COMPUTABLE RAMSEY'S THEOREM FOR PAIRS NEEDS INFINITELY MANY Π_2^0 SETS

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ABSTRACT. In [1], Theorem 4.2, Jockusch proves that for any computable k-coloring of pairs of integers, there is an infinite Π_2^0 homogeneous set. The proof uses a countable collection of Π_2^0 sets as potential infinite homogeneous sets. In a remark preceding the proof, Jockusch states without proof that it can be shown that there is no computable way to prove this result with a finite number of Π_2^0 sets. We provide a proof of this latter fact.

1. INTRODUCTION

In [1], Jockusch initiated the study of the effective content of Ramsey's theorem, stated below. This study has proved to be enormously fruitful in effective combinatorics, and also in reverse mathematics. In Theorem 4.2 of [1], Jockusch proves that for any computable k-coloring of pairs of integers, there is an infinite Π_2^0 homogeneous set. Before this proof, he makes the remark that even for 2-colorings of pairs of integers (basic recursive partitions, in his language), it can be shown that there is no uniform computable way to take an index for an arbitrary computable coloring, and to produce a finite number of indices of Π_2^0 sets with the property that one of those Π_2^0 sets will be an infinite homogeneous set for that coloring.

The proof of this fact appears to have been lost, and recently Jockusch has asked for a proof, which we present here.

2. Definitions

Definition 2.1. A k-coloring of the *n*-element subsets of \mathbb{N} is a function $c : [\mathbb{N}]^n \to k$, from the set of unordered *n*-element subsets of \mathbb{N} , to k.

We think of such a coloring as a rule that assigns a color to every *n*-element subset of \mathbb{N} , using up to *k* different colors.

Ramsey's theorem is then the following theorem of combinatorics.

Theorem 2.2 (Ramsey's Theorem). For any $n, k \ge 1$, and any k-coloring $c : [\mathbb{N}]^n \to k$, there exists an infinite subset $H \subseteq \mathbb{N}$ such that $c \upharpoonright [H]^n$ is a constant function.

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We call such an H an infinite homogeneous set for c.

In this paper, we will be primarily concerned with the case when n = k = 2.

In [1], Jockusch proves the following.

Theorem 2.3 (Jockusch, [1], Theorem 4.2). If $c : [\mathbb{N}]^2 \to k$ is a computable k-coloring of pairs, then there exists an infinite Π_2^0 homogeneous set for c.

We prove the following.

Theorem 2.4. There does not exist a partial computable f with the property that for any e, if e is the code of a total computable 2-coloring $c : [\mathbb{N}]^2 \to 2$ of pairs, then f(e) halts, producing the code for a finite set $\{a_0, a_1, \ldots, a_k\}$ of indices for Π_2^0 sets with the property that at least one of those Π_2^0 sets is an infinite homogeneous subset for c.

Indeed, we prove slightly more: that there is no such f where f(e) is the code for a c.e. set $W_{f(e)}$ enumerating finitely many codes for Π_2^0 sets.

3. Trains

Definition 3.1. An *n*-train is a sequence of distinct sets of size $n, \tau_0, \tau_1, \ldots, \tau_m$ such that for every $a \in \tau_{i+1} \setminus \tau_i, a > \tau_i$.

For instance

$$\{1, 2, 3\}, \{2, 3, 5\}, \{5, 7, 9\}, \{5, 9, 12\}$$

is a 3-train.

If $0 \leq k < n$ we write $\tau(k)$ for the k + 1-st element of τ in the usual ordering of N.

Our main tool is the following combinatorial lemma. We color pairs from R, B, and if $\iota \in \{R, B\}$, we write $\overline{\iota}$ for the opposite color: $\overline{R} = B$, $\overline{B} = R$.

Theorem 3.2. Suppose that for each j < n, $\tau_0^j, \tau_1^j, \ldots, \tau_{m_j}^j$ is an n+1-train. Let $c : [\mathbb{N}]^2 \to \{R, B\}$ be given. Then there is a coloring $c^* : \bigcup \tau_i^j \to \{R, B\}$ such that for each τ_i^j on which $c \upharpoonright [\tau_i^j]^2 = \iota$ homogeneously, there is an $a \in \tau_i^j$ with $c^*(a) = \overline{\iota}$.

Proof. We define an ordering \prec on the sets τ_i^j : we say $\tau_i^j \prec \tau_{i'}^{j'}$ if, taking k < n+1 largest such that $\tau_i^j(k) \neq \tau_{i'}^{j'}(k)$, we have $\tau_i^j(k) > \tau_{i'}^{j'}(k)$. (This is the opposite of the reverse lexicographic order, which we choose not to refer to as the reverse reverse lexicographic order.) Note that for a fixed j and i' < i, we have that $\tau_i^j(n) > \tau_{i'}^j(n)$, and so $\tau_i^j \prec \tau_{i'}^j$.

For each r, let j_r, i_r be such that $\tau_{i_r}^{j_r}$ is the r-th element in this ordering. We define the coloring c^* in stages, considering $\tau_{i_r}^{j_r}$ at the r-th stage. We let $c_0 = \emptyset$. At stage r, we meet the rth requirement: that if $c \upharpoonright [\tau_{i_r}^{j_r}]^2 = \iota$ homogeneously, then there is at least one $a \in \tau_{i_r}^{j_r}$ such that $c^*(a) = \overline{\iota}$. Suppose we have constructed a partial function c_r^* and, for some set of s < r, chosen values a_s so that:

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- at stage r, c_r^* is only defined on a_s for s < r,
- if a_s is defined then $a_s = \tau_{i_s}^{j_s}(k)$ for some k > 0, $c \upharpoonright [\tau_{i_s}^{j_s}]^2 = \iota_s$ homogeneously, and $c_r^*(a_s) = \overline{\iota}_s$,
- for each s < r, if $c \upharpoonright [\tau_{i_s}^{j_s}]^2 = \iota_s$ homogeneously then there is an $s' \leq s$ so that $a_{s'} \in \tau_{i_s}^{j_s}$ with $c_r^*(a_{s'}) = \overline{\iota}_s$, and
- if s' < s < r, $a_{s'}, a_s$ are both defined, and $a_{s'} \in \tau_{i_s}^{j_s}$ then $c_r^*(a_s) \neq c_r^*(a_{s'})$.

The first clause asserts that we meet our requirements in order. The second asserts that if we acted at stage s, then we acted because there was a requirement to meet, and we acted to meet that requirement. It furthermore asserts that we did not act with the smallest element of $\tau_{i_s}^{j_s}$. The third clause asserts that each earlier requirement has been met. The final clause asserts that if a requirement was already met, then we did not act again to meet it.

We make the following crucial observation: suppose $s' < s \leq r$, $j_{s'} = j_s$, $a_{s'} \in \tau_{i_s}^{j_s}$, and $c \upharpoonright [\tau_{i_s}^{j_s}]^2 = \iota_s$ homogeneously. Then $c_r^*(a_{s'}) = \overline{\iota}_s$. (This implies that a_s is not defined.) To see this, set $j = j_{s'} = j_s$ and observe that every $a \in \tau_{i_{s'}}^j \setminus \tau_{i_s}^j$ must have $a > \tau_{i_s}^j$. Since $\tau_{i_{s'}}^j(0) < a_{s'}$, we must have $\tau_{i_{s'}}^j(0) \in \tau_{i_s}^j$, and therefore $c(\tau_{i_{s'}}^j(0), a_{s'}) = \iota$. Therefore $c_r^*(a_{s'}) = \overline{\iota}$.

We now attempt to construct $c_{r+1}^* \supseteq c_r^*$. If $c \upharpoonright [\tau_{i_r}^{j_r}]^2$ is not homogeneous, we have no commitment regarding $c^* \upharpoonright \tau_{i_r}^{j_r}$, so set $c_{r+1}^* = c_r^*$. Suppose $c \upharpoonright [\tau_{i_r}^{j_r}]^2 = \iota$ homogeneously. If there is an s < r such that a_s is defined, $a_s \in \tau_{i_r}^{j_r}$, and $c^*(a_s) = \overline{\iota}$, then again we may set $c_{r+1}^* = c_r^*$. So suppose there is no such a_s . By the observation, if s < r, $j_s = j_r$, and

So suppose there is no such a_s . By the observation, if s < r, $j_s = j_r$, and a_s is defined, we have $a_s \notin \tau_{i_r}^{j_r}$.

If s' < s < r with $j_{s'} = j_s$ and $a_{s'}, a_s$ both defined, the observation implies that $a_{s'} \notin \tau_{i_s}^{j_s}$. Therefore $a_{s'} > \tau_{i_s}^{j_s}$. But $\tau_{i_s}^{j_s}(n) \ge \tau_{i_r}^{j_r}(n)$, so $a_{s'} \notin \tau_{i_r}^{j_r}$.

So c_r^* is defined on at most n-1 elements of $\tau_{i_r}^{j_r}$ —at most one for each j < n other than j_r . In particular, there are at least two elements in $\tau_{i_r}^{j_r}$ on which c_r^* is undefined; taking the larger to be a_r , we set $c_r^*(a_r) = \overline{\iota}$, and we have $a_r = \tau_{i_r}^{j_r}(k)$ for some k > 0.

We define c^* to be any extension of $\bigcup_r c_r^*$ to a total function on $\bigcup_r \tau_i^j$. \Box

4. Construction

Theorem 4.1. Fix finitely many Π_2 functionals given by formulas $\forall x \exists y R_0(z, x, y, c)$, ..., $\forall x \exists y R_{n-1}(z, x, y, c)$ depending on a coloring c. There is a computable c so that for each j < n, the set

$$S_j = \{ z \mid \forall x \exists y R_j(z, x, y, c) \}$$

fails to be an infinite homogeneous set for c.

Proof. We describe how, for a given b, we define c(a, b) for all a < b. Fix the value b and suppose we have defined c(a, a') for all a < a' < b. For

each j < n we define an n + 1-train by taking the set τ_i^j for $i \leq b$ to be the n + 1 smallest elements a < b such that $\forall x < i \exists y < bR_j(a, x, y, c)$ (where the computation is always true if c is not yet sufficiently defined to interpret $R_j(a, x, y, c)$). Let c^* be given by the theorem above, and extend c^* to be defined on all a < b by defining it arbitrarily where it is not already defined. Set $c(a, b) = c^*(a)$ for all a < b.

Suppose that for some j < n, the set S_j is infinite. Let τ^j be the n + 1 smallest elements of S_j . We claim that, for b sufficiently large, there is always some i so that $\tau_i^j = \tau^j$. For every $a < \tau^j(n)$ such that $a \notin \tau^j$, there is some i such that $\exists x \leq i \forall y \neg R_j(a, x, y, c)$, so certainly for every b, if $i' \geq i$ and $a \in \tau_i^j$, either $a \in \tau^j$ or $a > \tau^j$. Let i be large enough to witness this bound for all $a < \tau^j(n)$.

For each $a \in \tau^j$ and each $x \leq i$, there is some y such that $R_j(a, x, y, c)$. If b is big enough to bound these finitely many values of y, it must be the case that $\tau_i^j = \tau^j$. Therefore for all sufficiently large $b, \tau_i^j = \tau^j$.

Since S_j is infinite, let *b* be some element of S_j sufficiently large so that $\tau_i^j = \tau^j$. If $c \upharpoonright [\tau^j] = \iota$ then there is some $a \in \tau^j$ with $c(a, b) = c^*(a) = \overline{\iota}$. Therefore S_j is not homogeneous.

We can now prove our main theorem:

Theorem 4.2. There is no partial computable f such that for any e, if e is the code of a total computable 2-coloring $c : [\mathbb{N}]^2 \to 2$ of pairs, then f(e) halts, producing the code for a c.e. set $W_{f(e)}$ enumerating a finite set $\{a_0, a_1, \ldots, a_{n-1}\}$ of indices for Π_2^0 sets with the property that at least one of those Π_2^0 sets is an infinite homogeneous subset for c.

Proof. Let f be a partial computable function such that for any e, if e is the code of a total computable 2-coloring $c : [\mathbb{N}]^2 \to 2$ of pairs, then f(e) halts, producing the code for a c.e. set $W_{f(e)}$ enumerating a set $\{a_0, a_1, \ldots, a_{n-1}\}$ of indices for Π_2^0 sets.

We define a coloring c as follows. Via the recursion theorem, we obtain the code for c, and begin evaluating f(e). If f(e) has not halted after bsteps, we define c(a, b) for a < b arbitrarily. If f(e) has halted, then we begin enumerating $W_{f(e)}$. If $W_{f(e)}$ is empty after b steps, we continue to define c(a, b) for a < b arbitrarily. Each time that $W_{f(e)}$ enumerates a new element, we continue the construction of c as in the proof of the previous theorem, assuming that $W_{f(e)}$ will never enumerate any new elements.

If $W_{f(e)}$ is indeed finite, then at some point this assumption will be true, and we will be able to conclude that no a_i is the code for an infinite homogeneous Π_2^0 subset for c. Note that c always produces a total computable 2-coloring, whether or not f(e) halts, so the recursion theorem must produce a value e on which f(e) does halt. \Box

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References

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