Elementary classes of finite vc-dimension

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Abstract

Let \mathcal{U} be a saturated model of inaccessible cardinality, and let $\mathcal{D} \subseteq \mathcal{U}$ be arbitrary. Let $\langle \mathcal{U}, \mathcal{D} \rangle$ denote the expansion of \mathcal{U} with a new predicate for \mathcal{D} . Write $e(\mathcal{D})$ for the collection of subsets $\mathcal{C} \subseteq \mathcal{U}$ such that $\langle \mathcal{U}, \mathcal{C} \rangle \equiv \langle \mathcal{U}, \mathcal{D} \rangle$. We prove that if the vc-dimension of $e(\mathcal{D})$ is finite then \mathcal{D} is externally definable.

Let \mathcal{U} be a saturated model of signature *L*, and let *T* denote its theory and κ its cardinality. We require that κ is uncountable, inaccessible, and larger than |L|. There is no blanket assumption on *T*. Throughout the following *z* is a tuple of variables of finite length and the letters \mathcal{D} and \mathcal{C} denote arbitrary subsets of $\mathcal{U}^{|z|}$. As usua, I the letters *A*, *B*, ... denote subsets of \mathcal{U} of small cardinality.

Recall that \mathcal{D} is **externally definable** if $\mathcal{D} = \mathcal{D}_{p,\varphi}$ for some global type $p \in S_x(\mathcal{U})$ and some $\varphi(x, z) \in L$, where

$$\mathcal{D}_{p,\varphi} = \{a \in \mathcal{U}^{|z|} : \varphi(x,a) \in p\}$$

Externally definable sets are ubiquitous in model theory, though they mainly appear in the form of global φ -types (in fact, they are in one-to-one correspondence with these). One important fact about externally definable sets has been proved by Shelah in [Sh], generalizing a theorem of Baisalov and Poizat in [BP]. Assume *T* is NIP and let \mathcal{U}^{Sh} be the model obtained by expanding \mathcal{U} with a new predicate for each externally definable set. Then Th(\mathcal{U}^{Sh}) has quantifier elimination. A few proofs of this result are available, see [Pi] and [CS]. The proof in [CS], by Chernikov and Simon, is relevant to us because it introduces the notion of *honest definition* that will find an application here. The Shelah expansion of groups with NIP has been studied in [CPS].

To any set \mathcal{D} we associate an expansion of \mathcal{U} with a new |z|-ary predicate for $z \in \mathcal{D}$. We denote this expansion by $\langle \mathcal{U}, \mathcal{D} \rangle$. We denote by $e(\mathcal{D}/A)$ the set $\{\mathcal{C} : \langle \mathcal{U}, \mathcal{C} \rangle \equiv_A \langle \mathcal{U}, \mathcal{D} \rangle\}$. We would like to know if there there are conditions on $e(\mathcal{D}/A)$ that characterize externally definable sets. Note that there are straightforward conditions that characterize definable sets. For example, \mathcal{D} is definable if and only if $|e(\mathcal{D}/A)| = 1$ for some *A*.

By adapting some ideas in [CS] (see also [Z]), in Corollary 12 we prove a sufficient condition for \mathcal{D} to be externally definable, namely that it suffices that for some set of parameters *A* the vc-dimension of $e(\mathcal{D}/A)$ is finite. Though in general this is not a necessary condition, it characterizes external definability when *T* is NIP (see Corollary 13). Finally, in the last two sections we use $e(\mathcal{D})$ in an attempt to generalize the notion of non-dividing to sets.

1 Notation

Let *L* be a first-order language. We consider formulas build inductively from the symbols in *L* and the atomic formulas $t \in \mathcal{X}$, where \mathcal{X} is some second-order variable and *t* is a tuple of terms. For the the time being, the logical connectives are first-order only (in the last section we will add second-order quantification). The set of all formulas is itself denoted by *L* or, if parameters from *A* are allowed, by L(A). When a second-order parameter is included (we never need more than one) we write $L(A; \mathcal{D})$. When $\varphi(\mathcal{X}) \in L(A)$ and $\mathcal{D} \subseteq \mathcal{U}^{|z|}$, we write $\varphi(\mathcal{D})$ for the formula obtained by replacing \mathcal{X} by \mathcal{D} in $\varphi(\mathcal{X})$. The truth of $\varphi(\mathcal{D})$ is defined in the obvious way. Warning: the meaning of $\varphi(\mathcal{D})$ depends on whether the formula is presented as $\varphi(\mathcal{X})$ or as $\varphi(x)$ (see the first paragraph of Section 2).

We write $\mathbb{C} \equiv_A \mathcal{D}$ if the equivalence $\varphi(\mathbb{C}) \leftrightarrow \varphi(\mathcal{D})$ holds for all $\varphi(\mathfrak{X}) \in L(A)$. Then the class $e(\mathcal{D}/A)$ defined in the introduction coincides with the set $\{\mathbb{C} \subseteq \mathcal{U}^{|z|} : \mathbb{C} \equiv_A \mathcal{D}\}$.

We say that M is $L(A; \mathbb{C})$ -saturated if every finitely consistent type $p(x) \subseteq L(A; \mathbb{C})$ is realized in M. If \mathbb{C} is such that \mathcal{U} is $L(A; \mathbb{C})$ -saturated for every A, we say that \mathbb{C} is saturated. In other words, \mathbb{C} is saturated if the expansion $\langle \mathcal{U}, \mathbb{C} \rangle$ is a saturated model.

1 Proposition For every \mathcal{D} and every A there is a saturated \mathcal{C} such that $\mathcal{C} \equiv_A \mathcal{D}$. Moreover, if \mathcal{D} and \mathcal{C} are both saturated, then there is $f \in \operatorname{Aut}(\mathcal{U}/A)$ that takes \mathcal{D} to \mathcal{C} .

Proof We prove that there is $\mathcal{C} \equiv_A \mathcal{D}$ such that expansion $\langle \mathcal{U}, \mathcal{C} \rangle$ is saturated. As κ is a large inaccessible cardinal, there is a model $\langle \mathcal{U}', \mathcal{D}' \rangle \equiv_A \langle \mathcal{U}, \mathcal{D} \rangle$ that is saturated and of cardinality κ . Then there is an isomorphism $f : \mathcal{U}' \to \mathcal{U}$ that fixes *A*. Then $f[\mathcal{D}'] = \mathcal{C}$ is the required saturated subset of \mathcal{U} . The second claim is clear by back-and-forth.

Let Δ be a set of formulas and let $\langle I, <_I \rangle$ be a linearly ordered set. We say that the sequence $\langle a_i : i \in I \rangle$ is **indiscernible in** Δ if for every integer k and two increasing tuples $i_1 <_I \cdots <_I i_k$ and $j_1 <_I \cdots <_I j_k$ and formula $\varphi(x_1, \dots, x_k) \in \Delta$, we have $\varphi(a_{i_1}, \dots, a_{i_k}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_k})$. When $\Delta = L(A)$ we say that $\langle a_i : i \in I \rangle$ is *A*-indiscernible.

We denote by $o(\mathcal{D}/A)$ the set $\{f[\mathcal{D}] : f \in \operatorname{Aut}(\mathcal{U}/A)\}$, that is, the orbit of \mathcal{D} under $\operatorname{Aut}(\mathcal{U}/A)$. If $o(\mathcal{D}/A) = \{\mathcal{D}\}$ we say that \mathcal{D} is invariant over A. A global type $p \in S_x(\mathcal{U})$ is invariant over A if for every $\varphi(x, z)$ the set $\mathcal{D}_{p,\varphi}$ is invariant over A. The main fact to keep in mind about global A-invariant types is that any sequence $\langle a_i : i < \lambda \rangle$ such that $a_i \models p_{\uparrow A, a_{\uparrow i}}$ is an A-indiscernible sequence.

We assume that the reader is familiar with basic facts concerning NIP theories as presented, e.g., in [Sim, Chapter 2].

2 Approximations

The set $\mathcal{D} \cap A^{|z|}$ is called the **trace** of \mathcal{D} over *A*. For every formula $\psi(z) \in L(\mathcal{U})$ we define $\psi(A) = \psi(\mathcal{U}) \cap A^{|z|}$, that is, the trace over *A* of the definable set $\psi(\mathcal{U}) = \{a \in \mathcal{U}^{|z|} : \psi(a)\}$.

A set \mathcal{D} is called **externally definable** if there are a global type $p \in S_x(\mathcal{U})$ and a formula $\varphi(x, z)$ such that $\mathcal{D} = \{a : \varphi(x, a) \in p\}$. Equivalently, a set \mathcal{D} is externally definable if it is the trace over \mathcal{U} of a set which is definable in some elementary extension of \mathcal{U} . This explains the terminology.

We prefer to deal with external definability in a different, though equivalent, way.

2 Definition We say that \mathbb{D} is **approximable** by the formula $\varphi(x, z)$ if for every finite *B* there is a $b \in \mathcal{U}^{|x|}$ such that $\varphi(b, B) = \mathbb{D} \cap B^{|z|}$. We may call the formula $\varphi(x, z)$ the **sort** of \mathbb{D} . If in addition we have that $\varphi(b, \mathcal{U}) \subseteq \mathbb{D}$, we say that \mathbb{D} is **approximable from below**. If $\mathbb{D} \subseteq \varphi(b, \mathcal{U})$ we say that \mathbb{D} is **approximable from above**.

Approximability from below is an adaptation to our context of the notion of *having an honest definition* in [CS]. The following proposition is clear by compactness.

- **3 Proposition** For every \mathcal{D} the following are equivalent:
 - 1. \mathcal{D} is approximable;
 - 2. \mathcal{D} is externally definable.
- **4 Example** Let *T* be the theory a dense linear orders without endpoints and let $\mathcal{D} \subseteq \mathcal{U}$ be an interval. Then \mathcal{D} is approximable both from below and from above by the formula $x_1 < z < x_2$. Now let *T* be the theory of the random graph. Then every $\mathcal{D} \subseteq \mathcal{U}$ is approximable and, when \mathcal{D} has small cardinality, it is approximable from above but not from below.

In Definition 2, the sort $\varphi(x, z)$ is fixed (otherwise any set would be approximable) but this requirement of uniformity may be dropped if the sets *B* are allowed to be infinite.

- **5 Proposition** For every \mathcal{D} the following are equivalent:
 - 1. \mathcal{D} is approximable;
 - 2. for every *B* of cardinality $\leq |T|$ there is $\psi(z) \in L(\mathcal{U})$ such that $\psi(B) = \mathcal{D} \cap B^{|z|}$.

Similarly, the following are equivalent:

- 3. \mathcal{D} is approximable from below;
- 4. for every $B \subseteq \mathcal{D}$ of cardinality $\leq |T|$ there is $\psi(z) \in L(\mathcal{U})$ such that $B^{|z|} \subseteq \psi(\mathcal{U}) \subseteq \mathcal{D}$.

Proof To prove $2 \Rightarrow 1$, for a contradiction assume 2 and $\neg 1$. For each formula $\psi(x, z) \in L$ choose a finite set *B* such that $\psi(b, B) \neq \mathcal{D} \cap B^{|z|}$ for every $b \in \mathcal{U}^{|x|}$. Let *C* be the union of all these finite sets. Clearly $|C| \leq |T|$. By 2 there are a formula $\varphi(x, z)$ and a tuple *c* such that

 $\varphi(c, C) = \mathcal{D} \cap C^{|z|}$, contradicting the definition of *C*.

The implication $1\Rightarrow2$ is obtained by compactness and the equivalence $3\Leftrightarrow4$ is proved similarly.

6 Proposition If \mathcal{D} is approximable of sort $\varphi(x, z)$ then so is any \mathcal{C} such that $\mathcal{C} \equiv \mathcal{D}$. The same holds for approximability from below and from above.

Proof If the set \mathcal{D} is approximable by $\varphi(x, z)$ then for every *n*

$$\forall z_1, \dots, z_n \exists x \bigwedge_{i=1}^n \big[\varphi(x, z_i) \leftrightarrow z_i \in \mathcal{D} \big].$$

So the same holds for any $\mathcal{C} \equiv \mathcal{D}$. As for approximability from below, add the conjunct $\forall z [\varphi(x, z) \rightarrow z \in \mathcal{D}]$ to the formula above, and similarly for approximability from above. \Box

3 The Vapnik-Chervonenkis dimension

We say that $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$ shatters $B \subseteq \mathcal{U}^{|z|}$ if every $H \subseteq B$ is the trace over *B* of some set $\mathcal{D} \in u$. The vc-dimension of *u* is finite if there is some $n < \omega$ such that no set of size *n* is shattered by *u*.

7 Proposition The following are equivalent:

- 1. $e(\mathcal{D}/A)$ has finite vc-dimension;
- 2. $o(\mathcal{C}/A)$ has finite vc-dimension for some (any) saturated $\mathcal{C} \equiv_A \mathcal{D}$.

Proof 1 \Rightarrow 2. Clear because $o(C/A) \subseteq e(D/A)$.

2⇒1. Let \mathcal{C} be any saturated set such that $\mathcal{C} \equiv_A \mathcal{D}$. Let *B* be a finite set that is shattered by $e(\mathcal{D}/A)$, namely such that every $H \subseteq B$ is the trace of some $\mathcal{C}_H \equiv_A \mathcal{D}$. By Proposition 1, we can require that all these sets \mathcal{C}_H are saturated. Then they all belong to $o(\mathcal{C}/A)$. It follows that if $e(\mathcal{D}/A)$ has infinite vC-dimension so does $o(\mathcal{C}/A)$.

We say that a sequence of sentences $\langle \varphi_i : i < \omega \rangle$ converges if the truth value of φ_i is eventually constant.

8 Lemma Assume that $o(\mathcal{D}/A)$ has finite vc-dimension and let $\langle a_i : i < \omega \rangle$ be any *A*-indiscernible sequence. Then $\langle a_i \in \mathcal{D} : i < \omega \rangle$ converges.

Proof Negate the conclusion and let $\langle a_i : i \in \omega \rangle$ witness this. We show that $o(\mathcal{D}/A)$ shatters $\{a_i : i < n\}$ for arbitrary n, hence that $o(\mathcal{D}/A)$ has infinite VC-dimension. Fix some $H \subseteq n$, and for every h < n pick some a_{i_h} such that $a_{i_h} \in \mathcal{D}$ if and only if $h \in H$. We also require that $i_0 < \cdots < i_{n-1}$. Let $f \in \operatorname{Aut}(\mathcal{U}/A)$ be such that $f : a_{i_0}, \ldots a_{i_{n-1}} \mapsto a_0, \ldots a_{n-1}$. Then $a_h \in f[\mathcal{D}]$ if and only if $h \in H$.

We abbreviate $\mathcal{U} \sim \mathcal{C}$ as $\neg \mathcal{C}$. We write \neg^i for $\neg \dots (i \text{ times}) \dots \neg$ and abbreviate $\neg^i (\cdot \in \cdot)$ as \notin^i . The following lemmas adapt some ideas from [CS, Section 1] to our context.

9 Lemma Assume that C is saturated and that o(C/A) has finite vC-dimension. Let $M \leq U$ be an L(A; C)-saturated. Then every global *A*-invariant type p(z) contains a formula $\psi(z) \in L(M)$ such that either $\psi(U) \subseteq C$ or $\psi(U) \subseteq \neg C$.

Proof By lemma 8 there is no infinite sequence $\langle b_i : i < \omega \rangle$ such that

1. $b_i \models p(z)|_{A,b_{\uparrow i}} \land z \notin^i \mathbb{C}.$

Let *n* be the maximal length of a sequence $\langle b_i : i < n \rangle$ that satisfies 1. Then

$$p(z)|_{A,b_{\restriction n}} \to z \notin^n \mathbb{C}.$$

As *M* is $L(A; \mathbb{C})$ -saturated, we can assume further that $b_i \in M$. Also, by saturation we can replace $p(z)|_{A,b_{\uparrow n}}$ with some formula $\psi(z)$. Then, if *n* is even, $\psi(\mathcal{U}) \subseteq \mathbb{C}$, and if *n* is odd $\psi(\mathcal{U}) \subseteq \neg \mathbb{C}$.

Notice that $p(z) \in S(M)$ is finitely satisfied in $A \subseteq M$ if and only if it contains the type

$q(z) = \{\neg \varphi(z) \in L(M) : \varphi(A) = \varnothing\}.$

With this notation in mind, we can state the following lemma.

10 Lemma Assume C is saturated and o(C/A) has finite VC-dimension. Then there are two formulas $\psi_i(z)$, where i < 2, such that $\psi_i(z) \to z \notin^i C$ and, if q(z) is the type defined above, $q(z) \to \psi_0(z) \lor \psi_1(z)$.

Proof Let *M* be an $L(A; \mathbb{C})$ -saturated model. By definition, for every $a \models q(z)$ the type tp(a/M) is finitely satisfiable in *A* so it extends to a global invariant type. By Lemma 9, $q(\mathcal{U})$ is covered by formulas $\psi(z) \in L(M)$ such that either $[\psi(z) \rightarrow z \in \mathbb{C}]$ or $[\psi(z) \rightarrow z \notin \mathbb{C}]$. The conclusion follows by compactness.

11 Theorem Assume C is saturated and o(C/A) has finite vC-dimension for some A. Then C is approximable from below and from above.

Proof Let $B \subseteq C$ be given. Enlarging *A* if necessary, we can assume that $B \subseteq A$. Let *M* and $q(z) \subseteq L(M)$ be as in # above. Trivially $A \subseteq q(U)$, hence $B \subseteq \psi_0(U) \subseteq C$. The set *B* has arbitrary (small) cardinality. Then by Lemma 5, *C* is approximable from below.

As for approximation from above, observe that this is equivalent to $\neg C$ being approximable from below. As $\neg C$ is also saturated and $o(\neg C/A)$ has finite vc-dimension, approximability from above follows.

12 Corollary Assume e(D/A) has finite VC-dimension for some A. Then D is approximable from below and from above.

Proof Let $\mathcal{C} \equiv_A \mathcal{D}$ be saturated. As $o(\mathcal{C}/A)$ also has finite vC-dimension, from Theorem 11 it follows that \mathcal{C} is approximable from below and from above. Then by Proposition 6 the same conclusion holds for \mathcal{D} .

Recall that a formula $\varphi(x, z) \in L$ is NIP if $\{\varphi(a, \mathcal{U}) : a \in \mathcal{U}^{|x|}\}$ has finite vC-dimension. If this is the case, $\{\mathcal{D}_{p,\varphi} : p \in S_x(\mathcal{U})\}$, that is, the set of externally definable sets of sort $\varphi(x, z)$, also has finite vC-dimension. Now, observe that if \mathcal{D} is any externally definable set and $\mathcal{C} \equiv \mathcal{D}$ then \mathcal{C} is also externally definable and has the same sort as \mathcal{D} . Hence, if $\varphi(x, z)$ is NIP, $e(\mathcal{D}) \subseteq \{\mathcal{D}_{p,\varphi} : p \in S_x(\mathcal{U})\}$ has finite vC-dimension.

The theory *T* is NIP if in \mathcal{U} every formula is NIP. Hence we obtain the following characterization of externally definable sets in a NIP theory:

13 Corollary *Il T is* NIP *then the following are equivalent:*

- 1. \mathcal{D} is approximable from below (in particular, externally definable);
- *2.* $e(\mathcal{D})$ has finite VC-dimension.

We conclude by mentioning the following corollary, which is a version of Proposition 1.7 of [CS] stated with different terminology. Note that it is not necessary to require that T is NIP.

14 Corollary If \mathcal{D} is approximable by a NIP formula, then \mathcal{D} is approximable from below.

Proof If \mathcal{D} is approximable of sort $\varphi(x, z)$, by Proposition 6, so are all sets in $e(\mathcal{D})$. If $\varphi(x, z)$ is NIP, then $e(\mathcal{D})$ has finite vc-dimension and Corollary 12 applies.

Observe that, given a formula $\varphi(x, z)$ that approximates \mathcal{D} , the proof of Corollary 14 does not give explicitly the formula $\psi(x, z)$ that approximates \mathcal{D} from below.

4 Lascar invariance

The content of the second part of the paper is only loosely connected to the previous sections. We introduce the notion of a *pseudo-invariant set* which is connected to nondividing but it is sensible for arbitrary subsets of \mathcal{U} . We assume that the reader is familiar with basic facts concerning Lascar strong types and dividing (see e.g., [Sim], [Cas], [TZ]) though in this section we will recall everything we need.

If $o(\mathcal{D}/A) = \{\mathcal{D}\}$ we say that \mathcal{D} is **invariant over** A. We say that \mathcal{D} is **invariant** tout court if it is invariant over some A. We say that \mathcal{D} is **Lascar invariant over** A if it is invariant over every model $M \supseteq A$.

15 Proposition There are at most $2^{2^{|L(A)|}}$ sets \mathcal{D} that are Lascar invariant over *A*.

Proof Let *N* be a model containing *A* of cardinality $\leq |L(A)|$. Every Lascar invariant set over *A* is invariant over *N*. The proposition follows as $|N| \leq |L(A)|$, and there are at most $2^{2^{|N|}}$ sets invariant over *N*.

16 Proposition For every \mathcal{D} and every $A \subseteq M$ the following are equivalent:

- 1. \mathcal{D} is Lascar invariant over A;
- 2. every set in $o(\mathcal{D}/A)$ is *M*-invariant;
- 3. $o(\mathcal{D}/A)$ has cardinality < κ ;
- 4. every endless A-indiscernible sequence is indiscernible in $L(A; \mathcal{D})$;
- 5. $c_0 \in \mathcal{D} \leftrightarrow c_1 \in \mathcal{D}$ for every *A*-indiscernible sequence $c = \langle c_i : i < \omega \rangle$.

Proof The implication $1 \Rightarrow 2$ is clear because all sets in $o(\mathcal{D}/A)$ are Lascar invariant over *A*. To prove $2\Rightarrow 3$ it suffices to note that there are fewer than κ sets that are invariant over *M*.

We now prove $3 \Rightarrow 4$. Assume $\neg 4$. Then we can find an *A*-indiscernible sequence $\langle c_i : i < \kappa \rangle$ and a formula $\varphi(x) \in L(A; \mathcal{D})$ such that $\varphi(c_0) \nleftrightarrow \varphi(c_1)$. Define

 $E(x, y) \Leftrightarrow \psi(x) \leftrightarrow \psi(y)$ for every $\mathcal{C} \in o(\mathcal{D}/A)$ and every $\psi(x) \in L(A; \mathcal{C})$.

Then E(x, y) is an *A*-invariant equivalence relation. As $\neg E(c_0, c_1)$, indiscernibility over *A* implies that $\neg E(c_i, c_j)$ for every $i < j < \kappa$. Then E(x, y) has κ equivalence classes. As κ is inaccessible, this implies $\neg 3$.

The implication $4\Rightarrow5$ is trivial. We prove $5\Rightarrow1$. Suppose $a \equiv_M b$ for some $M \supseteq A$. Let p(z) be a global coheir of tp(a/M) = tp(b/M). Let $c = \langle c_i : i < \omega \rangle$ be a Morley sequence of p(z) over M, a, b. Then both a, c and b, c are A-indiscernible sequences. So from 5 we obtain $a \in \mathcal{D} \leftrightarrow c_0 \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$ and, as M is arbitrary, 1 follows.

As the number of *M*-invariant sets is at most $2^{2^{|M|}}$, we obtain the following corollary.

17 Corollary For every \mathcal{D} the following are equivalent:

- 1. $o(\mathcal{D}/A)$ has cardinality < κ ;
- 2. $o(\mathcal{D}/A)$ has cardinality $\leq 2^{2^{|L(A)|}}$.

5 Dividing

Though Definition 18 below does not make any assumptions on \mathcal{B} and $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$, it yields a workable notion only when \mathcal{B} is invariant and u is closed in a sense that we will explain. Moreover, for the proof of Lemma 22 we need κ to be a Ramsey cardinal, so this will a blanket assumption throughout this section.

18 Definition Let $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$ and let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$. We say that u **locally covers** \mathcal{B} if for every $\mathcal{K} \subseteq \mathcal{B}$ of cardinality κ and every integer k there is a $\mathcal{D} \in u$ such that $k \leq |\mathcal{K} \cap \mathcal{D}|$.

The subsets of $\mathcal{P}(\mathcal{U}^{|z|})$ that are definable by formulas $\varphi(\mathcal{X}) \in L(A)$ form a base of clopen sets for a topology. The proposition below implies that this topology is compact.

19 Proposition Let $p(\mathfrak{X}) \subseteq L(A)$ be finitely consistent, that is, for every $\varphi(\mathfrak{X})$ conjunction of formulas in $p(\mathfrak{X})$ there is a $\mathfrak{D} \subseteq \mathfrak{U}^{|z|}$ such that $\varphi(\mathfrak{D})$. Then there is a set \mathfrak{C} such that $p(\mathfrak{C})$.

Proof The proposition follows from the fact that every saturated model is resplendent, see [Poi, Théorème 9.17]. But the reader may prefer to prove it directly by adapting the argument used in the proof of Proposition 1.

Notice that the topology introduced above is not T_0 because there are $C \neq D$ such that $C \equiv D$. However, it is immediate that taking the Kolmogorov quotient (i.e. quotienting by the equivalence relation \equiv) gives a Hausdorff topology. Then there is no real need to distinguish between *compactness* and *quasi-compactness*.

We will say that the set $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$ is **closed** if it is closed in the topology introduced above. In other words, *u* is closed if $u = \{\mathcal{D} : p(\mathcal{D})\}$ for some $p(\mathcal{X}) \subseteq L$.

- **20 Remark** We may read Definition 18 as a generalization of non-dividing. Let us recall the definition of dividing. We say that the formula $\varphi(x, b)$ divides over A if there there is an infinite set $\mathcal{K} \subseteq o(b/A)$ such that $\{\varphi(x, c) : c \in \mathcal{K}\}$ is k-inconsistent for some k. By compactness, there is no loss of generality if we require $|\mathcal{K}| = \kappa$. Let $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$ contain the externally definable sets of sort $\varphi(x, z)$. Then the requirement that $\{\varphi(x, c) : c \in \mathcal{K}\}$ is k-inconsistent can be rephrased as $|\mathcal{K} \cap \mathcal{D}| < k$ for every $\mathcal{D} \in u$. So we may conclude that the following are equivalent:
 - 1. the formula $\varphi(x, b)$ does not divide over *A*;
 - 2. u locally covers o(b/A).

Incidentally, note that o(b/A) is *A*-invariant and that *u* is a closed set.

We now need to use second-order quantifiers. The set of formulas containing secondorder quantifiers is denoted by L^2 , or $L^2(A; \mathcal{D})$ when parameters occur. Second-order quantifiers are interpreted to range over $\mathcal{P}(\mathcal{U}^{|z|})$. The following fact is immediate but noteworthy.

- **21** Fact Every formula $\varphi(x) \in L^2(A)$ is *A*-invariant and consequently any *A*-indiscernible sequence is indiscernible in $L^2(A)$.
- **22 Lemma** Let $u \subseteq \mathcal{P}(\mathcal{U}^{|z|})$ be a closed set and let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be an *A*-invariant set. Then the following are equivalent:
 - 1. u locally covers \mathcal{B} ;
 - 2. every *A*-indiscernible sequence $\langle a_i : i < \omega \rangle \subseteq \mathcal{B}$ is contained in some $\mathcal{D} \in u$.

Proof 1⇒2. Let $p(\mathfrak{X}) \in L$ be such that $u = \{\mathcal{D} : p(\mathcal{D})\}$. Assume $\neg 2$ and fix an *A*-indiscernible sequence $\langle a_i : i < \omega \rangle \subseteq \mathcal{B}$ such that $p(\mathfrak{X}) \cup \{a_i \in \mathfrak{X} : i < \omega\}$ is inconsistent. By compactness there are some i_1, \ldots, i_k and some $\varphi(\mathfrak{X}) \in p$ such that

$$\forall \mathcal{X} \left[\varphi(\mathcal{X}) \to \neg \bigwedge_{n=1}^{k} a_{i_n} \in \mathcal{X} \right].$$

Extend $\langle a_i : i < \omega \rangle$ to an *A*-indiscernible sequence $\langle a_i : i < \kappa \rangle$. By indiscernibility, every $\mathcal{D} \in u$ contains fewer than *k* elements of $\{a_i : i < \kappa\} \subseteq \mathcal{B}$. Hence $\neg 1$.

2⇒1. Assume ¬1 and fix $\mathcal{K} \subseteq \mathcal{B}$ of cardinality κ and an integer k such that $|\mathcal{K} \cap \mathcal{D}| < k$ for every $\mathcal{D} \in u$. As κ is a Ramsey cardinal, there is an *A*-indiscernible $\langle a_i : i < \kappa \rangle \subseteq \mathcal{K}$. Then $\langle a_i : i < \kappa \rangle$ may not be contained in any $\mathcal{D} \in u$, hence ¬2.

We say that \mathcal{D} is **pseudo-invariant** over *A* if $e(\mathcal{D})$ locally covers o(b/A) for every $b \in \mathcal{D}$.

23 Proposition If \mathcal{D} is Lascar invariant over A, then for every $\varphi(w) \in L(A; \mathcal{D})$ the set $\varphi(\mathcal{U})$ is pseudo-invariant over A.

Proof Fix $\varphi(w) \in L(A; \mathcal{D})$ and let $b \in \varphi(\mathcal{U})$. Let $\langle a_i : i < \omega \rangle \subseteq o(b/A)$ be an indiscernible sequence and fix some $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $fa_0 = b$. Then $\langle fa_i : i < \omega \rangle$ is indiscernible in $L(A; \mathcal{D})$ by Proposition 16. Then $\langle fa_i : i < \omega \rangle \subseteq \varphi(\mathcal{U})$. Hence $\langle a_i : i < \omega \rangle \subseteq f^{-1}[\varphi(\mathcal{U})]$. Clearly, $f^{-1}[\varphi(\mathcal{U})] \in e(\varphi(\mathcal{U}))$, so the proposition follows from Lemma 22.

- **24 Proposition** Let $e(\mathcal{D})$ have finite vc-dimension. Then the following are equivalent:
 - 1. \mathcal{D} is Lascar invariant over A;
 - 2. $\varphi(\mathcal{U})$ is pseudo-invariant over *A* for every $\varphi(w) \in L(A; \mathcal{D})$;
 - *3.* $\mathcal{D} \times \neg \mathcal{D}$ *is pseudo-invariant over A.*

Proof $1 \Rightarrow 2$ holds for any \mathcal{D} by Proposition 23 and $2 \Rightarrow 3$ is obvious.

3⇒1. Assume ¬1. By Proposition 16, there is an *A*-indiscernible sequence $\langle a_i : i < \omega \rangle$ such that $a_0 \in \mathcal{D} \nleftrightarrow a_1 \in \mathcal{D}$, say $a_0 \in \mathcal{D}$ and $a_1 \notin \mathcal{D}$. Assume 2 for a contradiction. Then by Lemma 22 there is $\mathcal{C} \equiv \mathcal{D}$ such that $\langle a_{2i}a_{2i+1} : i < \omega \rangle \subseteq \mathcal{C} \times \neg \mathcal{C}$. By Lemma 8, $e(\mathcal{C}) = e(\mathcal{D})$ has infinite vC-dimension contradicting the assumptions.

The hypothesis of finite vC-dimension is necessary. Assume *T* is the theory of dense linear orders without endpoints. Let \mathcal{D} be a discretely ordered subset of \mathcal{U} of cardinality κ . Then \mathcal{D} is not invariant and $e(\mathcal{D})$ has infinite vC-dimension. One can verify that $\mathcal{D} \times \neg \mathcal{D}$ is pseudo-invariant over \emptyset directly from the definition.

It is well known that under the hypothesis that *T* is NIP, Lascar invariance of global types is equivalent to non-dividing (equivalently, non-forking), see [Sim, Proposition 5.21]. Then, when *T* is NIP, a global type p(x) does not divide over *A* if and only if $\mathcal{D}_{p,\varphi} \times \neg \mathcal{D}_{p,\varphi}$ is pseudo-invariant over *A* for every $\varphi(x, z)$.

However, pseudo-invariance is too strong a requirement to coincide with non-dividing in general. A counter-example may be found even when *T* is simple. Let *T* be the theory of the random graph and let \mathcal{D} be a complete subgraph of \mathcal{U} . Let p(x) be the unique global type that contains

 $\{r(x,a): a \in \mathcal{D}\} \cup \{\neg r(x,a): a \notin \mathcal{D}\} \cup \{x \neq a: a \in \mathcal{U}\}.$

Then p(x) does not fork over the empty set. On the other hand, \mathcal{D} is not pseudo-invariant: let $\langle a_i : i < \omega \rangle$ be an indiscernible sequence such that $a_0 \in \mathcal{D} \land \neg r(a_0, a_1)$. As every $\mathcal{C} \equiv \mathcal{D}$ is a complete graph, no such \mathcal{C} may contain $\langle a_i : i < \omega \rangle$.

References

- [AK] Noga Alon and Daniel J. Kleitman. Piercing convex sets and the Hadwiger-Debrunner (p, q)-problem. Adv. Math., vol.96 (1992) no.1 p.103–112.
- [BP] Yerzhan Baisalov and Bruno Poizat. Paires de structures o-minimales. The Journal of Symbolic Logic, vol.63 (1998) no.2 p.570–578.
- [Cas] Enrique Casanovas. Simple theories and hyperimaginaries. Lecture Notes in Logic vol.39, Cambridge University Press (2011)
- [CS] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs. arXiv:1007.4468 Israel J. Math. vol.194 (2013) no.1 p.409–425.
- [CS2] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs II. arXiv:1202.2650 (2012)
- [CPS] Artem Chernikov, Anand Pillay, Pierre Simon. External definability and groups in NIP theories. arXiv:1307.4794 (2014)
- [Mat] Jiří Matoušek. Bounded VC-dimension implies a fractional Helly theorem. Discrete Comput. Geom., vol.31 (2004) no.2 p.251–255.
- [Pi] Anand Pillay. On externally definable sets and a theorem of Shelah. In Felgner Festchrift, Studies in Logic. College Publications, (2007)
- [Poi] Poizat, Bruno. Cours de théorie des modèles. Bruno Poizat, (1985)
- [Sh] Saharon Shelah. Dependent first order theories, continued. arXiv:math/0504197
 Israel Journal of Mathematics, vol.173 (2009) no.1 p.1–60.
- [Sim] Pierre Simon. A Guide to NIP theories. arXiv:1202.2650 (2014)
- [TZ] Katrin Tent and Martin Ziegler. A course in model theory. Lecture Notes in Logic vol.40, Cambridge University Press (2012)
- [Z] Martin Ziegler. Chernikov and Simon's proof of Shelah's theorem. Unpublished notes available on the author's homepage (2010)

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