

Intuitionistic fixed point theories over set theories

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Abstract

In this paper we show that the intuitionistic fixed point theory $\text{FiX}^i(T)$ over set theories T is a conservative extension of T if T can manipulate finite sequences and has the full foundation schema.

1 Intuitionistic fixed point theory over set theories T

For a theory T in a language \mathcal{L} , let $\mathcal{Q}(X, x)$ be an X -positive formula in the language $\mathcal{L} \cup \{X\}$ with an extra unary predicate symbol X . Introduce a fresh unary predicate symbol Q together with the axiom stating that Q is a fixed point of $\mathcal{Q}(X, x)$:

$$\forall x[\mathcal{Q}(x) \leftrightarrow \mathcal{Q}(Q, x)] \quad (1)$$

By the completeness theorem, it is obvious that the resulting extension of T is conservative over T , though it has a non-elementary speed-up over T when T is a recursive theory containing the elementary recursive arithmetic EA , cf. [3].

When T has an axiom schema, e.g., $T = \text{PA}$, the Peano arithmetic with the complete induction schema, let us define the fixed point extension $\text{FiX}(\text{PA})$ to have the induction schema for any formula with the fixed point predicate Q . Then $\text{FiX}(\text{PA})$ is stronger than PA , e.g., $\text{FiX}(\text{PA})$ proves the consistency of PA . For the proof-theoretic strength of the fixed point theory $\text{FiX}(\text{PA})$, see [2, 10, 16].

On the other side, W. Buchholz [15] shows that an *intuitionistic* fixed point theory over the intuitionistic (Heyting) arithmetic HA for strongly positive formulae $\mathcal{Q}(X, x)$ is proof-theoretically reducible to HA . In a language of arithmetic strongly positive formulae with respect to X are generated from arithmetic formulae and atomic ones $X(t)$ by means of positive connectives $\vee, \wedge, \exists, \forall$. Then Riede and Strahm [18] extends the result to the intuitionistic fixed point theory $\text{FiX}^i(\text{HA})$ for strictly positive formulae $\mathcal{Q}(X, x)$, in which the predicate symbol X does not occur in the antecedent φ of implications $\varphi \rightarrow \psi$ nor in the scope

of negations \neg . Indeed as shown in [5] $\text{FiX}^i(\text{HA})$ is a conservative extension of HA .

However this might mislead us. Namely one might think that the conservation holds for the fixed point extensions because the theory $T = \text{HA}$ is intuitionistic. Actually this is not the case. For example, the intuitionistic fixed point theory $\text{FiX}^i(\text{PA})$ over the *classical* arithmetic PA is a conservative extension of PA . For, if $\text{FiX}^i(\text{PA})$ proves an arithmetical sentence A , then $\text{FiX}^i(\text{HA})$ proves $B \rightarrow A$ for a PA -provable sentence B . Since $\text{FiX}^i(\text{HA})$ is conservative over HA , we see that HA proves $B \rightarrow A$, and $\text{PA} \vdash A$.

Our proof in [5] is a proof-theoretic one by showing that the fixed point axiom (1) is eliminable quickly. The crux is that the *underlying logic* is intuitionistic.

Digression. Let $\widehat{ID}^i(\text{acc})$ be an intuitionistic theory obtained from $\widehat{ID}^i(\text{strict}) = \text{FiX}^i(\text{HA})$ by restricting $\mathcal{Q}(X, x)$ to *accessible formulas*, i.e., $\mathcal{Q}(X, x) \equiv (A(x) \wedge \forall y(B(y, x) \rightarrow X(y)))$ for arithmetical formulas A, B . The following Lemma 1.1 is shown in [18].

Lemma 1.1 ([18])

1. $\widehat{ID}^i(\text{strict})$ is conservative over $\widehat{ID}^i(\text{acc})$ with respect to almost negative formulas.
2. The classical theory $\widehat{ID}(\text{acc})$ is interpretable in the classical arithmetic PA .

Lemma 1.1.1 is shown by a recursive realizability interpretation following Buchholz [13], and the interpretation in the proof of Lemma 1.1.2 is done by a diagonalization argument. Specifically it is observed that there is an arithmetical fixed point for accessible operators, classically. Then they conclude that $\widehat{ID}^i(\text{strict})$ is conservative over the intuitionistic arithmetic HA with respect to negative formulas.

Let us try to prove the full conservation result in [5] along the line in [18]. The intuitionistic version of Lemma 1.1.2 is easy to see, which says that $\widehat{ID}^i(\text{acc})$ is a conservative extension of HA . Let $A(x)$ and $B(y, x)$ be arithmetical formulae. Let $y <_B x : \Leftrightarrow B(y, x)$ and $y \leq_B^* x$ denote its reflexive and transitive closure. Namely $y \leq_B^* x$ iff there exists a non-empty sequence (x_n, \dots, x_0) such that $x_n = y$, $x_0 = x$ and $\forall i < n (x_{i+1} <_B x_i)$. Then $A^*(x) : \Leftrightarrow \forall y \leq_B^* x A(y)$ is an arithmetical fixed point for accessible operators $\mathcal{Q}(X, x) \equiv (A(x) \wedge \forall y(B(y, x) \rightarrow X(y)))$ provably in HA , i.e., $\text{HA} \vdash \forall x[A^*(x) \leftrightarrow (A(x) \wedge \forall y <_B x A^*(y))]$. The problem is to extend Lemma 1.1.1 to all arithmetical formulae, which means that $\widehat{ID}^i(\text{strict})$ is conservative over $\widehat{ID}^i(\text{acc})$ with respect to any arithmetical formulae. If a combination of realizability interpretation and forcing works as in [12], then it would yield the full conservativity. However it is hard to show the soundness of the forcing stating that if $\widehat{ID}^i(\text{acc})\mathbf{a} \vdash A$, then

$\widehat{ID}^i(\text{acc}) \vdash \forall p \exists q \supset p(q \Vdash A)$ since $p \Vdash \forall y (B(y, x) \rightarrow Q(y))$ is not an accessible formula, but strictly positive.

In this paper we extend the observation in [5] for set theories T .

Let T be a set theory in the language $\{\in, =\}$.

Fix an X -strictly positive formula $Q(X, x)$ in the language $\{\in, =, X\}$ with an extra unary predicate symbol X . In $Q(X, x)$ the predicate symbol X occurs only strictly positive. The language of $\text{FiX}^i(T)$ is $\{\in, =, Q\}$ with a fresh unary predicate symbol Q . The axioms in $\text{FiX}^i(T)$ consist of the following:

1. All provable sentences in T (in the language $\{\in, =\}$).
2. Foundation schema for any formula φ in the language $\{\in, =, Q\}$:

$$\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) \quad (2)$$

3. Fixed point axiom (1).

The underlying logic in $\text{FiX}^i(T)$ is defined to be the intuitionistic (first-order predicate) logic (with equality). $\forall x, y (x = y \rightarrow Q(x) \rightarrow Q(y))$ is an axiom.

In this paper we show the following Theorem 1.2 for a weak base set theory BS defined in the next section 2.

Theorem 1.2 $\text{FiX}^i(T)$ is a conservative extension of any set theory $T \supset \text{BS}$.

We need Theorem 1.2 in [6–9] for proof-theoretic analyses of set theories for weakly compact cardinals, first-order reflecting ordinals, ZF and second-order indescribable cardinals. In these analyses, a provability relation $\mathcal{H} \vdash_c^\alpha \Gamma$ derived from operator controlled derivations is defined to be a fixed point of a strictly positive formula.

Let us mention the contents of the paper. In section 2 a weak base theory BS is introduced, and it is shown that BS can manipulate finite sequences and partially define truth. In section 3 a class of codes *Code* and a binary relation \prec on it are defined, and it is shown that the transfinite induction schema with respect to \prec is provable in BS up to *each* code. The order type of the well founded relation \prec is the next epsilon number to the order type of the class of ordinals in the universe. In section 4 a sequent calculus for $\text{FiX}^i(T)$ is introduced, and in section 5 Theorem 1.2 is proved by a finitary analysis of the proofs in the sequent calculus for $\text{FiX}^i(T)$.

2 Basic set theory BS

In this section we introduce a basic set theory BS, and show that BS can manipulate finite sequences of sets, thereby can encode syntax, and define truth partially.

Consider the following functions \mathcal{F}_i ($i < 9$), $\mathcal{F}_0(x, y) = \{x, y\}$, $\mathcal{F}_1(x, y) = \cup x$, $\mathcal{F}_2(x, y) = x \setminus y$, $\mathcal{F}_3(x, y) = \{u \cup \{v\} : u \in x, v \in y\}$, $\mathcal{F}_4(x, y) = \text{dom}(x) =$

$\{u \in \cup \cup x : \exists v \in \cup \cup x (\langle u, v \rangle \in x)\}$, $\mathcal{F}_5(x, y) = \text{rng}(x) = \{v \in \cup \cup x : \exists u \in \cup \cup x (\langle u, v \rangle \in x)\}$, $\mathcal{F}_6(x, y) = \{\langle v, u \rangle \in y \times x : v \in u\}$, $\mathcal{F}_7(x, y) = \{\langle u, v, w \rangle : \langle u, v \rangle \in x, w \in y\}$, and $\mathcal{F}_8(x, y) = \{\langle u, w, v \rangle : \langle u, v \rangle \in x, w \in y\}$, where $\langle v, u \rangle = \{v, \{v, u\}\}$ and $\langle u, v, w \rangle = \langle u, \langle v, w \rangle \rangle$.

Note that each \mathcal{F}_i is *simple* in the sense that for any Δ_0 -formula $\varphi(z)$, $\varphi(\mathcal{F}_i(x, y))$ is Δ_0 . For each i , $\mathcal{F}_i(x, y, z)$ denotes a Δ_0 -formula stating $\mathcal{F}_i(x, y) = z$.

Definition 2.1 BS is the set theory in the language $\{\in, =\}$. Its axioms are Extensionality, Foundation schema, and $\{\forall x, y \exists z \mathcal{F}_i(x, y, z) : i < 9\}$.

A set-theoretic function $f : V^n \rightarrow V$ is Σ_1^{BS} -definable if there exists a Σ_1 -formula $\varphi(x_1, \dots, x_n, y)$ for which $\text{BS} \vdash \forall x_1, \dots, x_n \exists! y \varphi(x_1, \dots, x_n, y)$, and $f(x_1, \dots, x_n) = y$ iff $V \models \varphi(x_1, \dots, x_n, y)$.

A relation $R \subset V^n$ is Δ_1^{BS} if there exist Σ_1 -formulae φ, ψ such that $\text{BS} \vdash \forall x_1, \dots, x_n [\varphi(x_1, \dots, x_n) \leftrightarrow \neg \psi(x_1, \dots, x_n)]$, and $(x_1, \dots, x_n) \in R$ iff $V \models \varphi(x_1, \dots, x_n)$.

A formula is said to be $\Delta_0(\omega)$ iff every quantifier occurring in it is a bounded quantifier $\exists m < n, \forall m < n$ with bounds $n \in \omega$.

Proposition 2.2 1. The Cartesian product $a \times b$ is a Σ_1^{BS} -function with a Δ_0 -graph.

2. BS proves Δ_0 -Separation: $\text{BS} \vdash \forall a, b \exists c [c = \{x \in a : \varphi(x, b)\}]$ for each Δ_0 -formula φ .

3. $\omega \times V \ni (n, a) \mapsto {}^n a, {}^n a$ are Σ_1^{BS} -functions. $z = {}^{<n} a$ is Δ_0 , and $z = {}^n a$ is Δ_1^{BS} .

4. The class of Δ_1^{BS} -relations is closed under propositional connectives \neg, \vee and bounded quantifications $\exists m < n, \forall m < n$ with bounds $n \in \omega$.

For Σ_1^{BS} -functions f and $\Delta_0(\omega)$ -formula $\theta(y)$, $\theta(f(\vec{x}))$ is Δ_1^{BS} .

5. The class of Σ_1^{BS} -functions is closed under compositions and primitive recursion on ω . The latter means that if $g : V^n \rightarrow V$ and $h : \omega \times V^{n+1} \rightarrow V$ are Σ_1^{BS} -functions, then so is the function $f : \omega \times V^n \rightarrow V$ defined by $f(0, \vec{x}) = g(\vec{x})$ and $f(n+1, \vec{x}) = h(n, \vec{x}, f(n, \vec{x}))$ for $\vec{x} = x_1, \dots, x_n$.

6. For the transitive closure $\text{trcl}(a)$, $x \in \text{trcl}(a)$ is Δ_1^{BS} .

Proof. 2.2.1. Let $G(a) = \mathcal{F}_3(\{\emptyset\}, a) = \{\{x\} : x \in a\}$. Then by $\mathcal{F}_3(G(a), b) = \{\{x, y\} : x \in a, y \in b\}$, we have $a \times b = \{\{\{x\}, \{x, y\}\} : x \in a, y \in b\} = \mathcal{F}_3(G(G(a)), \mathcal{F}_3(G(a), b))$. $a \times b = c$ is a Δ_0 -formula.

2.2.2. Standard, cf. [11], pp. 63-67 using Proposition 2.2.1.

2.2.3. Noting ${}^{n+1} a = \{x \cup \{y\} : x \in {}^n a, y \in \{n\} \times a\} = \mathcal{F}_3({}^n a, \{n\} \times a)$, BS proves the existence of ${}^n a$ by induction on $n \in \omega$. Next observe that ${}^{<n} a = z$

is Δ_0 since $x \in {}^{<n}a$ as well as $x \in {}^ma$ is Δ_0 and for $n > 0$, ${}^{<n}a = z$ iff $z \subset {}^{<n}a$ and $\{\emptyset\} = {}^0a \subset z$ and $\forall m < n-1 \forall x \in z \cap {}^ma \forall b \in a[x \cup \{\langle m, b \rangle\}] \in z$. Therefore $z = {}^na$ iff $z = (({}^{<n+1}a) \setminus ({}^{<n}a))$.

2.2.4. For Σ_1 -formula $\exists x \theta(m, x)$ with Δ_0 -matrix θ , BS proves that $\forall m < n \exists x \theta(m, x) \leftrightarrow \exists y \forall m < n \exists x \in y \theta(m, x)$ by induction on $n \in \omega$.

2.2.5. Let the function f be defined from Σ_1^{BS} -functions g, h by $f(0, \vec{x}) = g(\vec{x})$ and $f(n+1, \vec{x}) = h(n, \vec{x}, f(n, \vec{x}))$. Then $f(n, \vec{x}) = y$ iff there exists a function F with $\text{dom}(F) = n+1$ such that $F(0) = g(\vec{x})$, $\forall i < n [F(i+1) = h(i, \vec{x}, F(i))]$ and $y = F(n)$. By induction on $n \in \omega$ BS proves $\forall n \in \omega \forall \vec{x} \exists! y [f(n, \vec{x}) = y]$. Moreover $f(n, \vec{x}) = y$ is Σ_1 by Proposition 2.2.4.

2.2.6. From Proposition 2.2.5 we see that $(n, a) \mapsto \cup^{(n)}a$ is a Σ_1^{BS} -function, where $\cup^{(0)}a = a$ and $\cup^{(n+1)}a = \cup(\cup^{(n)}a)$. Hence $x \in \text{trcl}(a) \Leftrightarrow \exists n \in \omega (x \in \cup^{(n)}a) \Leftrightarrow \forall b (\cup b \subset b \wedge a \subset b \rightarrow x \in b)$. \square

From Proposition 2.2 we see that BS can encode syntax, e.g., formulae in the language $\{\in, =\}$. Let $[Fml] \subset \omega$ denote the set of codes $[\varphi]$ of formulae φ in $\{\in, =\}$.

We can assume that $[Fml]$ is Δ_1^{BS} , and manipulations on it, e.g., $([\varphi], [\psi]) \mapsto [\varphi \vee \psi]$, $[\varphi \vee \psi] \mapsto \langle [\varphi], [\psi] \rangle$, are all Σ_1^{BS} . Moreover for $x \in [Fml]$, let $\text{var}(x)$ denote the set $\{n \in \omega : v_n \text{ occurs freely in } x\}$, and $\text{ass}(x, y)$ the set of function $f : \text{var}(x) \rightarrow y$. Both $x \mapsto \text{var}(x)$ and $(x, y) \mapsto \text{ass}(x, y)$ are Σ_1^{BS} -functions. Let $\models [\varphi][a]$ denote the satisfaction relation for formulae φ and $a \in \text{ass}([\varphi], y)$ for a y .

For formula φ in $\{\in, =\}$, $[SbFML](\varphi)$ denotes the finite set of codes of subformulae of φ .

Lemma 2.3 *For each formula φ in the language $\{\in, =\}$, the satisfaction relation $\{(x, a) : x \in [SbFML](\varphi), a \in \text{ass}(x), \models x[a]\}$ for subformulae of φ is BS-definable in such a way that BS proves that $\varphi(v_0, \dots, v_{m-1}) \leftrightarrow \models [\varphi(v_0, \dots, v_{m-1})][a]$ for $a(i) = v_i$, $\models [\varphi_0 \vee \varphi_1][a] \leftrightarrow \models [\varphi_0][a_0] \vee \models [\varphi_1][a_1]$ for $a_i = a \upharpoonright \text{var}([\varphi_i])$ and subformulae φ_i , $\models [\exists v_m \varphi][a] \leftrightarrow \exists b \models [\varphi][a \cup \{\langle m, b \rangle\}]$ for subformula $\exists v_m \varphi$, and similarly for \wedge, \forall .*

Proof. It suffices to Δ_1^{BS} -define the satisfaction relation for subformulae of a given Δ_0 -formula φ . This is seen as in [19], p.613 using Propositions 2.2.3 and 2.2.4. Note that we don't need the existence of transitive closures to bound range y of the assignments $a : \text{var}(x) \rightarrow y$ since there are only finitely many subformulae of the given φ . \square

3 Codes

Let us define a class *Code* of codes and a binary relation \prec on it recursively. It is shown that the transfinite induction schema with respect to \prec is provable in

BS up to *each* code.

The class *Code* of codes together with the relation \prec is essentially a notation system of ‘ordinals’ whose order type is the next epsilon number to the order type of the class of ordinals in the universe V . To define such a notation system, we need at least ordinal addition $\alpha + \beta$ and exponentiation with base, say ω , ω^α at hand. However BS is too weak to Δ_1 -define $\alpha + \beta$ and ω^α , since it lacks Δ_0 -Collection. In other words, the order type Λ of the class of ordinals in the well founded universe $V \models \text{BS}$ need not to be an epsilon number nor even an additive principal number, which is closed under $\alpha + \beta$. Indeed, $L_\alpha \models \text{BS}$ for any limit ordinal α .

Instead of Δ_0 -Collection, we collect formal expressions called *products* $\bar{a}_1 \times \cdots \times \bar{a}_n$ of codes \bar{a}_i for $a_i \in V \cup \{V\}$ first, and then collect formal expressions called *sums* $\alpha_1 \# \cdots \# \alpha_n$ of products α_i . Intuitively $\#$ denotes the natural (commutative) sum, and \times the natural product, if the code \bar{a} is replaced by the ordinal $2\text{rank}(a)$. Each sum is defined to be smaller than a code Ω , which is interpreted as the least additive principal number $(\Lambda + 1)^\omega$ above Λ . Then introduce formal expressions $\Omega^\alpha \beta$, which is intended to be an exponential function. These three operations \times , $\#$ and $(\alpha, \beta) \mapsto \Omega^\alpha \beta$ on codes are needed in the ordinal assignment to proofs defined in Definition 4.5. The relation \prec on codes is well founded, but not a linear ordering. For our proof-theoretic analysis, the linearity of \prec is dispensable, the base Ω can be replaced by 2, and \bar{a} by $\text{rank}(a)$. Definitions 3.2 and 3.5 simplify the matters.

First let us define a class *Sum* and a relation \prec_p on it. $\ell(\alpha)$ is the *length* of $\alpha \in \text{Sum}$.

Definition 3.1 Let $\bar{a} := \langle 0, a \rangle$ for $a \in V$, and $\bar{V} := \langle 1, 0 \rangle$. $\ell(\bar{a}) := 0$.

1. A *product* is either $\bar{1}$ or $\bar{a}_1 \times \cdots \times \bar{a}_n := \langle 2, \bar{a}_1, \dots, \bar{a}_n \rangle$ for $a_1, \dots, a_n \in V \cup \{V\}$ with $a_i \neq 0, 1$ and $n > 0$. *Prod* denotes the class of all products.
 $\ell(\bar{a}_1 \times \cdots \times \bar{a}_n) = \max\{\ell(\bar{a}_1), \dots, \ell(\bar{a}_n)\} + 1$. When $n = 0$, let $\alpha_1 \times \cdots \times \alpha_n := \bar{1}$.
2. A *sum of products* is a set $\alpha_1 \# \cdots \# \alpha_n := \langle 3, \alpha_1, \dots, \alpha_n \rangle$ with $\alpha_i \in \text{Prod}$ and $n \geq 0$. *Sum* denotes the class of all sums of products.
 $\ell(\alpha_1 \# \cdots \# \alpha_n) = \max\{\ell(\alpha_1), \dots, \ell(\alpha_n)\} + 1$.
When $n = 0$, let $\alpha_1 \# \cdots \# \alpha_n := \bar{0}$.

Prod is a subclass of *Sum*.

Let us introduce some operations and ‘computation rules’ on sums.

1. \times and $\#$ are defined to be commutative, i.e., $\alpha_1 \# \cdots \# \alpha_n = \alpha_{\pi(1)} \# \cdots \# \alpha_{\pi(n)}$ and $\alpha_1 \times \cdots \times \alpha_n = \alpha_{\pi(1)} \times \cdots \times \alpha_{\pi(n)}$ for any permutation $\pi \in n!$.
These means that $\alpha_1 \# \cdots \# \alpha_n$ and $\alpha_1 \times \cdots \times \alpha_n$ are actually multisets of products and codes \bar{a} .

2. $\bar{0}$ is the zero element. $\alpha_1 \# \cdots \# \alpha_n \# \bar{0} = \alpha_1 \# \cdots \# \alpha_n$, and $\alpha_1 \times \cdots \times \alpha_n \times \bar{0} := \bar{0}$.
3. $\bar{1}$ is the unit for \times , $\alpha_1 \times \cdots \times \alpha_n \times \bar{1} := \alpha_1 \times \cdots \times \alpha_n$.
4. Associative laws $(\alpha_1 \# \cdots \# \alpha_n) \# (\beta_1 \# \cdots \# \beta_m) := \alpha_1 \# \cdots \# \alpha_n \# \beta_1 \# \cdots \# \beta_m$ for $\{\alpha_1, \dots, \alpha_n\} \cup \{\beta_1, \dots, \beta_m\} \subset Prod$, and $(\alpha_1 \times \cdots \times \alpha_n) \times \beta := \alpha_1 \times \cdots \times \alpha_n \times \beta$ for $\{\alpha_1, \dots, \alpha_n\} \cup \{\beta\} \subset \{\bar{a} : a \in V\} \cup \{\bar{V}\}$.
5. Distributive laws $(\alpha_1 \# \cdots \# \alpha_n) \times \beta := (\alpha_1 \times \beta) \# \cdots \# (\alpha_n \times \beta)$ where either $\{\alpha_1, \dots, \alpha_n\} \subset Prod$, $\beta \in \{\bar{a} : a \in V\} \cup \{\bar{V}\}$.

Therefore any combination of products by $\#$ and \times is equal to (reduced to) a sum of products.

Definition 3.2 $\alpha \prec_p \beta$ for $\alpha, \beta \in Sum$ is defined recursively as follows. Let $\alpha \preceq_p \beta :\Leftrightarrow (\alpha \prec_p \beta) \vee (\alpha = \beta)$.

1. $\bar{0} \prec_p \alpha$ for any sum $\alpha \neq \bar{0}$.
2. For $\gamma \in Prod$, $\{\gamma_i : 0 \leq i \leq n\} \subset Prod \cup \{\bar{0}\}$ and $n > 0$,

$$a \in^{(n)} b \in V \cup \{V\} \& \forall i \leq n (\gamma_i \preceq_p \gamma) \Rightarrow (\gamma_0 \times \bar{a}) \# \gamma_1 \# \cdots \# \gamma_n \prec_p \gamma \times \bar{b}$$
where $a \in^{(n+1)} b :\Leftrightarrow a \in \cup^{(n)} b$ with $\cup^{(0)} b = b$ and $\cup^{(n+1)} b = \cup(\cup^{(n)} b)$.
3. $\alpha_0 \prec_p \alpha_1 \& \beta_0 \preceq_p \beta_1 \Rightarrow \alpha_0 \# \beta_0 \prec_p \alpha_1 \# \beta_1$.

Definition 3.2.2 says that if $a = a_n \in a_{n-1} \in \cdots \in a_1 \in b$, then $(\gamma \times a) \# (\gamma \cdot n) \prec_p \gamma \times b$ for $\gamma \cdot n = \gamma \# \cdots \# \gamma$ (n times γ).

Proposition 3.3 The relation \prec_p on Sum is transitive.

Next let us define the class of codes $Code$. $\ell(\alpha)$ is the *length* of $\alpha \in Code$.

Definition 3.4 1. $\bar{0} \in Code$. $\ell(\bar{0}) = 0$.

2. $PCode \subset Code$.
3. $\alpha \in Code \& \bar{0} \neq \beta \in Sum \Rightarrow \Omega^\alpha \beta := \langle 4, \alpha, \beta \rangle \in PCode$.
 $\ell(\Omega^\alpha \beta) = \max\{\ell(\alpha), \ell(\beta)\} + 1$.
4. $\alpha_1, \dots, \alpha_n \in PCode \& n > 0 \Rightarrow \alpha_1 \# \cdots \# \alpha_n := \langle 3, \alpha_1, \dots, \alpha_n \rangle \in Code$.
 $\ell(\alpha_1 \# \cdots \# \alpha_n) = \max\{\ell(\alpha_1), \dots, \ell(\alpha_n)\} + 1$.

Let us introduce some operations and ‘computation rules’ on codes.

1. Again $\#$ is defined to be commutative, and $\bar{0}$ is the zero element.
2. $\bar{1}$ is the unit, $\Omega^\beta := \Omega^\beta \bar{1}$, $\Omega^{\bar{0}} = \bar{1}$ and $\alpha_1 \times \cdots \times \alpha_n := \alpha_1 \times \cdots \times \alpha_n \times \bar{1}$ for $\alpha_i \in \{\bar{a} : a \in V\} \cup \{\bar{V}\}$. Also $\alpha = \Omega^{\bar{0}} \alpha$. Thus $Sum \subset Code$.

3. When $n = 1$, $\alpha_1 \# \dots \# \alpha_n$ is identified with $\alpha_1 \in PCode$.
4. Exponential law $\Omega^\gamma(\Omega^\beta \alpha) := \Omega^{\gamma \# \beta} \alpha$ for $\alpha \in Sum$.
5. Associative laws for $\#$ and Distributive laws $\Omega^\beta(\alpha_1 \# \dots \# \alpha_n) := \Omega^\beta \alpha_1 \# \dots \# \Omega^\beta \alpha_n$ where $\{\alpha_1, \dots, \alpha_n\} \subset PCode$.

Next we define a binary relation \prec on *Code* recursively as follows.

Definition 3.5 1. $\bar{0} \prec \alpha$ for any code $\alpha \neq \bar{0}$.

2. Let $\bar{0} \notin \{\beta_i : i < n\} \cup \{\beta\} \subset Sum$ and $\{\alpha_i : i < n\} \cup \{\alpha\} \subset Code$ with $\alpha_i \neq \alpha_j$ ($i \neq j$).

$$\forall i < n [(\alpha_i, \beta_i) \prec_{lex} (\alpha, \beta)] \Rightarrow \Omega^{\alpha_0} \beta_0 \# \dots \# \Omega^{\alpha_{n-1}} \beta_{n-1} \prec \Omega^\alpha \beta$$

where for codes $\alpha, \gamma \in Code$ and sums $\beta, \delta \in Sum$

$$(\alpha, \beta) \prec_{lex} (\gamma, \delta) :\Leftrightarrow \alpha \prec \gamma \text{ or } (\alpha = \gamma \ \& \ \beta \prec_p \delta).$$

3. $\alpha_0 \prec \alpha_1 \ \& \ \beta_0 \preceq \beta_1 \Rightarrow \alpha_0 \# \beta_0 \prec \alpha_1 \# \beta_1$.

The following Proposition 3.6 is easily seen.

Proposition 3.6 1. Both *Code* and \prec are Δ_1^{BS} .

2. The relation \prec on *Code* is transitive.
3. For $\alpha, \beta \in Sum$, $\alpha \prec_p \beta \Leftrightarrow \alpha \prec \beta$, where $\alpha = \Omega^{\bar{0}} \alpha$.
4. $\bar{0}$ is the least element.
5. For $a \in b \in V \cup \{V\}$, sums of products $\gamma, \gamma', \delta \preceq \gamma \Rightarrow (\gamma \times a) \# \gamma' \prec (\gamma \times b) \# \delta$.
6. $\beta \prec \alpha \# \beta$ if $\alpha \neq \bar{0}$.
7. $\gamma \prec \alpha \# \beta \Rightarrow \gamma \prec \alpha \vee \exists \beta_0 \prec \beta (\gamma = \alpha \# \beta_0)$.
8. $\delta \prec \bar{a}_1 \times \dots \times \bar{a}_n \Rightarrow \exists i \leq n \exists b \in a_i [\delta \preceq (\bar{b} \prod_{j \neq i} \bar{a}_j) \# \prod_{j \neq i} \bar{a}_j]$ for $\delta \in Sum$ and $a_i \in V \cup \{V\}$.
9. Both $(\alpha, \beta) \mapsto \alpha \# \beta$ and $(\alpha, \beta) \mapsto \Omega^\alpha \beta$ are monotonic in each argument.
10. $\alpha, \beta \prec \Omega^\alpha \beta$ if $\alpha, \beta \neq \bar{0}$.
11. $\alpha_1, \alpha_2 \prec \beta \Rightarrow \Omega^{\alpha_1} \# \Omega^{\alpha_2} \prec \Omega^\beta$.
12. $\beta_0 \prec \beta \Rightarrow \Omega^{\beta_0}(\alpha \# \alpha) \prec \Omega^\beta \alpha$.

For a binary relation $<$ and formulae φ , let

$$\begin{aligned} \text{Prg}[\varphi, <] & :\Leftrightarrow \forall x(\forall y < x \varphi(y) \rightarrow \varphi(x)) \\ \text{TI}[\varphi, <, a] & :\Leftrightarrow \text{Prg}[\varphi, <] \rightarrow \forall x < a \varphi(x) \\ \text{TI}[<, a] & := \{ \text{TI}[\varphi, <, a] : \varphi \text{ is a formula} \} \end{aligned}$$

$T \vdash \text{TI}[<, a]$ means that $T \vdash \text{TI}[\varphi, <, a]$ for any formula φ , and $T \vdash \text{TI}[<, a] \rightarrow \text{TI}[<, b]$ means that for any φ there exists a formula ψ such that $T \vdash \text{TI}[\psi, <, a] \rightarrow \text{TI}[\varphi, <, b]$.

Lemma 3.7 For each code $\alpha \in \text{Code}$, $\text{BS} \vdash \text{TI}[\prec, \alpha]$.

Lemma 3.7 is shown by metainduction on the length $\ell(\alpha)$ of codes α using the following Proposition 3.8.

Proposition 3.8 1. $\text{BS} \vdash \text{TI}[\prec_p, \alpha] \wedge \text{TI}[\prec_p, \beta] \rightarrow \text{TI}[\prec_p, \alpha \# \beta]$. Similarly for \prec .

2. $\text{BS} \vdash \text{TI}[\prec_p, \bar{V}]$, i.e., $\text{BS} \vdash \text{Prg}[\varphi, \prec_p] \rightarrow \forall a \in V \cup \{V\} \varphi(\bar{a})$ for any formula φ .

3. For any formula φ , $\text{BS} \vdash \forall n < \omega \forall \{a_i\}_{i < n} \subset V \cup \{V\} [\forall i < n (\forall x \prec \prod_{j \neq i} \bar{a}_j \varphi(x) \wedge \forall y \prec_p \bar{a}_i \forall x \prec y \prod_{j \neq i} \bar{a}_j \varphi(x)) \rightarrow \forall x \prec \prod_{i < n} \bar{a}_i \varphi(x)]$.

4. $\text{BS} \vdash \forall \alpha \in \text{Sum TI}[\prec_p, \alpha]$, i.e., $\text{BS} \vdash \text{Prg}[\varphi, \prec_p] \rightarrow \forall \alpha \in \text{Sum} \varphi(\alpha)$ for any formula φ .

5. $\text{BS} \vdash \text{Prg}[\varphi, \prec] \wedge \forall \alpha_0 \prec \alpha \forall \gamma \forall x \prec \Omega^{\alpha_0} \gamma \varphi(x) \wedge \forall \gamma \prec_p \beta \forall x \prec \Omega^\alpha \gamma \varphi(x) \rightarrow \forall x \prec \Omega^\alpha \beta \varphi(x)$.

6. $\text{BS} \vdash \text{TI}[\prec, \alpha] \rightarrow \text{TI}[\prec, \Omega^\alpha \beta]$.

Proof. 3.8.1. This follows from the fact $\text{BS} \vdash \text{Prg}[\varphi, \prec_p] \rightarrow \text{Prg}[\varphi_{\alpha \#}, \prec_p]$ for $\varphi_{\alpha \#}(x) :\Leftrightarrow \varphi(\alpha \# x)$ using Proposition 3.6.7.

3.8.2. This is seen from Foundation schema.

3.8.3. This is seen from Propositions 3.6.8 and 3.8.1.

3.8.4. By Proposition 3.8.1 it suffices to show $\text{BS} \vdash \text{Prg}[\varphi, \prec_p] \rightarrow \forall \alpha \in \text{Prod} \varphi(\alpha)$ for any formula φ . We show this by induction on the number n of components in products $\bar{a}_1 \times \dots \times \bar{a}_n$. The case $n = 1$, $\text{Prg}[\varphi, \prec_p] \rightarrow \forall a \in V \cup \{V\} \varphi(\bar{a})$ follows from Proposition 3.8.2. Let Prod_n denote the class of all products such that the number of components is at most n . Suppose $\text{Prg}[\varphi, \prec_p]$ and $\forall \alpha \in \text{Prod}_n \varphi(\alpha)$ for a formula φ . Let $a_i \in V \cup \{V\}$. Then by Proposition 3.8.3 we have $\forall i < n+1 \forall y \prec_p \bar{a}_i \forall x \prec_p y \prod_{j \neq i} \bar{a}_j \varphi(x) \rightarrow \forall x \prec_p \prod_{i < n+1} \bar{a}_i \varphi(x)$. In other words, $\forall i < n \forall y \prec_p \bar{a}_i \forall x \prec_p y \prod_{j \neq i} \bar{a}_j \varphi(x) \rightarrow \text{Prg}[\varphi_n, \prec_p]$, where $\varphi_{n-k}(y) :\Leftrightarrow \forall x \prec_p y \prod_{j \neq n-k} \bar{a}_j \varphi(x)$. Thus by 3.8.2 we have $\forall i < n \forall y \prec_p$

$\bar{a}_i \forall x \prec_p y \prod_{j \neq i} \bar{a}_j \varphi(x) \rightarrow \forall x \prec_p \prod_{i < n+1} \bar{a}_i \varphi(x)$. In this way we see $\forall i < n+1 - k \forall y \prec_p \bar{a}_i \forall x \prec_p y \prod_{j \neq i} \bar{a}_j \varphi(x) \rightarrow \forall x \prec_p \prod_{i < n+1} \bar{a}_i \varphi(x)$ by induction on $k \leq n+1$. Hence $\forall x \prec_p \prod_{i < n+1} \bar{a}_i \varphi(x)$, i.e., $\forall \alpha \in \text{Prod}_{n+1} \varphi(\alpha)$.

3.8.5. This is seen from Proposition 3.6.8 and Definition 3.5.

3.8.6. Suppose $\text{Prg}[\varphi, \prec]$ and $\forall \alpha_0 \prec \alpha \forall \beta \forall x \prec \Omega^{\alpha_0} \beta \varphi(x)$. Then by Proposition 3.8.5 we have $\text{Prg}[\varphi_{\Omega^\alpha}, \prec_p]$, where $\varphi_{\Omega^\alpha}(\beta) :\Leftrightarrow \forall x \prec \Omega^\alpha \beta \varphi(x)$. Hence by Proposition 3.8.4 $\forall \beta \forall x \prec \Omega^\alpha \beta \varphi(x)$. Thus we have shown $\text{Prg}[\varphi, \prec] \rightarrow \text{Prg}[\text{j}[\varphi], \prec]$, where $\text{j}[\varphi](\alpha_0) :\Leftrightarrow \forall \beta \forall x \prec \Omega^{\alpha_0} \beta \varphi(x)$. Hence by $TI[\prec, \alpha]$ we have $\forall \beta \forall x \prec \Omega^\alpha \beta \varphi(x)$. \square

Lemma 3.7 is now seen by metainduction on the length $\ell(\alpha)$ of codes α using Propositions 3.8.1, 3.8.4 and 3.8.6.

4 Finitary analysis of $\text{FiX}^i(T)$

When the set theory T is sufficiently strong, e.g., when T comprises Kripke-Platek set theory, we could prove Theorem 1.2 as in [5], i.e., first the finitary derivations of set-theoretic sentences φ in $\text{FiX}^i(T)$ are embedded to infinitary derivations of a sequent $\theta \Rightarrow \varphi$ for a provable sentence θ in T , then partial cut-elimination is possible. This results in a Δ_1 -definable infinitary derivation of the same sequent $\theta \Rightarrow \varphi$ in which there occur no fixed point formulae. The depth of the derivation is bounded by an exponential ordinal tower. Then transfinite induction shows that $\theta \Rightarrow \varphi$ is true. By formalizing the infinitary arguments straightforwardly in T we would see that the end formula φ is true in T . To formalize the infinitary analysis in a weaker theory T , we need a finitary treatment of it as in [14].

Let us take another route in terms of Gentzen-Takeuti's finitary analyses of finite derivations as in [20] since its formalization in a weak (set) theory is a trivial matter.

In what follows we work in a set theory $T \supset \text{BS}$.

$\alpha, \beta, \gamma, \dots$ range over codes in Code , while a, b, c, \dots over sets in the universe V . A, B, C, \dots denote formulae in the language $\mathcal{L}_V := \{\in, =, Q\} \cup \{\bar{a} : a \in V\}$, where $\bar{a} := \langle 0, a \rangle$ is the name (individual constant) for the set a . A term is either a name or a variable. ι, ν, \dots denote terms.

Let us introduce a sequent calculus for transfinite induction schema (2) and the fixed point axiom (1). Logical connectives are $\vee, \wedge, \rightarrow, \exists, \forall$. $\neg A := (A \rightarrow \perp)$.

A *sequent* is a pair of a finite set Γ of formulae, and a formula A , denoted $\Gamma \Rightarrow A$. Its intended meaning is the implication $\bigwedge \Gamma \rightarrow A$. Γ is the *antecedent*, and A the *succedent* of the sequent $\Gamma \Rightarrow A$. For finite sets Γ, Δ and a formula A , $\Gamma, \Delta := \Gamma \cup \Delta$ and $\Gamma, A := \Gamma \cup \{A\}$.

\perp stands ambiguously for false atomic sentences $\bar{a} \in \bar{b}$ for $a \notin b$, and $\bar{a} = \bar{b}$ for $a \neq b$.

The *initial sequents* are

$$\Gamma, \iota = \nu, A(\iota) \Rightarrow A(\nu); \quad \Gamma, \perp \Rightarrow A$$

The *inference rules* are (LQ) , (RQ) , $(L\vee)$, $(R\vee)$, $(L\wedge)$, $(R\wedge)$, $(L\rightarrow)$, $(R\rightarrow)$, $(L\exists)$, $(R\exists)$, $(L\forall)$, $(R\forall)$, (cut) , $(chain)$, (ind) , (Rep) and (E) .

$$\frac{\Gamma, Q(\iota), \mathcal{Q}(Q, \iota) \Rightarrow C}{\Gamma, Q(\iota) \Rightarrow C} (LQ); \quad \frac{\Gamma \Rightarrow \mathcal{Q}(Q, \iota)}{\Gamma \Rightarrow Q(\iota)} (RQ)$$

$$\frac{\Gamma, A_0 \vee A_1, A_0 \Rightarrow C \quad \Gamma, A_0 \vee A_1, A_1 \Rightarrow C}{\Gamma, A_0 \vee A_1 \Rightarrow C} (L\vee); \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} (R\vee) \quad (i = 0, 1)$$

$$\frac{\Gamma, A_0 \wedge A_1, A_i \Rightarrow C}{\Gamma, A_0 \wedge A_1 \Rightarrow C} (L\wedge) \quad (i = 0, 1); \quad \frac{\Gamma \Rightarrow A_0 \quad \Gamma \Rightarrow A_1}{\Gamma \Rightarrow A_0 \wedge A_1} (R\wedge)$$

$$\frac{\Gamma, A \rightarrow B \Rightarrow A \quad \Gamma, A \rightarrow B, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C} (L\rightarrow); \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} (R\rightarrow)$$

$$\frac{\Gamma, \exists x B(x), B(y) \Rightarrow C}{\Gamma, \exists x B(x) \Rightarrow C} (L\exists); \quad \frac{\Gamma \Rightarrow B(\iota)}{\Gamma \Rightarrow \exists x B(x)} (R\exists)$$

The *eigenvariable* y in $(L\exists)$ does not occur in the lower sequent $\Gamma, \exists x B(x) \Rightarrow C$.

$$\frac{\Gamma, \forall x B(x), B(\iota) \Rightarrow C}{\Gamma, \forall x B(x) \Rightarrow C} (L\forall); \quad \frac{\Gamma \Rightarrow B(y)}{\Gamma \Rightarrow \forall x B(x)} (R\forall)$$

The *eigenvariable* y in $(R\forall)$ does not occur in the lower sequent $\Gamma \Rightarrow \forall x B(x)$.

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C}{\Gamma, \Delta \Rightarrow C} (cut)$$

where A is the *cut formula* of the (cut) .

$$\frac{\Gamma_k \Rightarrow A_k \quad \cdots \quad \Gamma_1 \Rightarrow A_1 \quad \Delta, A_k, \dots, A_1 \Rightarrow C}{\Gamma, \Delta \Rightarrow C} (chain)$$

where $\Gamma = \Gamma_k \cup \dots \cup \Gamma_1$, and A_k, \dots, A_1 ($k > 0$) is a non-empty list of *strictly positive* formulae.

The inference rule $(chain)$ is a series of several (cut) 's with the strictly positive *cut formulae* A_k, \dots, A_1 . Writing $\mathbf{\Gamma}$ for the list $\Gamma_k, \dots, \Gamma_1$ and \mathbf{A} for the list A_k, \dots, A_1 , the inference rule is denoted

$$\frac{\mathbf{\Gamma} \Rightarrow \mathbf{A} \quad \Delta, \mathbf{A} \Rightarrow C}{\mathbf{\Gamma}, \Delta \Rightarrow C} (chain)$$

$$\frac{\Gamma, \forall y \in x A(y) \Rightarrow A(x) \quad A(\iota), \Gamma \Rightarrow C \quad \Gamma \Rightarrow \iota \in \nu}{\Gamma \Rightarrow C} \text{ (ind)}$$

The *eigenvariable* x does not occur in the lower sequent $\Gamma \Rightarrow C$.

$$\frac{\Gamma \Rightarrow A}{\Gamma, \Delta \Rightarrow A} \text{ (Rep)}; \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A} \text{ (E)}$$

This inference rule (E) is called the height rule in [1], and its meaning is explained in Definition 4.4 as in [14].

A *proof* in this sequent calculus is a finite labelled tree according to the above initial sequents and inference rules. s, t, u, \dots denote the nodes in proof trees. $s : \Gamma \Rightarrow A$ indicates that the sequent $\Gamma \Rightarrow A$ is the label of the node s . The label $\Gamma \Rightarrow A$ of s is denoted $Seq(s)$.

Suppose that a $\{\in, =\}$ -sentence φ is provable in $\text{FiX}^i(T)$. Then there exists a T -provable sentence θ such that the sequent $\theta \Rightarrow \varphi$ is provable in the sequent calculus. In what follows fix φ, θ and a proof P_0 of $\theta \Rightarrow \varphi$.

Definition 4.1 A proof in the sequent calculus is said to enjoy the *pure variable condition* if

1. any eigenvariables (of $(L\exists), (R\forall), (ind)$) are distinct from each other,
2. any eigenvariable does not occur in its end sequent, and
3. if a free variable occurs in an upper sequent of an inference rule but not in the lower sequent, then the variable is one of the eigenvariables of the inference rule.

Without loss of generality we can assume that any proof enjoys the pure variable condition. Otherwise rename the eigenvariables to satisfy (1) and (2) in Definition 4.1, then replace the redundant free variables by an individual constant, e.g., the empty set $\bar{\emptyset}$ to satisfy (3).

Definition 4.2 The *end-piece* of a proof tree P is a collection of nodes in P such that any inference rule below it is one of $(cut), (chain), (Rep)$ and (E) .

If a proof enjoys the pure variable condition and its end sequent consists solely of sentences, no free variable occurs in its end-piece.

Definition 4.3 The *depth* $dp(A) < \omega$ of a formula A is defined as follows.

1. $dp(A) = 0$ if A is Q -free, i.e., the fixed point predicate Q does not occur in A .

In what follows consider the case when Q occurs in A .

2. $dp(A) = 2$ if A is strictly positive (with respect to Q).

In what follows consider the case when Q occurs in A , and A is not strictly positive.

3. $dp(A) = \max\{dp(A_0), dp(A_1)\} + 1$ if $A \equiv (A_0 \vee A_1), (A_0 \wedge A_1), (A_0 \rightarrow A_1)$.
4. $dp(A) = dp(A_0) + 1$ if $A \equiv (\exists x A_0), (\forall x A_0)$.

Note that $dp(A) \neq 1$.

Let P be a proof in the sequent calculus, and s a node in the proof tree P . We assign the *height* $h(s; P) < \omega$ recursively as follows.

Definition 4.4 1. $h(s; P) = 0$ if $Seq(s)$ is the end sequent of P .

In what follows let $Seq(s)$ be an upper sequent of an inference rule J with the lower sequent $Seq(s_0)$.

2. $h(s; P) = h(s_0; P) + 1$ if J is an (E) .
3. $h(s; P) = \max\{h(s_0; P), 2\}$ if J is a $(chain)$ with its *rightmost* upper sequent $Seq(s)$.

$$\frac{s : \Gamma \Rightarrow A \quad s : \Delta, A \Rightarrow C}{s_0 : \Gamma, \Delta \Rightarrow C} \text{ (chain)}$$

4. $h(s; P) = h(s_0; P)$ in all other cases.

Note that for upper sequents $s = s_k, \dots, s_1$ of a $(chain)$ other than the rightmost one s , we have $h(s_i; P) = h(s_0; P)$, i.e., the height is the same. A proof P is said to be *height-normal* if the following four conditions hold.

1. For any $(chain)$ occurring in P

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C}{s : \Gamma, \Delta \Rightarrow C} \text{ (chain)}$$

$h(s; P) = 0$, in other words there is neither (E) nor no rightmost upper sequent of $(chain)$ below any $(chain)$.

2. For any (cut) occurring in P

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C}{s : \Gamma, \Delta \Rightarrow C} \text{ (cut)}$$

$$h(s; P) \geq dp(A).$$

3. For any (ind) occurring in P

$$\frac{\Gamma, \forall y \in x A(y) \Rightarrow A(x) \quad A(\iota), \Gamma \Rightarrow C \quad \Gamma \Rightarrow \iota \in \nu}{s : \Gamma \Rightarrow C} \text{ (ind)}$$

$$h(s; P) \geq dp(\forall y \in \nu A(y)).$$

4. Any $(chain)$ and (E) in P is in the end-piece.

Without loss of generality we can assume that the given sequent calculus proof P_0 of $\theta \Rightarrow \varphi$ does not contain any (*chain*), and is height-normal. Otherwise add some inference rules (*E*) at the end of the proof.

Let P be a height-normal proof in the sequent calculus, and s be a node in the proof tree P . We assign a code $o(s; P) \in \text{Code}$ recursively as follows. For $n > 0$, $\bar{1} \cdot n := \bar{1} \# \dots \# \bar{1}$ with n times $\bar{1}$'s.

Definition 4.5 1. $o(s; P) = \bar{1} \cdot 2$ if S is an initial sequent.

In what follows let $\text{Seq}(s)$ be the lower sequent of an inference rule J with its upper sequents $\{s_i : \text{Seq}(s_i)\}_{i < m}$.

2. $o(s; P) = o(s_0; P) \# \bar{1}$ if J is one of the inference rules (LQ), (RQ), (RV), ($L\wedge$), ($R\rightarrow$), ($L\exists$), ($R\exists$), ($L\forall$), and ($R\forall$).
3. $o(s; P) = o(s_0; P) \# o(s_1; P)$ if J is one of the inference rules (LV), ($R\wedge$), and ($L\rightarrow$).
4. $o(s; P) = o(s_0; P) \# o(s_1; P)$ if J is a (*cut*).
5. $o(s; P) = \Omega_2(o(s_{m-1}; P))(o(s_0; P) \# \dots \# o(s_{m-2}; P))$ if J is a (*chain*) where $\Omega_2(\alpha) := \Omega^{\Omega^\alpha}$:

$$\frac{s_0 : \Gamma_0 \Rightarrow A_0 \quad \dots \quad s_{m-2} : \Gamma_{m-2} \Rightarrow A_{m-2} \quad s_{m-1} : \Delta, A_0, \dots, A_{m-2} \Rightarrow C}{s : \Gamma, \Delta \Rightarrow C} \text{ (chain) } J$$

with $\Gamma = \bigcup_{i < m-1} \Gamma_i$.

6. $o(s; P) = ((o(s_0; P) \# \bar{1} \cdot 6) \times mj(\nu)) \# o(s_1; P) \# o(s_2; P)$ if J is an (*ind*):

$$\frac{s_0 : \Gamma, \forall y \in x A(y) \Rightarrow A(x) \quad s_1 : A(\iota), \Gamma \Rightarrow C \quad s_2 : \Gamma \Rightarrow \iota \in \nu}{s : \Gamma \Rightarrow C} \text{ (ind) } J$$

where for terms ν ,

$$mj(\nu) := \begin{cases} \bar{a} & \text{if } \nu = \bar{a} \text{ with } a \in V \\ \bar{V} & \text{if } \nu \text{ is a variable} \end{cases}$$

7. $o(s; P) = o(s_0; P)$ if J is a (*Rep*).

8. $o(s; P) = \Omega^{o(s_0; P)}$ if J is an (*E*).

Finally let $o(P) = o(s_{\text{end}}; P)$ for the end sequent s_{end} of P .

The role of operations $\#$, \times and $\Omega^\alpha \beta$ in 'ordinal' assignment $o(s; P)$ are as follows. The sum $\alpha \# \beta$ collects two subproofs together, and \times is needed to multiply ν in transfinite induction (*ind*) up to ν , cf. **Case 2** in section 5. Exponentiation is used first in the rule (*E*), i.e., to measure an increase of ordinal depths in lowering cut rank, and second in the rule (*chain*). The assignment $\Omega^{\Omega^\alpha}(\alpha_k \# \dots \# \alpha_1)$ in (*chain*) comes from Lemma 9 in [5], which in turn is

inspired by the quick cut-elimination strategy in [4, 17] along Kleene-Brouwer ordering of infinitary derivations. Lexicographic comparing, i.e., multiplication of Ω^{Ω^α} and $\alpha_k \# \dots \# \alpha_1$ is used in **Case 9**, and a doubly exponential Ω^{Ω^α} is needed in **Case 6** and **Case 7**, once multiplications are introduced. Note that when exponent α decreases, one can duplicate multiplier β in $\Omega^\alpha \beta$, cf. Proposition 3.6.12.

Since any *(chain)* and *(E)* in P is in the end-piece, $o(s; P)$ is in *Sum* if s is above the end-piece.

A formula in \mathcal{L}_V is said to be an *instance* of a formula A if it is obtained from A by substituting terms for free variables.

Definition 4.6 $ISbfml(P_0)$ denotes the class of all instances of subformulae of formulae occurring in P_0 .

Call a proof *restricted* (with respect to P_0) if it is height-normal, enjoys the pure variable condition, any formula occurring in it is in $ISbfml(P_0)$, and its end sequent consists solely of Q -free sentences.

For $\alpha \in Code$ let $\tau(\alpha)$ denote the formula stating that for any restricted proof P if $o(P) \preceq \alpha$, then its end sequent is true. Note here that the satisfaction relation for the Q -free formulae in $Sbfml(P_0)$ (the set of subformulae of formulae occurring in P_0) or equivalently the partial truth definition for the Q -free sentences in $ISbfml(P_0)$ is BS-definable, a fortiori T -definable by Lemma 2.3.

We show the following Lemma 4.7.

Lemma 4.7 T proves that $\tau(\alpha)$ is progressive, i.e.,

$$T \vdash \forall \alpha \in Code [\forall \beta \prec \alpha \tau(\beta) \rightarrow \tau(\alpha)].$$

Then Theorem 1.2 is seen as follows. Lemmata 3.7 and 4.7 yields $\tau(o(P_0))$, and hence the end sequent $\theta \Rightarrow \varphi$ of P_0 is true in T . Therefore $T \vdash \varphi$.

5 Proof of Lemma 4.7

In this section we show the Lemma 4.7. We work in T .

Let P be a restricted proof of a sequent $\Gamma_0 \Rightarrow A_0$. Suppose as the IH(=Induction Hypothesis) that the end sequents of restricted proofs with smaller codes are true. We need to show that $\Gamma_0 \Rightarrow A_0$ is true. It suffices to show that there are restricted proofs P_i ($i \in I$) of sequents S_i such that $o(P_i) \prec o(P)$ for any $i \in I$ and if all of S_i are true, then so is $\Gamma_0 \Rightarrow A_0$.

Case 1. The case when there exists an initial sequent in the end-piece of P .

Since there are no free variables in the end-piece, any initial sequent in it is either $\Lambda, \perp \Rightarrow A$ or $\Lambda, A \Rightarrow A$.

If the end sequent $\Gamma_0 \Rightarrow A_0$ itself is an initial sequent, i.e., $\{\perp, A_0\} \cap \Gamma_0 \neq \emptyset$, then there is nothing to prove. In what follows assume that this is not the case.

Consider first the case that an initial sequent $\Lambda, \perp \Rightarrow A$ is in the end-piece. Then the formula \perp in the antecedent has to vanish somewhere as a cut formula. Let P be the following:

$$\frac{\frac{\Gamma \Rightarrow \mathbf{A} \quad \frac{\frac{\vdots Q}{s_0 : \Gamma \Rightarrow \perp} \quad \frac{\Lambda, \perp \Rightarrow A}{\Delta, \mathbf{A}, \perp \Rightarrow C}}{s : \Gamma, \Gamma, \Delta \Rightarrow C} \text{ (chain)}}{\Gamma_0 \Rightarrow A_0}$$

Let Q_C denote the proof obtained from the subproof Q of $s_0 : \Gamma \Rightarrow \perp$ by replacing \perp by C in the succedents of sequents $\Gamma' \Rightarrow \perp$ in Q . Let P' be the following:

$$\frac{\frac{\vdots Q_C}{s_0 : \Gamma \Rightarrow C}}{s : \Gamma, \Gamma, \Delta \Rightarrow C} \text{ (Rep)} \quad \Gamma_0 \Rightarrow A_0$$

Then it is clear that P' is restricted. Moreover $o(s; P') = o(s_0; P') = o(s_0; P) \prec o(s; P)$ by Propositions 3.6.6 and 3.6.10. Hence $o(P') \prec o(P)$ by Proposition 3.6.9. From IH we see that $\Gamma_0 \Rightarrow A_0$ is true.

The case when \perp vanishes at a (*cut*) is similar.

Next consider the case that an initial sequent $\Lambda, A \Rightarrow A$ is in the end-piece. Then one of the formulae A has to vanish somewhere as a cut formula of J , which is either a (*chain*) or a (*cut*). Suppose J is a (*chain*), and let P be one of the followings:

$$\frac{\frac{\Gamma \Rightarrow \mathbf{A} \quad \frac{\frac{\vdots Q}{s_0 : \Gamma \Rightarrow A} \quad \frac{\Lambda, A \Rightarrow A}{s_1 : \Delta, \mathbf{A}, A \Rightarrow A}}{s : \Gamma, \Gamma, \Delta \Rightarrow A}}{\Gamma_0 \Rightarrow A_0} J \quad \frac{\frac{\Gamma \Rightarrow \mathbf{A} \quad \frac{\frac{\vdots Q}{s_1 : \Gamma, A \Rightarrow A} \quad \frac{\Lambda, A \Rightarrow A}{s_0 : \Delta, \mathbf{A}, A \Rightarrow C}}{s : \Gamma, \Gamma, A, \Delta \Rightarrow C}}{\Gamma_0 \Rightarrow A_0} J$$

Let P' be the followings:

$$\frac{\frac{\vdots Q}{s_0 : \Gamma \Rightarrow A}}{s : \Gamma, \Gamma, \Delta \Rightarrow A} \text{ (Rep)} \quad \frac{\Gamma \Rightarrow \mathbf{A} \quad \frac{\frac{\vdots Q}{s_0 : \Delta, \mathbf{A}, A \Rightarrow C}}{s : \Gamma, \Gamma, A, \Delta \Rightarrow C}}{J} \quad \Gamma_0 \Rightarrow A_0$$

In the right hand side J denotes two consecutive (E)'s if \mathbf{A} is the empty list, and an (*chain*) otherwise. In each case P' is restricted. Moreover $o(s_0; P') =$

$o(s_0; P)$ and $o(s_1; P) \neq \bar{0}, \bar{1}$. Hence $o(s; P') \prec o(s; P)$ by Proposition 3.6.10 when \mathbf{A} is the empty list in the right hand side, and $o(P') \prec o(P)$. From IH we see that $\Gamma_0 \Rightarrow A_0$ is true.

The case when A vanishes at a (*cut*) is similar.

Case 2. The case when there exists a lower sequent of an (*ind*) in the end-piece of P . Let P be the following:

$$\frac{\frac{\frac{\vdots Q_0(x)}{s_0 : \Gamma, \forall y \in x A(y) \Rightarrow A(x)} \quad \frac{\frac{\vdots Q_1}{s_1 : A(\bar{a}), \Gamma \Rightarrow C} \quad s_2 : \Gamma \Rightarrow \bar{a} \in \bar{b}}{s : \Gamma \Rightarrow C} \quad (ind)}{\Gamma_0 \Rightarrow A_0}$$

If the formula $\bar{a} \in \bar{b}$ is false, i.e., $a \notin b$, then replace $\bar{a} \in \bar{b}$ by C in the succedents of the proof of $s_2 : \Gamma \Rightarrow \bar{a} \in \bar{b}$:

$$\frac{\frac{s_2 : \Gamma \Rightarrow C}{s : \Gamma \Rightarrow C} \quad (Rep)}{\Gamma_0 \Rightarrow A_0}$$

We have $o(s; P) = (\gamma_0 \# \bar{1} \cdot 6) \times \bar{b} \# \gamma_1 \# \gamma_2$ for $\gamma_i = o(s_i; P)$. Since $o(s; P') = o(s_2; P') = o(s_2; P) = \gamma_2 \prec o(s; P)$, we obtain $o(P') \prec o(P)$.

Assume $\bar{a} \in \bar{b}$ is true, and let P' be the following:

$$\frac{\frac{\frac{\vdots Q_0(x)}{s_0 : \Gamma, \forall y \in x A(y) \Rightarrow A(x)} \quad A(z), \Gamma \Rightarrow A(z) \quad \Gamma, z \in a \Rightarrow z \in a}{s' : \Gamma, z \in \bar{a} \Rightarrow A(z)} \quad (ind) \quad \frac{\frac{\frac{\vdots Q_0(a)}{\Gamma \Rightarrow \forall y \in \bar{a} A(y)} \quad (R \rightarrow, R\forall) \quad \frac{\frac{\vdots Q_1}{s_a : \Gamma, \forall y \in \bar{a} A(y) \Rightarrow A(\bar{a})} \quad \vdots Q_1}{s_1 : A(\bar{a}), \Gamma \Rightarrow C}}{\Gamma \Rightarrow A(\bar{a})} \quad (cut)}{\frac{s' : \Gamma, z \in \bar{a} \Rightarrow A(z) \quad \Gamma \Rightarrow A(\bar{a})}{s : \Gamma \Rightarrow C} \quad (cut)} \quad \Gamma_0 \Rightarrow A_0$$

where the proof $Q_0(a)$ is obtained from the subproof $Q_0(x)$ of P by substituting the constant \bar{a} for the eigenvariable x , and renaming free variables for the pure variable condition for P' . The last two inference rules leading to $s : \Gamma \Rightarrow C$ are (*cut*)'s.

It is easy to see that $\gamma'_0 = o(s_a; P') \preceq o(s_0; P) = \gamma_0$ from $\bar{a} \prec mj(x) = \bar{V}$ for $a \in V$ and Proposition 3.6.9. We have $o(s'; P') = (\gamma_0 \# \bar{1} \cdot 6) \times \bar{a} \# \bar{1} \cdot 4$. By Proposition 3.6.5 we have $(\gamma_0 \# \bar{1} \cdot 6) \times \bar{a} \# \bar{1} \cdot 6 \# \gamma'_0 \prec (\gamma_0 \# \bar{1} \cdot 6) \times \bar{b}$. Hence we obtain $o(s; P') = (\gamma_0 \# \bar{1} \cdot 6) \times \bar{a} \# \bar{1} \cdot 6 \# \gamma'_0 \# \gamma_1 \prec (\gamma_0 \# \bar{1} \cdot 6) \times \bar{b} \# \gamma_1 \# \gamma_2 = o(s; P)$. This yields $o(P') \prec o(P)$.

In the following two cases inference rules introducing Q -free formulae and (*cut*) with Q -free cut formulae are pushed down to the end of proofs.

Case 3. The case when there exists a lower sequent of an explicit inference rule in the end-piece of P , where an inference rule J is explicit in P iff its major (principal) formula is in the antecedents (succedents) of any sequent below it when the formula is in the antecedent (succedent) of the lower sequent of J , resp.

Let J be such an inference rule. J is one of the inference rules (LV) , (RV) , $(L\wedge)$, $(R\wedge)$, $(L\rightarrow)$, $(R\rightarrow)$, $(L\exists)$, $(R\exists)$, $(L\forall)$, and $(R\forall)$, but neither of (LQ) and (RQ) , since the fixed point predicate Q does not occur in the end sequent of P .

Consider the cases when J is either an $(R\forall)$ or an $(L\rightarrow)$. For the first case let P be the following:

$$\frac{\begin{array}{c} \vdots Q(y) \\ s_0 : \Gamma \Rightarrow A(y) \end{array}}{s : \Gamma \Rightarrow \forall x A(x)} (R\forall)$$

$$\vdots$$

$$\Gamma_0 \Rightarrow \forall x A(x)$$

For each $a \in V$, let P_a be the following:

$$\frac{\begin{array}{c} \vdots Q(a) \\ s_a : \Gamma \Rightarrow A(\bar{a}) \end{array}}{s : \Gamma \Rightarrow A(\bar{a})} (Rep)$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A(\bar{a})$$

Since $o(s; P_a) = o(s_a; P_a) \preceq o(s_0; P) \prec o(s; P)$, we have $o(P_a) \prec o(P)$. By IH $\Gamma_0 \Rightarrow A(\bar{a})$ is true for any $a \in V$. Hence so is $\Gamma_0 \Rightarrow \forall x A(x)$.

For the second case let P be the following:

$$\frac{\Gamma, B \rightarrow C \Rightarrow B \quad \Gamma, B \rightarrow C, C \Rightarrow A_1}{s : \Gamma, B \rightarrow C \Rightarrow A_1} (L\rightarrow)$$

$$\vdots Q$$

$$\Gamma_0 \Rightarrow A_0$$

where $(B \rightarrow C) \in \Gamma_0$.

Let P_C be the following:

$$\frac{\Gamma, B \rightarrow C, C \Rightarrow A_1}{s : \Gamma, B \rightarrow C, C \Rightarrow A_1} (Rep)$$

$$\vdots$$

$$\Gamma_0, C \Rightarrow A_0$$

Since $o(s; P') \prec o(s; P)$, we obtain $o(P_C) \prec o(P)$, and $\Gamma_0, C \Rightarrow A_0$ is true by IH.

Next let P_B be the following:

$$\frac{\Gamma, B \rightarrow C \Rightarrow B}{s : \Gamma, B \rightarrow C \Rightarrow B} (Rep)$$

$$\vdots Q_B$$

$$\Gamma_0 \Rightarrow B$$

where the trunk Q_B is obtained from the trunk Q of P as follows. If in Q , A_1 vanishes as a cut formula,

$$\frac{s : \Gamma, B \rightarrow C \Rightarrow A_1 \quad \vdots \quad \Gamma_1 \Rightarrow A \quad \Gamma_1, B \rightarrow C \Rightarrow A_1 \quad \Delta, A, A_1 \Rightarrow D}{\Gamma_1, \Gamma_1, B \rightarrow C, \Delta \Rightarrow D} (chain)$$

then this part turns to

$$\frac{s : \Gamma, B \rightarrow C \Rightarrow B \quad \vdots \quad \Gamma_1, B \rightarrow C \Rightarrow B}{\Gamma_1, \Gamma_1, B \rightarrow C, \Delta \Rightarrow B} (Rep)$$

This pruning step is iterated when D vanishes below. Clearly we have $o(P_B) \prec o(P)$, and $\Gamma_0 \Rightarrow B$ is true by IH.

Since both $\Gamma_0, C \Rightarrow A_0$ and $\Gamma_0 \Rightarrow B$ are true, and $(B \rightarrow C) \in \Gamma_0$, so is $\Gamma_0 \Rightarrow A_0$.

Case 4. The case when there exists a cut formula A_1 in the end-piece of P such that A_1 is a Q -free formula.

Let P be the following:

$$\frac{\Gamma \Rightarrow A \quad \Gamma_1 \Rightarrow A_1 \quad \Delta, A, A_1 \Rightarrow C}{\Gamma, \Gamma_1, \Delta \Rightarrow C} (chain)$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A_0$$

Let P_r be the following which is obtained from P as for P_B in the **Case 3**.

$$\frac{\Gamma_1 \Rightarrow A_1}{\Gamma, \Gamma_1, \Delta \Rightarrow A_1} (Rep)$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A_1$$

And let P_ℓ be the following:

$$\frac{\Gamma \Rightarrow A \quad \Delta, A, A_1 \Rightarrow C}{\Gamma, \Gamma_1, \Delta, A_1 \Rightarrow C} J$$

$$\vdots$$

$$\Gamma_0, A_1 \Rightarrow A_0$$

where J denotes two consecutive (E) 's if \mathbf{A} is the empty list, and a *(chain)* otherwise.

Obviously both P_r and P_ℓ are restricted, and $o(P_r), o(P_\ell) \prec o(P)$. IH says that both $\Gamma_0 \Rightarrow A_1$ and $\Gamma_0, A_1 \Rightarrow A_0$ are true. Hence so is $\Gamma_0 \Rightarrow A_0$.

The case when A_1 is a cut formula of a *(cut)* is similar.

Case 5. The case when there exists a *(cut)* J_0 in the end-piece of P such that for its lower sequent $s : \Gamma, \Delta \Rightarrow C$ and cut formula A , $h(s; P) > d := dp(A) > 0$. Let J be the uppermost (E) below J_0 . Note that here is no *(chain)* between J_0 and J since P is height-normal. Let P be the following.

$$\frac{\frac{s_1 : \Gamma \Rightarrow A \quad s_2 : \Delta, A \Rightarrow C}{s : \Gamma, \Delta \Rightarrow C} \text{ (cut) } J_0}{\begin{array}{c} \vdots \\ t : \Gamma_1 \Rightarrow C_1 \\ u : \Gamma_1 \Rightarrow C_1 \end{array} \text{ (E) } J} \Gamma_0 \Rightarrow A_0$$

Let P' be obtained from P by lowering the *(cut)* J_0 below the (E) J :

$$\frac{\frac{\frac{s_1 : \Gamma \Rightarrow A}{s : \Gamma, \Delta \Rightarrow A} \text{ (Rep)} \quad \frac{s_2 : \Delta, A \Rightarrow C}{s : \Gamma, \Delta, A \Rightarrow C} \text{ (Rep)}}{\begin{array}{c} \vdots \\ t_1 : \Gamma_1 \Rightarrow A \\ u_1 : \Gamma_1 \Rightarrow A \end{array} \text{ (E)} \quad \begin{array}{c} \vdots \\ t_2 : \Gamma_1, A \Rightarrow C_1 \\ u_2 : \Gamma_1, A \Rightarrow C_1 \end{array} \text{ (E)}}{u : \Gamma_1 \Rightarrow C_1 \text{ (cut)}} \Gamma_0 \Rightarrow A_0$$

Let $\alpha_i = o(s_i; P) = o(s_i; P')$ for $i = 1, 2$. Then for some β , $o(t; P) = \beta \# \alpha_1 \# \alpha_2$, and $o(u; P) = \Omega^{\beta \# \alpha_1 \# \alpha_2}$. On the other side for some $\beta' \preceq \beta$, $o(t_1; P') = \beta' \# \alpha_1$. The case $\beta' \prec \beta$ happens when a pruning is performed. Also $o(t_2; P') = \beta \# \alpha_2$. Hence $o(u_1; P') \preceq \Omega^{\beta \# \alpha_1}$ and $o(u_2; P') = \Omega^{\beta \# \alpha_2}$. Now we claim that $o(u; P') \preceq \Omega^{\beta \# \alpha_1 \# \Omega^{\beta \# \alpha_2}} \prec \Omega^{\beta \# \alpha_1 \# \alpha_2} = o(u; P)$, which follows from Proposition 3.6.11.

Hence $o(P') \prec o(P)$, and we see that $\Gamma_0 \Rightarrow A_0$ is true from IH.

In the following cases, adjacent *(cut)*'s are first collected into *(chain)*, **Case 6.** This as well as the analysis of strictly positive cut formula in **Case 9** prolongs *(chain)*. In **Case 7**, *(cut)* with strictly positive cut formula is replaced by *(chain)*, thereby *(chain)* is introduced in proofs.

Case 6. The case when there exists a *(cut)* J_0 in the end-piece of P such that its lower sequent $s : \Gamma_1, \Delta_1 \Rightarrow C$ is the rightmost upper sequent of a *(chain)* J .

Let P be the following with $\Delta = \Delta_0 \cup \Delta_1$:

$$\frac{s : \Gamma \Rightarrow \mathbf{A} \quad \frac{s_0 : \Delta_0, \mathbf{A} \Rightarrow A_0 \quad t : \Delta_1, \mathbf{A}, A_0 \Rightarrow C}{\Delta, \mathbf{A} \Rightarrow C} (cut) J_0}{s : \Gamma, \Delta \Rightarrow C} (chain) J$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A_0$$

Since P is height-normal, we have $2 = h(t; P) \geq dp(A_0)$. On the other side $h(t; P) \leq dp(A_0)$ by virtue of **Case 5**. Hence $dp(A_0) = 2$, i.e., the predicate Q occurs in A_0 and A_0 is strictly positive.

Let P' be the following:

$$\frac{s : \Gamma \Rightarrow \mathbf{A} \quad \frac{s_0 : \Delta_0, \mathbf{A} \Rightarrow A_0}{\Gamma, \Delta_0 \Rightarrow A_0} (chain) \quad \frac{t : \Delta_1, \mathbf{A}, A_0 \Rightarrow C}{\Delta_1, \mathbf{A}, A_0 \Rightarrow C} (Rep)}{s : \Gamma, \Delta \Rightarrow C} (chain)$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A_0$$

Observe that $2 = h(s_0; P') = h(s_0; P) = h(t; P) = h(t; P')$. Let $\alpha = o(s; P) = o(s; P')$, $\alpha_0 = o(s_0; P) = o(s_0; P')$ and $\beta = o(t; P) = o(t; P')$. Then $o(s; P) = \Omega_2(\alpha_0 \# \beta) \sum \alpha$ and $o(s; P') = \Omega_2(\beta)(\sum \alpha \# (\Omega_2(\alpha_0) \sum \alpha))$. $o(s; P') \prec o(s; P)$ is seen from Proposition 3.6.11. Hence $o(P') \prec o(P)$, and we see that $\Gamma_0 \Rightarrow A_0$ is true from IH.

Case 7. The case when there exists a *(cut)* with a strictly positive cut formula A in the end-piece of P . Let J be a lowest such *(cut)*. By virtue of **Case 5** we have $h(t; P) = dp(A) = 2$, and by **Case 6** there is no rightmost upper sequent of any *(chain)* below J . Hence there are two consecutive *(E)*'s below J by **Case 4** and **Case 5**. Furthermore the two consecutive *(E)*'s is immediately below the lowest J , i.e., there is no left upper sequent of any *(chain)* between J and *(E)*'s since P is height-normal. Let P be the following:

$$\frac{\frac{u_0 : \Gamma \Rightarrow A \quad u_1 : \Delta, A \Rightarrow C}{t : \Gamma, \Delta \Rightarrow C} (cut) J}{s : \Gamma, \Delta \Rightarrow C} (E)^2$$

$$\vdots$$

$$\Gamma_0 \Rightarrow A_0$$

Let P' be the following:

$$\begin{array}{c}
\vdots \\
\frac{u'_0 : \Gamma \Rightarrow A}{t_0 : \Gamma, \Delta \Rightarrow A} \text{ (Rep)} \\
\frac{s_0 : \Gamma, \Delta \Rightarrow A}{s' : \Gamma, \Delta \Rightarrow C} \text{ (E)}^2 \quad \frac{\vdots}{u'_1 : \Delta, A \Rightarrow C} \text{ (Rep)} \\
\frac{t_1 : \Gamma, \Delta, A \Rightarrow C}{s' : \Gamma, \Delta \Rightarrow C} \text{ (chain)} \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

We have $h(s; P) = h(s'; P') = h(s_0; P') = 0$ and $h(t_0; P') = h(t_1; P') = h(t; P) = 2$. Let $\alpha_i = o(u_i; P)$ for $i = 0, 1$. Then $o(u'_0; P') = \alpha_0$, $o(t_1; P') = o(u'_1; P') = \alpha_1$, and $o(s_0; P') = \Omega_2(\alpha_0)$. Hence $o(s'; P') = \Omega_2(\alpha_1)\Omega_2(\alpha_0) = \Omega^{\Omega^{\alpha_1} \# \Omega^{\alpha_0}} \prec \Omega_2(\alpha_0 \# \alpha_1) = o(s; P)$ by Proposition 3.6.11. Therefore $o(P') \prec o(P)$, and by IH $\Gamma_0 \Rightarrow A_0$ is true.

By virtue of **Case 1-Case 3** we can assume that any topmost sequent in the end-piece of P is a lower sequent of an implicit inference rule other than (ind) , (cut) , $(chain)$, (Rep) and (E) such that the fixed point predicate Q occurs in its major formula. Call temporarily such an inference rule *boundary* of P if its lower sequent is in the end-piece, but not its upper sequents. We then claim that there is an inference J such that J is either a (cut) or a $(chain)$, and one of its cut formula A comes from major formulae of boundaries.

$$\begin{array}{c}
\overline{\Gamma_1 \Rightarrow A} \text{ } J_\ell \quad \overline{\Delta_1, A \Rightarrow C_1} \text{ } J_r \\
\vdots \quad \vdots \\
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow A \quad \Delta, A, A \Rightarrow C}{\Gamma, \Gamma, \Delta \Rightarrow C} J \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

where both J_ℓ and J_r are boundaries, A in their lower sequents are their major formulae, and the formula A is in the succednets [antecedents] of any sequents between J_ℓ and J [between J_r and J], resp.

The claim is seen as in [20] (the existence of a suitable cut).

In what follows pick such rules J , J_ℓ and J_r with the formula A , which is a cut formula of J . By virtue of **Case 7**, J is a (cut) iff $dp(A) > 2$.

Case 8. The case when $dp(A) > 2$ and J is a (cut) . For example consider the

case when A is a formula $\forall x D(x)$. Let P be the following:

$$\begin{array}{c}
\frac{\Gamma_1 \Rightarrow D(y)}{\Gamma_1 \Rightarrow \forall x D(x)} (R\forall) J_\ell \quad \frac{\Delta_1, \forall x D(x), D(a) \Rightarrow C_1}{\Delta_1, \forall x D(x) \Rightarrow C_1} (L\forall) J_r \\
\vdots \qquad \qquad \qquad \vdots \\
\frac{u_0 : \Gamma \Rightarrow \forall x D(x) \quad u_1 : \Delta, \forall x D(x) \Rightarrow C}{s : \Gamma, \Delta \Rightarrow C} (cut) J \\
\vdots \\
\frac{v : \Gamma_2 \Rightarrow B}{t : \Gamma_2 \Rightarrow B} (E) J_0 \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

By virtue of **Case 5** we can assume that $h(s; P) = dp(\forall x D(x)) = d + 1$ with $d = dp(D(a)) > 2$. J_0 denotes the uppermost (E) below J with $h(t; P) = d$.

Let P' be the following:

$$\begin{array}{c}
\frac{\Gamma_1 \Rightarrow D(a)}{\Gamma_1 \Rightarrow D(a)} (Rep) \quad \frac{\Delta_1, \forall x D(x), D(a) \Rightarrow C_1}{\Delta_1, \forall x D(x), D(a) \Rightarrow C_1} (Rep) \\
\vdots \qquad \qquad \qquad \vdots \\
\frac{u'_0 : \Gamma \Rightarrow D(a)}{s_\ell : \Gamma, \Delta \Rightarrow D(a)} (Rep) \quad \frac{u_0 : \Gamma \Rightarrow \forall x D(x) \quad u'_1 : \Delta, \forall x D(x), D(a) \Rightarrow C}{s_r : \Gamma, \Delta, D(a) \Rightarrow C} (cut) \\
\vdots \qquad \qquad \qquad \vdots \\
\frac{v_\ell : \Gamma_2 \Rightarrow D(a)}{t_\ell : \Gamma_2 \Rightarrow D(a)} (E) \quad \frac{v_r : \Gamma_2, D(a) \Rightarrow B}{t_r : \Gamma_2, D(a) \Rightarrow B} (E) \\
\frac{t' : \Gamma_2 \Rightarrow B}{\Gamma_0 \Rightarrow A_0} (cut)
\end{array}$$

We have $o(s; P) = \alpha_0 \# \alpha_1$ where $\alpha_i = o(u_i; P)$ for $i = 0, 1$. On the other hand we have $o(s_\ell; P') = \alpha'_0 = o(u'_0; P') \prec \alpha_0 = o(u_0; P') \prec o(s_r; P')$ and $\alpha'_1 = o(u'_1; P') \prec \alpha_1$. Hence $o(s_\ell; P') \prec o(s_r; P') \prec o(s; P)$, and $o(u_\ell; P') \prec o(u_r; P') \prec o(u; P)$. Thus for $o(t_\ell; P') = \Omega^{o(u_\ell; P')}$, $o(t_r; P') = \Omega^{o(u_r; P')}$, and $\Omega^{o(u; P)} = o(t; P)$, we obtain $o(t'; P') = o(t_\ell; P') \# o(t_r; P') \prec o(t; P)$. Therefore $o(P') \prec o(P)$, and by IH $\Gamma_0 \Rightarrow A_0$ is true.

The other cases are seen similarly.

Case 9. The case when $dp(A) = 2$ and J is a (*chain*).

First consider the case when A is an implicational formula $D \rightarrow E$, where

E is strictly positive and D is Q -free. Let P be the following:

$$\begin{array}{c}
\frac{s_4 : \Gamma_1, D \Rightarrow E}{s_3 : \Gamma_1 \Rightarrow D \rightarrow E} (R \rightarrow) J_\ell \quad \frac{s_6 : \Delta_1, D \rightarrow E \Rightarrow D \quad s_7 : \Delta_1, D \rightarrow E, E \Rightarrow C_1}{s_5 : \Delta_1, D \rightarrow E \Rightarrow C_1} (L \rightarrow) J_r \\
\vdots \quad \vdots \\
\frac{s : \Gamma \Rightarrow A \quad s_1 : \Gamma \Rightarrow D \rightarrow E \quad s_2 : \Delta, A, D \rightarrow E \Rightarrow C}{s_0 : \Gamma, \Gamma, \Delta \Rightarrow C} (chain) J \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

Let P_ℓ be the following:

$$\begin{array}{c}
\frac{s_6 : \Delta_1, D \rightarrow E \Rightarrow D}{\Delta_1, D \rightarrow E \Rightarrow D} (Rep) \\
\vdots \\
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow D \rightarrow E \quad s_{2\ell} : \Delta, A, D \rightarrow E \Rightarrow D}{s_\ell : \Gamma, \Gamma, \Delta \Rightarrow D} (chain) \\
\vdots \\
\Gamma_0 \Rightarrow D
\end{array}$$

Let P_r be the following:

$$\begin{array}{c}
\frac{s_4 : \Gamma_1, D \Rightarrow E}{\Gamma_1, D \Rightarrow E} (Rep) \quad \frac{s_7 : \Delta_1, D \rightarrow E, E \Rightarrow C_1}{\Delta_1, D \rightarrow E, E \Rightarrow C_1} (Rep) \\
\vdots \quad \vdots \\
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow D \rightarrow E \quad s_{1r} : \Gamma, D \Rightarrow E \quad s_{2r} : \Delta, A, D \rightarrow E, E \Rightarrow C}{s_r : \Gamma, \Gamma, \Delta, D \Rightarrow C} (chain) \\
\vdots \\
\Gamma_0, D \Rightarrow A_0
\end{array}$$

Let $\alpha_i = o(s_i; P)$ for $i = 4, 6, 7$. In P , $o(s_3; P) = \alpha_4 \# \bar{1}$, $o(s_5; P) = \alpha_6 \# \alpha_7$, and $o(s_0; P) = \Omega_2(o(s_2; P))(\sum \alpha \# o(s_1; P))$ for $\alpha = o(s; P)$. On the other side in P_ℓ and P_r , $\alpha_6 = o(s_6; P_\ell)$, $\alpha_4 = o(s_4; P_r)$ and $\alpha_7 = o(s_7; P_r)$, and hence $o(s_{2\ell}; P_\ell) \prec o(s_2; P)$, $o(s_{1r}; P_r) \prec o(s_1; P)$ and $o(s_{2r}; P_r) \prec o(s_2; P)$. Moreover $o(s_\ell; P_\ell) = \Omega_2(o(s_{2\ell}; P_\ell))(\sum \alpha \# o(s_1; P))$ and $o(s_r; P_r) = \Omega_2(o(s_{2r}; P_r))(\sum \alpha \# o(s_1; P) \# o(s_{1r}; P_r))$.

We see $o(s_\ell; P_\ell), o(s_r; P_r) \prec o(s_0; P)$ from Proposition 3.6.12. From these we see that $o(P_\ell), o(P_r) \prec o(P)$, and by IH both $\Gamma_0 \Rightarrow D$ and $\Gamma_0, D \Rightarrow A_0$ are true. Therefore $\Gamma_0 \Rightarrow A_0$ is true.

Next consider the case when $A \equiv Q(a)$ for the fixed point predicate Q . Let

P be the following:

$$\begin{array}{c}
\frac{\Gamma_1 \Rightarrow \mathcal{Q}(Q, a)}{\Gamma_1 \Rightarrow Q(a)} (RQ) J_\ell \quad \frac{\Delta_1, Q(a), \mathcal{Q}(Q, a) \Rightarrow C_1}{\Delta_1, Q(a) \Rightarrow C_1} (LQ) J_r \\
\vdots \quad \vdots \\
\frac{s : \Gamma \Rightarrow \mathbf{A} \quad s_1 : \Gamma \Rightarrow Q(a) \quad s_2 : \Delta, \mathbf{A}, Q(a) \Rightarrow C}{s_0 : \Gamma, \Gamma, \Delta \Rightarrow C} (chain) J \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

Let P' be the following:

$$\begin{array}{c}
\frac{\Gamma_1 \Rightarrow \mathcal{Q}(Q, a)}{\Gamma_1 \Rightarrow Q(Q, a)} (Rep) \quad \frac{\Delta_1, Q(a), \mathcal{Q}(Q, a) \Rightarrow C_1}{\Delta_1, Q(a), \mathcal{Q}(Q, a) \Rightarrow C_1} (Rep) \\
\vdots \quad \vdots \\
\frac{s : \Gamma \Rightarrow \mathbf{A} \quad s_1 : \Gamma \Rightarrow Q(a) \quad s'_1 : \Gamma \Rightarrow Q(Q, a) \quad s_2 : \Delta, \mathbf{A}, Q(a), \mathcal{Q}(Q, a) \Rightarrow C}{s_0 : \Gamma, \Gamma, \Delta \Rightarrow C} (chain) \\
\vdots \\
\Gamma_0 \Rightarrow A_0
\end{array}$$

We have $o(s_0; P) = \Omega_2(o(s_2; P))(\sum \alpha \# o(s_1; P))$ for $\alpha = o(\mathbf{s}; P)$. Also $o(\mathbf{s}; P') = o(\mathbf{s}; P)$, $o(s'_1; P') \prec o(s_1; P) = o(s_1; P')$, $o(s_2; P') \prec o(s_2; P)$, and $o(s_0; P') = \Omega_2(o(s_2; P'))(\sum \alpha \# o(s_1; P) \# o(s'_1; P'))$. Hence $o(s_0; P') \prec o(s_0; P)$ from Proposition 3.6.12. Therefore $o(P') \prec o(P)$, and by IH $\Gamma_0 \Rightarrow A_0$ is true.

The other cases are seen similarly. This completes a proof of Lemma 4.7, and of Theorem 1.2.

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