

Baire Spaces and Infinite Games

Fred Galvin and Marion Scheepers

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Abstract

It is well known that if the nonempty player of the Banach-Mazur game has a winning strategy on a space, then that space is Baire in all powers even in the box product topology. The converse of this implication may also be true: We know of no consistency result to the contrary. In this paper we establish the consistency of the converse relative to the consistency of the existence of a proper class of measurable cardinals.¹

A topological space is said to be *Baire* if the intersection of any countable collection of dense open subsets is dense. This property is associated with game theory through the following classical game, called the *Banach-Mazur game*²:

A topological space X is given. Two players, White (or ONE) and Black (or TWO) play the following game: There is an inning per finite ordinal. White starts the game by choosing a nonempty open set $B_0 \subseteq X$, and Black responds by choosing a nonempty open set $B_1 \subseteq B_0$. In the $n + 1^{\text{st}}$ inning White chooses a nonempty open set $B_{2n+2} \subseteq B_{2n+1}$, and Black responds with a nonempty open set $B_{2n+3} \subseteq B_{2n+2}$, and so on. In this way the two players produce a *play*

$$B_0, B_1, \dots, B_{2n}, B_{2n+1}, \dots \quad (n \in \omega).$$

White wins this play if $\bigcap_{n < \omega} B_n = \emptyset$, and Black wins otherwise. The symbol $\text{BM}(X)$ denotes this game. Two of the central results about this game are:

Theorem 1 (Banach and Mazur, Oxtoby). *X is a Baire space if, and only if, White has no winning strategy in $\text{BM}(X)$.*

Theorem 2 (White [10], Theorem 3 (4)). *If Black has a winning strategy in the Banach-Mazur game on a space, then all powers of that space, considered in the box product topology, are Baire spaces.*

We conjecture that the converse of Theorem 2 is true. This conjecture is the motivation for the rest of the paper.

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²For the history of, and a good bibliography for the Banach-Mazur game, see [8].

In Section 1 of the paper we make the conjecture more precise, and relate it to another conjecture regarding the classical Gale-Stewart game. We formulate a conjecture regarding existence of winning strategies in a variant of the Gale-Stewart game, and show that this second conjecture implies the first one (Theorem 4). In Section 2 we introduce another class of games, invented by Richard Laver, and show how existence of winning strategies in Laver's game implies the existence of winning strategies in the above variants of the Gale-Stewart game (Theorem 6). In Section 3 we introduce yet another class of games based on the cut-and-choose games introduced by Ulam, and show how existence of winning strategies in these games imply existence of winning strategies in Laver's games (Theorem 10). In Section 4 we study a game on ideals and explain Baumgartner's result that existence of winning strategies in the game on ideals implies the existence of winning strategies in the cut-and-choose games (Theorem 13). In Section 5 we show how truth of some instances of the original two conjectures can be obtained from a hypothesis about the existence of certain large cardinals. We discuss how the full conjectures 1 and 2 follow from the hypothesis that there is a proper class of large cardinals of precise type. In Section 6 we return to ZFC and give a number of examples of the scenario that Conjecture 2 is about. Finally, in Section 7, we state some open problems.

We assume the consistency of ZFC, the base theory for this paper. Any other hypotheses or axioms used will be explicitly identified. Also, all products of topological spaces in this paper are assumed to carry the box product topology.

1 Two conjectures and multiboard games

A family \mathcal{B} of nonempty open subsets of a topological space X is said to be a π -base if there is for each nonempty open subset U of X an element $V \in \mathcal{B}$ such that $V \subseteq U$. We define the π -weight, denoted $\pi(X)$, to be the minimal cardinal λ such that X has a π -base of cardinality λ .

For λ and κ positive cardinal numbers, we define:

$\mathbf{B}(\lambda, \kappa)$: If X is a space with $\pi(X) \leq \lambda$ and the power X^κ , considered in the box product topology, is Baire, then Black has a winning strategy in the game $\mathbf{BM}(X)$.

Here is a more precise statement of our conjecture:

Conjecture 1. *For each $\lambda \geq 1$ there is a $\kappa \geq 1$ such that $\mathbf{B}(\lambda, \kappa)$ is true.*

We are not aware of any (consistent) examples that disprove Conjecture 1. One of the attractive consequences of Conjecture 1 is:

Proposition 3. *Assume Conjecture 1. If X is a Baire space and if Y is a space for which all powers, considered in the box product topology, are Baire spaces, then $X \times Y$ is a Baire space.*

Proof. Let X and Y be as given. By Conjecture 1, TWO has a winning strategy in $\mathbf{BM}(Y)$. By Theorem 3(9) of [10], $X \times Y$ is a Baire space. \square

We now show that this conjecture follows from another plausible conjecture for which we are also not aware of any (consistent) counterexamples.

Let a cardinal $\lambda \geq 2$ as well as a set $A \subseteq {}^\omega\lambda$ be given. The classical Gale-Stewart game $\text{GS}_\lambda(A)$ between two players, ONE and TWO, is played as follows: The players play an inning per finite ordinal. In inning $n < \omega$ ONE first chooses $\xi_{2n} \in \lambda$, and then TWO responds by choosing $\xi_{2n+1} \in \lambda$. In this way the players construct a play $p = (\xi_n : n < \omega)$. ONE wins this play if $p \in A$, and otherwise, TWO wins the play.

Let, in addition to the cardinal $\lambda \geq 2$, also a cardinal $\kappa \geq 1$ be given. Then the κ -board Gale-Stewart game $\text{GS}_\lambda(A, \kappa)$ is defined as follows: Players ONE and TWO play an inning per finite ordinal. In inning n ONE first selects $(f_{2n}(\alpha) : \alpha < \kappa) \in {}^\kappa\lambda$, and then TWO responds with $(f_{2n+1}(\alpha) : \alpha < \kappa) \in {}^\kappa\lambda$. In this way the players construct a play $(f_n : n < \omega)$, consisting of a sequence of members of ${}^\kappa\lambda$. ONE wins this play if there is an $\alpha < \kappa$ for which $(f_n(\alpha) : n < \omega)$ is a member of A . Else, TWO wins.

Intuitively, the game $\text{GS}_\lambda(A, \kappa)$ can be interpreted as the two players playing the Gale-Stewart game $\text{GS}_\lambda(A)$ on κ boards simultaneously. The rules are that a player must make a move on each of the boards before the opponent responds by also making a move on each of the boards. ONE wins the multiboard version if there is at least one board on which the play produced on that board is a win for ONE of $\text{GS}_\lambda(A)$. TWO wins the multiboard version if TWO wins each play produced on each of the boards. In this multiboard notation the classical Gale-Stewart game would be denoted $\text{GS}_\lambda(A, 1)$. However, for this case we shall continue to write $\text{GS}_\lambda(A)$ instead.

Assume that $A \subseteq {}^\omega\lambda$ is such that the game $\text{GS}_\lambda(A)$ is undetermined³. Then there are the following two cases for the multiboard version:

- (A) For all κ , $\text{GS}_\lambda(A, \kappa)$ is undetermined.
- (B) There is a cardinal $\kappa_0 > 1$ such that, for all κ with $1 \leq \kappa < \kappa_0$, $\text{GS}_\lambda(A, \kappa)$ is undetermined, but $\text{GS}_\lambda(A, \kappa_0)$ is determined⁴. In this case $\text{GS}_\lambda(A, \kappa)$ is determined for all $\kappa \geq \kappa_0$.

We do not know of a cardinal $\lambda \geq 2$ and a set $A \subseteq {}^\omega\lambda$ such that $\text{GS}_\lambda(A, \kappa)$ is undetermined for all κ . We conjecture that case (A) does not occur. More precisely, define:

$D(\lambda, \kappa)$: For each set $A \subseteq {}^\omega\lambda$ the game $\text{GS}_\lambda(A, \kappa)$ is determined.

Conjecture 2. *For each cardinal $\lambda \geq 2$ there exists a cardinal $\kappa \geq 1$ such that $D(\lambda, \kappa)$ holds.*

Theorem 4. *Let cardinal numbers $\lambda \geq 2$ and $\kappa \geq 1$ be given. Then $D(\lambda, \kappa)$ implies $B(\lambda, \kappa)$.*

³It is well known that in ZFC there are such examples.

⁴In this case player ONE would have the winning strategy.

Proof. Let a topological space X of π -weight λ be given, and assume that \mathcal{B} is a π -base of cardinality λ for X . Also assume that in the box product topology X^κ is a Baire space. We must show that Black has a winning strategy in $\mathbf{BM}(X)$.

Identify \mathcal{B} with λ , and define a subset A of ${}^\omega\mathcal{B}$ to be the set of sequences $(B_n : n < \omega)$ such that:

1. $(\forall n)(B_{2n+2} \subseteq B_{2n+1})$;
2. $\bigcap_{n < \omega} B_n = \emptyset$, or $(\exists n < \omega)(B_{2n+1} \not\subseteq B_{2n})$.

By the hypothesis $\mathbf{D}(\lambda, \kappa)$, the game $\mathbf{GS}_\lambda(A, \kappa)$ is determined.

Claim: Player ONE does not have a winning strategy in the game $\mathbf{GS}_\lambda(A, \kappa)$.

For let σ be a strategy for ONE. Define a corresponding strategy F for White in $\mathbf{BM}(X^\kappa)$ as follows:

In inning 0, σ calls on ONE to play an $f_0 = \sigma(\emptyset)$ where f_0 is a function $f_0 : \kappa \rightarrow \mathcal{B}$. This defines the open subset $B_0 = F(\emptyset) := \prod_{\alpha < \kappa} \sigma(\emptyset)(\alpha)$ of X^κ .

We may assume the players of $\mathbf{BM}(X^\kappa)$ play elements of the π -base obtained by taking products of κ elements chosen from the π -base of X . Now Black responds in $\mathbf{BM}(X^\kappa)$ with such an element $B_1 \subseteq B_0$. B_1 can be described as $B_1 := \prod_{\alpha < \kappa} f_1(\alpha)$, where $f_1 \in {}^\kappa\mathcal{B}$.

Considering f_1 as TWO's response in $\mathbf{GS}_\lambda(A, \kappa)$, apply ONE's strategy there to find $f_2 = \sigma(f_1) \in {}^\kappa\mathcal{B}$, and then define White's move in the game $\mathbf{BM}(X^\kappa)$ as $B_2 = F(B_1) := \prod_{\alpha < \kappa} \sigma(f_1)(\alpha)$, and so on, as depicted in the following diagram:

$\mathbf{GS}_\lambda(A, \kappa)$		$\mathbf{BM}(X^\kappa)$	
ONE	TWO	White	Black
$\sigma(\emptyset)$		$F(\emptyset) = \prod_{\alpha < \kappa} \sigma(\emptyset)(\alpha)$	
$\sigma(f_1)$	f_1	$F(B_1) = \prod_{\alpha < \kappa} \sigma(f_1)(\alpha)$	$B_1 = \prod_{\alpha < \kappa} f_1(\alpha)$
\vdots	f_3	\vdots	$B_3 = \prod_{\alpha < \kappa} f_3(\alpha)$
\vdots	\vdots	\vdots	\vdots

Figure 1: Playing a Banach-Mazur game using a multiboard Gale-Stewart game

Since X^κ is a Baire space, White has no winning strategy in $\mathbf{BM}(X^\kappa)$. But then consider an F -play of $\mathbf{BM}(X^\kappa)$ lost by White:

$$F(X^\kappa), B_1, F(B_1), B_3, \dots, F(B_1, \dots, B_{2n-1}), B_{2n+1}, \dots$$

We have for each n that

- $B_{2n+1} = \prod_{\alpha < \kappa} f_{2n+1}(\alpha)$ where f_{2n+1} is a function from κ to \mathcal{B} , and

- $F(B_1, \dots, B_{2n-1}) = \prod_{\alpha < \kappa} \sigma(f_1, \dots, f_{2n-1})(\alpha)$.

Since Black wins this play of $\text{BM}(X^\kappa)$ it follows that $\bigcap_{n < \omega} B_n \neq \emptyset$. This in turn implies that for each $\alpha < \kappa$ we have that $\bigcap_{n < \omega} f_{2n+1}(\alpha) \neq \emptyset$. But then the σ -play

$$\sigma(\emptyset), f_1, \sigma(f_1), f_3, \sigma(f_1, f_3), \dots$$

of $\text{GS}_\lambda(A, \kappa)$ is lost by ONE, completing the proof of the claim.

Thus, ONE has no winning strategy in the game $\text{GS}_\lambda(A, \kappa)$. By $\text{D}(\lambda, \kappa)$ TWO has a winning strategy in $\text{GS}_\lambda(A, \kappa)$, and thus also has a winning strategy, say τ , in $\text{GS}_\lambda(A)$. Now use the winning strategy τ of TWO to define a winning strategy F for Black in the Banach-Mazur game on X :

When White plays B_0 in $\text{BM}(X)$, Black responds with $F(B_0) := \tau(B_0)$. Upon White's next move B_2 Black responds with $F(B_0, B_2) := \tau(B_0, B_2)$, and so on. As τ is a winning strategy for TWO in $\text{GS}_\lambda(A)$ it follows that $\bigcap_{n < \omega} B_n$ is nonempty, and thus F is a winning strategy for Black in $\text{BM}(X)$. \square

2 Laver games

We now introduce a class of games invented by Richard Laver. For cardinals $\lambda \geq 2$ and $\kappa \geq 1$, the Laver game $\text{LG}(\lambda, \kappa)$ is as follows: Player ONE chooses $f_0 : \kappa \rightarrow \lambda$, then player TWO chooses $f_1 : \kappa \rightarrow \lambda$, then player ONE chooses $f_2 : \kappa \rightarrow \lambda$, and so on. ONE wins the play (f_0, f_1, f_2, \dots) of $\text{LG}(\lambda, \kappa)$ if for each $S \subseteq {}^\omega \lambda$ such that TWO does not have a winning strategy in $\text{GS}_\lambda(S)$, there exists $\xi \in \kappa$ such that $(f_0(\xi), f_1(\xi), f_2(\xi), \dots) \in S$. Otherwise, TWO wins.

Lemma 5. *Let cardinals $\lambda \geq 2$ and $\kappa \geq 1$, and a sequence $(f_n : n < \omega)$ of elements of ${}^\kappa \lambda$ be given. Define $Y = \{(f_n(\xi) : n < \omega) : \xi \in \kappa\}$. The following are equivalent:*

- For each $S \subseteq {}^\omega \lambda$ such that TWO does not have a winning strategy in $\text{GS}_\lambda(S)$, there exists $\xi \in \kappa$ such that $(f_0(\xi), f_1(\xi), f_2(\xi), \dots) \in S$;
- TWO has a winning strategy in $\text{GS}_\lambda({}^\omega \lambda \setminus Y)$;
- There is strategy σ for player TWO in the game structure (meaning that the winning condition is unspecified) GS_λ such that every possible σ -play of GS_λ is in Y .

Proof. (a) \Rightarrow (b) : Assume (a). Put $S = {}^\omega \lambda \setminus Y$. Suppose that contrary to (b), TWO does not have a winning strategy in $\text{GS}_\lambda(S)$. By (a) it follows that $Y \cap S$ is nonempty, a contradiction.

(b) \Rightarrow (c) : Assume (b) and let σ be a winning strategy for TWO in $\text{GS}_\lambda({}^\omega \lambda \setminus Y)$. Then σ is a strategy as in (c) for TWO.

(c) \Rightarrow (a) : Let σ be a strategy for TWO in the game structure GS_λ as in (c). Consider any set $S \subseteq {}^\omega \lambda$ for which TWO does not have a winning strategy in $\text{GS}_\lambda(S)$. Now let TWO use the strategy σ to play this game. Since σ is not a winning strategy for TWO, there is a play against σ that

is won by ONE, say $(\nu_0, \nu_1, \dots, \nu_{2n}, \nu_{2n+1}, \dots)$. Since ONE wins this play, $(\nu_0, \nu_1, \dots, \nu_{2n}, \nu_{2n+1}, \dots) \in S$. Since this play is a σ -play, it is according to (c) a member of Y , and thus of the form $(f_n(\xi) : n < \omega)$ for some $\xi \in \kappa$. \square

Let $L(\lambda, \kappa)$ denote the statement: ONE has a winning strategy in $LG(\lambda, \kappa)$.

Theorem 6. *For cardinal numbers λ and κ , $L(\lambda, \kappa)$ implies $D(\lambda, \kappa)$.*

Proof. Let σ be a winning strategy for ONE in $LG(\lambda, \kappa)$. Let $S \subseteq {}^\omega\lambda$ be given. We must show that $GS_\lambda(S, \kappa)$ is determined. If TWO has a winning strategy in $GS_\lambda(S)$ there is nothing to prove. Thus, assume that TWO does not have a winning strategy in $GS_\lambda(S)$.

Now ONE of $GS_\lambda(S, \kappa)$ uses the strategy σ of ONE of $LG(\lambda, \kappa)$ to play $GS_\lambda(S, \kappa)$. Let $(f_0, f_1, \dots, f_{2n}, f_{2n+1}, \dots)$ be a σ -play of $GS_\lambda(S, \kappa)$. Since this play is a winning play for ONE of $LG(\lambda, \kappa)$, and since TWO does not have a winning strategy in $GS_\lambda(S)$, there exists $\xi \in \kappa$ for which $(f_n(\xi) : n < \omega)$ is in S . But then this σ -play is also a winning play for ONE in the game $GS_\lambda(S, \kappa)$. \square

We leave the proof of the following partial converse to Theorem 6 to the reader:

Theorem 7. *If $D(\lambda, \kappa)$, then $L(\lambda, 2^{\lambda^{\aleph_0}} \cdot \kappa)$.*

Thus, when $\kappa \geq 2^{\lambda^{\aleph_0}}$, the games $D(\lambda, \kappa)$ and $L(\lambda, \kappa)$ are equivalent. This suggests that $L(\lambda, \kappa)$ is not much (if at all) stronger than $D(\lambda, \kappa)$.

Theorem 8. *TWO has a winning strategy in $LG(2, 2^{\aleph_0})$.*

Proof. Let the set of “boards” be some totally imperfect subset X of the Cantor space ${}^\omega\{0, 1\}$ with $|X| = 2^{\aleph_0}$; thus, at move n , ONE chooses a function $f_n : X \rightarrow \{0, 1\}$, and then TWO chooses a function $g_n : X \rightarrow \{0, 1\}$. TWO’s winning strategy π is very simple: at move n he chooses the function $g_n(x) = x(n)$. Assume towards a contradiction that ONE wins some π -play $(f_0, g_0, f_1, g_1, \dots)$ of $LG(2, 2^{\aleph_0})$, and define

$$Y = \{(f_0(x), g_0(x), f_1(x), g_1(x), \dots) : x \in X\}.$$

Then, by the implication (a) \Rightarrow (c) of Lemma 5, there is a strategy σ for TWO in GS_2 such that every σ -play of GS_2 is in Y .

For each $t \in {}^\omega\{0, 1\}$ let $y_t \in Y$ be the σ -play of GS_2 in which t_n is the sequence of ONE’s moves, and let $x_t \in X$ be such that

$$\begin{aligned} y_t &= (f_0(x_t), g_0(x_t), f_1(x_t), g_1(x_t), \dots) \\ &= (t(0), x_t(0), t(1), x_t(1), \dots) \\ &= (t(0), \sigma(t(0)), t(1), \sigma(t(0), t(1)), \dots). \end{aligned}$$

Then the function $t \mapsto x_t$ is a one-to-one continuous mapping of ${}^\omega\{0, 1\}$ into X ; but this is impossible because X is totally imperfect. \square

3 Cut-and-choose games and the conjectures

Again, let $\lambda \geq 2$ and $\kappa \geq 1$ be cardinals. Consider the following cut-and-choose game $\text{UG}(\lambda, \kappa)$ between players White and Black⁵:

First, White chooses a partition of κ into λ pieces, and then Black chooses one of these, S_0 . Then Black chooses a partition of S_0 into λ pieces, and White chooses one of these, S_1 . Then White chooses a partition of S_1 into λ pieces, and Black chooses one of these, say S_2 , and so on for ω innings. White wins if $\bigcap_{n < \omega} S_n \neq \emptyset$. Else, Black wins.

Alternately this game can be described as follows: White chooses a function $f_0 \in {}^\kappa\lambda$. Then Black chooses a $\xi_0 \in \lambda$. Then Black chooses a $f_1 \in {}^\kappa\lambda$, and then White chooses a $\xi_1 \in \lambda$. Then White chooses an $f_2 \in {}^\kappa\lambda$. Then Black chooses a $\xi_2 \in \lambda$, and an $f_3 \in {}^\kappa\lambda$, and so on. A play

$$f_0, (\xi_0, f_1), (\xi_1, f_2), (\xi_2, f_3), \dots$$

is won by White if there is an $\alpha < \kappa$ such that for each n , $f_n(\alpha) = \xi_n$. We define:

$\text{U}(\lambda, \kappa)$: White has a winning strategy in $\text{UG}(\lambda, \kappa)$.

The following observation will be useful later:

Lemma 9. *If $\lambda, \lambda', \kappa$ and κ' are cardinal numbers such that $1 \leq \lambda' \leq \lambda$ and $\kappa' \geq \kappa \geq 1$, then $\text{U}(\lambda, \kappa)$ implies $\text{U}(\lambda', \kappa')$.*

Here is how this cut-and-choose game is related to the Laver game:

Theorem 10. *For all cardinals $\lambda \geq 2$ and $\kappa \geq 1$, $\text{U}(\lambda, \kappa)$ implies $\text{L}(\lambda, \kappa)$.*

Proof. Let $\lambda \geq 2$ and $\kappa \geq 1$ be given cardinal numbers. Assume that $\text{U}(\lambda, \kappa)$ holds. We must show that $\text{L}(\lambda, \kappa)$ holds.

Recall that a play of $\text{LG}(\lambda, \kappa)$ is a sequence (f_0, f_1, f_2, \dots) in ${}^\kappa\lambda$ where f_0, f_2, f_4, \dots are chosen by ONE and f_1, f_3, f_5, \dots by TWO. By Lemma 5, ONE wins the play (f_0, f_1, f_2, \dots) just in case there is a strategy σ for TWO in GS_λ such that every σ -play of GS_λ has the form $(f_0(\xi), f_1(\xi), f_2(\xi), \dots)$ for some $\xi \in \kappa$.

Let φ be a winning strategy for White in $\text{UG}(\lambda, \kappa)$, for which we will use the following notation. On the first move, White plays f_0^φ . For $n > 0$, if Black's first n moves are $(\alpha_0, f_1), \dots, (\alpha_{2n-2}, f_{2n-1})$, White's next move is

$$(\alpha_{2n-1}^\varphi((\alpha_0, f_1), \dots, (\alpha_{2n-2}, f_{2n-1})), f_{2n}^\varphi((\alpha_0, f_1), \dots, (\alpha_{2n-2}, f_{2n-1}))).$$

Now we define a strategy Φ for ONE in $\text{LG}(\lambda, \kappa)$. It will be convenient to describe it in the form $f_{2n} = \Phi(f_0, f_1, f_2, \dots, f_{2n-1})$, i.e., ONE's next move is given as a function of the total history, including the moves of both players. Namely, we define $\Phi(\emptyset) = f_0^\varphi$, and for $n > 0$, $\xi \in \kappa$, we define

$$\Phi(f_0, f_1, \dots, f_{2n-1})(\xi) = f_{2n}^\varphi((f_0(\xi), f_1), (f_2(\xi), f_3), \dots, (f_{2n-2}(\xi), f_{2n-1}))(\xi)).$$

⁵The game $\text{UG}(2, \kappa)$ was invented by Ulam: See pp. 346-347 of [9].

Suppose (f_0, f_1, f_2, \dots) is a Φ -play of $\text{LG}(\lambda, \kappa)$; we have to show that this play is won by ONE. First, we define a strategy σ for TWO in GS_λ , by setting

$$\sigma(\alpha_0, \alpha_2, \dots, \alpha_{2n}) = \alpha_{2n+1}^\varphi((\alpha_0, f_1), (\alpha_2, f_3), \dots, (\alpha_{2n}, f_{2n+1})).$$

Now let $(\alpha_0, \alpha_1, \alpha_2, \dots)$ be any σ -play of GS_λ ; we have to find $\xi \in \kappa$ such that $f_n(\xi) = \alpha_n$ for all n .

For $n < \omega$ let $\hat{f}_{2n} = f_{2n}^\varphi((\alpha_0, f_1), (\alpha_2, f_3), \dots, (\alpha_{2n-2}, f_{2n-1}))$. Then

$$(\hat{f}_0, (\alpha_0, f_1), (\alpha_1, \hat{f}_2), (\alpha_2, f_3), (\alpha_3, \hat{f}_4), \dots)$$

is a φ -play of $\text{UG}(\lambda, \kappa)$. Since φ is a winning strategy for White in $\text{UG}(\lambda, \kappa)$, there exists $\xi \in \kappa$ such that $\hat{f}_{2n}(\xi) = \alpha_{2n}$ and $f_{2n+1}(\xi) = \alpha_{2n+1}$ for all $n < \omega$.

Finally, we prove by induction on n that $f_{2n}(\xi) = \alpha_{2n}$ for all $n < \omega$. This is clear for $n = 0$, since $f_0 = f_0^\varphi = \hat{f}_0$. Now suppose that $n > 0$, and $f_{2k}(\xi) = \alpha_{2k}$ for all $k < n$; then

$$\begin{aligned} f_{2n}(\xi) &= \Phi(f_0, f_1, f_2, \dots, f_{2n-1})(\xi) \\ &= f_{2n}^\varphi((f_0(\xi), f_1), (f_2(\xi), f_3), \dots, (f_{2n-2}(\xi), f_{2n-1}))(\xi) \\ &= f_{2n}^\varphi((\alpha_0, f_1), (\alpha_1, f_3), \dots, (\alpha_{2n-2}, f_{2n-1}))(\xi) \\ &= \hat{f}_{2n}(\xi) = \alpha_{2n}. \end{aligned}$$

□

Theorem 11. ⁶ *If White has a winning strategy in $\text{UG}(2, \kappa)$, then $\kappa > 2^{\aleph_0}$.*

Proof. By Theorem 10, if White has a winning strategy in $\text{UG}(2, \kappa)$, then ONE has a winning strategy in $\text{LG}(2, \kappa)$. Then Theorem 8 implies that $\kappa > 2^{\aleph_0}$. □

4 $\lambda = \aleph_0$ and Baumgartner's condition

For cardinals $\lambda \geq 2$ and $\kappa \geq 1$ we now have the following implications:

$$\text{U}(\lambda, \kappa) \xrightarrow{\text{Th 10}} \text{L}(\lambda, \kappa) \xrightarrow{\text{Th 6}} \text{D}(\lambda, \kappa) \xrightarrow{\text{Th 4}} \text{B}(\lambda, \kappa)$$

This sequence of implications gives us deeper insight into Conjecture 1 and Conjecture 2. Before considering the general cases of these conjectures in the next section, we first explore the special case when $\lambda = \aleph_0$. The results in this section explore the constraints on κ under which White has a winning strategy in the game $\text{UG}(\aleph_0, \kappa)$.

Theorem 12. *Let κ be an infinite cardinal number. Then White has a winning strategy in $\text{UG}(\aleph_0, \kappa)$ if, and only if, White has a winning strategy in $\text{UG}(2, \kappa)$.*

Proof. We show that if White has a winning strategy in $\text{UG}(2, \kappa)$, then White has a winning strategy in $\text{UG}(\aleph_0, \kappa)$.

Let σ be a winning strategy for White in $\text{UG}(2, \kappa)$. If p is a partial σ -play of $\text{UG}(2, \kappa)$, let $\mathcal{S}(p)$ be the collection of all sets $S \subseteq \kappa$ such that some subset of S is chosen in some σ -play p' which is a continuation of p .

⁶Theorem 11 was independently discovered by S. Hechler.

Claim 1. Any set S in $\mathcal{S}(p)$ can be partitioned into \aleph_0 pairwise disjoint sets, each of which belongs to $\mathcal{S}(p)$.

Proof of Claim 1. It is enough to show that each member of $\mathcal{S}(p)$ contains two disjoint members of $\mathcal{S}(p)$. Suppose $S \in \mathcal{S}(p)$, and let the σ -play p' be a continuation of p in which White chooses a set $S_i \subseteq S$. After choosing the set S_i White cuts it into two pieces, both of which are of course members of $\mathcal{S}(p)$.

Claim 2. If $\bigcup_{n < \omega} S_n = S \in \mathcal{S}(p)$, then $S_n \in \mathcal{S}(p)$ for some n .

Proof of Claim 2 Assume the contrary, $S_n \notin \mathcal{S}(p)$ for all n . Starting from p , since S is in $\mathcal{S}(p)$, Black has a sequence of moves which forces White (using σ) to choose a subset of S . Next, by presenting White with the choice of a set contained in S_0 and a set disjoint from S_0 , Black forces White to choose a set disjoint from S_0 . Continuing in this way, he forces White to choose a set disjoint from S_n for every n . Thus Black wins a σ -play of $\text{UG}(2, \kappa)$, contradicting the assumption that σ is a winning strategy for White.

To win $\text{UG}(\aleph_0, \kappa)$, White starts by cutting κ into \aleph_0 pieces, each of which belongs to $\mathcal{S}(\emptyset)$. Black chooses one of them, call it S_0 , and White chooses a partial σ -play p_0 of $\text{UG}(2, \kappa)$ which ends up at a subset of S_0 . Next, Black cuts S_0 into \aleph_0 pieces. Since S_0 is in $\mathcal{S}(p_0)$, White can choose a piece S_1 which belongs to $\mathcal{S}(p_0)$. White extends p_0 to a partial σ -play p_1 which ends up at a subset of S_1 , cuts it into \aleph_0 pieces each belonging to $\mathcal{S}(p_1)$, and so on. Since White wins the imagined σ -play of $\text{UG}(2, \kappa)$, White also wins the play of $\text{UG}(\aleph_0, \kappa)$. \square

Recall that a cardinal κ is *measurable* if it is uncountable and there is a κ -complete nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\kappa)$. See Jech [3] for more information on these notions. C. Gray and R.M. Solovay proved (unpublished) that if it is consistent that there is an infinite cardinal number κ such that White has a winning strategy in the game $\text{UG}(2, \kappa)$, then it is consistent that there is a measurable cardinal. M. Magidor proved that if it is consistent that there is a measurable cardinal, then it is consistent that there is an infinite cardinal κ such that White has a winning strategy in the game $\text{UG}(2, \kappa)$.

M. Magidor (unpublished) also showed that if it is consistent that there is a measurable cardinal κ , then it is consistent that κ is measurable and White has a winning strategy in $\text{UG}(2, \kappa)$. Then R.M. Solovay (unpublished) proved that if it is consistent that there is a measurable cardinal κ , then it is consistent⁷ that κ is measurable, but White does not have a winning strategy in the game $\text{UG}(2, \kappa)$. Then Laver proved that if it is consistent that there is a measurable cardinal then it is consistent that there is a successor κ of an infinite regular cardinal such that White has a winning strategy in the game $\text{UG}(2, \kappa)$. Laver's argument went unpublished also, but subsequently appeared in [2]. We now give an exposition of ideas of Baumgartner and of Laver that culminates in the consistency, under appropriate large cardinal hypotheses, of Conjectures 1 and 2.

⁷The model is of the form $\mathbb{L}[\mathcal{U}]$ where \mathcal{U} witnesses the measurability of κ .

Let S be a set. A nonempty family $J \subseteq \mathcal{P}(S)$ is said to be an *ideal* on S if: $A \cup B$ is an element of J whenever A and B are elements of J ; If $B \in J$ and $A \subseteq B$ then $A \in J$. J is said to be a *proper* ideal if $J \neq \mathcal{P}(S)$. J is said to be a *free* ideal if $\bigcup J = S$. For an ideal J on a set S , the symbol J^+ denotes the set $\mathcal{P}(S) \setminus J$. The ideal $J \subseteq \mathcal{P}(S)$ is said to be *atomless* if for each $X \in J^+$ there is a partition $X = A \cup B$ such that A and B are disjoint, and both are elements of J^+ .

For J a proper ideal on set S the game $G(J)$ of length ω is played as follows: In the first inning White chooses a set $W_0 \subseteq S$ with $W_0 \in J^+$, and Black responds with $B_0 \subseteq W_0$ and $B_0 \in J^+$. In the $n + 1^{st}$ inning White chooses $W_n \subseteq B_{n-1}$ with $W_n \in J^+$, and Black responds with $B_n \subseteq W_n$ and $B_n \in J^+$. A play is won by Black if $\bigcap_{n < \omega} B_n \neq \emptyset$; else, White wins. This game was studied in [2]. Banach considered the special case of $J = \{X \subseteq S : |X| < |S|\}$. Banach's game is recorded in Problem 67 of the Scottish Book [6]. Schreier [7] showed that White has a winning strategy in Banach's game.

Theorem 13 (Baumgartner). *If there is an atomless proper ideal J on κ such that Black has a winning strategy in $G(J)$, then White has a winning strategy in $UG(\aleph_0, \kappa)$.*

Proof. Assume that κ is an infinite cardinal number and that $J \subset \mathcal{P}(\kappa)$ is an atomless proper ideal on κ for which Black has a winning strategy in the game $G(J)$. We must show that then White has a winning strategy in the game $UG(\aleph_0, \kappa)$. By Theorem 12 it is sufficient to show that White has a winning strategy in $UG(2, \kappa)$.

Let F be a winning strategy for Black in the game $G(J)$. Define a strategy for White in the game $UG(2, \kappa)$ as follows:

Since J is atomless, White's first move is (C_0, D_0) where both C_0 and D_0 are in J^+ , are disjoint, and have union κ .

Suppose that Black selects one of these, S_0 , and partitions it into two disjoint sets (C_1, D_1) . To decide White's response to Black's move, do the following: Since S_0 is in J^+ select $S_1 \in \{C_1, D_1\}$ in J^+ and assign $B_0 = S_1$ to White of the game $G(J)$. Compute the response $B_1 := F(B_0)$ of Black of the game $G(J)$, using Black's winning strategy F .

Since B_1 is in J^+ and $B_1 \subseteq B_0$ and J is atomless, choose a partition (C_2, D_2) of S_1 with both $C_2 \cap B_1$ and $D_2 \cap B_1$ in J^+ .

Suppose that Black chooses $S_2 \in \{C_2, D_2\}$, and partitions it into two disjoint sets (C_3, D_3) . To decide White's response to Black's move in the game $UG(2, \kappa)$, do the following: Since $S_2 \cap B_1$ is in J^+ , select $S_3 \in \{C_3, D_3\}$ with $S_3 \cap B_1 \in J^+$. Assign $B_2 = S_3 \cap B_1$ to White of the game $G(J)$. Compute the response $B_3 := F(B_0, B_2)$ of Black in the game $G(J)$.

As J is an atomless ideal and B_3 is in J^+ White of the game $UG(2, \kappa)$ chooses a partition (C_4, D_4) of S_3 such that both $C_4 \cap B_3$ and $D_4 \cap B_3$ are in J^+ , and so on.

UG(2, κ)		G(J)	
White	Black	White	Black
$\kappa; (C_0, D_0)$	$S_0; (C_1, D_1)$	$B_0 = S_1$	$B_1 := F(B_0)$
$S_1; (C_2, D_2)$	$S_2; (C_3, D_3)$	$B_2 = S_3 \cap B_1$	$B_3 := F(B_0, B_2)$
$S_3; (C_4, D_4)$	$S_4; (C_5, D_5)$	$B_4 = S_5 \cap B_3$	$B_5 := F(B_0, B_2, B_4)$
$S_5; (C_6, D_6)$			
\vdots	\vdots	\vdots	\vdots

Figure 2: Playing UG(2, κ) using G(J).

To see that this strategy for White in the game UG(2, κ) is a winning strategy, consider a play according to the strategy:

$$(C_0, D_0), (S_0; (C_1, D_1)), (S_1; (C_2, D_2)), (S_2; (C_3, D_3)), (S_3, (C_4, D_4)), \dots$$

Corresponding to this play there is an F -play $B_0, B_1, B_2, B_3, \dots$ of G(J) where $B_{2n} \subseteq S_{2n+1}$ for all n . Since F is a winning strategy for Black in G(J) we have $\bigcap_{n < \omega} S_n \supseteq \bigcap_{n < \omega} B_n \neq \emptyset$, whence the strategy for White in UG(2, κ) is a winning strategy. \square

We shall say that an ideal $J \subseteq \mathcal{P}(\kappa)$ satisfies *Baumgartner's condition* if it is an atomless proper ideal for which Black has a winning strategy in the game G(J). A free ideal J on a set S is said to be *precipitous* if White does not have a winning strategy in the game G(J)⁸. Thus, ideals satisfying Baumgartner's condition are precipitous ideals.

Theorem 14. *If it is consistent that there is an atomless ideal J on a cardinal κ such that Black has a winning strategy in the game G(J), then it is consistent that there is a measurable cardinal.*

Proof. If κ is a cardinal and J is an atomless ideal on κ such that Black has a winning strategy in G(J), then $J \subseteq \mathcal{P}(\kappa)$ is a precipitous ideal. Thus, by Lemmas 2.1 and 2.2 of [4], there is an inner model in which there is a measurable cardinal. \square

We now describe a class of ideals that satisfy Baumgartner's condition. When $J \subseteq \mathcal{P}(\kappa)$ is a free ideal, a family $\mathcal{F} \subseteq J^+$ is said to be *dense* if there is for each element X of J^+ an element Y of \mathcal{F} such that $Y \subseteq X$.

⁸The notion of a precipitous ideal has another definition that was shown in Theorem 2 of [2] to be equivalent to White not having a winning strategy in the game G(J).

Theorem 15. *Let $J \subseteq \mathcal{P}(\kappa)$ be a proper free ideal. If there is a dense family $\mathcal{F} \subseteq J^+$ with the property that $\bigcap_{n < \omega} X_n \neq \emptyset$ for any sequence $(X_n : n < \omega)$ of elements of \mathcal{F} such that for each n , $X_{n+1} \subseteq X_n$, then Black has a winning strategy in the game $G(J)$.*

Proof. Black, in fact, has a very simple winning strategy, namely when White plays a set $X \in J^+$, then Black responds with $F(X) \in \mathcal{F}$ such that $F(X) \subseteq X$. \square

Laver proved a result which implies

Theorem 16 (Laver). *If it is consistent that there is a measurable cardinal κ then it is consistent that CH holds and there is an atomless ideal J on \aleph_2 and a dense family $\mathcal{F} \subseteq J^+$ such that for each sequence $(X_n : n < \omega)$ of sets in J^+ with $X_n \supseteq X_{n+1}$ for all n , the set $\bigcap_{n < \omega} X_n$ is in J^+ .*

We shall explain a proof of Laver's more general version of this result later in the paper.

As a result of the facts given in this section of the paper we can now conclude:

Theorem 17. *If it is consistent that there is a measurable cardinal, then it is consistent that $2^{\aleph_0} = \aleph_1$ and $U(\aleph_0, \aleph_2)$ holds.*

Thus, the consistency of the existence of a measurable cardinal implies the consistency of each of the special cases $B(\aleph_0, (2^{\aleph_0})^+)$ and $D(\aleph_0, (2^{\aleph_0})^+)$ of Conjectures 1 and 2, respectively.

5 The general case of the conjectures

Let κ and λ be infinite cardinals. Define:

$I(\lambda, \kappa)$: There is a free proper ideal J on κ such that

1. J is λ^+ -complete;
2. each $X \in J^+$ has a partition into λ disjoint sets, each in J^+ ;
3. there is a family $\mathcal{F} \subseteq J^+$ such that:
 - (a) for each $X \in J^+$ there exists a $Y \in \mathcal{F}$ such that $Y \subseteq X$ (\mathcal{F} is dense), and
 - (b) every descending ω -sequence of sets from \mathcal{F} has a nonempty intersection.

If we make the convention that for a cardinal λ the symbol λ^+ denotes the least *infinite* cardinal greater than λ (so that $2^+ = \aleph_0$), then $I(2, \kappa)$ implies that there is an ideal on κ which satisfies Baumgartner's condition, whence $U(\aleph_0, \kappa)$ holds.

Theorem 18 (Laver). *For infinite cardinals κ and λ , $I(\lambda, \kappa)$ implies $U(\lambda, \kappa)$.*

Proof. Let J be an ideal witnessing $\mathbb{I}(\lambda, \kappa)$. Define a strategy F for White in $\text{UG}(\lambda, \kappa)$ as follows: As κ is a member of J^+ , choose a partition $\kappa = \bigcup_{\xi < \lambda} X_\xi$ such that each X_ξ is a member of J^+ . Define White's first move, $F(\emptyset)$, to be this partition.

When Black responds with ξ_0 and a partition $\langle X_{\xi_0, \xi} : \xi < \lambda \rangle$ of X_{ξ_0} , White chooses $X_{\xi_0, \xi_1} \in J^+$, which exists by the λ^+ -completeness of J , and $F_0 \in \mathcal{F}$ such that $F_0 \subseteq X_{\xi_0, \xi_1}$. White then plays a partition $\langle X_{\xi_0, \xi_1, \xi} : \xi < \lambda \rangle$ of X_{ξ_0, ξ_1} so that $F_0 \cap X_{\xi_0, \xi_1, \xi} \in J^+$ for all $\xi < \lambda$. Next Black responds with ξ_2 and $\langle X_{\xi_0, \xi_1, \xi_2, \xi} : \xi < \lambda \rangle$, etc. We end up with a descending ω -sequence

$$X_{\xi_0} \supseteq \cdots \supseteq X_{\xi_0, \xi_1, \dots, \xi_n} \supseteq \cdots$$

with the properties that, for all n ,

$$X_{\xi_0, \xi_1, \dots, \xi_{2n+1}} \supseteq F_n \text{ and } F_n \cap X_{\xi_0, \xi_1, \dots, \xi_{2n+1}, \xi_{2n+2}} \in J^+.$$

Since the sets F_n are decreasing and in \mathcal{F} , we can find $\alpha \in \bigcap_{n < \omega} F_n$. Then $\alpha \in X_{\xi_0, \xi_1, \dots, \xi_{2n+1}}$ for all n , and thus this is a winning play for White. \square

For cardinals $\lambda \geq 2$ and $\kappa \geq 1$ we now have the following implications:

$$\mathbb{I}(\lambda, \kappa) \xrightarrow{\text{Th 18}} \mathbb{U}(\lambda, \kappa) \xrightarrow{\text{Th 10}} \mathbb{L}(\lambda, \kappa) \xrightarrow{\text{Th 6}} \mathbb{D}(\lambda, \kappa) \xrightarrow{\text{Th 4}} \mathbb{B}(\lambda, \kappa)$$

This sequence of implications permits us to establish via Theorem 19 the consistency, modulo the consistency of the existence of certain large cardinals, of Conjecture 2, and thus of Conjecture 1. The proof of Theorem 19 appears in large part in the proof of Theorem 4 of [2].

Towards the proof of Theorem 19 recall: For a filter \mathcal{F} on a set S , \mathcal{F}^+ denotes the set $\{Y \subseteq S : (\forall X \in \mathcal{F})(X \cap Y \neq \emptyset)\}$. The filter \mathcal{F} on a cardinal $\kappa > \aleph_0$ is said to be *normal* if for each regressive function f on a set $X \in \mathcal{F}^+$ there is a set $Y \subseteq X$ such that Y is in \mathcal{F}^+ and f is constant on Y . If a cardinal number κ is measurable, then it carries a normal κ -complete nonprincipal ultrafilter.

Theorem 19 (Laver). *Let λ be an infinite cardinal, and let κ be a measurable cardinal with $\kappa > \lambda$. Lévy collapse κ to size λ^{++} . In the resulting model there is an ideal on $\kappa = \lambda^{++}$ witnessing $\mathbb{I}(\lambda, \kappa)$.*

Proof. In V , let $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ be a normal ultrafilter on κ . In the generic extension, the witnessing ideal \mathcal{I} will be the dual of the filter that is generated by the ground model normal ultrafilter \mathcal{U} .

Let $\mathbb{P} = \text{Coll}(\lambda^+, < \kappa)$ be the Lévy collapse. Then \mathbb{P} has the κ -chain condition and is λ^+ -closed. Let G be a \mathbb{P} -generic filter over V . In $V[G]$, let

$$\mathcal{H} = \{X \subseteq \kappa : \exists U \in \mathcal{U} (U \subseteq X)\}$$

be the filter generated by \mathcal{U} .

Lemmas 20 and 21 give the details of Lemma 1 of [2].

Lemma 20. \mathcal{H} is λ^+ -complete.

Proof. Working in V , let $\dot{\mathcal{H}}$ be a \mathbb{P} -name for the filter \mathcal{H} of the extension, and let $(\tau_\alpha: \alpha < \lambda)$ be a \mathbb{P} -name for a sequence of elements of \mathcal{H} .

Consider any $q \in \mathbb{P}$. For each $\alpha < \lambda$, choose a $q_\alpha \in \mathbb{P}$ and a $U_\alpha \in \mathcal{U}$ such that for $\alpha < \beta < \lambda$ we have $q_\beta < q_\alpha < q$ and $q_\alpha \Vdash U_\alpha \subseteq \tau_\alpha$. Since \mathbb{P} is λ^+ -closed, fix a $p \in \mathbb{P}$ such that for all $\alpha < \lambda$ we have $p < q_\alpha$. Also put $U = \bigcap_{\alpha < \lambda} U_\alpha$. Since \mathcal{U} is κ -complete and $\kappa > \lambda$, U is a member of \mathcal{U} . Then

$$p \Vdash \forall \alpha < \lambda (U \subseteq \tau_\alpha)$$

and therefore

$$p \Vdash \bigcap_{\alpha < \lambda} \tau_\alpha \in \dot{\mathcal{H}}.$$

Thus, for each $q \in \mathbb{P}$ there is a $p < q$ in \mathbb{P} that forces that $\dot{\mathcal{H}}$ is λ^+ -complete. \square

Call a set $X \subseteq \kappa$ an \mathcal{H} -positive set if for each $U \in \mathcal{U}$ we have $U \cap X \neq \emptyset$.

Lemma 21. \mathcal{H} is normal.

Proof. Suppose otherwise. Then choose an \mathcal{H} -positive set $X \subseteq \kappa$ and a regressive function $f: X \rightarrow \kappa$ that is not constant on any \mathcal{H} -positive set. Then for each $\alpha < \kappa$ there is a $U \in \mathcal{U}$ such that $U \cap f^{-1}(\{\alpha\}) = \emptyset$.

Choose a condition $q \in G$ that, in V , forces the above of \mathbb{P} -names \dot{X} and \dot{f} for X and f . Work in V . For each $\alpha < \kappa$, choose a maximal antichain $A_\alpha = \{p_\gamma^\alpha: \gamma < \beta\} \subseteq \mathbb{P}$ below q such that there is a sequence $(U_\gamma^\alpha: \gamma < \beta)$ of elements of \mathcal{U} with the property that, for each $\gamma < \beta$, we have

$$p_\gamma^\alpha \Vdash U_\gamma^\alpha \cap \dot{f}^{-1}(\{\alpha\}) = \emptyset.$$

Since \mathcal{U} is κ -complete and \mathbb{P} has the κ -chain condition, the set $U_\alpha = \bigcap_{\gamma < \beta} U_\gamma^\alpha$ is a member of \mathcal{U} , and each member p of A_α satisfies $p \Vdash U_\alpha \cap \dot{f}^{-1}(\{\alpha\}) = \emptyset$ and, therefore, the same is forced by q .

We may assume that for each $\alpha < \beta < \kappa$, the relation $U_\beta \subseteq U_\alpha$ holds. By normality of \mathcal{U} , the set $U = \bigcap_{\alpha < \kappa} U_\alpha$ is in \mathcal{U} , so $q \Vdash \dot{X} \cap U \neq \emptyset$.

Consider $\eta \in U$. For all $\alpha < \eta$, we have $\eta \in U_\alpha$ and so $q \Vdash \dot{f}(\eta) \neq \alpha$. But this contradicts that f is regressive. \square

Now Lemma 2 of [2] proves

Lemma 22. The ground model filter $\mathcal{U} \subseteq \mathcal{I}^+$ is a σ -complete dense family in \mathcal{I}^+ .

Since $\kappa = \lambda^{++}$, fix for each $\alpha < \kappa$ an injection $f_\alpha: \alpha \rightarrow \lambda^+$. Let Z be an element of \mathcal{I}^+ and, for each $\eta < \lambda^+$ and each $\mu < \kappa$, define

$$S_\eta^\mu = \{\alpha \in Z: f_\alpha^{-1}(\eta) = \mu\}.$$

Lemma 23. For some $\eta < \lambda^+$, we have $|\{\mu < \kappa: S_\eta^\mu \in \mathcal{I}^+\}| = \kappa$.

Proof. For $\eta < \lambda^+$ put $T_\eta = \{\mu < \kappa : S_\eta^\mu \in \mathcal{I}^+\}$. If for each such η we have $|T_\eta| < \kappa$, then $|\bigcup_{\eta < \lambda^+} T_\eta| < \kappa$. Consider $\beta \in \kappa \setminus \bigcup_{\eta < \lambda^+} T_\eta$. Since Z is a member of \mathcal{I}^+ , it has cardinality κ . We may assume that each element of Z is larger than $\max\{\beta, \lambda^+\}$.

Consider an $\alpha \in Z$. Now $f_\alpha(\beta)$ is defined and less than $\lambda^+ < \alpha$. Thus, $f : Z \rightarrow \kappa$ defined by $f(\alpha) = f_\alpha(\beta)$ is a regressive function. Then there is an $\eta < \lambda^+$ such that $f^{-1}(\{\eta\})$ is in \mathcal{I}^+ . But then, since $\beta = f_\alpha^{-1}(\{\eta\})$ for any $\alpha \in f^{-1}(\{\eta\})$, we have that S_η^β is \mathcal{H} -positive, and $\beta \in T_\eta$, a contradiction. \square

It follows that the ideal \mathcal{I} dual to \mathcal{H} witnesses $\mathfrak{l}(\lambda, \kappa)$: λ^+ completeness holds by Lemma 20. That each \mathcal{I} -positive set can be partitioned into λ many positive sets holds by Lemma 23. Finally, \mathcal{U} plays the role of \mathcal{F} in the definition of $\mathfrak{l}(\lambda, \kappa)$. To see this, note first that \mathcal{U} is closed under decreasing ω -sequences, since \mathbb{P} is λ^+ -closed in V . \square

Here is a scenario under which Conjecture 2 would be true:

Lemma 24. *Assume that there is a proper class K of cardinal numbers such that for each $\lambda \in \mathsf{K}$ there is a cardinal number $f(\lambda)$ such that $\mathfrak{l}(\lambda, f(\lambda))$ holds. Then Conjecture 2 is true.*

Proof. If $\lambda' \leq \lambda$ and κ are cardinal numbers, then $\mathfrak{l}(\lambda, \kappa)$ implies $\mathfrak{l}(\lambda', \kappa)$ \square

Corollary 25. *If there is a proper class K of cardinal numbers such that for each $\lambda \in \mathsf{K}$ there is a cardinal number $f(\lambda)$ such that $\mathfrak{l}(\lambda, f(\lambda))$ holds, then Conjecture 1 is true.*

Proof. By Theorem 4 and Lemma 24. \square

Theorem 26 (Laver). *If it is consistent that there is a proper class of measurable cardinals, then it is consistent that there is a proper class of regular cardinals λ such that $\mathfrak{l}(\lambda, \lambda^{++})$ holds.*

Proof. Here is an outline of the argument given by Laver in a personal communication dated May 2, 1995: “Start with a class of measurable cardinals, none of which is a limit of the others. Collapse them by upwards Easton forcing to be the successors of regular cardinals. Note that the next measurable cardinal is still measurable when it is time to collapse it.” We leave the details of the construction to the reader. \square

As a result we find

Corollary 27. *If it is consistent that there is a proper class of measurable cardinals, then it is consistent that Conjecture 2 holds.*

Corollary 28. *If it is consistent that there is a proper class of measurable cardinals, then it is consistent that Conjecture 1 holds.*

6 Examples of multiboard games

In this section we give in ZFC a number of *ad hoc* illustrations of the phenomenon that ONE has a winning strategy in a multiboard version of an undetermined Gale–Stewart game.

Theorem 29. *Let $0 < \kappa \leq \aleph_0$ be a cardinal number.*

- (1) *There is a set $A_\kappa \subseteq {}^\omega 2$ such that $\text{GS}_2(A_\kappa, n)$ is undetermined for $0 < n < \kappa$, while ONE has a winning strategy in $\text{GS}_2(A_\kappa, \kappa)$.*
- (2) *There is a set $A_\kappa \subseteq {}^\omega \omega$ such that $\text{GS}_\omega(A_\kappa, n)$ is undetermined for $0 < n < \kappa$, while ONE has a winning strategy in $\text{GS}_\omega(A_\kappa, \omega)$ which ensures that TWO wins on fewer than κ boards. (Hence ONE has a winning strategy in $\text{GS}_\omega(A_\kappa, \kappa)$.)*

Proof. As the proofs of (1) and (2) are similar, we prove only (2).

Proof of (2):

For $x = (x_n : n < \omega)$ in ${}^\omega \omega$, define $B(x) := \{x_{2n} + x_{2n+1} : x_{2n} > 0\}$. For each $\mathcal{B} \subseteq \mathcal{P}(\omega)$, define $Y(\mathcal{B}) := \{x \in {}^\omega \omega : B(x) \notin \mathcal{B}\}$.

The set A_κ used in the proof of (2) is, for an appropriately chosen \mathcal{B} , of the form $Y(\mathcal{B})$. In preparation for the construction of A_κ we now introduce the following concepts:

For two infinite subsets A and B of ω we say that A is *interlaced with* B if for some $k \in \omega$ we have

$$a_0 < b_0 < a_1 < b_1 < \dots < a_n < b_n < \dots$$

where $(a_n : n < \omega)$ and $(b_n : n < \omega)$ are the increasing enumerations of $A \setminus k$ and $B \setminus k$. We write $I(A, B)$ to denote that A is interlaced with B . Observe that this is a symmetric binary relation on the power set of ω . For $A \subseteq \omega$, define

$$I(A) := \{B \subseteq \omega : I(A, B)\}.$$

Note that if A is finite then $I(A)$ is empty. Also note that if $I(A)$ is nonempty, then it has at least \aleph_0 elements. For example, if A is the set of even numbers then $I(A)$ is countable. If $A = \{a_n : n \in \omega\}$ where $a_0 < a_1 < \dots < a_n < \dots$, and if $a_{n+1} - a_n > 1$ for all but finitely many n and $a_{n+1} - a_n > 2$ for infinitely many n , then $I(A)$ has 2^{\aleph_0} elements.

For an ordinal $\alpha \leq \omega$, let G^α be the following game structure of length ω between players ONE and TWO. First ONE chooses an $f_0 \in {}^\alpha \omega$, then TWO chooses an $f_1 \in {}^\alpha \omega$, then ONE chooses an $f_2 \in {}^\alpha \omega$, then TWO chooses an $f_3 \in {}^\alpha \omega$, and so on.

For a play $F = \langle f_n : n < \omega \rangle \in {}^\omega ({}^\alpha \omega)$ of G^α , define for each $i \in \alpha$:

$$B_i^F := B((f_0(i), f_1(i), f_2(i), \dots)) = \{f_{2n}(i) + f_{2n+1}(i) : f_{2n}(i) > 0\}.$$

Lemma 30. *There is a strategy σ for ONE in G^ω such that, if F is a σ -play of G^ω , then $I(B_i^F, B_j^F)$ holds for all $i, j \in \omega$ with $i \neq j$.*

Proof. Let $(j_n : n < \omega) = (0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, 0, \dots)$. ONE's strategy is to choose f_{2n} so that $f_{2n}(i) = 0$ for $i \neq j_n$, while $f_{2n}(j_n) = f_{2n-2}(j_{n-1}) + f_{2n-1}(j_{n-1}) + 1$ if $n > 0$, and $f_0(j_0) = 1$. □

Lemma 31. *Suppose that $0 < \kappa \leq \omega$. If $\mathcal{B} \subseteq \mathcal{P}(\omega)$ is such that for any sequence $(B_i : i \in \kappa)$ of elements of \mathcal{B} there are $i < j < \kappa$ for which $I(B_i, B_j)$ fails, then there is a strategy σ for ONE in $\text{GS}_\omega(Y(\mathcal{B}), \kappa)$ which ensures that TWO wins on fewer than κ boards.*

Proof. Let κ and \mathcal{B} be as in the hypotheses. Let σ be the strategy from Lemma 30 for ONE. We show that σ has the claimed property. For let F be a σ -play. Consider the sequence $(B_i^F : i < \omega)$. For any $i \neq j$ we have $I(B_i^F, B_j^F)$. Thus, as $\kappa \leq \omega$, for any $J \subseteq \omega$ with $|J| = \kappa$, considering $(B_i^F : i \in J)$, there is a B_i^F not in \mathcal{B} , meaning $x \in Y(\mathcal{B})$ where $B_i^F = B(x)$. In particular, fewer than κ of the sets B_i^F are members of \mathcal{B} and thus fewer than κ of the corresponding sequences are not members of $Y(\mathcal{B})$. It follows that if ONE plays the game $\text{GS}_\omega(Y(\mathcal{B}), \kappa)$ according to strategy σ , then TWO wins on fewer than κ boards. □

It follows that for \mathcal{B} as in Lemma 31, TWO does not have a winning strategy in $\text{GS}_\omega(Y(\mathcal{B}))$. Therefore the games $\text{GS}_\omega(Y(\mathcal{B}), n)$ for $0 < n < \kappa$, if they are not wins for ONE, are undetermined.

Towards the rest of the proof of Theorem 29 (2) we now define: For $A, B \subseteq \omega$ we say that $D(A, B)$ holds if either of A or B is finite, or else there are for each $k \in \omega$ elements $a_1 < a_2 < a_3$ of $A \setminus k$ for which there is no element b of B such that $a_1 < b < a_3$. We obviously have:

Lemma 32. *If $D(A, B)$ holds, then $I(A) \cap I(B) = \emptyset$.* □

Lemma 33. *Let $0 < n < \omega$ as well as a strategy σ for ONE in G^n be given. For each $t \in {}^\omega 2$ there is a σ -play F_t such that if $s, t \in {}^\omega 2$ are different, then for all $i, j \in n$, $D(B_i^{F_s}, B_j^{F_t})$ holds.* □

The proof of Lemma 33 is left to the reader.

Lemma 34. *Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $|\mathcal{A}| < 2^{\aleph_0}$. If $0 < n < \omega$ and σ is a strategy for ONE in G^n , then there is a σ -play F of G^n such that $I(B_i^F) \cap \mathcal{A} = \emptyset$ for all $i \in n$.*

Proof. Lemmas 32 and 33. □

Lemma 35. *Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $|\mathcal{A}| < 2^{\aleph_0}$. If $0 < n < \omega$ and σ is a strategy for ONE in G^n , then there is a $\mathcal{B} \subseteq \mathcal{P}(\omega)$ such that $|\mathcal{B}| \leq n$, $I(B) \cap \mathcal{A} = \emptyset$ for all $B \in \mathcal{B}$, and σ is not a winning strategy for ONE in $\text{GS}_\omega(Y(\mathcal{B}), n)$.*

Proof. Choose F as in Lemma 34 and let $\mathcal{B} = \{B_i^F : i \in n\}$. □

Now we can complete the proof of item (2) of Theorem 29:
 Let $\{(n_\xi, \sigma_\xi) : \xi < 2^{\aleph_0}\}$ enumerate the set

$$\{\langle n, \sigma \rangle : 0 < n < \omega \text{ and } \sigma \text{ a strategy for ONE in } G^n\}.$$

Using Lemma 35, construct a sequence $(\mathcal{B}_\xi : \xi < 2^{\aleph_0})$ so that:

- $\mathcal{B}_\xi \subseteq \mathcal{P}(\omega)$;
- $|\mathcal{B}_\xi| \leq n_\xi$;
- $I(B_1, B_2)$ does not hold if $B_1 \in \mathcal{B}_{\xi_1}$, $B_2 \in \mathcal{B}_{\xi_2}$, and $\xi_1 \neq \xi_2$, and
- σ_ξ is not a winning strategy for ONE in $\text{GS}_\omega(Y(\mathcal{B}_\xi), n_\xi)$.

For $0 < \kappa \leq \omega$ let

$$A_\kappa = Y\left(\bigcup\{\mathcal{B}_\xi : n_\xi < \kappa\}\right).$$

Clearly, for $0 < n < \kappa$, ONE cannot have a winning strategy in $\text{GS}_\omega(A_\kappa, n)$, and the rest follows from Lemma 31. \square

Consider next the Morton Davis game $\mathbf{G}^*(Y)$, see Davis [1]: In inning $n < \omega$ ONE first chooses a finite sequence o_n of zeroes and ones; TWO responds with $t_n \in \{0, 1\}$. ONE wins a play if

$$o_1 \frown (t_1) \frown o_2 \frown (t_2) \frown \cdots \frown o_n \frown (t_n) \frown \cdots \in Y;$$

else, TWO wins. Evidently this can be coded as a Gale-Stewart game $\text{GS}_\omega(S)$ for some $S \subseteq {}^\omega\omega$.

Theorem 36 (Davis). *Let $Y \subseteq {}^\omega\{0, 1\}$ be given. Then TWO has a winning strategy in $\mathbf{G}^*(Y)$ if, and only if, Y is countable, and ONE has a winning strategy in $\mathbf{G}^*(Y)$ if, and only if, Y contains a nonempty perfect subset.* \square

Theorem 37. *There is a set $S \subseteq {}^\omega\omega$ such that for $1 \leq \kappa < 2^{\aleph_0}$ the game $\text{GS}_\omega(S, \kappa)$ is undetermined.*

Proof. If $2^{\aleph_0} = \aleph_1$, the result follows from Lemma 38. Thus we may assume that $2^{\aleph_0} > \aleph_1$. Choose a set $Y \subseteq {}^\omega\{0, 1\}$ with $\aleph_0 < |Y| < 2^{\aleph_0}$. Since Y is uncountable, Theorem 36 implies that TWO has no winning strategy in the game $\mathbf{G}^*(Y)$.

We show that for $\kappa < 2^{\aleph_0}$, ONE has no winning strategy in the κ -board game $\mathbf{G}^*(Y, \kappa)$.

Fix $\kappa < 2^{\aleph_0}$, and consider any strategy σ for ONE. For $\alpha < \kappa$ and $t \in {}^\omega\{0, 1\}$, consider the play of $\mathbf{G}^*(Y, \kappa)$ in which ONE follows σ and, for all $n < \omega$, in inning n , TWO plays $t(n)$ on each board. For each α , let $f_\alpha(t)$ be the element on board α resulting from ONE applying the strategy σ , and TWO following t . For fixed α , the function

$$f_\alpha : {}^\omega\{0, 1\} \rightarrow {}^\omega\{0, 1\}$$

is one-to-one. Thus for each α , we have $|\{t \in {}^\omega\{0,1\} : f_\alpha(t) \in Y\}| \leq |Y|$. But then we have

$$\left| \bigcup_{\alpha \in \kappa} \{t \in {}^\omega\{0,1\} : f_\alpha(t) \in Y\} \right| \leq \kappa \cdot |Y| < 2^{\aleph_0}.$$

Thus, if TWO follows $q \in {}^\omega\{0,1\} \setminus (\bigcup_{\alpha \in \kappa} \{t \in {}^\omega\{0,1\} : f_\alpha(t) \in Y\})$, this defeats ONE's strategy σ . \square

Lemma 38. *There is a set $S \subseteq {}^\omega\{0,1\}$ such that the game $\text{GS}_2(S, \kappa)$ is undetermined for every nonzero cardinal κ such that $2^\kappa \leq 2^{\aleph_0}$.*

Proof. The construction is a modified version of the usual diagonalization argument showing that there is some S with $\text{GS}_2(S)$ undetermined.

Let $\text{H}(S, \kappa)$ be the variant of $\text{GS}_2(S, \kappa)$ in which TWO is required to make the same move on all κ boards. Thus, at move n , ONE chooses a function $f_n : \kappa \rightarrow \{0,1\}$, and then TWO chooses a *constant* function $g_n : \kappa \rightarrow \{0,1\}$. It will suffice to construct a set $S \subseteq {}^\omega\{0,1\}$ so that TWO has no winning strategy in $\text{GS}_2(S)$ while, for each nonzero cardinal κ with $2^\kappa \leq 2^{\aleph_0}$, ONE has no winning strategy in $\text{H}(S, \kappa)$.

Observe that, if $2^\kappa \leq 2^{\aleph_0}$, then there are only 2^{\aleph_0} strategies for ONE in $\text{H}(S, \kappa)$. Let $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ be the set of all strategies for TWO in $\text{GS}_2(S)$, and let $\{(\kappa_\alpha, \sigma_\alpha) : \alpha < 2^{\aleph_0}\}$ be the set of all pairs (κ, σ) where κ is a nonzero cardinal with $2^\kappa \leq 2^{\aleph_0}$ and σ is a strategy for ONE in $\text{H}(S, \kappa)$.

Let $\lambda = \text{cf } 2^{\aleph_0}$. We are going to define, for each $\alpha < 2^{\aleph_0}$, a point $s_\alpha \in {}^\omega\{0,1\}$ and a set $T_\alpha \subseteq {}^\omega\{0,1\}$ with $|T_\alpha| \leq \kappa_\alpha < \lambda$. The point s_α is decreed to be a winning play for ONE, and is chosen so as to defeat TWO's strategy τ_α in $\text{GS}_2(S)$; the elements of T_α are decreed to be winning for TWO, are are chosen so as to defeat ONE's strategy σ_α in $\text{H}(S, \kappa_\alpha)$. Of course we require that $s_\alpha \notin T_\beta$ for all $\alpha, \beta < 2^{\aleph_0}$.

Let $\alpha < 2^{\aleph_0}$ and suppose that s_β and T_β have been defined for all $\beta < \alpha$. Since $\alpha < 2^{\aleph_0}$ and $|T_\beta| < \lambda$ for all β , we have $|\bigcup_{\beta < \alpha} T_\beta| < 2^{\aleph_0}$. Inasmuch as there are 2^{\aleph_0} different τ_α -plays of $\text{GS}_2(S)$, we can choose a τ_α -play s_α which is not in any T_β with $\beta < \alpha$.

Next, choose a sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots) \in {}^\omega\{0,1\}$ which differs from all the sequences $(s_\beta(1), s_\beta(3), s_\beta(5), \dots)$ for $\beta \leq \alpha$. Let $(f_0, g_0, f_1, g_1, f_2, g_2, \dots)$ be the σ_α -play of $\text{H}(S, \kappa_\alpha)$ in which TWO plays the constant function $g_n(\xi) = \varepsilon_n$ for every $n < \omega$. Let T_α consist of the sequences

$$\begin{aligned} t_{\alpha, \xi} &= (f_0(\xi), g_0(\xi), f_1(\xi), g_1(\xi), f_2(\xi), g_2(\xi), \dots) \\ &= (f_0(\xi), \varepsilon_0, f_1(\xi), \varepsilon_1, f_2(\xi), \varepsilon_2, \dots) \end{aligned}$$

for $\xi \in \kappa_\alpha$. Then $|T_\alpha| \leq \kappa_\alpha$, and $s_\beta \notin T_\alpha$ for all $\beta \leq \alpha$.

It is clear from the construction that the set $S = \{s_\alpha : \alpha < 2^{\aleph_0}\}$ has the desired properties. \square

7 Open problems

It is consistent, relative to the consistency of the existence of a proper class of measurable cardinals, that Conjecture 2, and thus Conjecture 1, are true.

Question 39. *Is Conjecture 1 a theorem of ZFC?*

Question 40. *Is Conjecture 2 a theorem of ZFC?*

We saw also that if it is consistent that there is a measurable cardinal, then it is consistent that for each $S \subseteq {}^\omega\omega$ the game $\text{GS}_\omega(S, (2^{\aleph_0})^+)$ is determined. It is not clear if this result is optimal.

Question 41. *Is there a set $S \subseteq {}^\omega\omega$ such that $\text{GS}_\omega(S, 2^{\aleph_0})$ is undetermined?*

Recall the Morton Davis game $\text{G}^*(S)$ introduced right before Theorem 36. The κ -board version $\text{G}^*(S, \kappa)$ was considered in the proof of Theorem 37:

Question 42. *Fix κ with $1 < \kappa \leq 2^{\aleph_0}$. Characterize those subsets $S \subseteq {}^\omega\{0, 1\}$ for which ONE has a winning strategy in $\text{G}^*(S, \kappa)$.*

We also do not know the answer to the following questions:

Question 43. *Is there a set $S \subseteq {}^\omega\omega$ such that $\text{GS}_\omega(S, 2^{\aleph_0})$ is determined but $\text{GS}_\omega(S, \aleph_0)$ is undetermined?*

Question 44. *If, for some cardinal λ , there is a cardinal κ such that $\text{D}(\lambda, \kappa)$ holds, is then $\kappa \geq (2^\lambda)^+$?*

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F. Galvin
 Department of Mathematics,
 University of Kansas,
 Lawrence, KS 66045
 e-mail: bof@sunflower.com

M. Scheepers
 Department of Mathematics,
 Boise State University,
 Boise, ID 83725
 e-mail: mscheepe@boisestate.edu