

BETWEEN POLISH AND COMPLETELY BAIRE

ANDREA MEDINI AND LYUBOMYR ZDOMSKYY

ABSTRACT. All spaces are assumed to be separable and metrizable. Consider the following properties of a space X .

- (1) X is Polish.
- (2) For every countable crowded $Q \subseteq X$ there exists a crowded $Q' \subseteq Q$ with compact closure.
- (3) Every closed subspace of X is either scattered or it contains a homeomorphic copy of 2^ω .
- (4) Every closed subspace of X is a Baire space.

While (4) is the well-known property of being *completely Baire*, properties (2) and (3) have been recently introduced by Kunen, Medini and Zdomskyy, who named them the *Miller property* and the *Cantor-Bendixson property* respectively. It turns out that the implications (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) hold for every space X . Furthermore, it follows from a classical result of Hurewicz that all these implications are equivalences if X is coanalytic. Under the axiom of Projective Determinacy, this equivalence result extends to all projective spaces. We will complete the picture by giving a ZFC counterexample and a consistent definable counterexample of lowest possible complexity to the implication (i) \leftarrow (i + 1) for $i = 1, 2, 3$. For one of these counterexamples we will need a classical theorem of Martin and Solovay, of which we give a new proof, based on a result of Baldwin and Beaudoin. Finally, using a method of Fischer and Friedman, we will investigate how changing the value of the continuum affects the definability of these counterexamples. Along the way, we will show that every uncountable completely Baire space has size continuum.

1. INTRODUCTION

All spaces are assumed to be separable and metrizable. Recall that a space is *crowded* if it is non-empty and it has no isolated points. Recall that a space is *scattered* if it has no crowded subspaces. We will write $X \approx Y$ to mean that the spaces X and Y are homeomorphic. For all undefined topological notions, see [14]. The aim of this paper is to investigate the following topological properties.

Definition 1.1 (Kunen, Medini, Zdomskyy). A space X has the *Miller property* (briefly, the MP) if for every countable crowded $Q \subseteq X$ there exists a crowded $Q' \subseteq Q$ with compact closure.

Definition 1.2 (Kunen, Medini, Zdomskyy). A space X has the *Cantor-Bendixson property* (briefly, the CBP) if every closed subspace of X is either scattered or it contains a homeomorphic copy of 2^ω .

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Definition 1.3. A space X is *completely Baire*¹ (briefly, CB) if every closed subspace of X is a Baire space.

While CB spaces are well-known (see for example [4], [11] or [12]), the MP and the CBP have only recently been introduced in [10], inspired respectively by a remark from [15] and by the classical Cantor-Bendixson derivative. In particular, the MP turned out to be very useful in the study of the countable dense homogeneity of filters on ω (viewed as subspaces of 2^ω through characteristic functions).

As the title suggests, the MP and the CBP are intermediate in strength between the property of being Polish and the property of being CB (this is the content of Section 2). However, in Section 3, we will construct ZFC counterexamples to the reverse implications.

It follows from a result of Marciszewski in [12] that under combinatorial assumptions on X (namely, when $X = \mathcal{F}$ is a filter on ω) the three properties defined above become equivalent (see Theorem 10 in [10]). In Section 5, using a classical result of Hurewicz and Corollary 5.5, we will show that these properties also become equivalent under definability assumptions on X . In Section 7, we will prove that our results are sharp, by constructing consistent definable counterexamples of lowest possible complexity (see Theorem 5.1). For one of these counterexamples (namely, Proposition 7.1), we will employ a classical result of Martin and Solovay, of which we will give a new, topological proof in Section 8, using a result of Baldwin and Beaudoin. Section 4 contains preliminary material for the remainder of the article, and Section 6 contains preliminary material for Section 7.

Finally, in Section 9, we will investigate how changing the value of the continuum affects the definability of these counterexamples, using a method of Fischer and Friedman. As a byproduct of this investigation, we will show that every CB space is either countable or has size \mathfrak{c} (see Theorem 9.9). This dichotomy, which is well-known for Polish spaces, seems to be of independent interest.

2. ARBITRARY SPACES

The following theorem gives a complete picture of the relationships among the properties that we are interested in, if one disregards the issue of definability.

Theorem 2.1. *Consider the following conditions on a space X .*

- (1) X is Polish
- (2) X has the MP.
- (3) X has the CBP.
- (4) X is CB.

The implications (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) hold for every space X . There exists a ZFC counterexample to the implication (i) \leftarrow (i + 1) for $i = 1, 2, 3$.

Proof. The implication (1) \rightarrow (2) is the content of Proposition 2.3. The implication (2) \rightarrow (3) is straightforward. In order to prove the implication (3) \rightarrow (4), assume that the space X is not CB. By Corollary 1.9.13 in [14], it follows that X contains a closed subspace Q homeomorphic to the rationals \mathbb{Q} . It is clear that Q witnesses that X does not have the CBP. The counterexamples are given by Proposition 3.1, Proposition 3.3 and Proposition 3.4. \square

¹Some authors use ‘hereditarily Baire’ or even ‘hereditary Baire’ instead of ‘completely Baire’.

Lemma 2.2. *The space ω^ω has the MP.*

Proof. Fix a countable crowded subset Q of ω^ω . We will construct finite subsets F_n of Q for $n \in \omega$. Start by choosing any singleton $F_0 \subseteq Q$. Now assume that F_0, \dots, F_n are given. Given any $x \in F_0 \cup \dots \cup F_n$, using the fact that Q is crowded, it is possible to pick $x' \in Q$ such that $x' \neq x$ and $x' \upharpoonright (n+1) = x \upharpoonright (n+1)$. Then let $F_{n+1} = \{x' : x \in F_0 \cup \dots \cup F_n\}$. In the end, let $Q' = \bigcup_{n \in \omega} F_n$.

It is easy to check that $Q' \subseteq Q$ is crowded. Now let $g : \omega \rightarrow \omega$ be defined by $g(n) = \max\{x(n) : x \in F_n\}$. Notice that $K = \{x \in \omega^\omega : x(n) \leq g(n) \text{ for all } n \in \omega\}$ is compact. Furthermore, our construction guarantees that $Q' \subseteq K$. Therefore Q' has compact closure. \square

Proposition 2.3. *Every Polish space X has the MP.*

Proof. The statement is vacuously true if X is empty, so assume that X is non-empty. By Exercise 7.14 in [7], there exists a continuous map $f : \omega^\omega \rightarrow X$ that is open and surjective. Fix a countable crowded $Q \subseteq X$. It is not hard to construct a countable crowded $R \subseteq f^{-1}[Q]$ such that $f \upharpoonright R$ is injective. This implies that $f[R']$ is crowded for every crowded $R' \subseteq R$. Since ω^ω has the MP by Lemma 2.2, there exists a crowded $R' \subseteq R$ with compact closure K . Let $Q' = f[R']$. It is clear that $Q' \subseteq f[K]$ is the desired subset of Q . \square

3. ZFC COUNTEREXAMPLES

Recall that a space X is a λ -set if every countable subset of X is \mathbf{G}_δ . Recall that a space $X \subseteq 2^\omega$ is a λ' -set if $X \cup C$ is a λ -set for every countable subset C of 2^ω . It is well-known that a λ' -set of size ω_1 exists in ZFC (see Theorem 5.5 in [16] and the argument that follows it).

Proposition 3.1. *Let $Y \subseteq 2^\omega$ be an uncountable λ' -set. Then $X = 2^\omega \setminus Y$ has the MP but it is not Polish.*

Proof. Notice that X cannot be a \mathbf{G}_δ subset of 2^ω , otherwise Y would be an uncountable \mathbf{F}_σ subset of 2^ω , hence it would contain a copy of 2^ω . In order to show that X has the MP, let $Q \subseteq X$ be crowded. Since Y is a λ' -set, the set Q is a \mathbf{G}_δ subset of $Y \cup Q$. This means that there exists a \mathbf{G}_δ subset G of 2^ω such that $Q \subseteq G \subseteq X$. Therefore, by Proposition 2.3, there exists a crowded $Q' \subseteq Q$ such that Q' has compact closure in G , hence in X . \square

The existence of Y as in the next proposition is due to Brendle (take the complement of the set of branches of the tree given by Theorem 2.2 in [2]). We will also need the following lemma, which can be safely assumed to be folklore.

Lemma 3.2. *Fix a countable dense subset D of 2^ω , and let $Z = 2^\omega \setminus D$. Let $N \subseteq Z$ be a copy of ω^ω that is closed in Z . Then $D' = \text{cl}(N) \cap D$ is crowded, where the closure is taken in 2^ω .*

Proof. First observe that $D' = \emptyset$ would imply that N is a closed, hence compact, subset of 2^ω . Since this contradicts the fact that $N \approx \omega^\omega$, it follows that D' is non-empty. Now assume, in order to get a contradiction, that x is an isolated point of D' . Let U be an open subset of $\text{cl}(N)$ such that $U \cap D = \{x\}$. This would imply that $(U \cap N) = U \setminus \{x\}$ is a non-empty locally compact open subset of N , contradicting again the fact that $N \approx \omega^\omega$. \square

Proposition 3.3. *Let $Y \subseteq \omega^\omega$ be such that the following conditions hold.*

- (1) *For every copy K of 2^ω in ω^ω there exists a copy $K' \subseteq K$ of 2^ω such that $K' \subseteq Y$.*
- (2) *There exists a closed copy N of ω^ω in ω^ω such that $N' \not\subseteq Y$ whenever $N' \subseteq N$ is a closed copy of ω^ω in ω^ω .*

Fix a countable dense subset D of 2^ω and identify ω^ω with $2^\omega \setminus D$. Then the subspace $X = Y \cup D$ of 2^ω has the CBP but not the MP.

Proof. Throughout this proof, cl will denote closure in 2^ω . First, we will show that X has the CBP. Let C be a closed subspace of X that is not scattered. Then there exists a crowded $C' \subseteq C$. Notice that $K = \text{cl}(C')$ is a copy of 2^ω . Since D is countable, there exists a copy $K' \subseteq K$ of 2^ω such that $K' \subseteq \omega^\omega$. Hence, by condition (1), there exists a copy $K'' \subseteq K'$ of 2^ω such that $K'' \subseteq Y \subseteq X$.

Now assume, in order to get a contradiction, that X has the MP. Fix N as in condition (2). Let $Q = \text{cl}(N) \cap D$, and notice that Q is crowded by Lemma 3.2. Therefore, by the MP, there exists a crowded $Q' \subseteq Q$ with compact closure K in X . Notice that K is a copy of 2^ω . But then $N' = K \setminus D \subseteq N$ would be a closed copy of ω^ω in ω^ω , contradicting our assumptions on N . \square

Recall that a subset X of an uncountable Polish space Z is a *Bernstein set* if $X \cap K \neq \emptyset$ and $(Z \setminus X) \cap K \neq \emptyset$ for every copy K of 2^ω in Z . It is easy to see that Bernstein sets exist in ZFC (use the same method as in the proof of Example 8.24 in [7]). Since $2^\omega \approx 2^\omega \times 2^\omega$, every Bernstein set has size \mathfrak{c} .

Proposition 3.4. *Let X be a Bernstein set in some uncountable Polish space Z . Then X is CB but it does not have the CBP.*

Proof. The space X does not have the CBP because X itself is a non-scattered closed subspace of X containing no copies of 2^ω . Now assume, in order to get a contradiction, that X is not CB. By Corollary 1.9.13 in [14], it follows that X contains a closed subspace Q homeomorphic to the rationals \mathbb{Q} . Let $G = \text{cl}(Q) \setminus Q$, where the closure is taken in Z . It is easy to realize that G is an uncountable \mathbf{G}_δ subset of Z . Therefore G contains a copy of 2^ω . Since $G \cap X = \emptyset$, this contradicts the fact that X is a Bernstein set. \square

4. PRELIMINARIES ABOUT DEFINABILITY

Our reference for descriptive set theory will be [7]. In this section, Γ will always denote one of the (boldface) projective pointclasses Σ_n^1 , Π_n^1 or Δ_n^1 , where n is a non-zero natural number. It is well-known how to define a *subset* of complexity Γ of a given Polish space. Since it seems to be slightly less well-known that this can be easily extended to arbitrary *spaces*, we will recall the following definition (which coincides with the one given at the end of page 315 in [7]). We will say that a space X *embeds* in a space Z if there exists a subspace X' of Z such that $X' \approx X$.

Definition 4.1. Let X be a space and Γ a pointclass. We will say that X is a *space of complexity Γ* (briefly, a Γ space) if there exists a Polish space in which X embeds as a subset of complexity Γ . We will say that X is a *projective space* if it is a space of complexity Γ for some Γ .

The following ‘reassuring’ proposition, which can be safely assumed to be folklore, shows that the choice of the Polish space in the above definition is irrelevant.

Proposition 4.2. *Let X be a space. The following are equivalent.*

- (1) X is a Γ space.
- (2) X is a Γ subset of every Polish space in which it embeds.

Proof. The implication (2) \rightarrow (1) follows from the standard fact that every space embeds in the Polish space $[0, 1]^\omega$. In order to prove (1) \rightarrow (2), we will proceed by induction. Clearly, it will be enough to deal with the cases $\Gamma = \Sigma_n^1$ and $\Gamma = \Pi_n^1$. Notice that the case $\Gamma = \Sigma_1^1$ is trivial, because such sets are by definition continuous images of a Polish space, and this property is preserved by homeomorphisms.

Now assume that the result holds for $\Gamma = \Sigma_n^1$, and let X be a Π_n^1 space. Assume that X is a subspace of a Polish space Z , and that X is a Π_n^1 subset of Z . Let X' be a subspace of a Polish space Z' , and assume that $h : X \rightarrow X'$ is a homeomorphism. We will show that X' is a Π_n^1 subset of Z' . By Lavrentiev's Theorem (see Theorem 3.9 in [7]), there exists a homeomorphism $f : G \rightarrow G'$ that extends h , where $G \supseteq X$ is a G_δ subset of Z and $G' \supseteq X'$ is a G_δ subset of Z' . It will be enough to show that $Z' \setminus X' = (Z' \setminus G') \cup (G' \setminus X')$ is a Σ_n^1 subset of Z' . But $Z' \setminus G'$ is a Σ_n^1 subset of Z' because it is an F_σ , and $G' \setminus X'$ is a Σ_n^1 subset of Z' by the inductive hypothesis, being homeomorphic to the Σ_n^1 space $G \setminus X = (Z \setminus X) \cap G$.

Finally, assume that the result holds for $\Gamma = \Pi_n^1$, and let X be a Σ_{n+1}^1 space. Assume that X is a subspace of a Polish space Z , and that X is a Σ_{n+1}^1 subset of Z . This means that $X = \pi[Y]$, where Y is a Π_n^1 subset of $Z \times \omega^\omega$ and $\pi : Z \times \omega^\omega \rightarrow Z$ is the projection on the first coordinate. Let X' be a subspace of a Polish space Z' , and assume that $h : X \rightarrow X'$ is a homeomorphism. We will show that X' is a Σ_{n+1}^1 subset of Z' . Observe that the function $h \times \text{id} : X \times \omega^\omega \rightarrow X' \times \omega^\omega$ defined by $(h \times \text{id})(x, w) = \langle h(x), w \rangle$ is a homeomorphism, and let $Y' = (h \times \text{id})[Y]$. Notice that Y' is a Π_n^1 subset of $Z' \times \omega^\omega$ by the inductive hypothesis, being homeomorphic to the Π_n^1 space Y . It is clear that $X' = \pi'[Y']$, where $\pi' : Z' \times \omega^\omega \rightarrow Z'$ is the projection on the first coordinate. \square

5. DEFINABLE SPACES

As we mentioned in the introduction, the properties that we are interested in become equivalent under certain definability assumptions. The following theorem, which can be viewed as a ‘definable’ analogue of Theorem 2.1, makes this precise. Theorem 5.1 also states that, under the axiom of Projective Determinacy, these properties become equivalent for every projective space. This is the reason why the *definable* counterexamples that we obtain could not have been constructed in ZFC alone. Also notice that these counterexamples are optimal, in the sense that their complexity is as low as possible.

Theorem 5.1. *Consider the following conditions on a space X .*

- (1) X is Polish
- (2) X has the MP.
- (3) X has the CBP.
- (4) X is CB.

If X is Π_1^1 then (1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow (4). Under the axiom of Projective Determinacy this holds whenever X is projective. If X is Σ_1^1 then (2) \leftrightarrow (3) \leftrightarrow (4). There exists a consistent Σ_1^1 counterexample to the implication (1) \leftarrow (2). There exists a consistent Δ_2^1 counterexample to the implication (i) \leftarrow (i + 1) for $i = 2, 3$.

Proof. By Theorem 2.1, in order to prove the first statement, it will be enough to show that (4) \rightarrow (1) whenever X is $\mathbf{\Pi}_1^1$. This is exactly what a classical theorem of Hurewicz states (see Corollary 21.21 in [7]). Since under the axiom of Projective Determinacy this theorem extends to every projective space (see Exercise 28.20 in [7]), the second statement holds. In order to prove the third statement, it will be enough to show that (4) \rightarrow (2) whenever X is $\mathbf{\Sigma}_1^1$. This is the content of Corollary 5.5. The fourth statement follows from Proposition 7.1. The fifth statement follows from Proposition 7.2 (for the case $i = 2$) and Proposition 7.3 (in the case $i = 3$). \square

The following two results are well-known. For a proof of Theorem 5.2, see Corollary 21.23 in [7]. Recall that a subset A of ω^ω is *Miller-measurable* if for every closed copy N of ω^ω in ω^ω there exists a closed copy $N' \subseteq N$ of ω^ω in ω^ω such that $N' \subseteq A$ or $N' \subseteq \omega^\omega \setminus A$.

Theorem 5.2 (Kechris; Saint Raymond). *Assume that A is a $\mathbf{\Sigma}_1^1$ subset of ω^ω . Then (exactly) one of the following alternatives holds.*

- (1) *There exist compact subsets K_n of ω^ω for $n \in \omega$ such that $A \subseteq \bigcup_{n \in \omega} K_n$.*
- (2) *There exists a closed copy N of ω^ω in ω^ω such that $N' \subseteq A$.*

Corollary 5.3. *Every $\mathbf{\Sigma}_1^1$ subset of ω^ω is Miller-measurable.*

Theorem 5.4. *Assume that $\mathbf{\Gamma}$ is a projective pointclass such that every $\mathbf{\Gamma}$ subset of ω^ω is Miller-measurable. Let X be a $\mathbf{\Gamma}$ space that is CB. Then X has the MP.*

Proof. Fix a compatible metric d on $[0, 1]^\omega$. Given $x \in [0, 1]^\omega$ and $\varepsilon > 0$, let $S(x, \varepsilon) = \{z \in [0, 1]^\omega : d(x, z) = \varepsilon\}$. Assume, without loss of generality, that X is a subspace of $[0, 1]^\omega$. Fix a countable crowded subset Q of X . Let $D \supseteq Q$ be a countable dense subset of $[0, 1]^\omega$. Since D is countable, there exist $\varepsilon_n > 0$ for $n \in \omega$ such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varepsilon_n : n \in \omega\} \cap \{d(x, y) : x, y \in D\} = \emptyset$. It is easy to check that

$$\left([0, 1]^\omega \setminus \bigcup \{S(x, \varepsilon_n) : x \in D, n \in \omega\}\right) \cap X$$

is a zero-dimensional $\mathbf{\Gamma}$ space containing Q . Therefore, we can assume without loss of generality that X is zero-dimensional. We will actually assume that X is a subspace of 2^ω . Throughout this proof, cl will denote closure in 2^ω .

Let $Z = \text{cl}(Q) \setminus Q$, and notice that $Z \approx \omega^\omega$. Assume, in order to get a contradiction, that $Z \setminus X$ contains a copy N of ω^ω that is closed in Z . Since $\text{cl}(Q) \approx 2^\omega$ and Q is a countable dense subset of $\text{cl}(Q)$, Lemma 3.2 shows that $Q' = \text{cl}(N) \cap Q$ is crowded. This contradicts the fact that X is CB because $Q' = \text{cl}(N) \cap X$ is also closed in X . Since $Z \cap X$ is a Miller-measurable subset of Z , it follows that $Z \cap X$ contains a copy N of ω^ω that is closed in Z . Once again, Lemma 3.2 shows that $Q' = \text{cl}(N) \cap Q$ is crowded. Furthermore, the closure of Q' in X is compact because $Q' \subseteq \text{cl}(N) = N \cup Q' \subseteq X$. Therefore X has the MP. \square

Corollary 5.5. *Let X be a $\mathbf{\Sigma}_1^1$ space. If X is CB then X has the MP.*

6. PRELIMINARIES ABOUT THE CONSTRUCTIBLE UNIVERSE

All the result in this section are well-known. The aim of this section is simply to collect the main results needed to give rigorous proofs of Proposition 7.2 and Proposition 7.3. However, we will assume some familiarity with the basic theory of L. Our references will be [6] and [9].

The following theorem essentially shows that if all the ‘ingredients’ of a construction by transfinite recursion are absolute, then the end result will be absolute as well. It is obtained by combining Theorem I.9.11 and the proof Theorem II.4.15 from [9] in the case $A = \omega_1$, $R = \in$. We will denote by OC the statement “Every ordinal is countable”.

Theorem 6.1. *Suppose $\varphi(x, s, y)$ is such that $\forall x, s \exists! y \varphi(x, s, y)$. Define $G(x, s)$ to be the unique y such that $\varphi(x, s, y)$. Then there exists a formula $\psi(x, y)$ such that the following are provable.*

- $\forall x \exists! y \psi(x, y)$. (In particular, $\psi(x, y)$ defines a function F , where $F(x)$ is the unique y such that $\psi(x, y)$ holds.)
- $\forall \alpha < \omega_1 [F(\alpha) = G(\alpha, F \upharpoonright \alpha)]$.

Assume that Φ is a collection of sentences in the language \mathcal{L}_\in of set theory such that $\mathbf{ZF} - \mathbf{P} + \mathbf{OC} \subseteq \Phi$. If M is a transitive model for Φ and G is absolute for M , then $F^M(\alpha)$ is defined for every $\alpha \in M \cap \omega_1$ and F is absolute for M .

For the proofs of the following three results, see Theorem II.6.22, Theorem II.5.10 and Lemma II.6.16 in [9].

Proposition 6.2. *If κ is a regular uncountable cardinal then $\mathbf{L}_\kappa \models \mathbf{ZF} - \mathbf{P}$.*

Proposition 6.3. *There exist arbitrarily large $\delta < \omega_1$ such that $\mathbf{L}_\delta \prec \mathbf{L}_{\omega_1}$.*

Proposition 6.4. *Let M be a transitive set such that $M \models \mathbf{ZF} - \mathbf{P}$, and let δ be the least ordinal such that $\delta \notin M$. Then $M \models \mathbf{V} = \mathbf{L}$ if and only if $M = \mathbf{L}_\delta$.*

Next, we recall some notation from the section of [6] entitled “Regularity properties in \mathbf{L} ” (which begins on page 167). Let $E_z = \{\langle m, n \rangle \in \omega \times \omega : x(\langle \langle m, n \rangle \rangle) = 0\}$ for $z \in \omega^\omega$, where $\langle \langle m, n \rangle \rangle = 2^m \cdot 3^n$. Let $M_z = \langle \omega, E_z \rangle$ be the structure with domain ω which interprets \in as the binary relation E_z . Whenever M_z is well-founded and extensional, denote by $\text{tr}(M_z)$ the transitive collapse of M_z , and let $\pi_z : M_z \rightarrow \text{tr}(M_z)$ the corresponding isomorphism.

For the proofs of the following three results, see Proposition 13.8 in [6].

Proposition 6.5. *Let $\varphi(x)$ be a formula in the language \mathcal{L}_\in of set theory. Then $\{\langle n, z \rangle \in \omega \times \omega^\omega : M_z \models \varphi(n)\}$ is a Borel set.*

Proposition 6.6. *Let Φ be a collection of sentences in the language \mathcal{L}_\in of set theory. Then $\{z \in \omega^\omega : M_z \models \Phi\}$ is a Borel set.*

Proposition 6.7. *Given $z \in \omega^\omega$ such that M_z is well-founded and extensional, define $R(z) = \{\langle n, x \rangle \in \omega \times \omega^\omega : \pi_z(n) = x\}$. Then there exists a Borel set $A \subseteq \omega \times \omega^\omega \times \omega^\omega$ such that $\langle n, x \rangle \in R(z) \leftrightarrow \langle n, x, z \rangle \in A$ for every $z \in \omega^\omega$ such that M_z is well-founded and extensional.*

7. CONSISTENT DEFINABLE COUNTEREXAMPLES

For our first counterexample, we will employ a classical theorem of Martin and Solovay (see Theorem 8.1), of which we will give a new proof in Section 8.

Proposition 7.1. *Assume $\mathbf{MA} + \neg\mathbf{CH} + \omega_1 = \omega_1^{\mathbf{L}}$. Then there exists a Σ_1^1 space that has the MP but is not Polish.*

Proof. Let $Y \subseteq 2^\omega$ be a \mathcal{N} -set of size ω_1 . The space $X = 2^\omega \setminus Y$ has the MP but is not Polish by Proposition 3.1, and it is Σ_1^1 by Theorem 8.1. \square

The proof of the following Proposition was inspired by the exposition in [8] (in particular, by Fact 1.2.11 and Fact 1.3.8). Next, we will introduce some terminology that will be needed in its proof. Let $D = \{x \in 2^\omega : \exists n \in \omega \forall m \geq n (x(m) = 0)\}$. We will identify ω^ω with the subspace $2^\omega \setminus D$ of 2^ω . For any given $T \subseteq 2^{<\omega}$, let $[T] = \{x \in 2^\omega : \forall n \in \omega (x \upharpoonright n \in T)\}$ be the set of branches through T . We will say that $C \subseteq 2^{<\omega}$ is a *code for a copy of 2^ω in ω^ω* if $[C]$ is crowded and $[C] \cap D = \emptyset$. In this case, one sees that $[C] \subseteq 2^\omega \setminus D$ is in fact a copy of 2^ω , and that every such copy can be obtained this way. We will say that $B \subseteq 2^{<\omega}$ is a *code for a closed copy of ω^ω in ω^ω* if $[B]$ is crowded and $[B] \cap D$ is dense in $[B]$. In this case, one sees that $[B] \cap (2^\omega \setminus D)$ is in fact a closed copy of ω^ω , and that every such copy can be obtained this way (see the proof of Lemma 3.2). It is easy to check that both notions, as well as $x \in [T]$, are absolute for transitive models of $\text{ZF} - \text{P}$.

Proposition 7.2. *Assume $\mathbb{V} = \mathbb{L}$. Then there exists a Δ_2^1 space that has the CBP but does not have the MP.*

Proof. It will be enough to construct a Δ_2^1 subset X of ω^ω that satisfies the following conditions.

- (1') For every copy K of 2^ω in ω^ω there exists a copy $K' \subseteq K$ of 2^ω such that $K' \cap X = \emptyset$.
- (2') There exists a closed copy N of ω^ω in ω^ω such that $N' \cap X \neq \emptyset$ whenever $N' \subseteq N$ is a closed copy of ω^ω in ω^ω .

In fact, it is clear that $Y = \omega^\omega \setminus X$ will be Δ_2^1 as well, and it will satisfy the requirements of Proposition 3.3.

First we describe the construction of such a set X , disregarding the definability requirements. Enumerate as $\{N_\alpha : \alpha < \omega_1\}$ all closed copies of ω^ω in ω^ω . Enumerate as $\{K_\alpha : \alpha < \omega_1\}$ all copies of 2^ω in ω^ω . For every $\alpha < \omega_1$, choose

$$x_\alpha \in N_\alpha \setminus \bigcup_{\beta < \alpha} K_\beta.$$

Notice that the above choice is always possible because $N_\alpha \approx \omega^\omega$ cannot be written as the union of countably many of its compact subspaces. Let $X = \{x_\alpha : \alpha < \omega_1\}$. One sees that condition (2') is satisfied by setting $N = \omega^\omega$. Furthermore, the intersection of X with each K_α is at most countable by construction. Since each $K_\alpha \approx 2^\omega \approx 2^\omega \times 2^\omega$, it follows that condition (1') is satisfied.

The rest of the proof is devoted to making the above construction definable. The formula that defines X will be

$$\exists \alpha [(\alpha \text{ is a countable ordinal}) \wedge (x = F(\alpha))],$$

where F is the function that will be given by Theorem 6.1. Once F is defined, we will denote the above formula by $\chi(x)$.

For the inductive step, we need to define $G(\alpha, s)$. Let C_α for $\alpha < \omega_1$ denote the α -th code for a copy of 2^ω in ω^ω according to the well-order $<_{\mathbb{L}}$. Let B_α for $\alpha < \omega_1$ denote the α -th code for a closed copy of ω^ω in ω^ω according to the well-order $<_{\mathbb{L}}$. If α is not a countable ordinal, simply let $G(\alpha, s) = \emptyset$. If α is a countable ordinal, let $G(\alpha, s) = x$, where x is uniquely defined by the following conditions. Recall that we are identifying ω^ω with the subspace $2^\omega \setminus D$ of 2^ω . Notice that we will not make use of the parameter s . However, such parameter is needed in general (consider for example Proposition 7.3).

- (1) $x \in \omega^\omega$.
- (2) $x \in [B_\alpha] \setminus \bigcup_{\beta < \alpha} [C_\beta]$.
- (3) $x \notin L_\alpha$.
- (4) x is the $<_L$ -least set satisfying (1), (2) and (3).

As in Section 6, we will denote by OC the statement “Every ordinal is countable”. Let Φ denote the set of sentences φ in the language \mathcal{L}_\in of set theory such that $L_{\omega_1} \models \varphi$. Notice that $\mathbf{ZF} - \mathbf{P} + \mathbf{V} = \mathbf{L} + \mathbf{OC} \subseteq \Phi$ (use Proposition 6.2 for $\mathbf{ZF} - \mathbf{P}$ and Proposition 6.4 for $\mathbf{V} = \mathbf{L}$). Furthermore, it is easy to check that the following sentences also belong to Φ .

- (A) “For every ordinal α there exists a set \mathcal{C} consisting of codes for copies of 2^ω in ω^ω , such that the order type of \mathcal{C} according to $<_L$ is at least α ”.
- (B) “For every ordinal α there exists a set \mathcal{B} consisting of codes for closed copies of ω^ω in ω^ω , such that the order type of \mathcal{B} according to $<_L$ is at least α ”.
- (C) “For every ordinal α there exists x satisfying (1) and (2)”.

We claim that G is well-defined and absolute for transitive models of Φ . In fact, since (A) and (B) guarantee that the functions $\alpha \mapsto C_\alpha$ and $\alpha \mapsto B_\alpha$ are well-defined, it will follow from (C) that G is well-defined too. At this point, absoluteness is easy to check.

Notice that, since we are not using the parameter s , the absoluteness of F immediately follows from the absoluteness of G . However, in general, one would have to use the second part of Theorem 6.1 to prove the absoluteness of F .

Let $\theta(x)$ denote the statement

$$\exists \delta < \omega_1 [(L_\delta \models \Phi) \wedge (x \in L_\delta) \wedge (L_\delta \models \chi(x))].$$

Next, we will show that $\chi(x)$ is equivalent to $\theta(x)$ for every x . First assume that $\chi(x)$ holds, and let $\alpha < \omega_1$ be such that $x = F(\alpha)$. By Proposition 6.3, there exists $\delta < \omega_1$ such that $L_\delta \models \Phi$ and $x \in L_\delta$. Notice that $\alpha < \delta$ by condition (3). Therefore $L_\delta \models F(\alpha) = x$ by the absoluteness of F . Since $L_\delta \models \mathbf{OC}$, it follows that $L_\delta \models \chi(x)$. The other direction simply uses the absoluteness of F .

Next, we will show that X is a Σ_2^1 space. It is easy to realize, using the transitive collapse and Proposition 6.4, that $\theta(x)$ is equivalent to

$$\begin{aligned} \exists z \in \omega^\omega [(M_z \text{ is well-founded}) \wedge (M_z \models \Phi) \wedge \\ \wedge (\exists n \in \omega ((\pi_z(n) = x) \wedge (M_z \models \chi(n))))], \end{aligned}$$

where we use the same notation of Section 6. The well-known (and easy to prove) fact that the set $\{z \in \omega^\omega : M_z \text{ is well-founded}\}$ is Π_1^1 , together with Proposition 6.5, Proposition 6.6 and Proposition 6.7, shows that the above statement defines a Σ_2^1 subset of ω^ω .

Finally, to see that X is Π_2^1 , let $\theta(x)$ denote the statement

$$\forall \delta < \omega_1 [((L_\delta \models \Phi) \wedge (x \in L_\delta)) \rightarrow (L_\delta \models \chi(x))]$$

and use the same kind of argument as above. \square

Proposition 7.3. *Assume $\mathbf{V} = \mathbf{L}$. Then there exists a Δ_2^1 space that is CB but does not have the CBP.*

Proof. Using the same method as in the proof of Proposition 7.2, one can show that under $\mathbf{V} = \mathbf{L}$ there exists a Δ_2^1 Bernstein set in ω^ω (this is well-known, see Fact 1.3.8 in [8]). Therefore, the desired conclusion follows from Proposition 3.4. \square

8. A NEW PROOF OF A THEOREM OF MARTIN AND SOLOVAY

The aim of this section is to give a new proof of the following classical result (see Theorem 23.3 in [17]), which is perhaps more transparent than the usual one. The main idea is that $\omega_1 = \omega_1^{\perp}$ implies the existence of *one* space of size ω_1 with the property that we want (see Proposition 8.2), while $\text{MA} + \neg\text{CH}$ implies that *all* spaces of size ω_1 are ‘the same’ for our purposes (see Lemma 8.3).

Recall that, given an infinite cardinal λ , a subset D of 2^ω is λ -dense if $|U \cap D| = \lambda$ for every non-empty open subset U of 2^ω . Given a space X , we will denote by X^* the space $X \setminus V$, where $V = \bigcup\{U : U \text{ is a countable open subset of } X\}$. It is easy to see that $V = X \setminus X^*$ is countable, and that every non-empty open subset of X^* is uncountable. Notice that, given any projective pointclass Γ , a space X is of complexity Γ if and only if X^* is of complexity Γ .

Theorem 8.1 (Martin, Solovay). *Assume $\text{MA} + \neg\text{CH} + \omega_1 = \omega_1^{\perp}$. Then every space of size ω_1 is Π_1^1 .*

Proof. By Proposition 8.2 there exists a Π_1^1 space D of size ω_1 . Since any two uncountable Polish spaces are Borel isomorphic (see Theorem 15.6 in [7]), we can assume that $D \subseteq 2^\omega$. By considering D^* , we can assume that every non-empty open subset of D is uncountable. In particular D is crowded, hence its closure in 2^ω is homeomorphic to 2^ω . In conclusion, we can assume without loss of generality that D is an ω_1 -dense subspace of 2^ω . Now let E be a space of size ω_1 . As above, we can assume that E is an ω_1 -dense subspace of 2^ω . An application of Lemma 8.3 concludes the proof. \square

The following proposition is well-known. Actually, it is possible to obtain a space with the additional property of not containing any copy of 2^ω (this is a classical result of Gödel, see Theorem 13.12 in [6]), but we will not need this stronger version.

Proposition 8.2. *Assume $\omega_1 = \omega_1^{\perp}$. Then there exists a Π_1^1 space of size ω_1 .*

Proof. It is well-known that the set $R = \omega^\omega \cap \mathbb{L} = (\omega^\omega)^\perp$ of all constructible reals is Σ_2^1 (see for example Theorem 13.9 in [6]). Also notice that R has size ω_1 by the assumption $\omega_1 = \omega_1^{\perp}$. Let $A \subseteq \omega^\omega \times \omega^\omega$ be a Π_1^1 set such that $\pi[A] = R$, where $\pi : \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$ is the projection on the first coordinate. By the Kondô Uniformization Theorem (see Theorem 12.3 in [6]), there exists a Π_1^1 set $D \subseteq A$ such that $\pi \upharpoonright D : D \rightarrow R$ is a bijection. In particular, the size of D is ω_1 . \square

The following result first appeared (in a more general form) as Lemma 3.2 in [1]. See also Theorem 2.1 and Corollary 2.2 in [13] for a simpler version of the proof.

Lemma 8.3 (Baldwin, Beaudoin). *Assume $\text{MA}(\sigma\text{-centered})$. Let $\lambda < \mathfrak{c}$ be an infinite cardinal. If D and E are λ -dense subsets of 2^ω then there exists a homeomorphism $f : 2^\omega \rightarrow 2^\omega$ such that $f[D] = E$.*

9. MODIFYING THE VALUE OF THE CONTINUUM

At this point, it is natural to wonder whether the counterexamples obtained in Section 7 are compatible with different values of the continuum. In the case of Proposition 7.1, it is clear that one can obtain arbitrarily large values of \mathfrak{c} by forcing over \mathbb{L} with the usual ccc poset that proves the consistency of MA . The next proposition show that $\mathfrak{c} = \omega_1$ is also possible.

Proposition 9.1. *The existence of a Σ_1^1 space that has the MP but is not Polish is compatible with CH.*

Proof. Let $Y \subseteq 2^\omega$ be a λ' -set of size ω_1 in a model $\mathbf{MA} + \neg\text{CH} + \omega_1 = \omega_1^1$, and notice that Y is Π_1^1 by Theorem 8.1. Now collapse \mathfrak{c} to ω_1 using a countably closed forcing poset. It is easy to check that Y will remain a λ' -set of size ω_1 in the extension. Furthermore, $X = 2^\omega \setminus Y$ will remain Σ_1^1 . An application of Proposition 3.1 concludes the proof. \square

The situation regarding Proposition 7.2 and Proposition 7.3 is more delicate. We will indicate how to obtain Δ_3^1 counterexamples in models of $\mathfrak{c} = \omega_2$ using a general method introduced by Fischer and Friedman in [5]. We will assume some familiarity with their article, and use the same notation. The general idea is to perform a countable support iteration $\langle \langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle, \langle \dot{Q}_\alpha : \alpha < \omega_2 \rangle \rangle$ of S -proper posets over \mathbf{L} , where S is a stationary subset of ω_1 that has been fixed in advance, as in Section 5 in [5]. Suppose that we have already defined $\langle \langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle \rangle$ for some $\alpha < \omega_2$. We will set $\dot{Q}_\alpha = \dot{Q}_\alpha^0 * \dot{Q}_\alpha^1$. Let \mathbb{Q}_α^0 be a proper poset of size ω_1 in $\mathbf{L}^{\mathbb{P}_\alpha}$. (There are no additional requirements on \mathbb{Q}_α^0 : this poset is “reserved” for future applications, as in the proofs of Theorem 2 and Theorem 3 in [5].) Suppose also that σ_α is a $\mathbb{P}_\alpha * \dot{Q}_\alpha^0$ -name for a real. Then there exists an S -proper poset \mathbb{Q}_α^1 of size ω_1 in $\mathbf{L}^{\mathbb{P}_\alpha * \dot{Q}_\alpha^0}$ such that, at the end of the construction, both $\{\sigma_\alpha^G : \alpha < \omega_2 \text{ is a limit}\}$ and $\{\sigma_\alpha^G : \alpha < \omega_2 \text{ is a successor}\}$ will be Σ_3^1 for every \mathbb{P}_{ω_2} -generic filter G over \mathbf{L} . In fact, this can be obtained by replacing $x * y$ with σ_α^G in items (1), (2) at the beginning of page 920 in [5], and by modifying the definition of ϕ_α in item (2) by specifying that X_α codes a limit (resp. successor) ordinal $\bar{\alpha} < \omega_2$ whenever $\alpha < \omega_2$ is a limit (resp. successor).

Proposition 9.2. *The existence of a Δ_3^1 space that has the CBP but not MP is compatible with $\neg\text{CH}$.*

Proof. We will construct a Δ_3^1 subset X of ω^ω satisfying the same conditions (1') and (2') that appear in the proof of Proposition 7.2. Start by fixing a bookkeeping function $F : \omega_2 \rightarrow \mathbf{H}(\omega_2)$ such that $\{\alpha < \omega_2 : \alpha \text{ is a limit and } F(\alpha) = x\}$ and $\{\alpha < \omega_2 : \alpha \text{ is a successor and } F(\alpha) = x\}$ are unbounded in ω_2 for each $x \in \mathbf{H}(\omega_2)$.

Assume that the iteration $\langle \langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle \rangle$ has already been defined for some $\alpha < \omega_2$. First assume that α is a limit. If $F(\alpha)$ is a \mathbb{P}_α -name for a code B for a closed copy of ω^ω in ω^ω , choose a poset \mathbb{Q}_α^0 adding an unbounded real, then let σ_α be a $\mathbb{P}_\alpha * \dot{Q}_\alpha^0$ -name for an element of ω^ω such that the following conditions are satisfied.

- (1) $\Vdash_{\mathbb{P}_\alpha * \dot{Q}_\alpha^0} \text{“}\sigma_\alpha \in [B]\text{”}$.
- (2) $\Vdash_{\mathbb{P}_\alpha * \dot{Q}_\alpha^0} \text{“}\sigma_\alpha \text{ is unbounded over } \omega^\omega \cap \mathbf{L}^{\mathbb{P}_\alpha}\text{”}$.

Otherwise, let \mathbb{Q}_α^0 be the trivial forcing and set $\sigma_\alpha = \langle 0, 0 \dots \rangle$. Now assume that α is a successor. Let \mathbb{Q}_α^0 be the trivial forcing. If $F(\alpha) = \tau$ is a \mathbb{P}_α -name for an element of ω^ω , proceed as follows, otherwise let $\sigma_\alpha = \langle 1, 1 \dots \rangle$. Define $\mathcal{B} = \{p \in \mathbb{P}_\alpha : p \Vdash \text{“}\tau \notin \{\sigma_\beta : \beta < \alpha \text{ and } \beta \text{ is a limit}\}\text{”}\}$ and $\mathcal{C} = \{p \in \mathbb{P}_\alpha : p \Vdash \text{“}\tau \in \{\sigma_\beta : \beta < \alpha \text{ and } \beta \text{ is a limit}\}\text{”}\}$. Since $\mathcal{B} \cup \mathcal{C}$ is dense in \mathbb{P}_α , we can fix a maximal antichain \mathcal{A} in \mathbb{P}_α such that $\mathcal{A} \subseteq \mathcal{B} \cup \mathcal{C}$. Let

$$\sigma_\alpha = \{\langle \tau, p \rangle : p \in \mathcal{A} \cap \mathcal{B}\} \cup \{\langle \langle 1, 1 \dots \rangle, p \rangle : p \in \mathcal{A} \cap \mathcal{C}\}.$$

This concludes the construction.

Let G be a \mathbb{P}_{ω_2} -generic filter over \mathbb{L} , then set $X = \{\sigma_\alpha^G : \alpha < \omega_2 \text{ is a limit}\}$. Notice that $\omega^\omega = X \cup \{\sigma_\alpha^G : \alpha < \omega_2 \text{ is a successor}\}$ by the successor case of our construction. Therefore X is a Δ_3^1 space. Using condition (2), it is easy to check that $|X \cap K| \leq \omega_1 < \mathfrak{c}$ for every copy K of 2^ω in ω^ω . This shows that condition (1') is satisfied. Finally, it is clear that condition (2') is satisfied with $N = \omega^\omega$. \square

Proposition 9.3. *The existence of a Δ_3^1 space that is CB but does not have the CBP is compatible with $\neg\text{CH}$.*

Proof. We will construct a Δ_3^1 Bernstein subset X of ω^ω . Fix F as in the proof of Proposition 9.2. Assume that the iteration $\langle\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle\rangle$ has already been defined for some $\alpha < \omega_2$. First assume that α is a limit. If $F(\alpha)$ is a \mathbb{P}_α -name for a code C for a copy of 2^ω in ω^ω , choose a poset \mathbb{Q}_α^0 adding a new real, then let σ_α be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$ -name for an element of ω^ω such that $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0} \text{“}\sigma_\alpha \in [C] \setminus \mathbb{L}^{\mathbb{P}_\alpha}\text{”}$. Otherwise, let \mathbb{Q}_α^0 be the trivial forcing and set $\sigma_\alpha = \langle 0, 0 \dots \rangle$. If α is a successor, proceed as in the proof of Proposition 9.2. This concludes the construction.

Let G be a \mathbb{P}_{ω_2} -generic filter over \mathbb{L} , then set $X = \{\sigma_\alpha^G : \alpha < \omega_2 \text{ is a limit}\}$. The same reasoning as in the the proof of Proposition 9.2 shows that X is a Δ_3^1 space. Now let K be a copy of 2^ω in ω^ω , coded by C . Assume that $F(\alpha)$ is a \mathbb{P}_α -name for C at a limit stage α of our construction. Clearly σ_α^G witnesses that $X \cap K \neq \emptyset$. Furthermore, since $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0} \text{“}[C] \setminus \mathbb{L}^{\mathbb{P}_\alpha}$ is infinite”, there exists a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$ -name τ for an element of ω^ω such that $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0} \text{“}\tau \in [C] \setminus \mathbb{L}^{\mathbb{P}_\alpha}$ and $\tau \neq \sigma_\alpha\text{”}$. It is easy to check that τ^G witnesses that $(\omega^\omega \setminus X) \cap K \neq \emptyset$. Therefore X is Bernstein set. \square

The following questions ask whether the counterexamples constructed in Proposition 9.2 and Proposition 9.3 are of lowest possible complexity. Question 9.5 only asks for a Π_2^1 counterexample, because Corollary 9.10 rules out the existence of Σ_2^1 counterexamples. Also observe that the existence of a Σ_2^1 Bernstein set (or, equivalently, a Π_2^1 Bernstein set) is not compatible with $\neg\text{CH}$. In fact, every Σ_2^1 space of size at least ω_2 contains a copy of 2^ω (see Proposition 13.7 in [6]).

Question 9.4. Is $\neg\text{CH}$ compatible with the existence of a Σ_2^1 or Π_2^1 space that has the CBP but not the MP?

Question 9.5. Is $\neg\text{CH}$ compatible with the existence of a Π_2^1 space that is CB but does not have the CBP?

The following corollary shows that none of the counterexamples mentioned in the above questions is compatible with the assumption $\mathfrak{d} > \omega_1$. For a proof of Theorem 9.6, see Theorem 6.1 in [3].

Theorem 9.6 (Brendle, Löwe). *The following are equivalent.*

- $\omega^\omega \cap \mathbb{L}[a]$ is not dominating for any $a \in \omega^\omega$.
- Every Σ_2^1 subset of ω^ω is Miller-measurable.

Corollary 9.7. *Assume that $\omega^\omega \cap \mathbb{L}[a]$ is not dominating for any $a \in \omega^\omega$. Let X be a CB space, and assume that X is Σ_2^1 or Π_2^1 . Then X has the MP.*

Proof. Simply apply Theorem 5.4. \square

Notice that the following theorem generalizes the classical fact that every uncountable Polish space has size \mathfrak{c} . In its proof, we will identify 2^ω with the power set of ω through characteristic functions.

Lemma 9.8. *Let X be CB space. Then every G_δ subset of X is CB.*

Proof. Throughout this proof, cl will denote closure in X . Let G be a G_δ subset of X . Let C be a closed subset of G . Notice that $\text{cl}(C)$ is CB because X is CB. Furthermore, the fact that $C = \text{cl}(C) \cap G$ shows that C is a G_δ subset of $\text{cl}(C)$. Since every G_δ subset of a CB space is Baire (see Proposition 1.2 in [4]), it follows that C is Baire. \square

Theorem 9.9. *Let X be an uncountable CB space. Then $|X| = \mathfrak{c}$.*

Proof. Using the classical Cantor-Bendixson derivative, we can assume that X is crowded. The same method that we used in the first paragraph of the proof of Theorem 5.4, together with Lemma 9.8, shows that X can be assumed to be a subspace of 2^ω . Since X is crowded, we can assume that X is dense in 2^ω . Since 2^ω is countable dense homogeneous (see Theorem 1.6.9 and Lemma 1.9.5 in [14]), we can also assume that $[\omega]^{<\omega} \subseteq X$.

Assume, in order to get a contradiction, that there exists $z \in [\omega]^\omega$ such that $[z]^\omega \cap X = \emptyset$. It is easy to check that $[z]^{<\omega}$ is a countable crowded closed subspace of X , which contradicts the fact that X is CB. Therefore, there exists a function $f : [\omega]^\omega \rightarrow [\omega]^\omega \cap X$ such that $f(z) \subseteq z$ for every $z \in [\omega]^\omega$. Fix an almost disjoint family \mathcal{A} of size \mathfrak{c} (see Lemma III.1.16 in [9]). It is easy to check that $f \upharpoonright \mathcal{A}$ is injective. Therefore $X \supseteq f[\mathcal{A}]$ has size \mathfrak{c} . \square

Corollary 9.10. *Assume $\neg\text{CH}$. Let X be a CB space, and assume that X is Σ_2^1 . Then X has the CBP.*

Proof. Assume, without loss of generality, that X is uncountable. Let C be a non-scattered closed subset of X . Using the classical Cantor-Bendixson derivative, we can assume that C is crowded. Since X is CB, it follows that C is uncountable. Therefore $|C| = \mathfrak{c} \geq \omega_2$ by Theorem 9.9. The well-known fact that every Σ_2^1 space of size at least ω_2 contains a copy of 2^ω (see Proposition 13.7 in [6]) concludes the proof. \square

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KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA, WÄHRINGER STRASSE 25, A-1090 WIEN, AUSTRIA
E-mail address: andrea.medini@univie.ac.at
URL: <http://www.logic.univie.ac.at/~medinia2/>

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA, WÄHRINGER STRASSE 25, A-1090 WIEN, AUSTRIA
E-mail address: lyubomyr.zdomskyy@univie.ac.at
URL: <http://www.logic.univie.ac.at/~lzdomsky/>