On uncountable hypersimple unidimensional theories

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Abstract

We extend the dichotomy between 1-basedness and supersimplicity proved in [S1]. The generalization we get is to arbitrary language, with no restrictions on the topology (we do not demand type-definability of the open set in the definition of essential 1-basedness from [S1]). We conclude that every (possibly uncountable) hypersimple unidimensional theory that is not s-essentially 1-based by means of the forking topology is supersimple. We also obtain a strong version of the above dichotomy in the case where the language is countable.

1 Introduction

Shelah has defined unidimensional theories as (stable) theories in which any two sufficiently large $|T|^+$ -saturated models of the same cardinality are isomorphic. For stable theories this definition is equivalent to the requirement that any two non-algebraic types are non-orthogonal. This requirement serves as the definition of unidimensionality for the larger class of simple theories. A problem posed by Shelah was whether any unidimensional stable theory is superstable. Around 1986 Hrushovski has solved the problem by answering it in the affirmative [H1].

Several years after the discovery of Kim [K] that the algebraic properties of forking (symmetry and transitivity) can be proved for simple theories (1996) and the development of the basic machinery [K,KP,HKP], there were several attempts to generalize the above result of Hrushovski to the simple case. A generalization of this proof along the the same lines seems very problematic because of the lack of definability of types, and so many of the results on definable groups in stable theories do not seem to generalize to simple theories in a direct way.

In 2003, we observed that any small simple unidimensional theory is supersimple [S3]. A bit later, Pillay [P] has proved that any countable hypersimple theory (i.e. a simple theory that eliminates hyperimaginaries) with the wnfcp (the weak non finite cover property) is supersimple; this proof builds on ideas from Hrushovski's old proof of the result for countable stable theories [H0] and some machinery from the theory of lovely-pairs [BPV]. This has been extended by Pillay [P1] to any countable low hypersimple theory using the result on elimination of the "there exists infinitely many" quantifier [S2]. In 2008, it has been proved that any countable hypersimple unidimensional theory is supersimple [S1]. An important notion that used in [S1] is the forking topology (or the τ^f -topology); this is a variant of the topology used in [H0] and [P]: for variables x and set A the forking topology on $S_x(A)$ is defined as the topology whose basis is the collection of all sets of the form $\mathcal{U} = \{a | \phi(a, y) \text{ forks over } A\}$, where $\phi(x, y) \in L(A)$.

The goal of this paper is to reduce the problem on supersimplicity of general hypersimple unidimensional theories (possibly uncountable) to the case where the theory is s-essentially 1-based by means of the forking topology, namely, any type internal in a SU-rank 1 type is s-essentially 1-based (a strong version of the notion "essentially 1-based" from [S1]) by means of the forking-topology. We do this by generalizing the dichotomy theorem from [S1] to any hypersimple theory (rather than a countable one) equipped with a projection-closed family of topologies, while its conclusion is strengthened to get that any type internal in a SU-rank 1 type is s-essentially 1-based (in [S1] we got only "essentially 1-based" in the conclusion), provided that no unbounded open supersimple is interpreted. This will ensure the existence of many stable formulas that witness forking. In [S1] we dealt with the remaining case by the development of a model theoretic Baire category theorem in which we analyze more complicated "forking sets" that are related to the forking topology. This theorem made an essential use of the existence of many stable formulas and the assumption that the language is countable.

We assume basic knowledge of simple theories; a good textbook on simple theories is [W]. Throughout this paper we work in a κ -saturated and κ -

strongly saturated model C, for some large κ , of a complete first order theory T.

2 The dichotomy

In this section we assume $T = T^{eq}$ is a hypersimple theory and we work in $\mathcal{C} = \mathcal{C}^{eq}$. First recall the definition of a projection-closed family of topologies.

Definition 2.1 A family

 $\Upsilon = \{\Upsilon_{x,A} \mid x \text{ is a finite sequence of variables and } A \subset \mathcal{C} \text{ is small} \}$

is said to be a projection-closed family of topologies if each $\Upsilon_{x,A}$ is a topology on $S_x(A)$ that refines the Stone-topology on $S_x(A)$, this family is invariant under automorphisms of \mathcal{C} and change of variables by variables of the same sort, the family is closed under product by the full Stone spaces $S_y(A)$ (where y is a disjoint tuple of variables) and closed by projections, namely whenever $\mathcal{U}(x,y) \in \Upsilon_{xy,A}, \exists y \mathcal{U}(x,y) \in \Upsilon_{x,A}$.

From now on Υ denotes a projection-closed family of topologies.

Definition 2.2 1) A type $p \in S(A)$ is said to be *s*-essentially 1-based over $A_0 \subseteq A$ by means of Υ if for every finite tuple \bar{c} from p and for every Υ -open set \mathcal{U} over $A\bar{c}$, with the property that a is independent from A over A_0 for every $a \in \mathcal{U}$, the set $\{a \in \mathcal{U} | Cb(a/A\bar{c}) \notin bdd(aA_0)\}$ is nowhere dense in the Stone-topology of \mathcal{U} . We say $p \in S(A)$ is *s*-essentially 1-based by means of Υ if p is s-essentially 1-based over A by means of Υ . 2) Let V be an A_0 -invariant set and let $p \in S(A_0)$. We say that p is analyzable

in V by s-essentially 1-based types by means of Υ if there exists $a \models p$ and there exists a sequence $(a_i \mid i \leq \alpha) \subseteq dcl(A_0 a)$ with $a_\alpha = a$ such that $tp(a_i/A_0 \cup \{a_j \mid j < i\})$ is V-internal and s-essentially 1-based over A_0 by means of Υ for all $i \leq \alpha$.

In [S1] we said that $p \in S(A)$ is essentially 1-based with respect to Υ , if 1) in Definition 2.2 holds with the additional requirement that \mathcal{U} is typedefinable. Before stating the main theorem, recall that for an A-invariant set \mathcal{U} and a type p over A, we say that \mathcal{U} is almost p-internal (over A) if tp(a/A) is almost *p*-internal for every $a \in \mathcal{U}$. Also, \mathcal{U} is said to be *unbounded* if it contains the solution set of some non-algebraic type (equivalently, its cardinality is $\geq \kappa$). We can now phrase the dichotomy.

Theorem 2.3 Let T be any hypersimple theory. Let Υ be a projection-closed family of topologies. Let p_0 be a partial type over \emptyset of SU-rank 1. Then, either there exists an unbounded Υ -open set (over some small set A) that is almost p_0 -internal (and in particular has finite SU-rank), or every complete type $p \in S(A)$ that is internal in p_0 is s-essentially 1-based over \emptyset by means of Υ . In particular, either there exists an unbounded Υ -open set that is almost p_0 -internal, or whenever $p \in S(A)$ and every non-algebraic extension of pis non-foreign to p_0 , p is analyzable in p_0 by s-essentially 1-based types by means of Υ .

Before proving the dichotomy, note the following easy generalization of [S1, Proposition 4.4] (recall the domination notation: $b \succeq_a c$ iff for any d if d independent from b over a then d is independent from c over a.)

Proposition 2.4 Let $q(x, y) \in S(\emptyset)$ and let $\chi(x, y, z)$ be an \emptyset -invariant set such that for all $(c, b, a) \models \chi(x, y, z)$ we have $b \succeq_a bc$. Then the set

$$\mathcal{U} = \{ (e, c, b, a) | e \in acl(Cb(cb/a)) \}$$

is relatively Stone-open inside the set

$$F = \{(e, c, b, a) \mid b \downarrow a , \models \chi(c, b, a), tp(cb) = q\}.$$

(where e is taken from a fixed sort too).

The proof of Proposition 2.4 is the same as in [S1], we write it for completeness. Let us recall the basic notion and fact that are needed for the proof. Recall that a set \mathcal{U} is said to be a basic τ_*^f -open set over C if there exists $\psi(x, y, C) \in L(C)$ such that $\mathcal{U} = \{a \mid \psi(x, aC) \text{ forks over } a\}$.

Fact 2.5 [S1, Lemma 4.3] Let C be any set and let $\mathcal{W} = \{(e, a) | e \in acl(Cb(C/a))\}$ (where e, a are taken from fixed sorts). Then \mathcal{W} is a τ_*^f -open set over C.

Proof: Note that since $q \in S(\emptyset)$, it is enough to show that for any fixed $c^*b^* \models q$ the set $\mathcal{U}^* = \{(e, a) \mid e \in acl(Cb(c^*b^*/a))\}$ is relatively Stone-open inside

$$F^* = \{(e, a) | \ b^* \, {\ \ } a \ , \models \chi(c^*, b^*, a) \}.$$

Now, by Fact 2.5, we know \mathcal{U}^* is a τ^f_* -open set over b^*c^* . Thus, for some $\psi_i(t_i; w, z, c^*b^*) \in L(c^*b^*)$ $(i \in I)$ we have $\mathcal{U}^* = \bigcup_i \mathcal{U}^*_{\psi_i}$ where

$$\mathcal{U}_{\psi_i}^* = \{ (e, a) \mid \psi_i(t_i; e, a, c^* b^*) \text{ forks over } ea \}.$$

Subclaim 2.6 For every $(e, a) \in F^*$ we have $(e, a) \in \mathcal{U}^*_{\psi_i}$ iff

$$\forall d(\psi_i(d; e, a, c^*b^*) \rightarrow da \not b^*) \land e \in acl(a)$$

Proof: Let $(e, a) \in F^*$. Assuming the left hand side we know $e \in acl(Cb(c^*b^*/a))$, hence $e \in acl(a)$. Let $d \models \psi_i(z; e, a, c^*b^*)$. If $da \perp b^*$, then $\frac{d \perp b^*}{a}$. Since $(e, a) \in F^*$, $b^* \succeq_a b^*c^*$ implies $\frac{d \perp b^*c^*}{ea}$, contradicting $(e, a) \in \mathcal{U}^*_{\psi_i}$. Assume now the right hand side. By a way of contradiction assume there exists $d \models \psi_i(t_i; e, a, c^*b^*)$ such that $\frac{d \perp b^*c^*}{ea}$. Since $e \in acl(a)$, this is equivalent to $\frac{d \perp b^*c^*}{a}$. Since $(e, a) \in F^*$ this is equivalent to $da \perp b^*$,

contradiction.

By Subclaim 2.6 we see that each of $\mathcal{U}_{\psi_i}^*$ and hence \mathcal{U}^* is Stone-open relatively inside F^* (since dependence in b^* is a Stone-open condition over b^*). \Box

Proof of Theorem 2.3 Υ will be fixed and we'll freely omit the phrase "by means of Υ ". To see the "In particular" part, work over A and assume that every $p' \in S(A')$, with $A' \supseteq A$, that is internal in p_0 , is s-essentially 1-based over A. Moreover, assume $p \in S(A)$ is non-algebraic and every nonalgebraic extension of p is non-foreign to p_0 . Then, for $a \models p$ there exists $a' \in dcl(Aa) \setminus acl(A)$ such that tp(a'/A) is p_0 -internal and thus s-essentially 1-based over A by our assumption. Thus, by repeating this process we get that p is analyzable in p_0 by s-essentially 1-based types. We now prove the main part. Assume there exists $p \in S(A)$ that is internal in p_0 , and p is not s-essentially 1-based over \emptyset . By the definition, there exist a finite tuple d of realizations of p and b that is independent from d over A, and a finite tuple $\overline{c} \subseteq p_0$ such that $d \in dcl(Ab\overline{c})$, and there exists a Υ -open set \mathcal{U} over Ad such that a is independent from A for all $a \in \mathcal{U}$ and $\{a \in \mathcal{U} | Cb(a/Ad) \not\subseteq acl(a)\}$ is not nowhere dense in the Stone-topology of \mathcal{U} . So, since Υ refines the Stone-topology, by intersecting \mathcal{U} with a definable set, we may assume that $\{a \in \mathcal{U} | Cb(a/Ad) \not\subseteq acl(a)\}$ is dense in the Stone-topology of \mathcal{U} . Now, for each (finite) subsequence \overline{c}_0 of \overline{c} , let

$$F_{\bar{c}_0} = \{ a \in \mathcal{U} | \exists b', \bar{c}'_0, \bar{c}'_1 \text{ s.t. } tp(b'\bar{c}'_0\bar{c}'_1/Ad) = tp(b\bar{c}_0(\bar{c}\setminus\bar{c}_0)/Ad) \text{ and } a \ \bigcup \ Ab'\bar{c}'_0 \ \}.$$

Note that since d is independent from b over A, any $a \in \mathcal{U}$ is independent from Ab' whenever tp(b'/Ad) = tp(b/Ad) and $\begin{array}{c} a \qquad \downarrow \qquad b'\\ Ad \end{array}$. Thus $F_{\langle\rangle} = \mathcal{U}$. Let \bar{c}_0^* be a maximal subsequence (with respect to inclusion) of \bar{c} such that $F_{\bar{c}_0^*}$ has non-empty Stone-interior in \mathcal{U} over Ad (note that $F_{\bar{c}}$ has no Stoneinterior relatively in \mathcal{U}). Let $\mathcal{U}^* = \bigcap_{\bar{c}_0^* \subset \bar{c}' \subseteq \bar{c}} \mathcal{U} \setminus F_{\bar{c}'}$. Note that each $F_{\bar{c}'}$ is Stone closed relatively in \mathcal{U} . Thus \mathcal{U}^* is Stone-dense and open in \mathcal{U} and therefore there exists a non-empty relatively Stone-open in \mathcal{U} set $W^* \subseteq F_{\bar{c}_0^*} \cap \mathcal{U}^*$.

Subclaim 2.7 W^* is a non-empty Υ -open set over Ad such that $\{a \in W^* | Cb(a/Ad) \not\subseteq acl(a)\}$ is dense in the Stone-topology of W^* and for every $a \in W^*$ we have: there exists $b'\bar{c}'_0\bar{c}'_1 \models tp(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ such that a is independent from $Ab'\bar{c}'_0$ over \emptyset and moreover, for every $b'\bar{c}'_0\bar{c}'_1 \models tp(b\bar{c}^*_0(\bar{c} \setminus \bar{c}^*_0)/Ad)$ such that a is independent from $Ab'\bar{c}'_0$ we necessarily have $\bar{c}'_1 \in acl(aAb'\bar{c}'_0)$.

Proof: As p_0 has SU-rank 1, this is a conclusion of our construction. \Box

Let us now define a set V over Ad by

$$V = \{ (e', b', \vec{c}'_0, \vec{c}'_1, a') | \text{ if } tp(b'\vec{c}'_0\vec{c}'_1/Ad) = tp(b\vec{c}^*_0(\vec{c}\setminus\vec{c}^*_0)/Ad) \text{ and } a' \, \bigcup \, Ab'\vec{c}'_0 \\ \text{ then } e' \in acl(Cb(Ab'\vec{c}'_0\vec{c}'_1/a')) \}.$$

Let $V^* = \{e' | \exists a' \in W^* \ \forall b', \vec{c}_0', \vec{c}_1' \ V(e', b', \vec{c}_0', \vec{c}_1', a') \}.$

Subclaim 2.8 V^* is a Υ -open set over Ad.

Proof: By Proposition 2.4 and Subclaim 2.7, there exists a Stone-open set V' over Ad such that for all $a' \in W^*$ and for all $e', b', \vec{c}_0, \vec{c}_1'$ we have $V'(e', b', \vec{c}_0, \vec{c}_1, a')$ if and and only if $V(e', b', \vec{c}_0, \vec{c}_1, a')$. Thus, we may replace V by V' in the definition of V^* . As Stone-open sets are closed under the \forall quantifier, the Υ topology refines the Stone-topology and closed under product by a full Stone-space and closed under projections, we conclude that V^* is a Υ -open set. \Box

Subclaim 2.9 For appropriate sort for e', the set V^* is unbounded and is almost p_0 -internal (over Ad) and thus has finite SU-rank over Ad.

Proof: First, note the following general observation.

Fact 2.10 Assume $d \in dcl(c)$. Then $Cb(d/a) \in dcl(Cb(c/a))$ for all a.

Let $a^* \in W^*$ be such that $Cb(a^*/Ad) \not\subseteq acl(a^*)$. Then $Cb(Ad/a^*) \not\subseteq acl(Ad)$. By Fact 2.10, there exists $e^* \notin acl(Ad)$ such that $e^* \in acl(Cb(Ab'\vec{c}_0\vec{c}_1/a^*))$ for all $b'\vec{c}_0\vec{c}_1 \models tp(b\vec{c}_0^*(\vec{c}\setminus\vec{c}_0^*)/Ad)$. In particular, $e^* \in V^*$. Thus, if we fix the sort for e' in the definition of V^* to be the sort of e^* , then V^* is unbounded. Now, let $e' \in V^*$. Then for some $a' \in W^*$, $\models V(e', \vec{c}_0, \vec{c}_1, b', a')$ for all b', \vec{c}_0, \vec{c}_1 . By Subclaim 2.7, there exists $b'\vec{c}_0'\vec{c}_1 \models tp(b\vec{c}_0^*(\vec{c}\setminus\vec{c}_0^*)/Ad)$ such that a' is independent from $Ab'\vec{c}_0$ over \emptyset . Thus, by the definition of V^* and $V, e' \in acl(Cb(Ab'\vec{c}_0'\vec{c}_1/a'))$. Since Ab' is independent from a' over $\emptyset, tp(e')$ is almost- p_0 -internal (as $Cb(Ab'\vec{c}_0'\vec{c}_1/a')$ is in the definable closure of any Morley sequence of $Lstp(Ab'\vec{c}_0'\vec{c}_1/a')$), and in particular tp(e'/Ad) is almost p_0 -internal (note that, in general, whenever q = tp(a/A) is internal in an \emptyset -invariant set \mathcal{R} then any extension of q is almost \mathcal{R} -internal) and therefore tp(e'/Ad) has finite SU-rank. \Box

Thus V^* is the required set. \Box

We now draw some consequences of the above dichotomy for countable languages.

Theorem 2.11 Let T be any countable hypersimple theory. Let Υ be a projection-closed family of topologies such that $\{a \in \mathcal{C}^x | a \notin acl(A)\} \in \Upsilon_{x,A}$ for all x and set A. Let p_0 be a partial type over \emptyset of SU-rank 1. Then, either there exists an unbounded type-definable Υ -open set over some small set that

is almost p_0 -internal and has **bounded** finite SU-rank, or every complete type $p \in S(A)$ that is internal in p_0 is essentially 1-based over \emptyset by means of Υ . In particular, either there exists an unbounded Υ -open set that is almost p_0 -internal and has **bounded** finite SU-rank, or whenever $p \in S(A)$, where A is countable, and every non-algebraic extension of p is non-foreign to p_0 , p is analyzable in p_0 by essentially 1-based types by means of Υ .

Proof: We go back to the proof of Theorem 2.3 (the main part); we start with $p \in S(A)$ that is p_0 -internal and not essentially 1-based over \emptyset and apply the same proof (but note that in the proof of Theorem 2.3 we assumed p is not s-essentially 1-based). So, now \mathcal{U} is assumed to be a **type-definable** Υ -open set over Ad.

Subclaim 2.12 We may assume W^* is type-definable and Υ -open over Ad and there exists $V^{**} \subseteq V^*$ that is unbounded, type-definable and Υ -open over Ad.

Proof: In the proof of Theorem 2.3 the set W^* is chosen to be a non-empty intersection of \mathcal{U} with a Stone-open set over Ad, so we could instead take it to be a non-empty intersection of \mathcal{U} with a definable subset of this Stoneopen set (and still $W^* \subseteq F_{\overline{c}_0^*} \cap \mathcal{U}^*$). Since \mathcal{U} is Υ -open and type-definable, W^* is type-definable and Υ -open over Ad. Now, by the definition of V^* and the proof of Subclaim 2.8 there exist a Stone open set V_0 over Ad such that $V^* = \{e' | \exists a' \in W^* (V_0(e', a'))\}$. From this we easily get the required set V^{**} (by replacing V_0 by a definable set and using the fact that W^* is typedefinable and that Υ is a projection-closed family of topologies). \Box

By the proof of Subclaim 2.9 we know that for all $e' \in V^{**}$ we have $e' \in acl(Cb(Ab'\vec{c}_0\vec{c}_1/a'))$ for some $a' \in W^*$ and some b', \vec{c}_0, \vec{c}_1 such that a' is independent from $Ab'\vec{c}_0$ over \emptyset and $b'\vec{c}_0'\vec{c}_1 \models tp(b\vec{c}_0^*(\vec{c} \setminus \vec{c}_0^*)/Ad)$. Let $q = tp(Ab\vec{c}_0^*)$. For every $\chi = \chi(x, y_0, ..., y_n, \vec{z}) \in L$ (for some $n < \omega$) such that $\forall y_0y_1...y_n \vec{z} \exists^{<\infty}x \ \chi(x, y_0, y_1, ..., y_n, \vec{z})$, and $m < \omega$ let

 $F_{\chi,m} = \{ e \in V^{**} | \models \chi(e, C_0, C_1, ...C_n, \bar{c}) \text{ for some } \bar{c} \in p_0^m \text{ and some } \emptyset - \text{independent sequence} \\ (C_i | i \le n) \text{ of realization of } q \text{ with } e \ igcup (C_i | i \le n) \}.$

By the aforementioned, we get that $V^{**} \subseteq \bigcup_{m,\chi} F_{m,\chi}$ (the union is over each m, χ as above). By the Baire category theorem applied to the Stone-topology of the Stone-closed set $V^{**} \setminus acl(Ad)$, there exists $\theta \in L(Ad)$ such that

$$\tilde{V} \equiv \theta^{\mathcal{C}} \cap (V^{**} \setminus acl(Ad)) \neq \emptyset \text{ and } \tilde{V} \subseteq F_{m^*,\chi^*}$$

for some m^*, χ^* as above. Clearly, \tilde{V} is unbounded, type-definable and Υ open (by the assumptions on Υ). Now, for every $a \in \tilde{V}$, $SU(a/Ad) \leq m^*$ and tp(a/Ad) is almost p_0 -internal (as tp(a) is almost p_0 -internal, and $SU(a) \leq$ m^* by the definition of F_{m^*,χ^*}). This completes the proof of the first part of the theorem. The rest follow easily by repeated applications of the first part (when working over A). \Box

Recall that T is PCFT if its forking-topologies is a projection-closed family of topologies, that is, whenever $\mathcal{U}(x, y)$ is a τ^{f} -open set over a small set A, $\exists y \mathcal{U}(x, y)$ is a τ^{f} -open set over A. Applying Theorem 2.11 for the special case of the forking-topologies we conclude the following.

Corollary 2.13 Let T be any countable hypersimple theory with PCFT. Let p_0 be a partial type over \emptyset of SU-rank 1. Then, either there exists a weaklyminimal formula that is almost p_0 -internal, or every complete type $p \in S(A)$ that is internal in p_0 is essentially 1-based over \emptyset by means of τ^f . In particular, either there exists a weakly-minimal formula that is almost p_0 -internal, or whenever $p \in S(A)$, where A is countable, and every non-algebraic extension of p is non-foreign to p_0 , p is analyzable in p_0 by essentially 1-based types by means of τ^f .

Proof: Our assumptions are clearly a special case of the assumptions of Theorem 2.11, thus we only need to prove the first part. By the conclusion of Theorem 2.11, we may assume that there exists a τ^f -open set \mathcal{U} of bounded finite SU-rank over some small set A that is almost p_0 -internal. Recall now [S0, Proposition 2.13]:

Fact 2.14 Let \mathcal{U} be an unbounded τ^f -open set over some set A. Assume \mathcal{U} has bounded finite SU-rank. Then there exists a set $B \supseteq A$ and $\theta(x) \in L(B)$ of SU-rank 1 such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup acl(B)$.

By Fact 2.14, there exists exists a weakly-minimal $\theta(x, b) \in L(B)$ for some small set $B \supseteq A$, such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup acl(B)$. Now, tp(a/B) is almost p_0 -internal for every $a \in \theta^{\mathcal{C}}$, and so tp(a/b) (b is the parameter of $\theta(x, b)$) is almost p_0 -internal over b for every $a \in \theta^{\mathcal{C}}$ (by taking non-forking extensions). \Box

We now state the main conclusion for uncountable hypersimple unidimensional theories.

Definition 2.15 We say that T is s-essentially 1-based if for every SU-rank 1 partial type p_0 over some A, every $p \in S(A)$ that is internal in p_0 is s-essentially 1-based by means of τ^f .

Corollary 2.16 Let T be a hypersimple unidimensional theory that is not s-essentially 1-based. Then T is supersimple.

Proof: First, recall the following fact [S1, Corollary 3.15] (an A-invariant set \mathcal{U} is called *supersimple* if $SU(a/A) < \infty$ for every $a \in \mathcal{U}$).

Fact 2.17 Let T be a hypersimple unidimensional theory and work in $C = C^{eq}$. Let $p \in S(A)$ and let \mathcal{U} be an unbounded τ^f -open set over A. Then p is analyzable in \mathcal{U} in finitely many steps. In particular, for such T the existence of an unbounded supersimple τ^f -open set over some small set A implies T is supersimple.

Now, assume T is a hypersimple unidimensional theory that is not s-essentially 1-based. By Theorem 2.3, there exists an unbounded τ^{f} -open set of finite SU-rank over some small set. By Fact 2.17, every complete type has finite SU-rank. \Box

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