trianglelefteq t \ \&\ t \in S_{\delta,\eta}\big) \text{ or } \big(\forall\alpha \leq \mathrm{lh}(t)\big)\big(t{\restriction}\alpha \notin A_\delta\big)\big\}.$$

It should be clear that $T^{\delta+1}$ is a condition in $\mathbb{Q}^\ell_\lambda$.

After the play is finished and sequences

$$\langle T_{\delta,\eta}, S_{\delta,\eta} : \delta < \lambda \ \&\ \eta \in Y_\delta\rangle \quad \text{and} \quad \langle A_\delta, T^\delta : \delta < \lambda\rangle$$

have been constructed, Generic lets

$$S = \bigcap_{\delta<\lambda} T^\delta \subseteq T.$$

Claim 3.6.1. *$S \in \mathbb{Q}^\ell_\lambda$ is aux-generic over $\bar{S} = \langle S_{\delta,\eta} : \delta < \lambda \ \&\ \eta \in Y_\delta\rangle$.*

Proof of the Claim. First note that the sequence $\langle T^\delta, F_\delta = T^\delta \cap {}^\delta\lambda : \delta < \lambda\rangle$ satisfies the assumptions of Lemma 3.5 and hence $S \in \mathbb{Q}^\ell_\lambda$.

Now we consider the three possible cases separately.

Case $\ell = 2$.
Let us describe a strategy $\mathbf{st}^*$ of COM in the game $\Game^{\rm aux}_{\bar{Y}}(S,\bar{S},\mathbb{Q}^2_\lambda,\leq,\leq_{\rm pr})$. It instructs COM to play as follows. Aside, COM picks also ordinals $\xi_\delta<\lambda$ so that after arriving at a stage $\delta<\lambda$, when a sequence $\langle(S_\alpha,A_\alpha,\eta_\alpha,S'_\alpha),\xi_\alpha:\alpha<\delta\rangle$ has been already constructed, she answers with $S_\delta,A_\delta,\eta_\delta$ (and ξ_δ) chosen so that the following demands are satisfied.

(A) $S_0=S$, $\xi_0={\rm lh}({\rm root}(S))+942$, $A_0=[\xi_0,\lambda)$ and $\eta_0=\langle\rangle$.

(B) If δ is a successor ordinal, say $\delta=\alpha+1$, then
$$\eta_\alpha\vartriangleleft\eta_\delta\in S'_\alpha\cap{}^\delta\lambda,\quad \xi_\delta=\xi_\alpha+\sup(\eta_\delta(i):i<\delta)+{\rm lh}\big({\rm root}\big((S'_\alpha)_{\eta_\delta}\big)\big)+942,$$
$A_\delta=[\xi_\delta,\lambda)$ and $S_\delta=(S'_\alpha)_{\eta_\delta}$. (Note: then we will also have $\eta_\delta\in S'_\delta$.)

(C) If δ is a limit ordinal, then $\eta_\delta=\bigcup\limits_{\alpha<\delta}\eta_\alpha$, $\xi_\delta=\sup(\xi_\alpha:\alpha<\delta)+942$, $A_\delta=[\xi_\delta,\lambda)$ and $S_\delta=\bigcap\limits_{\alpha<\delta}S_\alpha=\bigcap\limits_{\alpha<\delta}S'_\alpha=\bigcap\limits_{\alpha<\delta}\big(S'_\alpha\big)_{\eta_\delta}$. (Note: then we will also have $\eta_\delta\in S'_\delta$.)

Note that if $\langle(S_\alpha,A_\alpha,\eta_\alpha,S'_\alpha):\alpha<\lambda\rangle$ is a play in which COM follows $\mathbf{st}^*$ and $\delta\in\mathop{\triangle}\limits_{\alpha<\lambda}A_\alpha$ is a limit ordinal, then $\eta_\delta\in Y_\delta\cap S$ and it is a limit of splitting points in S'_α, so also $|{\rm succ}_{S'_\alpha}(\eta_\delta)|>1$ for all $\alpha<\delta$. Therefore, by (C), η_δ is a splitting node in S_δ (and in S as well). It follows from the description of the δth move of Generic in $\Game^{\rm main}_{\bar{Y}}(T,\mathbb{Q}^2_\lambda,\leq,\leq_{\rm pr})$, that
$$(T^\delta)_{\eta_\delta}\leq_{\rm pr}S_{\delta,\eta_\delta}=(T^{\delta+1})_{\eta_\delta}\leq_{\rm pr}(S)_{\eta_\delta}\leq_{\rm pr}S_\delta.$$
Consequently, $\mathbf{st}^*$ is a winning strategy for COM.
Case $\ell=3$.
The winning strategy $\mathbf{st}^*$ of COM in the game $\Game^{\rm aux}_{\bar{Y}}(S,\bar{S},\mathbb{Q}^3_\lambda,\leq,\leq_{\rm pr})$ is almost exactly the same as in the previous case. The only difference is that now COM shrinks the answers S'_α of INC to members of $\mathbb{Q}^{3,*}_\lambda$ pretending they were played in the game. The argument that this is a winning strategy is exactly the same as before (as $\mathbb{Q}^{3,*}_\lambda\subseteq\mathbb{Q}^2_\lambda$).
Case $\ell=4$. Similar. $\square$

$\square$

The forcing notions considered above can be slightly generalized by allowing the use of filters other than the club filter on λ. The forcing notions $\mathbb{Q}^{\bar{E}}_E$ of [11, Definition 1.11] and $\mathbb{P}^{\bar{E}}_E$ of [11, Definition 4.2] follow this pattern. However, to apply the iteration theorems of [11] we need to assume that the filter E controlling splittings along branches is concentrated on a stationary co-stationary set. Therefore the case of E being the club filter seems to be of a different character. Putting general filters on the splitting nodes *only* and controlling the splitting levels by the club filter leads to Definition 3.7.

The forcing notion $\mathbb{Q}^{2,\bar{E}}$ was studied by Brown and Groszek [1] who described when this forcing adds a generic of minimal degree.

Definition 3.7. Suppose that $\bar{E}=\langle E_t:t\in{}^{<\lambda}\lambda\rangle$ is a system of $(<\lambda)$–complete filters on λ. (These could be principal filters.) We define forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ for $\ell=1,2,3,4$ as follows:

(1) **A condition** in $\mathbb{Q}^{2,\bar{E}}$ is a complete λ–tree $T\subseteq{}^{<\lambda}\lambda$ such that
  (a) if $t\in T$, then $|{\rm succ}_T(t)|=1$ or ${\rm succ}_T(t)\in E_t$, and

(b) $(\forall t\in T)(\exists s\in T)(t\vartriangleleft s\ \&\ |\mathrm{succ}_T(s)|>1)$, and

$(c)^2$ if $\langle t_i : i<j\rangle\subseteq T$ is $\vartriangleleft$–increasing, $|\mathrm{succ}_T(t_i)|>1$ for all $i<j$ and $t=\bigcup\limits_{i<j} t_i$, then ($t\in T$ and) $|\mathrm{succ}_T(t)|>1$,

the order $\leq$ of $\mathbb{Q}^{2,\bar{E}}$ is the inverse inclusion, i.e., $T_1\leq T_2$ if and only if $T_2\subseteq T_1$.

(2) Forcings notions $\mathbb{Q}^{1,\bar{E}},\mathbb{Q}^{3,\bar{E}},\mathbb{Q}^{4,\bar{E}}$ are defined analogously, but the demand $(c)^2$ is replaced by the respective $(c)^\ell$:

$(c)^1$ for every λ–branch $\eta\in\lim_\lambda(T)$ the set $\{\alpha\in\lambda : |\mathrm{succ}_T(\eta{\restriction}\alpha)|>1\}$ contains a club of λ,

$(c)^3$ for some club $C\subseteq\lambda$ we have

$$(\forall t\in T)(\mathrm{lh}(t)\in C\ \Rightarrow\ |\mathrm{succ}_T(t)|>1),$$

$(c)^4$ $(\forall t\in T)(\mathrm{root}(T)\vartriangleleft t\ \Rightarrow\ |\mathrm{succ}_T(t)|>1)$.

(3) For $\ell=1,2,3,4$ we define a binary relation $\leq_{\mathrm{pr}}$ on $\mathbb{Q}^{\ell,\bar{E}}$ by $T_1\leq_{\mathrm{pr}} T_2$ if and only if $T_1\leq T_2$ and $\mathrm{root}(T_1)=\mathrm{root}(T_2)$.

Remark 3.8. Since in Definition 3.7 we allow the filters E_t to be principal, we may fit some classical forcings into our schema. If $E_t=\{\lambda\}$ for each $t\in {}^{<\lambda}\lambda$, then $\mathbb{Q}^{4,\bar{E}}$ is the λ–Cohen forcing $\mathbb{C}_\lambda$ (see Definition 4.2(1)) and $\mathbb{Q}^{2,\bar{E}}$ is the forcing $\mathbb{D}_\lambda$ from [8, Definition 4.9(b)]. If for each $t\in {}^{<\lambda}\lambda$ we let E_t be the filter of all subsets of λ including $\{0,1\}$, then the forcing notion $\mathbb{Q}^{2,\bar{E}}$ will be equivalent with Kanamori's λ–Sacks forcing of [6, Definition 1.1].

Proposition 3.9. *Let $\bar{E}=\langle E_t : t\in {}^{<\lambda}\lambda\rangle$ be a system of $({<}\lambda)$–complete filters on λ and $\ell\in\{1,2,3,4\}$.*

(1) *$(\mathbb{Q}^{\ell,\bar{E}},\leq,\leq_{\mathrm{pr}})$ is a forcing notion with λ–complete semi-purity. Moreover, the relations $(\mathbb{Q}^{\ell,\bar{E}},\leq)$ and $(\mathbb{Q}^{\ell,\bar{E}},\leq_{\mathrm{pr}})$ are $(<\lambda)$–complete.*

(2) *If λ is strongly inaccessible, $Y_\delta={}^\delta\delta$ for $\delta<\lambda$ and $\bar{Y}=\langle Y_\delta:\delta<\lambda\rangle$, then the forcing notions $(\mathbb{Q}^{\ell,\bar{E}},\leq,\leq_{\mathrm{pr}})$ for $\ell\in\{2,3,4\}$ are λ-semi–purely proper over $\bar{Y}$.*

Proof. Same as 3.4, 3.6. □

Close relatives of the forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ were considered in [10, Section B.8] and [11, Definition 4.6]. The modification now is that we consider trees branching into less than λ successor nodes (but there are many successors from the point of view of suitably complete filters).

Definition 3.10. Assume that

- λ is strongly inaccessible, $f:\lambda\longrightarrow\lambda$ is a increasing function such that each $f(\alpha)$ is a regular uncountable cardinal and $\prod\limits_{\xi<\alpha} f(\xi)^{|\alpha|}<f(\alpha)$ (for $\alpha<\lambda$),
- $\bar{F}=\langle F_t : t\in\bigcup\limits_{\alpha<\lambda}\prod\limits_{\xi<\alpha} f(\xi)\rangle$ where F_t is a ${<}f(\alpha)$–complete filter on $f(\alpha)$ whenever $t\in\prod\limits_{\xi<\alpha} f(\xi)$, $\alpha<\lambda$.

(1) We define a forcing notion $\mathbb{Q}^1_{f,\bar{F}}$ as follows.
A condition in $\mathbb{Q}^1_{f,\bar{F}}$ is a complete λ–tree $T\subseteq\bigcup\limits_{\alpha<\lambda}\prod\limits_{\xi<\alpha} f(\xi)$ such that

(a) for every $t\in T$, either $|\mathrm{succ}_T(t)|=1$ or $\mathrm{succ}_T(t)\in F_t$, and

(b) $(\forall t \in T)(\exists s \in T)(t \vartriangleleft s \ \& \ |\mathrm{succ}_T(s)| > 1)$, and

$(c)^1$ for every $\eta \in \lim_\lambda(T)$ the set $\{\alpha < \lambda : \mathrm{succ}_T(\eta{\restriction}\alpha) \in F_{\eta{\restriction}\alpha}\}$ contains a club of λ .

The order of $\mathbb{Q}^1_{f,\bar{F}}$ is the reverse inclusion.

(2) Forcing notions $\mathbb{Q}^\ell_{f,\bar{F}}$ for $\ell = 2,3,4$ are defined similarly, but the demand $(c)^1$ is replaced by the respective $(c)^\ell$:

$(c)^2$ if $\langle t_i : i < j\rangle \subseteq T$ is $\vartriangleleft$–increasing, $|\mathrm{succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup\limits_{i<j} t_i$, then ($t \in T$ and) $|\mathrm{succ}_T(t)| > 1$,

$(c)^3$ for some club C of λ we have

$$\big(\forall t \in T\big)\big(\mathrm{lh}(t) \in C \ \Rightarrow\ \mathrm{succ}_T(T) \in F_t\big).$$

$(c)^4$ $(\forall t \in T)(\mathrm{root}(T) \vartriangleleft t \ \Rightarrow\ |\mathrm{succ}_T(t)| > 1)$.

(3) For $\ell = 1,2,3,4$ and $\alpha < \lambda$ we define a binary relation $\le^\alpha_{\mathrm{pr}}$ on $\mathbb{Q}^\ell_{f,\bar{F}}$ by $T_1 \le^\alpha_{\mathrm{pr}} T_2$ if and only if either $T_1 = T_2$ or $T_1 \le T_2$, $\mathrm{root}(T_1) = \mathrm{root}(T_2)$ and $\mathrm{lh}(\mathrm{root}(T_2)) \ge \alpha$.

Proposition 3.11. *Assume $\lambda, f, \bar{F}$ are as in 3.10.*

(1) $(\mathbb{Q}^\ell_{f,\bar{F}}, \le, \bar{\le}_{\mathrm{pr}})$ *is a forcing notion with f–complete semi-purity.*

(2) *If $Y_\delta = \prod\limits_{\xi<\delta} f(\xi)$ for $\delta < \lambda$ and $\bar{Y} = \langle Y_\delta : \delta < \lambda\rangle$, then $\bar{Y}$ is an indexing sequence and the forcing notions $(\mathbb{Q}^\ell_{f,\bar{F}}, \le, \bar{\le}_{\mathrm{pr}})$ for $\ell \in \{2,3,4\}$ are f-semi–purely proper over $\bar{Y}$.*

Proof. Similar to 3.4, 3.6. □

Observation 3.12. *Let $\eta \in {}^\lambda\lambda$ and $Y_\alpha = \{\eta{\restriction}\alpha\}$ for $\alpha < \lambda$. Suppose that $(\mathbb{Q}, \le)$ is a strategically $(\le\lambda)$–complete forcing notion and let $\le^\alpha_{\mathrm{pr}}$ be $\le$ (for $\alpha < \lambda$). Then $\mathbb{Q}$ is λ–semi-purely proper over $\langle Y_\xi : \xi < \lambda\rangle$ and the club filter with $\langle \le^\alpha_{\mathrm{pr}} : \alpha < \lambda\rangle$ witnessing this.*

Corollary 3.13. *Let λ be a strongly inaccessible cardinal. Suppose that $\bar{E}$ is as in 3.7 and $f, \bar{F}$ are as in 3.10. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ be a λ–support iteration such that for every $\xi < \gamma$ the iterand $\underset{\sim}{\mathbb{Q}}_\xi$ is either strategically $(\le\lambda)$–complete, or it is one of $\mathbb{Q}^2_\lambda, \mathbb{Q}^3_\lambda, \mathbb{Q}^4_\lambda, \mathbb{Q}^{2,\bar{E}}, \mathbb{Q}^{3,\bar{E}}, \mathbb{Q}^{4,\bar{E}}, \mathbb{Q}^2_{f,\bar{F}}, \mathbb{Q}^3_{f,\bar{F}}, \mathbb{Q}^4_{f,\bar{F}}$. Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is λ–proper in the standard sense.*

4. Are $\mathbb{Q}^2_\lambda$, $\mathbb{Q}^1_\lambda$ very different?

The forcing notions $\mathbb{Q}^1_\lambda$ and $\mathbb{Q}^2_\lambda$ appear to be very close. In this section we will show that, consistently, they are equivalent, but also consistently, they may be different.

Lemma 4.1. *Assume $T \in \mathbb{T}^{\mathrm{club}}$ and consider $(T, \vartriangleleft)$ as a forcing notion. Let $\underset{\sim}{\eta}$ be a T–name for the generic λ–branch added by T. Suppose that*

$$\Vdash_T \text{ `` the set } \{\alpha < \lambda : |\mathrm{succ}_T(\underset{\sim}{\eta}{\restriction}\alpha)| > 1\} \text{ contains a club ''}.$$

Then there is $T^* \subseteq T$ such that $T^* \in \mathbb{Q}^2_\lambda$.

Proof. Let $\underset{\sim}{C}$ be a T–name for a club of λ such that

$$\Vdash_T \text{ `` } \big(\forall \alpha \in \underset{\sim}{C}\big)\big(|\mathrm{succ}_T(\underset{\sim}{\eta}{\restriction}\alpha)| > 1\big) \text{ ''},$$

and put

$$S=\big\{t\in T: \mathrm{lh}(t)\text{ is a limit ordinal and } t\Vdash_T \text{`` } \big(\forall\alpha<\mathrm{lh}(t)\big)\big(\underset{\sim}{C}\cap(\alpha,\mathrm{lh}(t))\neq\emptyset\big)\big\}\text{ ''}.$$

One easily verifies that

(i) if $t\in S$, then $t\Vdash_T$ “ $\mathrm{lh}(t)\in\underset{\sim}{C}$ ” and hence $|\mathrm{succ}_T(t)|>1$,
(ii) if a sequence $\langle t_\alpha:\alpha<\alpha^*\rangle\subseteq S$ is $\vartriangleleft$–increasing, $\alpha^*<\lambda$, then $\bigcup\limits_{\alpha<\alpha^*} t_\alpha\in S$,
(iii) $(\forall t\in T)(\exists s\in S)(t\vartriangleleft s)$.

Consequently we may choose $T^*\subseteq T$ so that $T^*\in\mathbb{T}^{\rm club}$ and for some fronts F_α of T^* (for $\alpha<\lambda$) we have

- $F_\alpha\subseteq S$, and if $\alpha<\beta<\lambda$, $t\in F_\beta$, then for some $s\in F_\alpha$ we have $s\vartriangleleft t$,
- if $\delta<\lambda$ is limit and $t_\alpha\in F_\alpha$ for $\alpha<\delta$ are such that $\alpha<\beta<\delta\ \Rightarrow t_\alpha\vartriangleleft t_\beta$, then $\bigcup\limits_{\alpha<\delta} t_\alpha\in F_\delta$,
- $|\mathrm{succ}_{T^*}(t)|>1$ if and only if $t\in\bigcup\limits_{\alpha<\lambda} F_\alpha$.

Then also $T^*\in\mathbb{Q}^{1,*}_\lambda=\mathbb{Q}^2_\lambda$ (remember Observations 3.2, 3.3). $\square$

Definition 4.2. (1) The λ–Cohen forcing notion $\mathbb{C}_\lambda$ is defined as follows:
a condition in $\mathbb{C}_\lambda$ is a sequence $\nu\in{}^{<\lambda}\lambda$,
the order $\leq$ **of** $\mathbb{C}_\lambda$ is the extension of sequences (i.e., $\nu_1\leq\nu_2$ if and only if $\nu_1\trianglelefteq\nu_2$).

(2) The axiom $\mathrm{Ax}^+_{\mathbb{C}_\lambda}$ is the following statement:
if $\underset{\sim}{S}$ is a $\mathbb{C}_\lambda$–name and $\Vdash_{\mathbb{C}_\lambda}$ “ $\underset{\sim}{S}$ is a stationary subset of λ ”, and $\mathcal{O}_\alpha\subseteq\mathbb{C}_\lambda$ are open dense sets (for $\alpha<\lambda$) then there is a $\trianglelefteq$–directed $\trianglelefteq$–downward closed set $H\subseteq\mathbb{C}_\lambda$ such that
   - $H\cap\mathcal{O}_\alpha\neq\emptyset$ for all $\alpha<\lambda$, and
   - the interpretation $\underset{\sim}{S}[H]$ of the name $\underset{\sim}{S}$ is a stationary subset of λ.

Lemma 4.3. *Let $T\in\mathbb{T}^{\rm club}$. Then the following conditions are equivalent:*

(a) *there is $T^*\subseteq T$ such that $T^*\in\mathbb{Q}^2_\lambda$,*
(b) *there is $T^*\subseteq T$ such that $T^*\in\mathbb{T}^{\rm club}$ and*

$$\Vdash_{\mathbb{C}_\lambda}(\forall\eta\in\lim_\lambda(T^*))(\textit{the set }\{\delta{<}\lambda:|\mathrm{succ}_{T^*}(\eta{\restriction}\delta)|>1\}\textit{ contains a club of }\lambda).$$

Proof. Assume (a). By the $({<}\lambda)$–completeness of $\mathbb{C}_\lambda$ we see that $\Vdash_{\mathbb{C}_\lambda} T^*\in\mathbb{Q}^2_\lambda$, and hence $\Vdash_{\mathbb{C}_\lambda} T^*\in\mathbb{Q}^1_\lambda$ (remember Observation 3.3). Consequently (b) follows.

Now assume (b). Since $(T^*,\trianglelefteq)$ (as a forcing notion) is isomorphic with $\mathbb{C}_\lambda$ we have

$$\Vdash_{T^*}(\forall\eta\in\lim_\lambda(T^*))(\text{the set }\{\delta{<}\lambda:|\mathrm{succ}_T(\eta{\restriction}\delta)|>1\}\text{ contains a club of }\lambda),$$

so in particular

$$\Vdash_{T^*}\text{ `` the set }\{\delta<\lambda:|\mathrm{succ}_T(\underset{\sim}{\eta}{\restriction}\delta)|>1\}\text{ contains a club of }\lambda\text{ '',}$$

where $\underset{\sim}{\eta}$ is a T^*–name for the generic λ–branch. It follows now from Lemma 4.1 that (a) holds. $\square$

Proposition 4.4. *Assume $\mathrm{Ax}^+_{\mathbb{C}_\lambda}$. Then $\mathbb{Q}^2_\lambda$ is a dense subset of $\mathbb{Q}^1_\lambda$ (so the forcing notions $\mathbb{Q}^1_\lambda,\mathbb{Q}^2_\lambda$ are equivalent).*

Proof. Let $T \in \mathbb{Q}^1_\lambda$ and let us consider $(T, \trianglelefteq)$ as a forcing notion. Let $\underset{\sim}{S}$ be a T–name given by

$$\Vdash_T \underset{\sim}{S} = \{\delta < \lambda : |\mathrm{succ}_T(\underset{\sim}{\eta}\restriction\delta)| > 1\}$$

where $\underset{\sim}{\eta}$ is a T–name for the generic λ–branch. Ask the following question

- Does $\Vdash_T$ " $\underset{\sim}{S}$ contains a club of λ " ?

If the answer is "yes", then by Lemma 4.1 there is $T^* \subseteq T$ such that $T^* \in \mathbb{Q}^2_\lambda$.

So assume that the answer to our question is "not". Then there is $t \in T$ such that

$$t \Vdash_T \text{" } \lambda \setminus \underset{\sim}{S} \text{ is stationary "}.$$

Let $\underset{\sim}{S}' = \{(\check{\alpha}, s) : s \in T \text{ and } \alpha = \mathrm{lh}(s) \text{ and } |\mathrm{succ}_T(s)| = 1\}$. Then $\underset{\sim}{S}'$ is a T–name for a subset of λ and $\Vdash_T \underset{\sim}{S}' = \lambda \setminus \underset{\sim}{S}$. Therefore, $t \Vdash_T$ " $\underset{\sim}{S}'$ is stationary " and since the forcing notion T above t is isomorphic with $\mathbb{C}_\lambda$, we may use the assumption of $\mathrm{Ax}^+_{\mathbb{C}_\lambda}$ to pick a $\trianglelefteq$–directed $\trianglelefteq$–downward closed set $H \subseteq T$ such that $t \in H$ and

- $H \cap \{s \in T : \mathrm{lh}(s) > \alpha\} \neq \emptyset$ for all $\alpha < \lambda$, and
- $\underset{\sim}{S}'[H]$ is stationary in λ.

Then for each $\alpha < \lambda$ the intersection $H \cap {}^\alpha\lambda$ is a singleton, say $H \cap {}^\alpha\lambda = \{\eta_\alpha\}$, and

- if $\alpha < \beta$ then $\eta_\alpha \vartriangleleft \eta_\beta$, and
- $\alpha \in \underset{\sim}{S}'[H]$ if and only if $|\mathrm{succ}_T(\eta_\alpha)| = 1$.

Let $\eta = \bigcup\limits_{\alpha<\lambda} \eta_\alpha$. Then $\eta \in \lim_\lambda(T)$ and the set $\{\alpha < \lambda : |\mathrm{succ}_T(\eta\restriction\alpha)| = 1\}$ is stationary, contradicting $T \in \mathbb{Q}^1_\lambda$. $\square$

Proposition 4.5. *It is consistent that $\mathbb{Q}^2_\lambda$ is not dense in $\mathbb{Q}^1_\lambda$.*

Proof. We will build a $({<}\lambda)$–strategically complete λ^+–cc forcing notion forcing that $\mathbb{Q}^2_\lambda$ is not dense in $\mathbb{Q}^1_\lambda$. It will be obtained by means of a $({<}\lambda)$–support iteration of length 2^λ. First, we define a forcing notion $\mathbb{Q}_0$:
a condition in $\mathbb{Q}_0$ is a tree $q \subseteq {}^{<\lambda}\lambda$ such that $|q| < \lambda$;
the order $\leq = \leq_{\mathbb{Q}_0}$ **of** $\mathbb{Q}_0$ is defined by
$q \leq q'$ if and only if $q \subseteq q'$ and $(\forall \eta \in q)(|\mathrm{succ}_q(\eta)| = 1 \ \Rightarrow\ |\mathrm{succ}_{q'}(\eta)| = 1)$.

Plainly, $\mathbb{Q}_0$ is a $({<}\lambda)$–complete forcing notion of size λ. Let $\underset{\sim}{T}_0$ be a $\mathbb{Q}_0$–name such that $\Vdash_{\mathbb{Q}_0}$ " $\underset{\sim}{T}_0 = \bigcup \underset{\sim}{G}_{\mathbb{Q}_0}$ ". Then

$$\Vdash_{\mathbb{Q}_0} \text{" } \underset{\sim}{T}_0 \in \mathbb{T}^{\mathrm{club}} \text{ and } \big(\forall \eta \in \underset{\sim}{T}_0\big)\big(|\mathrm{succ}_{\underset{\sim}{T}_0}(\eta)| > 1 \ \Rightarrow\ \mathrm{succ}_{\underset{\sim}{T}_0}(\eta) = \lambda\big) \text{ "}.$$

For a set $A \subseteq \lambda$ let $\mathbb{Q}^A$ be the forcing notion shooting a club through A. Thus
a condition in $\mathbb{Q}^A$ is a closed bounded set c included in A,
the order $\leq = \leq_{\mathbb{Q}^A}$ **of** $\mathbb{Q}^A$ is defined by
$c \leq c'$ if and only if $c = c' \cap \big(\max(c) + 1\big)$.

Now we inductively define a $({<}\lambda)$–support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < 2^\lambda \rangle$ and a sequence $\langle \underset{\sim}{A}_\xi, \underset{\sim}{\eta}_\xi : \xi < 2^\lambda \rangle$ so that the following demands are satisfied.

(i) $\mathbb{Q}_0$ is the forcing notion defined above, $\underset{\sim}{T}_0$ is the $\mathbb{Q}_0$–name for the generic tree added by $\mathbb{Q}_0$.

(ii) $\mathbb{P}_\xi$ is strategically $({<}\lambda)$–complete, satisfies λ^+–cc and has a dense subset of size 2^λ (for each $\xi \leq 2^\lambda$).

(iii) $\underset{\sim}{\eta}_\xi, \underset{\sim}{A}_\xi$ are $\mathbb{P}_\xi$–names such that

$$\Vdash_{\mathbb{P}_\xi} \text{" } \underset{\sim}{\eta}_\xi \in \lim_\lambda(\underset{\sim}{T}_0) \text{ and } \underset{\sim}{A}_\xi = \{\alpha < \lambda : |\mathrm{succ}_{\underset{\sim}{T}_0}(\underset{\sim}{\eta}_\xi\restriction\alpha)| > 1\} \text{ "}.$$

(iv) $\Vdash_{\mathbb{P}_\xi}$ " $\underset{\sim}{\mathbb{Q}}_\xi = \mathbb{Q}^{\underset{\sim}{A}_\xi}$ ".

(v) If $\underset{\sim}{\eta}$ is a $\mathbb{P}_{2^\lambda}$–name for a member of $\lim_\lambda(\underset{\sim}{T}_0)$, then for some $\xi < 2^\lambda$ we have $\Vdash_{\mathbb{P}_\xi}$ " $\underset{\sim}{\eta} = \underset{\sim}{\eta}_\xi$ ".

Clause (ii) will be shown soon, but with it in hand using a bookkeeping device we can take care of clause (v). Then the iteration $\bar{\underset{\sim}{\mathbb{Q}}}$ will be fully determined. So let us argue for clause (ii) (assuming that the iteration is constructed so that clauses (i), (iii) and (iv) are satisfied).

For $0 < \xi \leq 2^\lambda$ we let $\mathbb{P}^*_\xi$ consist of all conditions $p \in \mathbb{P}_\xi$ such that $0 \in \mathrm{dom}(p)$ and for some limit ordinal $\delta^p < \lambda$, for each $i \in \mathrm{dom}(p) \setminus \{0\}$ we have:

(a) $p(0) \subseteq {}^{\leq \delta^p+1}\lambda$ and for some $\eta_{p,i} \in {}^{\delta^p}\lambda \cap p(0)$, $p{\restriction}i \Vdash_{\mathbb{P}_i}$ " $\underset{\sim}{\eta}_i{\restriction}\delta^p = \eta_{p,i}$ ",
(b) $p(i)$ is a closed subset of $\delta^p + 1$ (not just a $\mathbb{P}_i$–name) and $\delta^p \in p(i)$,
(c) if $\beta \in p(i)$, then $|\mathrm{succ}_{p(0)}(\eta_{p,i}{\restriction}\beta)| > 1$.

Claim 4.5.1. (1) *If $p, p' \in \mathbb{P}^*_\xi$, then $p \leq_{\mathbb{P}_\xi} p'$ if and only if* $\mathrm{dom}(p) \subseteq \mathrm{dom}(p')$, $p(0) \leq_{\mathbb{Q}_0} p'(0)$ *and* $(\forall i \in \mathrm{dom}(p) \setminus \{0\})(p(i) = p'(i) \cap (\delta^p + 1))$.
(2) $|\mathbb{P}^*_\xi| = \lambda \cdot |\xi|^{<\lambda}$.
(3) *$\mathbb{P}^*_\xi$ is a $({<}\lambda)$–complete λ^+–cc subforcing of $\mathbb{P}_\xi$. Moreover, if $\langle p_\alpha : \alpha < \gamma\rangle$ is a $\leq_{\mathbb{P}_\xi}$–increasing sequence of members of $\mathbb{P}^*_\xi$, $\gamma < \lambda$, then there is $p \in \mathbb{P}^*_\xi$ such that $p_\alpha \leq_{\mathbb{P}_\xi} p$ for all $\alpha < \lambda$ and $\delta^p = \sup(\delta^{p_\alpha} : \alpha < \gamma)$.*
(4) *$\mathbb{P}^*_\xi$ is dense in $\mathbb{P}_\xi$. Moreover, for every $p \in \mathbb{P}_\xi$ and $\alpha < \lambda$ there is $q \in \mathbb{P}^*_\xi$ such that $p \leq_{\mathbb{P}_\xi} q$ and $\delta^q > \alpha$.*

Proof of the Claim. (1), (2), (3) Straightforward.
(4) Induction on $\xi \in (0, 2^\lambda]$.
Case $\xi = \xi_0 + 1$
Let $p \in \mathbb{P}_\xi$. Construct inductively a sequence $\langle p_n : n < \omega\rangle \subseteq \mathbb{P}^*_{\xi_0}$ such that for each $n < \omega$ we have

- $p{\restriction}\xi_0 \leq_{\mathbb{P}_{\xi_0}} p_n \leq_{\mathbb{P}_{\xi_0}} p_{n+1}$ and $\alpha < \delta^{p_n} < \delta^{p_{n+1}}$,
- for some closed set $c \subseteq \delta^{p_0}$ we have $p_0 \Vdash_{\mathbb{P}_{\xi_0}}$ " $p(\xi_0) = c$ ",
- for some sequence $\nu_n \in {}^{\delta^{p_n}}\lambda \cap p_{n+1}(0)$ we have $p_{n+1} \Vdash_{\mathbb{P}_{\xi_0}}$ "$\underset{\sim}{\eta}_{\xi_0}{\restriction}\delta^{p_n} = \nu_n$".

(The construction is clearly possible by our inductive hypothesis.) Now we define a condition $q \in \mathbb{P}^*_\xi$. We declare that $\mathrm{dom}(q) = \bigcup\limits_{n<\omega} \mathrm{dom}(p_n) \cup \{\xi_0\}$ and for $i \in \mathrm{dom}(q) \setminus \{0, \xi_0\}$ we set $\eta_{q,i} = \bigcup\{\eta_{p_n,i} : i \in \mathrm{dom}(p_n),\ n \in \omega\}$ and we also put $\eta_{q,\xi_0} = \bigcup\limits_{n<\omega} \nu_n$. We define

- $\delta^q = \sup\limits_{n<\omega} \delta^{p_n}$, $q(0) = \bigcup\limits_{n<\omega} p_n(0) \cup \{\eta_{q,i}, \eta_{q,i}{}^\frown\langle 0\rangle, \eta_{q,i}{}^\frown\langle 1\rangle : i \in \mathrm{dom}(q) \setminus \{0\}\}$,
- $q(i) = \bigcup\limits_{n<\omega} p_n(i) \cup \{\delta^q\}$ for $i \in \mathrm{dom}(q) \setminus \{0, \xi_0\}$ and
- $q(\xi_0) = c \cup \{\delta^q\}$.

One easily verifies that $q \in \mathbb{P}^*_\xi$ and it is stronger than p.
Case ξ is limit and $\mathrm{cf}(\xi) < \lambda$
Let $p \in \mathbb{P}_\xi$. Fix an increasing sequence $\langle \xi_\varepsilon : \varepsilon < \mathrm{cf}(\xi)\rangle \subseteq \xi$ cofinal in ξ and then use the inductive assumption (and properties of an iteration) to choose inductively a sequence $\langle p_\varepsilon : \varepsilon < \mathrm{cf}(\xi)\rangle$ such that for each $\varepsilon < \varepsilon' < \mathrm{cf}(\xi)$ we have

$$p_\varepsilon \in \mathbb{P}^*_{\xi_\varepsilon}, \quad \alpha < \delta^{p_\varepsilon} < \delta^{p_{\varepsilon'}} \quad \text{and} \quad p{\restriction}\xi_\varepsilon \leq_{\mathbb{P}_{\xi_\varepsilon}} p_\varepsilon \leq_{\mathbb{P}_{\xi_\varepsilon}} p_{\varepsilon'}{\restriction}\xi_\varepsilon.$$

Then define a condition $q \in \mathbb{P}^*_\xi$ as follows. Declare that $\mathrm{dom}(q) = \bigcup\limits_{\varepsilon<\mathrm{cf}(\xi)} \mathrm{dom}(p_\varepsilon)$ and for $i \in \mathrm{dom}(q) \setminus \{0\}$ set $\eta_{q,i} = \bigcup\{\eta_{p_\varepsilon,i} : i \in \mathrm{dom}(p_\varepsilon),\ \varepsilon < \mathrm{cf}(\xi)\}$. Put

- $\delta^q = \sup\limits_{\varepsilon<\mathrm{cf}(\xi)} \delta^{p_\varepsilon}$,
- $q(0) = \bigcup\limits_{\varepsilon<\mathrm{cf}(\xi)} p_\varepsilon(0) \cup \{\eta_{q,i}, \eta_{q,i}{}^\frown\langle 0\rangle, \eta_{q,i}{}^\frown\langle 1\rangle : i \in \mathrm{dom}(q)\setminus\{0\}\}$,
- $q(i) = \bigcup\limits_{\varepsilon<\mathrm{cf}(\xi)} p_\varepsilon(i) \cup \{\delta^q\}$ for $i \in \mathrm{dom}(q)\setminus\{0\}$.

Case ξ is limit and $\mathrm{cf}(\xi) \geq \lambda$
Immediate as then $\mathbb{P}_\xi = \bigcup\limits_{\zeta<\xi} \mathbb{P}_\zeta$. □

It follows from 4.5.1 that the clause (ii) of the construction of the iteration is satisfied. In particular, the limit $\mathbb{P}_{2^\lambda}$ is strategically $({<}\lambda)$–complete λ^+–cc and has a dense subset of size 2^λ. It should be also clear that $\Vdash_{\mathbb{P}_{2^\lambda}}$ " $\underset{\sim}{T}_0 \in \mathbb{Q}^1_\lambda$ " (remember (iii)–(v)).

Claim 4.5.2. $\Vdash_{\mathbb{P}_{2^\lambda}}$ *" $\underset{\sim}{T}_0$ contains no tree from $\mathbb{Q}^2_\lambda$ ".*

Proof of the Claim. Suppose towards contradiction that $\underset{\sim}{T}$ is a $\mathbb{P}_{2^\lambda}$–name such that

$$p \Vdash_{\mathbb{P}_{2^\lambda}} \text{" } \underset{\sim}{T} \in \mathbb{Q}^2_\lambda \text{ and } \underset{\sim}{T} \subseteq \underset{\sim}{T}_0 \text{ "} \qquad \text{for some } p \in \mathbb{P}_{2^\lambda}.$$

Note that, by 4.5.1(3,4),

(∗) if $p \leq_{\mathbb{P}_{2^\lambda}} q$, $\nu \in {}^{<\lambda}\lambda$, $q \Vdash_{\mathbb{P}_{2^\lambda}} \nu \in \underset{\sim}{T}$, and $\kappa < \lambda$ and $\underset{\sim}{\rho}_i$ are $\mathbb{P}_{2^\lambda}$–names for members of ${}^\lambda\lambda$ (for $i < \kappa$),
then there are $q^* \in \mathbb{P}^*_{2^\lambda}$ and $\nu^* \in q^*(0)$ such that $q \leq_{\mathbb{P}_{2^\lambda}} q^*$, $\nu \vartriangleleft \nu^*$ and

$$q^* \Vdash_{\mathbb{P}_{2^\lambda}} \text{" } \nu^* \in \underset{\sim}{T} \ \ \& \ \ |\mathrm{succ}_{\underset{\sim}{T}}(\nu^*)| > 1 \ \ \& \ \ (\forall i < \kappa)(\underset{\sim}{\rho}_i{\restriction}\mathrm{lh}(\nu^*) \neq \nu^*) \text{ ".}$$

Using (∗) repeatedly ω times we may construct a sequence $\langle p_n, \nu^*_n : n < \omega\rangle$ such that

- $p_n \in \mathbb{P}^*_{2^\lambda}$, $p \leq_{\mathbb{P}_{2^\lambda}} p_n \leq_{\mathbb{P}_{2^\lambda}} p_{n+1}$, $\delta^{p_n} < \delta^{p_{n+1}}$, and
- $\nu^*_n \in {}^{<\lambda}\lambda$, $\nu^*_n \in p_{n+1}(0)$, $\nu^*_n \vartriangleleft \nu^*_{n+1}$, and

$$p_{n+1} \Vdash_{\mathbb{P}_{2^\lambda}} \text{" } \nu^*_n \in \underset{\sim}{T} \ \& \ |\mathrm{succ}_{\underset{\sim}{T}}(\nu^*_n)| > 1 \ \& \ (\forall i{\in}\mathrm{dom}(p_n)\setminus\{0\})(\underset{\sim}{\eta}_i{\restriction}\mathrm{lh}(\nu^*_n) \neq \nu^*_n) \text{ ".}$$

Then we define a condition $q \in \mathbb{P}^*_\xi$: we declare that $\mathrm{dom}(q) = \bigcup\limits_{n<\omega} \mathrm{dom}(p_n)$ and for $i \in \mathrm{dom}(q)\setminus\{0\}$ we set $\eta_{q,i} = \bigcup\{\eta_{p_n,i} : i \in \mathrm{dom}(p_n),\ n \in \omega\}$. We also put $\nu^* = \bigcup\limits_{n<\omega} \nu^*_n$, $\delta^q = \sup\limits_{n<\omega} \delta^{p_n}$, and then:

- $q(0) = \bigcup\limits_{n<\omega} p_n(0) \cup \{\eta_{q,i}, \eta_{q,i}{}^\frown\langle 0\rangle, \eta_{q,i}{}^\frown\langle 1\rangle : i \in \mathrm{dom}(q)\setminus\{0\}\} \cup \{\nu^*, \nu^*{}^\frown\langle 0\rangle\}$,
- $q(i) = \bigcup\limits_{n<\omega} p_n(i) \cup \{\delta^q\}$ for $i \in \mathrm{dom}(q)\setminus\{0\}$.

Note that $\nu^* \notin \{\eta_{q,i}{\restriction}\mathrm{lh}(\nu^*) : i \in \mathrm{dom}(q)\setminus\{0\}\}$ and $\mathrm{lh}(\nu^*) \leq \delta^q = \mathrm{lh}(\eta_{q,i})$ for $i \in \mathrm{dom}(q)\setminus\{0\}$. Now we easily check that $q \in \mathbb{P}^*_{2^\lambda}$ is stronger than p and it forces that $\nu^* \in \underset{\sim}{T}$ is a limit of splitting points of $\underset{\sim}{T}$, but itself it is not a splitting point (even in $\underset{\sim}{T}_0$). A contradiction with $p \Vdash \underset{\sim}{T} \in \mathbb{Q}^2_\lambda$. □

□

Proposition 4.6. *Assume that the complete Boolean algebras* $\mathrm{RO}(\mathbb{Q}^1_\lambda)$ *and* $\mathrm{RO}(\mathbb{Q}^2_\lambda)$ *are isomorphic. Then $\mathbb{Q}^2_\lambda$ is a dense subset of $\mathbb{Q}^1_\lambda$.*

Proof. Since $\mathrm{RO}(\mathbb{Q}^1_\lambda)$ and $\mathrm{RO}(\mathbb{Q}^2_\lambda)$ are isomorphic, we may find $\mathbb{Q}^{3-\ell}_\lambda$–names $\underset{\sim}{H}_\ell, \underset{\sim}{\eta}_\ell$ (for $\ell = 1, 2$) such that

$(\boxdot)_1$ $\Vdash_{\mathbb{Q}^\ell_\lambda}$ " $\underset{\sim}{H}_{3-\ell} \subseteq \mathbb{Q}^{3-\ell}_\lambda$ is generic over $\mathbf{V}$ and $\underset{\sim}{\eta}_{3-\ell} \in {}^\lambda\lambda$ is the corresponding generic branch ",

$(\boxdot)_2$ if $G_\ell \subseteq \mathbb{Q}^\ell_\lambda$ is generic over $\mathbf{V}$ and $G_{3-\ell} = \underset{\sim}{H}_{3-\ell}[G_\ell]$, then $G_\ell = \underset{\sim}{H}_\ell[G_{3-\ell}]$.

Consider $\mathbb{Q}^1_\lambda \times \mathbb{Q}^2_\lambda$ with the product order and for $\ell = 1, 2$ put

$$R_\ell = \{(T_1, T_2) \in \mathbb{Q}^1_\lambda \times \mathbb{Q}^2_\lambda : T_\ell \Vdash_{\mathbb{Q}^\ell_\lambda} T_{3-\ell} \in \underset{\sim}{H}_{3-\ell}\}$$

and $R = R_1 \cap R_2$.

Claim 4.6.1. *R is a dense subset of both R_1 and R_2.*

Proof of the Claim. First note that

$(\boxdot)_3$ if $(T_1, T_2) \in \mathbb{Q}^1_\lambda \times \mathbb{Q}^2_\lambda$ and $T_1 \nVdash_{\mathbb{Q}^1_\lambda}$ " $T_2 \notin \underset{\sim}{H}_2$ ", then there is $T_1^* \geq T_1$ such that $(T_1^*, T_2) \in R_1$ (and symmetrically when the roles of 1 and 2 are interchanged).

Also,

$(\boxdot)_4$ if $(T_1, T_2) \in R_\ell$, $\ell = 1, 2$, then $T_{3-\ell} \nVdash_{\mathbb{Q}^{3-\ell}_\lambda} T_\ell \notin \underset{\sim}{H}_\ell$.

[Why? Assume towards contradiction that $T_{3-\ell} \Vdash_{\mathbb{Q}^{3-\ell}_\lambda} T_\ell \notin \underset{\sim}{H}_\ell$. Let $G_\ell \subseteq \mathbb{Q}^\ell_\lambda$ be a generic over $\mathbf{V}$ such that $T_\ell \in G_\ell$. Put $G_{3-\ell} = \underset{\sim}{H}_{3-\ell}[G_\ell]$. Then $G_{3-\ell} \subseteq \mathbb{Q}^{3-\ell}_\lambda$ is generic over $\mathbf{V}$ and $\underset{\sim}{H}_\ell[G_{3-\ell}] = G_\ell$. Since $(T_1, T_2) \in R_\ell$ we know $T_{3-\ell} \in \underset{\sim}{H}_{3-\ell}[G_\ell]$ and hence (by our assumption towards contradiction) $T_\ell \notin \underset{\sim}{H}_\ell[G_{3-\ell}] = G_\ell$, contradicting the choice of G_ℓ.]

Now suppose $(T_1, T_2) \in R_\ell$, $\ell \in \{1, 2\}$. Choose inductively a sequence $\langle (T_1^n, T_2^n) : n < \omega\rangle$ such that $(T_1^0, T_2^0) = (T_1, T_2)$ for all $n < \omega$:

- if n is even, then $(T_1^n, T_2^n) \in R_\ell$,
- if n is odd, then $(T_1^n, T_2^n) \in R_{3-\ell}$,
- $(T_1^n, T_2^n) \leq (T_1^{n+1}, T_2^{n+1})$.

By $(\boxdot)_3 + (\boxdot)_4$ there are no problems with carrying out the inductive process. Put

$$T_1^\omega = \bigcap_{n<\omega} T_1^n \quad \text{and} \quad T_2^\omega = \bigcap_{n<\omega} T_2^n.$$

Then T_ℓ^ω is the least upper bound of $\langle T_\ell^n : n < \omega\rangle$ and hence easily $(T_1^\omega, T_2^\omega) \in R$, $(T_1, T_2) \leq (T_1^\omega, T_2^\omega)$. $\square$

Claim 4.6.2. *Let $\ell \in \{1, 2\}$, $T \in \mathbb{Q}^\ell_\lambda$. Then there is $T^* \geq T$ such that for some $\nu \in {}^{<\lambda}\lambda$ we have*

$$\mathrm{lh}(\nu) = \mathrm{lh}(\mathrm{root}(T^*)) \quad \textit{and} \quad T^* \Vdash_{\mathbb{Q}^\ell_\lambda} \text{“ } \nu \vartriangleleft \underset{\sim}{\eta}_{3-\ell} \text{ ”}.$$

Proof of the Claim. By induction on $\alpha < \lambda$ choose a sequence $\langle T_\alpha : \alpha < \lambda\rangle$ so that for all $\alpha < \beta < \lambda$ we have

$(\boxdot)_5$ $T_\alpha \leq T_\beta$, $\mathrm{root}(T_\alpha) \vartriangleleft \mathrm{root}(T_\beta)$ and

$(\boxdot)_6$ $T_{\alpha+1}$ forces a value to $\underset{\sim}{\eta}_{3-\ell}{\restriction}\mathrm{lh}(\mathrm{root}(T_\alpha))$, and

$(\boxdot)_7$ if α is limit then $T_\alpha = \bigcap\limits_{\xi<\alpha} T_\xi$.

Let $\eta = \bigcup\limits_{\alpha<\lambda} \mathrm{root}(T_\alpha) \in {}^\lambda\lambda$. Then $\eta \in \lim_\lambda(T_\alpha)$ for each $\alpha < \lambda$ so the sets $\{\delta < \lambda : |\mathrm{succ}_{T_\alpha}(\eta{\restriction}\delta)| > 1\}$ contain clubs (for each $\alpha < \lambda$). Consequently we may pick limit $\delta < \lambda$ such that $|\mathrm{succ}_{T_\alpha}(\eta{\restriction}\delta)| > 1$ for all $\alpha < \delta$. Then also, by $(\boxdot)_7$, $\eta{\restriction}\delta = \mathrm{root}(T_\delta)$ and clearly (by $(\boxdot)_6$) T_δ forces a value to $\underset{\sim}{\eta}_{3-\ell}{\restriction}\delta$. $\square$

Claim 4.6.3. *Let $\ell\in\{1,2\}$, $T\in\mathbb{Q}^\ell_\lambda$. Then there is $T^*\geq T$ such that for every $t\in T^*$, for some $\nu\in {}^{<\lambda}\lambda$ we have*

$$\mathrm{lh}(\nu)=\mathrm{lh}(t)\quad\text{ and }\quad (T^*)_t\Vdash_{\mathbb{Q}^\ell_\lambda}\text{“ }\nu\vartriangleleft\underset{\sim}{\eta}_{3-\ell}\text{ ”}.$$

Proof of the Claim. We choose inductively conditions $T_\alpha\in\mathbb{Q}^\ell_\lambda$ and fronts F_α of T_α so that for all $\alpha<\beta<\lambda$:

$(\boxdot)_8$ $T_0=T$, $F_0=\{\langle\rangle\}$,

$(\boxdot)_9$ $T_\alpha\leq T_\beta$, $F_\alpha\subseteq T_\beta$ and $(\forall t\in F_\beta)(\exists i<\mathrm{lh}(t))(t\restriction i\in F_\alpha)$,

$(\boxdot)_{10}$ if α is limit, then $T_\alpha=\bigcap\limits_{\xi<\alpha}T_\xi$ and $F_\alpha=\big\{t\in T_\alpha:(\forall\xi<\alpha)(\exists i<\mathrm{lh}(t))(t\restriction i\in F_\xi)\text{ and }(\forall i<\mathrm{lh}(t))(\exists\xi<\alpha)(\exists j<\mathrm{lh}(t))(i<j\ \&\ t\restriction j\in F_\xi)\big\}$,

$(\boxdot)_{11}$ if $t\in F_\alpha$, $\zeta\in\mathrm{succ}_{T_\alpha}(t)$ then $t^\frown\langle\zeta\rangle\in T_{\alpha+1}$ and for some $s=s_{t,\zeta}\in F_{\alpha+1}$ we have

$$\mathrm{root}\big((T_{\alpha+1})_{t^\frown\langle\zeta\rangle}\big)=s\ \text{ and }\ (T_{\alpha+1})_s\text{ forces a value to }\underset{\sim}{\eta}_{3-\ell}\restriction\mathrm{lh}(s),$$

$(\boxdot)_{12}$ $F_{\alpha+1}=\{s_{t,\zeta}:t\in F_\alpha\ \&\ \zeta\in\mathrm{succ}_{T_\alpha}(t)\}$ (where $s_{t,\zeta}$ are determined by $(\boxdot)_{11}$).

It should be clear that the construction is possible (at successor stages use 4.6.2). Set $T^*=\bigcap\limits_{\alpha<\lambda}T_\alpha$. By Lemma 3.5 we know that $T^*\in\mathbb{Q}^\ell_\lambda$ and $F_\alpha\subseteq T^*$ are fronts of T^* (for $\alpha<\lambda$). Moreover,

$(\boxdot)_{13}$ if $s\in T^*$ and $|\mathrm{succ}_{T^*}(s)|>1$, then $s\in\bigcup\limits_{\alpha<\lambda}F_\alpha$.

It follows from $(\boxdot)_{11}+(\boxdot)_{12}+(\boxdot)_{10}$ that for every $t\in F_\alpha$ the condition $(T^*)_t$ forces a value to $\underset{\sim}{\eta}_{3-\ell}\restriction\mathrm{lh}(t)$. If $t\in T^*\setminus\bigcup\limits_{\alpha<\lambda}F_\alpha$, then choose the shortest $s\in\bigcup\limits_{\alpha<\lambda}F_\alpha$ such that $t\vartriangleleft s$. Then $(T^*)_t=(T^*)_s$ (remember $(\boxdot)_{13}$) and hence in particular the condition $(T^*)_t$ forces a value to $\underset{\sim}{\eta}_{3-\ell}\restriction\mathrm{lh}(t)$. $\square$

Now suppose that $T_1\in\mathbb{Q}^1_\lambda$. Use Claim 4.6.3 to choose a condition $T_1^*\in\mathbb{Q}^1_\lambda$ such that $T_1\leq T_1^*$ and

$(\boxdot)^{T_1^*,2}_{14}$ for every $t\in T_1^*$ the condition $(T_1^*)_t$ forces a value to $\underset{\sim}{\eta}_2\restriction\mathrm{lh}(t)$.

Then use Claim 4.6.1 to pick $(T_1',T_2')\in R$ so that $T_1^*\leq T_1'$. Note that then also $(\boxdot)^{T_1',2}_{14}$ holds. Apply Claim 4.6.3 to T_2' and $\ell=2$ to find a condition $T_2''\geq T_2'$ such that the suitable demand $(\boxdot)^{T_2'',1}_{14}$ holds, and then use Claim 4.6.1 again to choose $(T_1^+,T_2^+)\in R$ such that $T_1^+\geq T_1'$ and $T_2^+\geq T_2''$. Note that then

$(\boxdot)^{T_\ell^+,3-\ell}_{14}$ for every $t\in T_\ell^+$ the condition $(T_\ell^+)_t$ forces a value to $\underset{\sim}{\eta}_{3-\ell}\restriction\mathrm{lh}(t)$.

For $\ell=1,2$ and $t\in T_\ell^+$ let $\varphi_\ell(t)\in{}^{<\lambda}\lambda$ be such that $\mathrm{lh}(\varphi_\ell(t))=\mathrm{lh}(t)$ and $(T_\ell^+)_t\Vdash$“$\varphi_\ell(t)\vartriangleleft\underset{\sim}{\eta}_{3-\ell}$”. Since $(T_1^+,T_2^+)\in R$ we know that

$(\boxdot)^\ell_{15}$ $\varphi_\ell(t)\in T_{3-\ell}^+$ for each $t\in T_\ell^+$

and by $(\boxdot)_2$ we also have

$(\boxdot)_{16}$ $\varphi_1\circ\varphi_2$ is the identity on T_2^+ and $\varphi_2\circ\varphi_1$ is the identity on T_1^+. Moreover, if $t\in T_1^+$, then $((T_1^+)_t,(T_2^+)_{\varphi_1(t)})\in R$ and if $s\in T_2^+$, then $((T_1^+)_{\varphi_2(s)},(T_2^+)_s)\in R$.

Thus $\varphi_\ell:T_\ell^+\longrightarrow T_{3-\ell}^+$ is a bijection preserving levels and the extension relation $\vartriangleleft$, and φ_1 is the inverse of φ_2. Consequently, $t\in T_\ell^+$ is a splitting of T_ℓ^+ if and only if $\varphi_\ell(t)$ is a splitting in $T_{3-\ell}^+$. Therefore we may conclude that $T_1^+\in\mathbb{Q}^2_\lambda$. $\square$

Conclusion 4.7. It is consistent that the forcing notions $\mathbb{Q}^1_\lambda, \mathbb{Q}^2_\lambda$ are not equivalent.

Proof. By Propositions 4.5 and 4.6. $\square$

References

[1] Elizabeth T. Brown and Marcia J. Groszek. Uncountable superperfect forcing and minimality. *Ann. Pure Appl. Logic*, 144:73–82, 2006.

[2] Todd Eisworth. On iterated forcing for successors of regular cardinals. *Fundamenta Mathematicae*, 179:249–266, 2003.

[3] Sy-David Friedman, Radek Honzik, and Lyubomyr Zdomskyy. Fusion and large cardinal preservation. *Annals of Pure and Applied Logic*, accepted, 2012.

[4] Sy-David Friedman and Lyubomyr Zdomskyy. Measurable cardinals and the cofinality of the symmetric group. *Fundamenta Mathematicae*, 207:101–122, 2010.

[5] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[6] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19:97–114, 1980.

[7] Andrzej Roslanowski and Saharon Shelah. The last forcing standing with diamonds. *preprint*.

[8] Andrzej Roslanowski and Saharon Shelah. Iteration of λ-complete forcing notions not collapsing λ^+. *International Journal of Mathematics and Mathematical Sciences*, 28:63–82, 2001. math.LO/9906024.

[9] Andrzej Roslanowski and Saharon Shelah. Reasonably complete forcing notions. *Quaderni di Matematica*, 17:195–239, 2006. math.LO/0508272.

[10] Andrzej Roslanowski and Saharon Shelah. Sheva-Sheva-Sheva: Large Creatures. *Israel Journal of Mathematics*, 159:109–174, 2007. math.LO/0210205.

[11] Andrzej Roslanowski and Saharon Shelah. Lords of the iteration. In *Set Theory and Its Applications*, volume 533 of *Contemporary Mathematics (CONM)*, pages 287–330. Amer. Math. Soc., 2011. math.LO/0611131.

[12] Andrzej Roslanowski and Saharon Shelah. Reasonable ultrafilters, again. *Notre Dame Journal of Formal Logic*, 52:113–147, 2011. math.LO/0605067.

[13] Saharon Shelah. Not collapsing cardinals $\leq \kappa$ in $(<\kappa)$–support iterations. *Israel Journal of Mathematics*, 136:29–115, 2003. math.LO/9707225.

[14] Saharon Shelah. Successor of singulars: combinatorics and not collapsing cardinals $\leq \kappa$ in $(<\kappa)$-support iterations. *Israel Journal of Mathematics*, 134:127–155, 2003. math.LO/9808140.

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