Prefix and plain Kolmogorov complexity characterizations of 2-randomness: simple proofs

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Abstract

Joseph Miller [16] and independently Andre Nies, Frank Stephan and Sebastiaan Terwijn [18] gave a complexity characterization of 2-random sequences in terms of plain Kolmogorov complexity $C(\cdot)$: they are sequences that have infinitely many initial segments with $O(1)$-maximal plain complexity (among the strings of the same length).

Later Miller [17] showed that prefix complexity $K(\cdot)$ can also be used in a similar way: a sequence is 2-random if and only if it has infinitely many initial segments with $O(1)$-maximal *prefix* complexity (which is $n + K(n)$ for strings of length n).

The known proofs of these results are quite involved; in this paper we provide simple direct proofs for both of them.

In [16] Miller also gave a quantitative version of the first result: the $\mathbf{0}'$-randomness deficiency of a sequence ω equals $\liminf_n [n - C(\omega_1 \ldots \omega_n)] + O(1)$. (Our simplified proof can also be used to prove this.) We show (and this seems to be a new result) that a similar quantitative result is also true for prefix complexity: $\mathbf{0}'$-randomness deficiency equals $\liminf_n [n + K(n) - K(\omega_1 \ldots \omega_n)] + O(1)$.

Introduction

The connection between complexity and randomness is one of the basic ideas that motivated the development of algorithmic information theory and algorithmic randomness theory. However, at first the definition of complexity (plain complexity of a bit string, introduced by Ray Solomonoff [21] and Andrei Kolmogorov [11] as the minimal length of a program that produces this string) and the definition of randomness (given by Per Martin-Löf [15]) were given separately, and only later some connections between them became clear.

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Leonid Levin [13, 9] and later Gregory Chaitin [6] introduced a modified version of complexity, called *prefix* complexity and denoted usually by $K(\cdot)$, that corresponds to self-delimiting programs. It turned out (see the papers of Claus-Peter Schnorr [19], Levin [12], Chaitin [6]) that a bit sequence $\omega = \omega_1\omega_2\ldots$ is Martin-Löf random if and only if $\sup_n[n - K(\omega_1\ldots\omega_n)]$ is finite. Moreover, this supremum coincides with randomness deficiency (a quantitative version of Martin-Löf definition of randomness suggested by Levin and Peter Gacs, see [10]).

Let us recall the definition of randomness deficiency since it is less known compared to other notions of algorithmic information theory. By Ω we denote the Cantor space of infinite bit sequences.

- A *basic function* is a function $f\colon \Omega \to \mathbb{Q}^+$ whose value $f(\omega)$ is a non-negative rational number that depends on a *finite* initial prefix of ω of some length. Basic functions are constructive objects, so we can speak about computable sequences of basic functions.

- A *lower semicomputable function* is a function $f\colon \Omega \to \overline{\mathbb{R}}^+$ (values are non-negative reals and $+\infty$) that is a pointwise upper bound of a computable sequence of basic functions. Equivalent definition: a sum $\sum h_i(\cdot)$ where $h_i(\cdot)$ is a computable sequence of basic functions.

- A *randomness test* is a lower semicomputable function t such that the integral $\int t(\omega)dP(\omega)$ does not exceed 1. (Here P is the uniform Bernoulli measure on Cantor space that corresponds to independent fair coin tossings.)

- There exists a *universal* randomness test $u(\omega)$ that exceeds every other one (up to $O(1)$-factor). We fix some universal randomness test $\mathbf{u}$. Its logarithm $\log \mathbf{u}(\omega)$ is called the *randomness deficiency* of ω and denoted by $\mathbf{d}(\omega)$. The randomness deficiency is defined up to $O(1)$-additive term since different universal tests differ at most by a bounded factor.

The quantitative version of Schnorr–Levin theorem says that

$$\mathbf{d}(\omega) = \sup_n [n - K(\omega_1\ldots\omega_n)] + O(1).$$

So we can give an equivalent definition of randomness deficiency just as the supremum in the right-hand side of this equation.

This statement looks a bit counterintuitive. One can expect that a sequence is random if its initial segments (prefixes) have maximal possible complexity (among all strings of the same length). But the maximal prefix complexity for n-bit strings is $n + K(n)$, not n, up to $O(1)$ additive term. So why we compare $K(\omega_1\ldots\omega_n)$ to n, not to $n + K(n)$? Or why we consider prefix complexity and not the plain one, for which the maximal complexity of n-bit string is indeed n?

The obstacle here is an old Martin-Löf observation: for *every* sequence ω the difference $n - C(\omega_1\ldots\omega_n)$, as well as the difference $n + K(n) - K(\omega_1\ldots\omega_n)$, is unbounded. There are some workarounds, still: for example, instead of requiring that $n - C(\omega_1 \ldots \omega_n)$ is bounded for *all* n, we can require it to be bounded for *infinitely many* n, i.e., consider sequences such that $\liminf_n [n - C(\omega_1 \ldots \omega_n)]$ is finite.[1] It is easy to see that indeed this lim inf is finite for almost all sequences (except for a set of zero measure). What are these sequences?

The answer was found by Joseph Miller [16] and independently by Andre Nies, Frank Stephan and Sebastian Terwijn [18]. They proved that this class of sequences coincides with the class of 2-*random sequences*, i.e., the sequences that are Martin-Löf random even with an oracle for $\mathbf{0}'$ (the halting problem). The proof in [16] is quite involved, and the proof in [18] uses special tools from recursion theory (the low basis theorem). Some other approach was suggested in [3], and later Chris Conidis [7] showed that one can avoid low basis theorem in this way. Still Conidis' argument is a bit complicated. In Section 1 we provide a simple proof of Conidis' result thus giving a simple proof of Miller–Nies–Stephan-Terwijn characterization of 2-random sequences. Extending this argument and using an effective version of Fatou lemma, we get also a new simple proof for a quantitative version of this characterization from [16]:

$$\liminf [n - C(\omega_1 \ldots \omega_n)] = \mathbf{d}^{\mathbf{0}'}(\omega) + O(1).$$

In the right-hand side $\mathbf{d}^{\mathbf{0}'}$ stands for the randomness deficiency relativized to $\mathbf{0}'$; this deficiency is finite when ω is 2-random.

Later Miller [17] got a similar result for prefix complexity: a sequence ω is 2-random if and only if ω has infinitely many initial segments with $O(1)$-maximal *prefix* complexity (which is $n + K(n)$ for strings of length n), i.e., if

$$\liminf [n + K(n) - K(\omega_1 \ldots \omega_n)]$$

is finite. The original proof was even more complicated than the proof for plain complexity; it used van Lambalgen theorem about random pairs, Kučera – Slaman result about random lower semicomputable reals and some other tools. Some simplifications were found by Laurent Bienvenu and others (see Downey and Hirschfeldt [8]), but even with these simplifications the proof remains quite difficult. In Section 2 we present a much simpler proof.

Finally, in Section 3 we show that this result also has a quantitative version, thus completing the picture:

$$\begin{aligned} \mathbf{d}^{\mathbf{0}'}(\omega) &= \sup n - K^{\mathbf{0}'}(\omega_1 \ldots \omega_n) + O(1) \\ &= \liminf [n - C(\omega_1 \ldots \omega_n)] + O(1) \\ &= \liminf [n + K(n) - K(\omega_1 \ldots \omega_n)] + O(1)\,. \end{aligned}$$

It is not clear whether this quantitative version can be extracted from Miller's argument. One can raise the question whether the same initial segments have

[1] The other (may be, more natural) approach is to consider the so-called *monotone* complexity, or *a priori* complexity, that do not have this problem. We do not consider these complexities in our paper.

maximal plain or prefix complexity. In an upcomming paper we show this is not the case: for every 3-random sequence, there exist a c and infinitely many prefixes x such that $n - C(x) \leq c$ and $n + K(n) - K(x) \geq \log\log n - c$.

Section 1 and 2–3 are (mostly) independent, so the readers interested only in plain or prefix complexity can proceed directly to the corresponding part of the paper.

1 Plain complexity and 2-randomness

This section is devoted to the Miller–Nies–Stephan–Terwijn characterization of 2-random sequences in terms of plain complexity, and it's quantified form:

Theorem 1 (Miller)**.**

$$\mathbf{d}^{\mathbf{0}'}(\omega) = \liminf[n - C(\omega_1 \ldots \omega_n)] + O(1)\,.$$

First let us reproduce the proof of the easy direction ($\leq$). We assume that $\mathbf{d}^{\mathbf{0}'}(\omega)$ equals d, and show that $n - C(\omega_1 \ldots \omega_n) \geq d - O(1)$ for sufficiently large n. Since

$$\mathbf{d}^{\mathbf{0}'}(\omega) = \limsup n - K^{\mathbf{0}'}(\omega_1, \ldots, \omega_n)$$

(we omit $O(1)$ terms here and later) we may assume that

$$K^{\mathbf{0}'}(\omega_1 \ldots \omega_m) \leq m - d$$

for some m. Then we can use the additivity property[2] for plain complexity [1],

$$C(a, b) = K(a|\, C(a, b)) + C(b|a, C(a, b)),$$

for $a = \omega_1 \ldots \omega_m$ and $b = \omega_{m+1} \ldots \omega_n$. Then we have

$$C(\omega_1 \ldots \omega_n) \leq C(a, b) \leq K(a|\, C(a, b)) + C(b|\, C(a, b)).$$

The second term does not exceed $|b|$, i.e., $n - m$; it is enough to show, therefore, that the first term is bounded by $m - d$, i.e., by $K^{\mathbf{0}'}(\omega_1, \ldots, \omega_m)$. Indeed, the condition $C(a, b)$ tends to infinity as $n \to \infty$, and $\lim_N K(x|N) \leq K^{\mathbf{0}'}(x)$. (Indeed, we can approximate $\mathbf{0}'$ making N steps of enumeration, and for large N this is enough.)

Now we switch to the other direction ($\geq$). The qualitative version says that a sequence ω such that $n - C(\omega_1 \ldots \omega_n) \to \infty$, is not $\mathbf{0}'$-random, and we start by proving this version. So let us assume that $C(\omega_1 \ldots \omega_n) < n - c$ for all sufficiently large n. To show that ω is not Martin-Löf $\mathbf{0}'$-random, we need to cover ω by a $\mathbf{0}'$-effectively open set of small measure (uniformly).

Consider the set U_n of sequences α such that $C(\alpha_1 \ldots \alpha_n) < n - c$. This is an effectively open set (uniformly in n) that has measure at most 2^{-c} (since

[2] The direction ($\leq$) that we need is quite simple: $C(a, b) = C(a, b|\, C(a, b))$, and $C(u, v|w) \leq K(u|w) + C(v|w)$ by concatenation of the programs.

there are less than 2^{n-c} strings of complexity less than $n-c$). We know that our sequence ω belongs to all U_n for sufficiently large n (but we do not know the threshold for "sufficiently large"). It remains to apply the following result of Conidis [7] (for its applications and discussion see also [3] where this statement was mentioned as a conjecture, and the revised version [4]).

Theorem 2 (Conidis)**.** *Let $\varepsilon > 0$ be a rational number and let $U_0, U_1, \ldots$ be a sequence of uniformly effectively open sets of measure at most ε each. Then for every rational $\varepsilon' > \varepsilon$ there exists a $\mathbf{0}'$-effectively open set V of measure at most ε' that contains $\liminf_{n\to\infty} U_n = \bigcup_N \bigcap_{n\ge N} U_n$, and the $\mathbf{0}'$-enumeration algorithm for V can be effectively found given ε, ε', and the enumeration algorithm for U_i.*

Proof. Let us denote by $U_{k..l}$ the intersection $U_k \cap U_{k+1} \cap \ldots \cap U_l$. The set V will be constructed as $U_{1..k_1} \cup U_{k_1+1..k_2} \cup \ldots$ for some $\mathbf{0}'$-computable sequence $k_1 < k_2 < \ldots$; this guarantees that V is $\mathbf{0}'$-effectively open and that $\liminf U_i \subset V$. It remains to explain how we choose k_i such that V has measure at most ε'.

Let us fix an increasing computable sequence $\varepsilon < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon'$. There exists some k_1 such that for every $i > k_1$ the set

$$U_{1..k_1} \cup U_i$$

has measure at most ε_1. Indeed, if for some i the measure is greater than ε_1, then, adding U_i as a new term in the intersection (by increasing k_1 up to i), we decrease the measure of the intersection at least by $\varepsilon_1 - \varepsilon$. (If $A \cup B$ has measure greater than $\varepsilon_1 > \varepsilon$ while B itself thas measure at most ε, then $A \setminus B$ has measure at least $\varepsilon_1 - \varepsilon$, so the measure of A decreases at least by $\varepsilon_1 - \varepsilon$ after intersecting it with B.) If the newly found k_i does not satisfy the condition, we repeat the process. Each time this happens, the measure of the intersection decreases by at least $\varepsilon_1 - \varepsilon$, hence this can happen only finitely many times.

For similar reasons we can then find k_2 such that for every i the set

$$U_{1..k_1} \cup U_{k_1+1..k_2} \cup U_i$$

has measure at most ε_2 for every $i > k_2$. Indeed, the size of $U_{1..k_1} \cup U_i$ is bounded by ε_1, hence if the measure of the set above exceeds ε_2, then there is at least a $(\varepsilon_2 - \varepsilon_1)$-part of $U_{k_1+1..k_2}$ outside $U_{1..k_1} \cup U_i$ (in particular, outside U_i). Thus adding U_i as a new term in the intersection $U_{k_1+1..k_2}$ decreases its measure by at least $\varepsilon_2 - \varepsilon_1$; such a decrease may happen only finitely many times.

We continue this construction for k_3, k_4 etc. Note that this construction is $\mathbf{0}'$-computable and the union

$$V = U_{1..k_1} \cup U_{k_1+1..k_2} \cup U_{k_2+1..k_3} \cup \ldots$$

is an $\mathbf{0}'$-effectively open cover of $\liminf U_n$ of measure at most ε'. $\square$

A more careful analysis of this argument allows us to get the statement of Theorem 1 in weak form, with logarithmic precision. So we need to modify the argument. First, we formulate a version of Conidis' theorem with functions instead of sets (that also can be considered as a constructive version of Fatou's lemma).

Theorem 3. *Let $f_1, f_2, \ldots$ be a series of uniformly lower semicomputable functions on Cantor space such that $\int f_i(\omega)\, d\mu(x)$ does not exceed some rational $\varepsilon > 0$ for all i. Then for every $\varepsilon' > \varepsilon$ one can uniformly construct a lower $\mathbf{0}'$-semicomputable function φ such that*

$$\liminf\, f_n(\omega) \le \varphi(\omega) \quad \textit{for every } \omega, \quad \textit{and} \quad \int \varphi(\omega) d\mu(\omega) \le \varepsilon'.$$

We get the original Conidis' result when f_i are indicator functions of open sets. In fact, the proof remains almost the same. For each function f_i we consider the set U_i below its graph, i.e., the set of pairs (ω, u) in $\Omega \times \mathbb{R}$ such that $0 \le u \le f_i(\omega)$. The measure of this set equals $\int f_i(\omega)\, d\omega$. The intersection/union operations with these sets correspond to min/max operations with the functions. So the same construction as before gives the function

$$\varphi(\omega) = \sup(f_{1..k_1}(\omega), f_{k_1+1..k_2}(\omega), \ldots)$$

where

$$f_{k..l}(\omega) = \min(f_k(\omega), f_{k+1}(\omega), \ldots, f_l(\omega)).$$

It is easy to see that $\liminf_n f_n(\omega) \le \varphi(\omega)$ (note that lim inf operation on functions corresponds to the same operation on sets). Also functions $f_{i..j}$ are lower semicomputable (minimum of a finite family of lowersemicomputable functions is lower semicomputable), and the function φ is semicomputable with an oracle that computes the sequence k_i.

Theorem 3 is proved.

Now we use this theorem to show that if $C(\omega_1 \ldots \omega_n) < n - c$ for large n, then $\mathbf{d}^{\mathbf{0}'}(\omega) \ge c - O(1)$. For that we need to construct a $\mathbf{0}'$-lower semicomputable randomness test that exceeds 2^c on all those ω.

One may try to let $f_n(\omega)$ be equal to $2^{n - C(\omega_1 \ldots \omega_n)}$. Then for all ω in question we have $f_n(\omega) > 2^c$ for large n, and $\liminf f_n(\omega) \ge 2^c$. If the integrals $\int f_n(\omega)\, d\omega$ were bounded, we could finish the proof by applying Theorem 3. However, it is not the case: we know that $f_n(\omega)$ exceeds 2^k on a set of measure at most 2^{-k} (for every k), but this is not enough for the integral bound.

To fix the problem, we change the definition of f_n. For a binary string u, let us define the function $\chi_{x\Omega}$ that equals 1 on the extensions of x and equals 0 otherwise. Its integral is $2^{-|x|}$. Multiplying this function by $2^{|x|-m}$ for some m, we get a function with integral 2^{-m}. Then consider the sum

$$f_m(\omega) = \sum_{\{x \mid C(x) < m\}} 2^{|x|-m} \chi_{x\Omega}.$$

This sum contains less than 2^m terms; each has integral 2^{-m}, so the integral of the sum is bounded by 1. On the other hand, if $C(\omega_1 \ldots \omega_n) < n - c$ for all large enough c, the sum for $f_m(\omega)$ includes a term of size at least 2^c for all sufficiently large m.

This observation finished the proof of Theorem 1.

2 Prefix complexity and 2-randomness

In this section we provide a simple proof of the following result of Miller:

Theorem 4 (Miller)**.** *A sequence ω is 2-random (Martin-Löf random with oracle $\mathbf{0}'$) if and only if* $\liminf_n[n + K(n) - K(\omega_1 \ldots \omega_n)]$ *is finite.*

In the next section we will prove a quantitative version of this result: this lim inf equals $\mathbf{d}^{\mathbf{0}'}(\omega)$, and this will require a more complicated proof. However, in one of the directions the quantitative result is equally simple, so we start with this direction.

Let us prove that $\mathbf{d}^{\mathbf{0}'}(\omega) \le \liminf[n + K(n) - K(\omega_1 \ldots \omega_n)]$. We use almost the same argument as for Theorem 1. Since $\mathbf{d}^{\mathbf{0}'}(\omega)$ is equal to $\liminf_m[m - K^{\mathbf{0}'}(\omega_1 \ldots \omega_m)$ up to $O(1)$ additive term, we assume that $K^{\mathbf{0}'}(\omega_1 \ldots \omega_m) = m - d$ and show that $K(\omega_1 \ldots \omega_n) \le n + K(n) - d + O(1)$ for large n.

Let $a = \omega_1 \ldots \omega_m$ and $b = \omega_{m+1} \ldots \omega_n$. Using the bound for the prefix complexity of a pair $K(u, v) \le K(u) + K(v|u) + O(1)$ (also in the conditional version), we note that (up to $O(1)$-terms)

$$\begin{aligned} K(\omega_1 \ldots \omega_n) &\le K(n) + K(\omega_1 \ldots \omega_n|n) \le \\ &\le K(n) + K(a, b|n) \le \\ &\le K(n) + K(a|n) + K(b|a, n). \end{aligned}$$

It remains to note that

- the last term does not exceed $m-n$ (the condition is enough to reconstruct $m - n$, and the prefix complexity of a string when its length is given, is bounded by this length);
- for sufficiently large n the value of $K(a|n)$ does not exceed $K^{\mathbf{0}'}(a)$ (the required part of $\mathbf{0}'$ can be reconstructed during n enumeration steps).

So, for large n the right-hand side is bounded by

$$K(n) + K^{\mathbf{0}'}(a) + n - m \le K(n) + (m - d) + n - m = n + K(n) - d,$$

as required.

It remains to prove the (qualitative) statement in the other direction:

> *Let ω be a binary sequence such that $K(\omega_1 \ldots \omega_n) - (n + K(n)) \to -\infty$. Then ω is not 2-random.*

It will be done in the rest of the section, in several steps.

2.1 Slow convergence

Let us start with the following simple definition. Let a_i and b_i be two series with non-negative terms. We say that *a_i-tails are bounded by b_i-tails* if

$$(a_N + a_{N+1} + \ldots) \le c(b_N + b_{N+1} + \ldots)$$

for some c and all N. We assume here that $\sum a_i$ converges (but $\sum b_i$ may diverge). Reformulation: a_i-tails are *not* bounded by b_i-tails if the ratio

$$\frac{a_N + a_{N+1} + \ldots}{b_N + b_{N+1} + \ldots}$$

is unbounded.

Examples:

1. Let $\mathbf{m}(i)$ be the (discrete) a priori probability of i, the maximal (up to a constant) lower semicomputable converging series; we may let $\mathbf{m}(i) = 2^{-K(i)}$ (see e.g., [14] or [20]). Then the tails of every convergent computable series $\sum a_i$ are bounded by the tails of the series $\sum \mathbf{m}(i)$. Indeed, $a_i \le O(\mathbf{m}(i))$ implies the same relation for tails.

2. On the other hand, for every lower semicomputable series there exist a computable series with rational terms that has the same limit and has bigger tails (that bound the tails of the first one). Indeed, each lower semicomputable term can be split into a sum of a computable series, and we can add all the summands (for all terms) one by one; this delay can only increase the tails. Therefore, being bounded by tails of some convergent computable series is equivalent to being bounded by the tails of $\sum \mathbf{m}(i)$.

2.2 Lower semicomputable tests and 2-randomness

Remind from the introduction that Martin-Löf randomness can be defined using randomness tests (lower semicomputable non-negative functions on the Cantor space that have integral at most 1, see the Introduction). It turns out that lower semicomputable tests can be used in a more ingenious way to show that some sequence is *not 2-random* (not ML-random relative to the halting problem).

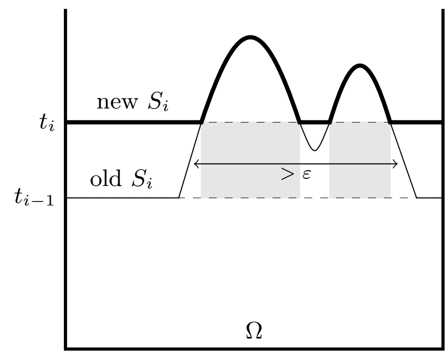
Let $f_i(\cdot)$ be a sequence of (uniformly) lower semicomputable non-negative functions on Ω. Assume that the sum $\sum_i \int f_i$ is finite. Thus $\sum_i f_i(\cdot)$ is a lower semicomputable test, and every sequence ω such that $\sum_i f_i(\omega)$ diverges, is not ML-random. Moreover, the following statement (where both the condition and the claim are weaker) is true:

Lemma 5. *If the tails of the series $\sum_i f_i(\omega)$ are not bounded by any computable series, then ω is not $\mathbf{0}'$-random.*

As we have seen, we may use for comparison the series $\sum_i \mathbf{m}(i)$ instead of computable series.

Proof. Without loss of generality we may assume that $f_1(\cdot)$, $f_2(\cdot)$, $\ldots$ is a computable sequence of basic functions (splitting each semicomputable term into a sum of computable terms, we only increase the tails).

To show that every ω with this property (very slow convergence) is not $\mathbf{0}'$-random, we need to construct for every rational $\varepsilon > 0$ a $\mathbf{0}'$-effectively open set of measure at most ε that covers (all such) ω. This construction goes as follows. Consider computable increasing sequences of basic functions $S_i \colon \Omega \to \mathbb{Q}$ and rational numbers t_i ("thresholds") constructed in the following way. We start with zero function S_0 and zero threshold t_0. Then for each $i = 1, 2, 3, \dots$ we do the following steps:



- First, let $S_i(\omega) = S_{i-1}(\omega) + f_i(\omega)$, and $t_i = t_{i-1}$.
- If after that the measure of the set $\{\omega | S_i(\omega) > t_i\}$ exceeds ε, increase t_i to get rid of this excess (minimally).
- Change S_i as follows: $S_i(\omega) := \max(S_i(\omega), t_i)$

If the two last "correction steps" were omitted, the sequence S_i would converge to $\sum_i f_i$. The correction steps make functions S_i bigger (small values of S_i are replaced by the threshold). Note that the second step is well defined, since S_i is a basic function, and t_i will be one of its finitely many values. The following two invariant relations are easy to check:

- The measure of the set $\{\omega | S_i(\omega) > t_i\}$ is bounded by ε. [Indeed, the second step restores this relation if it was destroyed by the first step, and the third step does not change the set in question, since the inequality is strict.]
- $\varepsilon t_i + \int_\Omega [S_i(\omega) - t_i]\, d\omega \le \sum_{k=0}^{i} \int_\Omega f_k(\omega) d\omega$. [Indeed, the first step increases the integral in the left-hand side by $\int_\Omega f_i$, and two other steps (combined) only decrease the lefthand side (the horizontal sections exceeding ε are replaced by ε, see the illustration).]

Since the right-hand side of the last inequality is bounded by assumption, the sequence t_i is a bounded (computable increasing) sequence, and its limit $T = \lim t_i$ is lower semicomputable (and therefore $\mathbf{0}'$-computable). The limit of S_i is some lower semicomputable function $S(\cdot)$.

Recall that we have to construct a $\mathbf{0}'$-effectively open set of small measure that covers all ω where tails of f_i exceed tails of all converging computable series. This set is defined as the set W_ε of all ω such that $S(\omega) > T$. We need to check that this set works:

- W_ε is $\mathbf{0}'$-effectively open (uniformly in ε), since T is $\mathbf{0}'$-computable and S is lower semicomputable (even without $\mathbf{0}'$-oracle).
- The measure of W_ε does not exceed ε. Indeed, if it does, then the measure of the set $\{\omega|S_i(\omega) > T\}$ would exceed ε for some i, which would immediately make the threshold greater than its limit value T.
- Finally, we need to show that $\omega \in W_\varepsilon$ if the tails of the series $\sum_i f_i(\omega)$ are not bounded by tails of any computable converging series. In our case we compare it with the convergence $t_i \to T$, i.e., with the series $\sum(t_i - t_{i-1})$. Indeed, our assumption guarantees that some tail $f_i(\omega) + f_{i+1}(\omega) + \dots$ exceeds the distance $T - t_{i-1}$, and this implies that $S(\omega) > T$ (since we add $f_n(\omega)$ at each step, starting from the same point t_{i-1}; additional increases are possible, too).

$\square$

2.3 Proof of Theorem 4

Now we are ready to finish the proof of Theorem 4 by applying Lemma 5 to the sum used in Gács' formula for the universal lower semicomputable test. We already mentioned the formula for randomness deficiency:

$$\mathbf{d}(\omega) = \sup_n [n - K(\omega_1 \dots \omega_n)] + O(1).$$

It is convenient to rewrite it in exponential form. Namely, let $\mathbf{m}(x)$ be the universal discrete semimeasure $\mathbf{m}(x) = 2^{-K(x)}$, and let $P(x)$ be the uniform measure of the interval $x\Omega$, i.e., $P(x) = 2^{-|x|}$. Then for the universal test $\mathbf{u}(\omega) = 2^{\mathbf{d}(\omega)}$ we get (up to $O(1)$-factors in both directions)

$$\mathbf{u}(\omega) = \max_{x \prec \omega} \frac{\mathbf{m}(x)}{P(x)}$$

where the maximum is taken over prefixes x of ω. Gacs [10] showed not only this formula, but also a similar formula where maximum is replaced by sum:

$$\mathbf{u}(\omega) = \sum_{x \prec \omega} \frac{\mathbf{m}(x)}{P(x)}$$

(See [2] for the details.) In fact, we only need to know that the right hand side of this formula has finite integral. For a fixed x the integral of the corresponding term is $\mathbf{m}(x)$, so the entire integral is $\sum_x \mathbf{m}(x) \le 1$.

To prove Theorem 4, we apply Lemma 5 to the sequence

$$f_i(\omega) = \mathbf{m}(x)/P(x) = 2^{i - K(\omega_1 \dots \omega_i)}$$

and our assumption says that the ratio $f_i(\omega)/\mathbf{m}(i)$ tends to infinity. (Recall that $\mathbf{m}(i) = 2^{-K(i)}$.) So the tails of the series $f_i(\omega)$ are not bounded by the tails of the series $\mathbf{m}(i)$ and therefore not bounded by tails of any computable converging series (being maximal, $\sum \mathbf{m}(i)$ has $O(1)$-bigger tails). The theorem is proven.

3 Prefix-free complexity: the quantitative result

This section is devoted to the quantitative version of the result of the previous section.

Theorem 6.

$$\mathbf{d}^{\mathbf{0}'}(\omega) = \liminf_i [i + K\,(i) - K\,(\omega_1 \dots \omega_i)] + O(1)\,.$$

In the previous section we already proved the $\le$-inequality; now we need to prove the reverse one. This follows from Lemma 7 and in its proof we use a quantitative version of Lemma 5.

Lemma 7. *Let $f_i(\cdot)$ be a series of lower semicomputable functions on the Cantor space such that $\sum_i \int f_i < \infty$. Then there exist a $\mathbf{0}'$-lower-semicomputable function $Q(\cdot)$ on Cantor space with finite integral such that*

$$\liminf_i \left[\frac{f_i(\omega)}{\mathbf{m}(i)}\right] \le O(Q(\omega))\,.$$

The $\ge$-inequality of Theorem 6 then follows from this lemma if we let (as before)

$$f_i(\omega) = \mathbf{m}(\omega_1 \dots \omega_i)/P(\omega_1 \dots \omega_i) = 2^{i - K\,(\omega_1 \dots \omega_i)}\,.$$

The lemma gives us a function $Q(\cdot)$ that is a $\mathbf{0}'$-lower semicomputable test (up to a constant: the integral of Q may exceed 1, but is finite) and

$$\log Q(\omega) \ge \liminf\,[\,(i - K\,(\omega_1 \dots \omega_i)) + K\,(i)\,] + O(1)$$

for every ω. Since $\mathbf{d}^{\mathbf{0}'}(\omega)$ is universal, we get the desired $\ge$-inequality.

It remains to prove Lemma 7. As we have done in Section 2, we convert functions $f_i \colon \Omega \to \mathbb{R}$ to sets in $\Omega \times \mathbb{R}$. Then we apply a version of Lemma 5 (Lemma 8 below) to functions defined on this space.

Let us first explain what are the changes in Lemma 5. We considered a sequence of functions $g_i(x)$ and then the set of points x where the ratios

$$\frac{g_i(x) + g_{i+1}(x) + \dots}{\mathbf{m}(i) + \mathbf{m}(i+1) + \dots}$$

are not bounded (we have changed the notation and write g_i instead of f_i to avoid confusion, since now the lemma is applied not to f_i but to other functions). The change is that now we consider a larger set of points where these ratios are not bounded by some specific constant (1, though any other constant would work), and cover it by a $\mathbf{0}'$-effectively open set of finite measure. (The entire space $\Omega \times \mathbb{R}$ now has infinite measure, so this makes sense.) Here is the exact statement:

Lemma 8. *Consider a sequence of uniformly lower semicomputable non-negative functions* $g_i\colon \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that* $\sum_i \int_{\Omega\times\mathbb{R}_{\geq 0}} g_i$ *is finite, where the integrals are taken with respect to the product of standard measures on Cantor space and* $\mathbb{R}_{\geq 0}$*. Then there exists a* $\mathbf{0}'$*-effectively open set* $W \subseteq \Omega \times \mathbb{R}_{\geq 0}$ *of finite measure that covers all points* z *such that*

$$g_i(z) + g_{i+1}(z) + \ldots > \mathbf{m}(i) + \mathbf{m}(i+1) + \ldots$$

for some i*.*

In this lemma we speak about effectively open sets and lower semicomputable functions for the space $\Omega \times \mathbb{R}_{\geq 0}$, so we need to define them formally. An *effectively open set* is a union of an enumerable family of basic open sets of the form $x\Omega \times (a, b)$ where $x\Omega$ is an interval in the Cantor space and (a, b) is an open interval with rational endpoints; the interval $[0, b)$ can also be used instead of (a, b). A *lower semicomputable function* $g\colon \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ can be defined as a function such that for every rational r the preimage $\{(\omega, u)|g(\omega, u) < r\}$ is effectively open uniformly in r. However, for the proof it is convenient to use an equivalent definition of lower semicomputable functions as pointwise limit of increasing computable sequences of basic functions. Here a *basic* function is a non-negative function $b(\omega, r)$ that depends only on some finite prefix of ω (of some length m) and for each of 2^m values of ω is a piecewise constant function of r that has finite support, and rational breakpoints and values. Such a function is a constructive object, so we can speak about computable sequences of basic functions in which the breakpoints and the number of breakpoints of each basic function are computable. Taking differences, we can also say that a lower semicomputable function is a sum of a series whose terms are basic functions.

Proof. We use the same construction as in the proof of Lemma 5 (see figure 3), but now the threshold ε is large; we will see later how large ε should be. Without loss, we can assume the functions g_i to be computable (rather than lower semicomputable) basic functions defined on $\Omega\times\mathbb{R}_{\geq 0}$; indeed, by delaying terms, the tails only increase, making the statement only stronger. The functions S_i are now basic functions too, and t_i are still rational numbers. Recall the construction: we first add g_i (was f_i) to S_{i-1}, then take minimal t_i such that the set $S_i(\cdot) > t_i$ has measure at most ε, and then let $S_i := \max(S_i, t_i)$. The choice of t_i now is a more difficult task, but since ε is rational, functions S_i are basic, and the set $S_i(\cdot) > t_i$ is non-increasing in t_i, the number t_i is rational and can be computed from i.

The construction of S_i and t_i depend on ε, so we use the notation S_i^ε and t_i^ε for them. The set W^ε where the function $\lim S_i^\varepsilon$ exceeds $T^\varepsilon = \lim t_i^\varepsilon$ is $\mathbf{0}'$-effectively open uniformly in ε. Note that the limit T^ε is finite and the set W^ε has measure at most ε (for every ε) for the same reasons as before; more precisely, $T^\varepsilon = O(1/\varepsilon)$. ($T^\varepsilon\varepsilon \leq \sum_i \int g_i(z)dz \leq O(1)$.) We need only to prove that for some ε the set W^ε contains all the points z such that

$$g_i(z) + g_{i+1}(z) + \ldots > \mathbf{m}(i) + \mathbf{m}(i+1) + \ldots$$

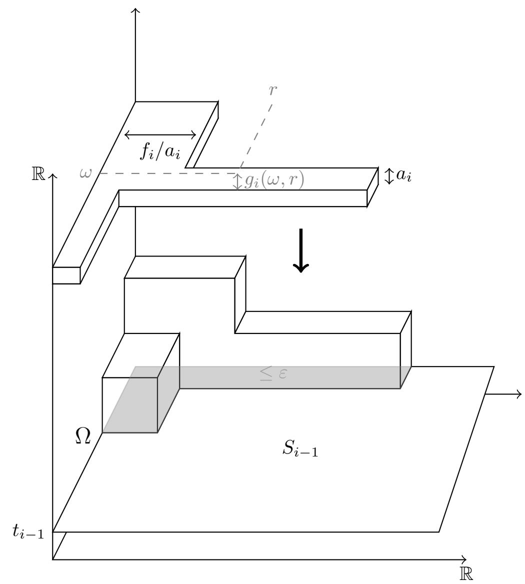


Figure 1: Constructing t_i and S_i, and choice of $g_i(\omega, r)$.

for some i.

This is guaranteed if

$$m(i) + m(i+1) + \ldots \geq \Delta t_i^\varepsilon + \Delta t_{i+1}^\varepsilon + \ldots$$

where Δt_i^ε is defined as the difference $t_i - t_{i-1}$ (in the construction for the corresponding value of ε). We show that $\Delta t_i^\varepsilon \leq m(i)$ for large ε. Since Δt_i^ε is computable (given i and ε) and

$$\sum_i \Delta t_i^\varepsilon = O(1/\varepsilon),$$

we can estimate Δt_i^ε:

$$\Delta t_i^\varepsilon = O(\mathbf{m}(i) 2^{K\,(\varepsilon)}/\varepsilon). \tag{$*$}$$

Indeed, the sum

$$\sum_{\varepsilon,i} 2^{-K\,(\varepsilon)} \varepsilon \Delta t_i^\varepsilon \leq \sum_\varepsilon 2^{-K\,(\varepsilon)} \varepsilon O(1/\varepsilon) = O\big(\sum_\varepsilon 2^{-K\,(\varepsilon)}\big)$$

is finite, so

$$2^{-K\,(\varepsilon)} \varepsilon \Delta t_i^\varepsilon \leq O(\mathbf{m}(i,\varepsilon)) \leq O(\mathbf{m}(i)).$$

Whatever the O-constant in $(*)$ is, we can ensure that $\Delta t_i^\varepsilon < \mathbf{m}(i)$ if we take ε large and simple enough, i.e., $\varepsilon = 2^k$ for large k. As we have seen, such ε finishes the proof of Lemma 8. $\square$

Using this result, we can now prove Lemma 7 (and therefore finish the proof of Theorem 6).

Proof. Let $a(i)$ be a computable sequence of rational numbers that converges slower than $\mathbf{m}(i)$ in the sense that $a(i)+a(i+1)+\ldots>\mathbf{m}(i)+\mathbf{m}(i+1)+\ldots$ for all i. By universality of $\mathbf{m}$, it suffices to prove the statement of the lemma where $\mathbf{m}(n)$ is replaced by $a(n)$, i.e., to construct Q such that

$$Q(\omega) \geq \liminf_i \left[\frac{f_i(\omega)}{a(i)}\right].$$

First we construct the functions $g_i(\omega,u)$ to which Lemma 8 is applied. (Remember that ω is a point in Cantor space, and u is a non-negative real number.) Consider the function $f_i/a(i)$ and the points below its graph, i.e., pairs (ω,u) such that $0\leq u<f_i(\omega)/a(i)$. The area of this "lower-graph" is $\int f_i/a(i)$. Then we consider the indicator function of this set multiplied by $a(i)$: let $g_i(\omega,u)$ be equal to $a(i)$ if $0\leq u<f_i(\omega)/a(i)$ and zero otherwise (see also figure 3). The integral of g_i (over $\Omega\times\mathbb{R}$) equals $\int f_i$, so the sum of integrals is finite. The functions are uniformly lower semicomputable.

Applying Lemma 8, we get a $\mathbf{0}'$-effectively open set $W\subset\Omega\times\mathbb{R}_{\geq 0}$ of finite measure that contains all pairs (ω,u) such that

$$g_i(\omega,u)+g_{i+1}(\omega,u)+\ldots>\mathbf{m}(i)+\mathbf{m}(i+1)+\ldots$$

Note that that includes all points (ω,u) such that

$$0\leq u<\liminf_i \left[\frac{f_i(\omega)}{a(i)}\right].$$

Indeed, for such ω and u the point (ω,u) is under the graph of $f_i/a(i)$ for large enough i, so $g_i(\omega,u)=a_i$ for large enough i and

$$g_i(\omega,u)+g_{i+1}(\omega,u)+\ldots=a(i)+a(i+1)+\ldots>\mathbf{m}(i)+\mathbf{m}(i+1)+\ldots$$

for large enough i.

Now, having the $\mathbf{0}'$-effectively open set W, we define the function Q as a maximal function such that the area under this function is in W:

$$Q(\omega)=\sup\{v|(\omega,u)\in W \text{ for all } u \text{ in } [0,v)\}.$$

Note that this function is lower semicomputable for every effectively open W with the same oracle; the area under its graph is included in W and therefore the integral of Q does not exceed the area of W and is finite. As we already noted, Q is an upper bound for lim inf in question. Lemma 7 is proved. □

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