Normal hyperimaginaries*

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December 3, 2013

Abstract

We introduce the notion of normal hyperimaginary and we develop its basic theory. We present a new proof of Lascar-Pillay's theorem on bounded hyperimaginaries based on properties of normal hyperimaginaries. However, the use of Peter-Weyl's theorem on the structure of compact Hausdorff groups is not completely eliminated from the proof. In the second part, we show that all closed sets in Kim-Pillay spaces are equivalent to hyperimaginaries and we use this to introduce an approximation of φ-types for bounded hyperimaginaries.

As usual, we work in the monster model $\mathfrak{C}$ of a complete theory T of language L. For background on hyperimaginaries we refer to [2]. Recall that a hyperimaginary is an equivalence class $e = a_E$ of a possibly infinite tuple a under a 0-type-definable equivalence relation E. We use the notation $E(x, y)$ for the partial type defining the equivalence relation E.

For a hyperimaginary e, let $\mathrm{Fix}(e) = \mathrm{Aut}(\mathfrak{C}/e)$ be the group of automorphisms of the monster model $\mathfrak{C}$ fixing e. A hyperimaginary d is *definable over* e if $f(d) = d$ for all $f \in \mathrm{Fix}(e)$. The *definable closure* $\mathrm{dcl}(e)$ of e is the class of all hyperimaginaries definable over e. Two hyperimaginaries e, d are *equivalent*, written $e \sim d$, if they are interdefinable, that is, if $\mathrm{dcl}(e) = \mathrm{dcl}(d)$. This notation can also be applied to the case where e or d are sequences of hyperimaginaries. If A is a set of hyperimaginaries $e \sim A$ means that $e \sim d$ for a sequence d enumerating A. In some cases we will be interested in automorphisms fixing A set-wise. We write $e \overset{\mathrm{sw}}{\sim} A$ to mean that $\mathrm{Fix}(e)$ is the set of all automorphisms f such that $f(A) = A$ (set-wise). If $(A_i : i \in I)$ is a sequence of sets, we write $e \overset{\mathrm{sw}}{\sim} (A_i : i \in I)$ meaning that $\mathrm{Fix}(e)$ is the set of all automorphisms f such that $f(A) = A$ for all $i \in I$.

The *cardinality* $|e|$ of a hyperimaginary e is the minimal cardinality of a set A of real elements (i.e., $A \subseteq \mathfrak{C}$) such that $e \in \mathrm{dcl}(A)$. In this case, for any cardinal $\kappa \geq |e|$ there is a 0-type-definable equivalence relation E on κ-tuples and there is a κ-tuple a such that $e \sim a_E$. The hyperimaginary e is called *finitary* if $|e| < \omega$. Equivalently, e is finitary if $e \sim a_E$ for some finite tuple a and some 0-type-definable equivalence relation E.

A hyperimaginary e is *bounded* if it has a small orbit (an orbit of cardinality smaller than the size of the monster model). We denote by $\mathrm{bdd}(\emptyset)$ the class of all bounded hyperimaginaries. There is a single hyperimaginary e which is interdefinable with $\mathrm{bdd}(\emptyset)$, in the sense that $\mathrm{dcl}(e) = \mathrm{bdd}(\emptyset)$. More generally, for any definably closed class $A \subseteq \mathrm{bdd}(\emptyset)$ there is a single $e \in A$ such that $A = \mathrm{dcl}(e)$. For any index set I, the relation $\equiv_{\mathrm{bdd}(\emptyset)}$ of having

*Partially supported by grants MTM 2011-26840 of Spanish Ministry of Economy and Competitiveness and 2009SGR-187 of Catalan Government

the same type over $\mathrm{bdd}(\emptyset)$ restricted to I-tuples is the smallest bounded (i.e., with a small number of classes) 0-type-definable equivalence relation on I-tuples. It is also called the *Kim-Pillay* equivalence relation and its classes are called KP-strong types. The set of all KP-classes of α-tuples is $\mathfrak{C}^\alpha/\mathrm{KP}$.

We see the class of all definably closed classes of hyperimaginaries as a lattice with the order of inclusion. Hence $\inf(A, B) = A \cap B$ and $\sup(A, B) = \mathrm{dcl}(A \cup B)$. By abuse of notation we write something like $\inf(e_1, e_2) \sim d$ or even $\inf(e_1, e_2) = d$ for hypermaginaries e_1, e_2, d to mean that $\inf(\mathrm{dcl}(e_1), \mathrm{dcl}(e_2)) = \mathrm{dcl}(d)$. Note that $\sup(e_i : i \in I) = \mathrm{dcl}(e_i : i \in I)$.

Lascar and Pillay proved in [3] that every bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries. Their proof rely on an application of Peter-Weyl's theorem on the structure of compact Hausdorff groups according to which each such group is an inverse limit of compact Lie groups. We seek for a purely model-theoretical proof of the same result, avoiding the use of Peter-Weyl's theorem. There are particular cases where the existence of such a sequence of finitary hyperimaginaries is easy to guarantee: normal hyperimaginaries and KP-classes (see Proposition 8 and Lemma 18 below)

1 Normal hyperimaginaries

The group $G = \mathrm{Aut}(\mathrm{bdd}(\emptyset))$ of elementary permutations of $\mathrm{bdd}(\emptyset)$ is a topological group, with a compact Hausdorff topology. Its closed subgroups are all subgroups of the form $\mathrm{Fix}_G(e) = \{f \in G : f(e) = e\}$ with $e \in \mathrm{bdd}(\emptyset)$. For a complete description of the topology see [3] or [4]. If we endow $\mathrm{Aut}(\mathfrak{C})$ with the topology of point-wise convergence (a basis of open sets is given by all sets of the form $\{f \in \mathrm{Aut}(\mathfrak{C}) : f(a) = b\}$ for all finite tuples $a, b \in \mathfrak{C}$) then $\mathrm{Aut}(\mathfrak{C})$ is a topological group and the canonical projection $\mathrm{Aut}(\mathfrak{C}) \to G$ is continuous. Notice that $G \cong \mathrm{Aut}(\mathfrak{C})/\mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(\emptyset))$. According to Peter-Weyl's theorem, there is a family $(G_i : i \in I)$ of normal closed subgroups G_i of G such that $\bigcap_{i\in I} G_i = \{1\}$ and each G/G_i is a compact Lie group, and hence it has the descending chain condition (DCC) on closed subgroups. Each G_i is of the form $\mathrm{Fix}_G(e_i)$ for some $e_i \in \mathrm{bdd}(\emptyset)$. Let $H_i = \mathrm{Fix}(e_i)$ be the corresponding subgroup of $\mathrm{Aut}(\mathfrak{C})$. Then $\bigcap_{i\in I} H_i = \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(\emptyset))$ and therefore $(e_i : i \in I)$ is interdefinable with any tuple enumerating $\mathrm{bdd}(\emptyset)$. Moreover the DCC of G/G_i translates as follows: there is no strictly ascending chain $(G_{i,j} : j < \omega)$ of closed subgroups $G_{i,j} \leq G_{i,j+1}$ of G extending G_i. This explains the following definitions:

Definition 1 A hyperimaginary e is *normal* if $\mathrm{Fix}(e)$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$. A hyperimaginary e is *DCC* if there is no sequence $(e_n : n < \omega)$ of hyperimaginaries $e_n \in \mathrm{dcl}(e)$ such that $e_n \in \mathrm{dcl}(e_{n+1})$ and $e_{n+1} \notin \mathrm{dcl}(e_n)$ for each $n < \omega$.

Peter-Weyl's theorem give us a sequence $(e_i : i \in I)$ of normal DCC hyperimaginaries $e_i \in \mathrm{bdd}(\emptyset)$ such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$. We will see that normal hyperimaginaries are bounded and that normal DCC hyperimaginaries are finitary. We will show that in order to prove Lascar-Pillay's theorem it is in fact enough to find a sequence $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$.

Definition 2 We call *Peter-Weyl's condition* the statement that there is a sequence $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$.

We have not found a proof of Peter-Weyl's condition avoiding the use of Peter-Weyl's theorem, but we can offer an easy-to-follow proof of Lascar-Pillay's theorem assuming this condition.

Proposition 3 *The following are equivalent for any hyperimaginary e:*

1. *e is normal.*
2. *For any $e' \equiv e$, $e' \in \mathrm{dcl}(e)$.*
3. *$e \sim (f(e) : f \in \mathrm{Aut}(\mathfrak{C}))$*
4. *e is equivalent to a sequence enumerating an orbit of a hyperimaginary.*

Proof: *1* $\Leftrightarrow$ *2*. By definition, e is normal iff for any $f, g \in \mathrm{Aut}(\mathfrak{C})$ such that $f(e) = e$, we have $g^{-1}fg(e) = e$, that is $f(g(e)) = g(e)$. Therefore, e is normal iff $\{g(e) : g \in \mathrm{Aut}(\mathfrak{C})\} \subseteq \mathrm{dcl}(e)$.

2 $\Rightarrow$ *3*. Clear, since $f(e) \equiv e$ for every $f \in \mathrm{Aut}(\mathfrak{C})$.

3 $\Rightarrow$ *4*. Obvious.

4 $\Rightarrow$ *2*. If e is equivalent to an enumeration of an orbit and $e' \equiv e$, then e' is equivalent to an enumeration of the same orbit and therefore $e' \in \mathrm{dcl}(e)$. $\square$

Remark 4 *Normal hyperimaginaries are bounded.*

Proof: Let e be normal. If $(e_i : i < \kappa)$ is a long enough sequence of different conjugates of e, then we can find $i < j < \kappa$ with $e_i \equiv_e e_j$. Since e_i, e_j are definable over e, $e_i = e_j$, a contradiction. $\square$

Proposition 5 *A hyperimaginary e is normal if and only if for any index set I, the equivalence relation $\equiv_e$ on I-tuples is 0-type-definable.*

Proof: Let $(e_j : j \in J)$ be a (bounded) orbit equivalent to e. Then $\equiv_e = \equiv_{(e_j : j \in J)}$, which is clearly invariant and type-definable, hence 0-type-definable.

If $\equiv_e$ is 0-type definable, then also $\equiv_e$ as a relation between hyperimaginaries is 0-type-definable. Let $f \in \mathrm{Fix}(e)$ and $g \in \mathrm{Aut}(\mathfrak{C})$ such that $g(e) = e'$. Then $e' \equiv_e f(e')$. If we apply g^{-1} we see that $e \equiv_e g^{-1}f(e')$ and hence $g^{-1}f(e') = e$. If we apply g we conclude that $f(e') = g(e) = e'$. Therefore $e' \in \mathrm{dcl}(e)$. $\square$

Remark 6 *If each e_i is normal, then $(e_i : i \in I)$ is normal.*

Lemma 7 *Let $e = a_E$ be normal.*

1. *$e \sim a_{\equiv_e}$.*
2. *For any tuple m enumerating a model, $e \sim m_{\equiv_e}$.*

Proof: *1.* If e is normal, then $\equiv_e$ is 0-type-definable and $a_{\equiv_e}$ is a hyperimaginary. Assume first $f \in \mathrm{Fix}(e)$. Then $a \equiv_e f(a)$ and therefore $f(a_{\equiv_e}) = a_{\equiv_e}$. For the other direction, assume now $f(a_{\equiv_e}) = a_{\equiv_e}$. Then $f(a) \equiv_e a$. Since $a_E = e$, $f(a_E) = e$, that is, $f(e) = e$.

2. Assume m enumerates a model. Clearly, $m_{\equiv_e} \in \mathrm{dcl}(e)$. On the other hand, if f fixes $m_{\equiv_e}$ then $m \equiv_e f(m)$ and there is some $g \in \mathrm{Fix}(e)$ such that $g(m) = f(m)$. It follows that fg^{-1} fixes point-wise a model and it is a strong automorphism, which implies it fixes every element of $\mathrm{bdd}(\emptyset)$. Hence $f(e) = fg^{-1}g(e) = fg^{-1}(e) = e$. $\square$

Proposition 8 *Every normal hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.*

Proof: Let e be normal. By the previous lemma, $\equiv_e$ is type-definable over $\emptyset$ and $e \sim a_{\equiv_e}$ for some tuple a. Let $a = (a_i : i < \kappa)$ and for each finite $X \subseteq \kappa$ let E^X be defined for κ-tuples b, c by

$$E^X(b,c) \Leftrightarrow b \restriction X \equiv_e c \restriction X.$$

If $e^X = a_{E^X}$, then each e^X is finitary and $e \sim (e^X : X \subseteq \kappa \text{ finite })$. □

Lemma 9 *Every normal DCC hyperimaginary is finitary.*

Proof: Let e be normal DCC. Choose, like in the proof of Proposition 8, a tuple $a = (a_i : i < \kappa)$ such that $e \sim a_{\equiv_e}$ and define E^X and e^X as in that proof. Clearly, $e^X \in \mathrm{dcl}(e)$ and if $X \subseteq Y$, then $e^X \in \mathrm{dcl}(e^Y)$. Since e is DCC, there is some finite X such that for all finite $Y \supseteq X$, $e^Y \in \mathrm{dcl}(e^X)$. It follows that $e \sim e^X$ and hence e is finitary. □

Proposition 10 *1. For any 0-type-definable equivalence relation on κ-tuples F, for any hyperimaginary e, if $E = \equiv_e$, then the relational product $E \circ F = F \circ E = E \circ F \circ E$ is an equivalence relation.*

2. Given normal e and $d \in \mathrm{bdd}(\emptyset)$, there are a κ-tuple m and a 0-type-definable equivalence relation F on κ-tuples such that, if E is the 0-type-definable equivalence relation $\equiv_e$ on κ-tuples, then $m_E \sim e$, $m_F \sim d$ and $m_{E \circ F} \sim \inf(e, d)$.

Proof: *1.* We must check symmetry and transitivity of $E \circ F$. For symmetry, assume $a \equiv_e bFc$ and choose an automorphism f such that $f(e) = e$ and $f(a) = b$. Let c' be such that $f(c') = c$. Then $ac' \equiv bc$ and therefore $F(a, c')$. Hence $c \equiv_e c'Fa$. Using now symmetry, for transitivity it is enough to prove that if $a \equiv_e bFc \equiv_e d$, then $aE \circ Fd$. Choose $f \in \mathrm{Fix}(e)$ such that $f(c) = d$. Then $a \equiv_e f(b)Fd$.

2. Let $d = a_G$ for a tuple a, and extend a to a tuple $m = (m_i : i < \kappa)$ enumerating a model. Let $I \subseteq \kappa$ be such that $a = (m_i : i \in I)$ and define F by

$$F(x, y) \leftrightarrow G(x \restriction I, y \restriction I).$$

It is a 0-type-definable equivalence relation and $m_F \sim d$. Let $E = \equiv_e$. By Lemma 7, $m_E \sim e$. It is clear that $m_{E\circ F} \in \mathrm{dcl}(m_E) \cap \mathrm{dcl}(m_F)$. Now we assume $e' \in \mathrm{dcl}(m_E) \cap \mathrm{dcl}(m_F)$ and we check that $e' \in \mathrm{dcl}(m_{E\circ F})$. For this purpose, let f be an automorphism fixing $m_{E\circ F}$. Then $E \circ F(m, f(m))$ and by symmetry $F \circ E(m, f(m))$. Let b be such that $F(m, b) \wedge E(b, f(m))$. Since $b \equiv_e f(m)$, there is an automorphism $g \in \mathrm{Fix}(e)$ such that $g(b) = f(m)$. Then $F(g(m), g(b))$, that is $F(m, g^{-1}f(m))$. Let $h = g^{-1}f$. Since h fixes m_F, $h(e') = e'$. Since $g \in \mathrm{Fix}(e)$, $m \equiv_e g(m)$ and hence g fixes m_E and $g(e') = e'$. Therefore $f(e') = gh(e') = g(e') = e'$. □

Remark 11 *Under the Galois correspondence,* $\inf(e, d)$ *corresponds to* $\sup(\mathrm{Fix}(e), \mathrm{Fix}(d))$ *in the lattice of closed subgroups. If* $\mathrm{Fix}(e)$ *is a normal subgroup, this* sup *is the product* $\mathrm{Fix}(e) \cdot \mathrm{Fix}(d)$ *(the product of two compact subgroups is compact, hence closed). So in Proposition 10,* $\mathrm{Fix}(e) \cdot \mathrm{Fix}(d) = \mathrm{Fix}(m_{E\circ F})$.

Remark 12 *To prove Peter-Weyl's condition it is enough to prove that for every finitary bounded hyperimaginary e there is a family $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $e \in \mathrm{dcl}(e_i : i \in I)$.*

Proof: There is a normal e such that $e \sim \operatorname{bdd}(\emptyset)$. Since e is equivalent to a family of finitary bounded hyperimaginaries and each finitary bounded hyperimaginary is definable over a family of finitary normal hyperimaginary, we conclude that e is definable over a family $(e_i : i \in I)$ of finitary normal hyperimaginaries. It follows that $e \sim (e_i : i \in I)$. $\square$

Corollary 13 (Lascar-Pillay) *Every bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.*

Proof: (Assuming Peter-Weyl's condition) Let d be a bounded hyperimaginary and choose a family $(e_i : i \in I)$ of finitary normal hyperimaginaries such that $(e_i : i \in I) \sim \operatorname{bdd}(\emptyset)$. Let $\kappa \geq |I|, |d|, |T|$, and for each $i \in I$ let E_i be the equivalence relation $\equiv_{e_i}$ on κ-tuples. Let E be the Kim-Pillay equivalence relation $\equiv_{\operatorname{bdd}(\emptyset)}$ on κ-tuples. We may assume that the family is closed under finite composition (that is, for any $i, j \in I$ there is some $k \in I$ such that $e_k \sim e_i e_j$), which implies $E = \bigcap_{i \in I} E_i$. Choose with Proposition 10 a 0-type-definable bounded equivalence relation F on κ-tuples and some κ-tuple m such that $d \sim m_F$, $e_i \sim m_{E_i}$ and $\inf(e_i, d) \sim m_{E_i \circ F}$. Since e_i is finitary, $\inf(e_i, d)$ is finitary too. We claim that $d \sim (\inf(e_i, d) : i \in I)$. Notice that $F = E \circ F$. Hence $d \sim m_{E \circ F}$ and it is enough to check that $m_{E \circ F} \in \operatorname{dcl}(m_{E_i \circ F} : i \in I)$. Let f be an automorphism fixing each $m_{E_i \circ F}$. Then for each $i \in I$ there is some a_i such that

$$E_i(m, a_i) \wedge F(a_i, f(m)).$$

By compactness there is some a such that $E(m, a) \wedge F(a, f(m))$. Hence f fixes $m_{E \circ F}$. $\square$

Remark 14 *The Galois correspondence provides another proof of Corollary 13 in terms of groups. Let d be a bounded hyperimaginary and let $(e_i : i \in I)$ be a family of finitary normal hyperimaginaries such that $(e_i : i \in I) \sim \operatorname{bdd}(\emptyset)$. As above, we may assume that the family is closed under finite composition. Let $H_i = \operatorname{Fix}(e_i)$, a closed normal subgroup of the Galois group of T. Under the Galois correspondence, the conditions on the e_i's means that $\bigcap_i H_i = \{1\}$, and for each i, j there is some k such that $H_i \cap H_j = H_k$. Let $H = \operatorname{Fix}(d)$, and consider $L_i = H.H_i$, a closed subgroup of the Galois group. Again, the Galois correspondence tells us that $L_i = \operatorname{Fix}(h_i)$ for some bounded hyperimaginary h_i, and certainly h_i is finitary since e_i is. Now $\bigcap_i L_i = \bigcap_i H.H_i = H.\bigcap_i H_i = H$, which means that $d \sim (h_i : i \in I)$.*

2 Local types of hyperimaginaries

Definition 15 Let e, d be hyperimaginaries. The *orbit* of e over d is the set $\mathcal{O}(e/d)$ of all hyperimaginaries e' such that $e \equiv_d e'$.

Remark 16 *Notice that for an automorphism f, the condition $f(\mathcal{O}(e/d)) = \mathcal{O}(e/d)$ is equivalent the conjunction of $e \equiv_d f(e)$ and $e \equiv_d f^{-1}(e)$.*

Next lemma is due to Buechler, Pillay and Wagner (Lemma 2.18 in [1]). It basically says that we can consider $\mathcal{O}(e/d)$ as a hyperimaginary if $e \in \operatorname{bdd}(d)$. In our Proposition 20 below we have generalized this fact to any closed set in a Kim-Pillay space. We apply this to some closed sets $\mathcal{O}_\varphi(e/d)$ obtaining thus some hyperimaginaries $h_{p,\varphi,d}$. For $d \in \operatorname{bdd}(\emptyset)$ and $p(x) = \operatorname{tp}(e/\emptyset)$ we understand $\operatorname{tp}(e/h_{p,\varphi,d})$ as an approximation to the φ-type of e over d.

Remark 17 *If $e \in \text{bdd}(d)$, then $\mathcal{O}(e/d)$ is $\overset{\text{sw}}{\sim}$-equivalent to some hyperimaginary h, in the sense that the automorphisms of the monster model fixing h is the set of automorphisms fixing set-wise $\mathcal{O}(e/d)$.*

Lemma 18 *If $e = (a_i : i < \omega)_{\text{KP}}$ and $e_n = (a_i : i \leq n)_{\text{KP}}$, then $e \sim (e_n : n < \omega)$.*

Proof: For every automorphism f, $f \in \text{Fix}(e)$ iff $(a_i : i < \omega) \equiv_{\text{bdd}(\emptyset)} (f(a_i) : i < \omega)$ iff $(a_i : i \leq n) \equiv_{\text{bdd}(\emptyset)} (f(a_i) : i \leq n)$ for all $n < \omega$ iff $f(e_n) = e_n$ for all $n < \omega$. □

Proposition 19 *If $d \in \text{bdd}(\emptyset)$, then $d \overset{\text{sw}}{\sim} (\mathcal{O}(e/d) : e \in \mathfrak{C}^\omega/\text{KP})$ and $d \overset{\text{sw}}{\sim} (\mathcal{O}(e/d) : e \in \mathfrak{C}^n/\text{KP}, n < \omega)$*

Proof: If $f \in \text{Fix}(d)$, then f permutes every orbit $\mathcal{O}(e/d)$.

Assume f permutes every $\mathcal{O}(e/d)$ for every countable KP-class $e \in \mathfrak{C}^\omega/\text{KP}$. It is well known that each hyperimaginary is equivalent to a sequence of countable hyperimaginaries. Hence $d \sim (d_i : i \in I)$, where every d_i is a countable hyperimaginary. Choose an ω-tuple a_i and a bounded 0-type-definable equivalence relation E_i such that $d_i = {a_i}_{E_i}$. By hypothesis, $f(a_{i\text{KP}}) \in \mathcal{O}(a_{i\text{KP}}/d)$ and therefore $f(a_{i\text{KP}}) = g_i(a_{i\text{KP}})$ for some $g_i \in \text{Fix}(d)$. Note that d_i is a union of KP-classes of ω-tuples. Since g_i fixes d_i, f permutes these KP-classes and then $f(d_i) = d_i$. Since f fixes each d_i, $f(d) = d$.

Assume now f permutes every $\mathcal{O}(e/d)$ for every finitary KP-class $e \in \mathfrak{C}^n/\text{KP}$. We show that f permutes $\mathcal{O}(e/d)$ for every countable KP-class $e \in \mathfrak{C}^\omega/\text{KP}$. Let $e = (a_i : i < \omega)_{\text{KP}}$ and let $e_n = (a_i : i \leq n)_{\text{KP}}$. Since $e_n \equiv_d f(e_n)$ for all $n < \omega$, $(e_n : n < \omega) \equiv_d (f(e_n) : n < \omega)$. Choose $g \in \text{Fix}(d)$ such that $g(e_n : n < \omega) = (f(e_n) : n < \omega)$. Then $f^{-1}g(e_n) = e_n$ for all $n < \omega$ and by Lemma 18 $f^{-1}g(e) = e$. It follows that $e \equiv_d f(e)$ and hence f permutes $\mathcal{O}(e/d)$. □

Proposition 20 *Every closed set C in a Kim-Pillay space is $\overset{\text{sw}}{\sim}$-equivalent to a hyperimaginary h_C, that is, the automorphisms of the monster model fixing set-wise C are the automorphisms fixing h_C.*

Proof: Let E be a bounded 0-type-definable equivalence relation on α-tuples and let $X = \mathfrak{C}^\alpha/E$ be the corresponding Kim-Pillay space. If $C \subseteq X$ is closed, then for some partial type $\pi(x,z)$, for some tuple b, $\pi(\mathfrak{C}, b) = \{a : a_E \in C\}$. For each formula $\theta(x,y) \in E(x,y)$ there is a maximal length $n = n_\theta < \omega$ of a sequence of tuples $(a_i : i < n)$ such that ${a_i}_E \in C$ and $\models \neg\theta(a_i, a_j)$ for all $i < j < n$. Let $(a_i^\theta : i < n_\theta)$ witness it, let $\Sigma_\theta(z,z')$ be the partial type

$$\exists(x_i : i < n_\theta)(\bigwedge_{i<j<n_\theta} \neg\theta(x_i,x_j) \wedge \bigwedge_{i<n_\theta} \pi(x_i,z) \wedge \bigwedge_{i<n_\theta} \pi(x_i,z'))$$

and

$$F(z,z') = \bigwedge_{\theta \in E} \Sigma_\theta(z,z')$$

Claim: For every automorphism f, $f(C) = C$ if and only if $\models F(b, f(b))$.
Proof of the claim: From left to right it is straightforward. For the other direction, assume $\models F(b, f(b))$ and choose $(c_i^\theta : i < n_\theta, \theta \in E)$ witnessing it. Let $a_E \in C$. Then $\models \pi(a,b)$, and hence $\models \pi(f(a), f(b))$. By maximality of n_θ, for every $\theta \in E$ there is some $i < n_\theta$ such that $\models \theta(f(a), c_i^\theta)$. By compactness, $E(f(a), x) \cup \pi(x, b)$ is consistent and therefore $f(a_E) \in C$. This shows that $f(C) \subseteq C$. By the same reason, $f^{-1}(C) \subseteq C$, that is, $C \subseteq f(C)$.

It follows from the claim that F defines a 0-type-definable equivalence relation on realizations of $p(x) = \text{tp}(b)$. By standar arguments it can be extended to a 0-type-definable equivalence relation defined for all tuples of the length of b. The hyperimaginary b_F satisfies the requirements. $\square$

Definition 21 Let e, d be hyperimaginaries. If $\varphi(x,y) \in L$, $\models \varphi(e,d)$ means that $\models \varphi(a,b)$ for some representatives a, b of e, d respectively. Notice that $e \equiv_d e'$ iff $\models \varphi(e,d) \Leftrightarrow \models \varphi(e',d)$ for all $\varphi(x,y) \in L$. Let $\mathcal{O}_\varphi(e/d) = \{e' : e' \equiv e \text{ and } \models \varphi(e',d)\}$. Let $p(x) = \mathrm{tp}(e)$ and assume $e \in \mathrm{bdd}(\emptyset)$. Then $e = a_E$ for some tuple a and some bounded 0-type-definable equivalence relation E. The set of all E-classes is a Kim-Pillay space and $\mathcal{O}_\varphi(e/d)$ defines a closed subset. By Lemma 20 there is some hyperimaginary $h_{p,\varphi,d}$ such that

$$h_{p,\varphi,d} \overset{\text{sw}}{\sim} \mathcal{O}_\varphi(e/d).$$

The equivalence relation $E(e,e')$ defined by $\models \varphi(e,d) \Leftrightarrow \models \varphi(e',d)$ is not, in general, type-definable. This is the reason why an adequate treatment of local types (or φ-types) is missing in the model theory of hyperimaginaries. The following results show that the types $\mathrm{tp}(e/h_{\mathrm{tp}(e),\varphi,d})$ are (for e bounded) a substitute for the φ-type of e over d and we apply this in Corollary 24 to obtain a new decomposition of a bounded hyperimaginary in terms of orbits.

Remark 22 *Let d be a hyperimaginary, $e \in \mathrm{bdd}(\emptyset)$, $p(x) = \mathrm{tp}(e)$ and $\varphi(x,y) \in L$.*

1. *If $e' \equiv_{h_{p,\varphi,d}} e$ then $\models \varphi(e,d) \Leftrightarrow \models \varphi(e',d)$.*
2. *$h_{p,\varphi,d} \in \mathrm{dcl}(d)$.*

Proof: Clear. $\square$

Proposition 23 *Let d be a hyperimaginary, $e \in \mathrm{bdd}(\emptyset)$ and $p(x) = \mathrm{tp}(e)$. For any $e' \models p$:*

$$e' \equiv_d e \textit{ if and only if } e' \equiv_{h_{p,\varphi,d}} e \textit{ for every } \varphi(x,y) \in L$$

Proof: By Remark 22. $\square$

Corollary 24 *If $d \in \mathrm{bdd}(\emptyset)$, then $d \overset{\text{sw}}{\sim} (\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d}) : e \in \mathfrak{C}^n/\mathrm{KP}, n < \omega)$.*

Proof: Let $d \in \mathrm{bdd}(\emptyset)$. If $f \in \mathrm{Fix}(d)$, then f fixes $h_{\mathrm{tp}(e),\varphi,d}$ and permutes $\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d})$. On the other hand, if $e \in \mathfrak{C}^n/\mathrm{KP}$ and f permutes all the orbits $(\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d})$, then $e \equiv_{h_{\mathrm{tp}(e),\varphi,d}} f(e)$ for all φ and by Proposition 23 $e \equiv_d f(e)$. Similarly, $e \equiv_d f^{-1}(e)$. It follows that f permutes $\mathcal{O}(e/d)$. By Proposition 19, $f(d) = d$. $\square$

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