The reverse mathematics of non-decreasing subsequences

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THE REVERSE MATHEMATICS OF NON-DECREASING SUBSEQUENCES

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Abstract. Every function over the natural numbers has an infinite subdomain on which the function is non-decreasing. Motivated by a question of Dzhafarov and Schweber, we study the reverse mathematics of variants of this statement. It turns out that this statement restricted to computably bounded functions is computationally weak and does not imply the existence of the halting set. On the other hand, we prove that it is not a consequence of Ramsey's theorem for pairs. This statement can therefore be seen as an arguably natural principle between the arithmetic comprehension axiom and stable Ramsey's theorem for pairs.

1. Introduction

A *non-decreasing subsequence* for a function $f : \mathbb{N} \to \mathbb{N}$ is a set $X \subseteq \mathbb{N}$ such that $f(x) \leq f(y)$ for every pair $x < y \in X$. Every function over $\mathbb{N} \to \mathbb{N}$ admits an infinite non-decreasing subsequence. Moreover, such a sequence can be computably, but not uniformly obtained from the function f. Indeed, given $f : \mathbb{N} \to \mathbb{N}$, either there is a value $y \in \mathbb{N}$ for which the set $S_y = \{x \in \mathbb{N} : f(x) = y\}$ is infinite, or for every $y \in \mathbb{N}$, there is a threshold $t \in \mathbb{N}$ after which $f(x) > y$ for every $x > t$. In the former case, the set S_y is an infinite f-computable non-decreasing subsequence for f, while in the latter case, one can f-computably thin out the set $\mathbb{N}$ to obtain an infinite, strictly increasing subsequence. This non-uniform argument can be shown to be necessary by Weihrauch reducing the limited principle of omniscience (LPO) to this statement [1].

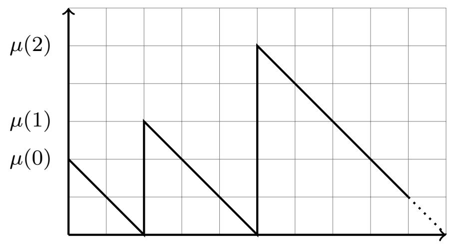
In this paper, we study the reverse mathematics of variants of this statement by considering various classes of non-computable functions over $\mathbb{N} \to \mathbb{N}$. This study is motivated by the following question of Dzhafarov and Schweber in MathOverflow [6] and taken up by Hirschfeldt in his open questions paper [7]. A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is *stable* if for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists. A set X is a *limit non-decreasing subsequence* for a stable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ if it is a non-decreasing subsequence of its limit function $\tilde{f} : \mathbb{N} \to \mathbb{N}$ defined by $\tilde{f}(x) = \lim_s f(x, s)$.

> Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable function such that $f(x, s+1) \leq f(x, s)$ for every $x, s \in \mathbb{N}$. Let X be an infinite limit non-decreasing subsequence for f. How complicated must such an X be? In particular, can it avoid computing the halting set?

Let LNS be the statement asserting the existence of an infinite limit non-decreasing subsequence for any such function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Liang Yu noticed that LNS implies the existence of a *diagonally non-computable* function, that is, a function $h : \mathbb{N} \to \mathbb{N}$ such that $h(e) \neq \Phi_e(e)$ for every $e \in \mathbb{N}$. We identify a natural strengthening of LNS that we call CNS, standing for "computably bounded non-decreasing subsequence". We prove that every computable instance of CNS admits low_2 solutions using the first jump control of Cholak, Jockusch and Slaman [2], and show that CNS is a computationally weak statement by proving that it does not imply weak weak König's lemma (WWKL). On the other hand, CNS is not a consequence of Ramsey's theorem for pairs (RT^2_2) and implies stable Ramsey's theorem for pairs (SRT^2_2). Finally, we separate LNS from CNS by proving that the former does not imply the stable ascending descending sequence principle (SADS) in reverse mathematics.

2. The weakness of non-decreasing subsequences

First, note that the general statement of the existence of a non-decreasing sequence for any function over $\mathbb{N} \to \mathbb{N}$ implies the existence of the halting set. Indeed, let μ be the *modulus* function of $\emptyset'$, that is, $\mu(x)$ is the minimal stage s such that $\emptyset'_s \upharpoonright x = \emptyset' \upharpoonright x$, and let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(x) = \mu(n) - k - 1$, where n and k are the unique solution to the equation $x = k + \sum_{j<n} \mu(j)$ satisfying $k < \mu(n)$.



The above argument uses finite decreasing sequences to ensure a sufficient amount of sparsity in the non-decreasing subsequences for f, to compute fast-growing functions. At first sight, such an argument does not seem to be applicable to LNS since the value of $\tilde{f}(x) = \lim_s f(x, s)$ is bounded by $f(x, 0)$ for every instance f of LNS. Therefore, one cannot force the solutions to have a hole of size more than $f(x, 0)$ starting from x. Actually, this computable bounding of the function $\tilde{f}$ is the essential feature of the weakness of the LNS statement. A function $f : \mathbb{N} \to \mathbb{N}$ is *computably bounded* if it is dominated by a computable function. Let CNS be the statement "Every computably bounded Δ^0_2 function admits an infinite non-decreasing subsequence." In particular, CNS generalizes LNS, in that every instance of LNS can be seen as the Δ^0_2 approximation of a computably bounded function. As a warm-up, we prove that CNS admits *cone avoidance.*

Theorem 2.1 Fix a set C and a set $A \not\leq_T C$. For every C-computably bounded function $f : \mathbb{N} \to \mathbb{N}$, there is an infinite set G non-decreasing for f such that $A \not\leq_T G \oplus C$.

Proof. Let $b : \mathbb{N} \to \mathbb{N}$ be a C-computable function bounding f. Assume that there is no infinite non-decreasing subsequence G such that $A \not\leq_T G \oplus C$, otherwise we are done. We will construct the set G using a variant of Mathias forcing.

Our forcing condition are pairs (F, X) where F is a finite set of integers non-decreasing for f, X is an infinite set such that $\max F < \min X$ and such that

$f(x) \leq f(y)$ for every $x \in F$ and $y \in X$. We furthermore require that $A \not\leq_T X \oplus C$. A condition $d = (E, Y)$ extends $c = (F, X)$ if d Mathias extends c, that is, $E \supseteq F$, $Y \subseteq X$ and $E \setminus F \subseteq X$. A set G *satisfies* a condition (F, X) if $F \subseteq G \subseteq F \cup X$. We start by proving that every sufficiently generic filter for this notion of forcing yields an infinite set.

Lemma 2.2 For every condition $c = (F, X)$, there is an extension $d = (E, Y)$ of c such that $|E| > |F|$.

Proof. Pick any $x \in X$. By strong cone avoidance of the infinite pigeonhole principle [5], there is an infinite set $Y \subseteq X \setminus [0, x]$ such that either f is constant over Y for some value smaller than $f(x)$, or $f(y) \geq f(x)$ for every $y \in Y$. In the former case, Y is an infinite non-decreasing subsequence for f such that $A \not\leq_T Y \oplus C$, contradicting our assumption. In the latter case, the condition $d = (F \cup \{x\}, Y)$ is the desired extension of c. □

A condition c *forces* a formula $\varphi(G)$ if $\varphi(G)$ holds for every infinite set G satisfying c. We now prove that $A \not\leq_T G \oplus C$ for every set G yielded by a sufficiently generic filter.

Lemma 2.3 For every condition $c = (F, X)$ and every Turing functional Γ, there is an extension d of c forcing $\Gamma^{G \oplus C} \neq A$.

Proof. For every $x \in \mathbb{N}$ and $i < 2$, let $\mathcal{F}_{x,i}$ be the $\Pi^{0,X\oplus C}_1$ class of all functions $g : \mathbb{N} \to \mathbb{N}$ dominated by b such that for every set $E \subset X$ non-decreasing for g,

$$\Gamma^{(F\cup E)\oplus C}(x) \uparrow \text{ or } \Gamma^{(F\cup E)\oplus C}(n) \downarrow \neq i$$

Also define $P = \{(x, i) : \mathcal{F}_{x,i} = \emptyset\}$. We have three outcomes. In the first case, $\mathcal{F}_{x,1-A(x)} \in P$ for some $x \in \mathbb{N}$. In other words, $\mathcal{F}_{x,1-A(x)} = \emptyset$. In particular, $f \notin \mathcal{F}_{x,1-A(x)}$, so there is a finite set $E \subseteq X$ non-decreasing for f, such that $\Gamma^{(F\cup E)\oplus C}(x) \downarrow = 1 - A(x)$. Apply strong cone avoidance avoidance of the infinite pigeonhole principle as in Lemma 2.2 to obtain an infinite set $Y \subseteq X$ such that $d = (F \cup E, Y)$ is a valid extension of c forcing $\Gamma^{G\oplus C}(x) \downarrow \neq A(x)$.

In the second case, there is some $x \in \mathbb{N}$ such that $\mathcal{F}_{x,A(x)} \notin P$. By the cone avoidance basis theorem [10], there is some $g \in \mathcal{F}_{x,A(x)}$ such that $A \not\leq_T g \oplus X \oplus C$. We can $g \oplus X$-computably thin out the set X to obtain an infinite set Y non-decreasing for g. In particular the condition $d = (F, Y)$ is a valid extension of c forcing $\Gamma^{G\oplus C}(x) \uparrow$ or $\Gamma^{G\oplus C}(x) \downarrow \neq A(x)$.

In the last case, for every $x \in \mathbb{N}$ and $i < 2$, $(x, i) \in P \leftrightarrow A(x) = i$. This case cannot happen, since otherwise $A \leq_T P \leq_T X \oplus C$, contradicting our assumption. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega)$, where $c_s = (F_s, X_s)$. The filter $\mathcal{F}$ yields a unique set $G = \bigcup_s F_s$. By Lemma 2.2, the set G is infinite, and by definition of a condition, G is non-decreasing for f. By Lemma 2.3, $A \not\leq_T G \oplus C$. This completes the proof. □

König's lemma asserts that every infinite, finitely branching tree admits an infinite path. Weak König's lemma (WKL) is the restriction of König's lemma to binary trees. WKL plays an important role in reverse mathematics as many statements happen to be equivalent to it [13]. It is therefore natural to compare

CNS and LNS to weak König's lemma. Actually, we will prove that CNS does not imply an even weaker variant of König's lemma, namely, weak weak König's lemma. A binary tree $T \subseteq 2^{<\mathbb{N}}$ has *positive measure* if $\lim_s \frac{|\{\sigma \in T : |\sigma| = s\}|}{2^s} > 0$. Weak weak König's lemma (WWKL) is the restriction of WKL to binary trees of positive measure. It can be seen as asserting the existence of a random real, in the sense of Martin-Löf [4]. Liu [11] introduced the notion of constant-bound enumeration avoidance to separate Ramsey's theorem for pairs from weak weak König's lemma. We shall use the same notion to separate CNS from WWKL.

A *k-enumeration* (or k-enum) of a class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is a sequence $D_0, D_1, \dots$ such that for each $n \in \mathbb{N}$, $|D_n| \leq k$, $(\forall \sigma \in D_n)|\sigma| = n$ and $\mathcal{C} \cap [D_n] \neq \emptyset$, where $[D_n]$ denotes the clopen set of reals in the Cantor space extending any string in D_n. A *constant-bound enumeration* (or c.b-enum) of $\mathcal{C}$ is a k-enum of $\mathcal{C}$ for some $k \in \mathbb{N}$. We now prove that CNS does not imply weak weak König's lemma over $\mathsf{RCA_0}$.

Theorem 2.4 Fix a set C and a class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ with no C-computable c.b-enum. For every C-computably bounded function $f : \mathbb{N} \to \mathbb{N}$, there is an infinite non-decreasing subsequence G such that $\mathcal{C}$ has no $G \oplus C$-computable c.b-enum.

Proof. Let $b : \mathbb{N} \to \mathbb{N}$ be a C-computable function bounding f. Again, assume that there is no infinite non-decreasing subsequence G such that $\mathcal{C}$ has no $G \oplus C$-computable c.b-enum, otherwise we are done. We will construct the set G using another variant of Mathias forcing.

Our forcing condition are tuples (F, X, S) where F is a finite set of integers, X is an infinite set such that $\max F < \min X$, and S is a finite collection of functions over $\mathbb{N} \to \mathbb{N}$ dominated by b, and such that $g(x) \leq g(y)$ for every $x \in F$, $y \in X$ and $g \in S$. We furthermore require that $\mathcal{C}$ has no $X \oplus C$-computable c.b-enum. A condition $d = (E, Y, T)$ extends $c = (F, X, S)$ if $E \supseteq F$, $Y \subseteq X$, $T \supseteq S$ and $E \setminus F$ is a non-decreasing subset of X for every $g \in S$. A set G *satisfies* a condition (F, X, S) if $F \subseteq G \subseteq F \cup X$ and $G \setminus F$ is non-decreasing for every $g \in S$. In particular, every infinite set satisfying the condition $(\emptyset, \omega, \{f\})$ is an infinite non-decreasing sequence for f.

Given a condition $c = (F, X, S)$, we let $\#(c)$ be the number of functions $g \in S$ such that g is not constant over X. We now prove that every sufficiently generic filter for this notion of forcing yields an infinite set.

Lemma 2.5 For every condition $c = (F, X, S)$, there is an extension $d = (E, Y, S)$ of c such that either $\#(d) < \#(c)$, or $|E| > |F|$.

Proof. Suppose that $S = \{g_0, \dots, g_{k-1}\}$. Pick any $x \in X$. By iteratively applying strong c.b-enum avoidance of the infinite pigeonhole principle [11], define a finite sequence $X = X_0 \supseteq X_1 \supseteq \dots \supseteq X_k$ of infinite sets such that for each $i < k$, $\mathcal{C}$ has no $X_{i+1} \oplus C$-computable c.b-enum and either there is some $n < g_i(x)$ such that $g_i(y) = n$ for each $y \in X_{i+1}$, or $g_i(y) \geq g_i(x)$ for each $y \in X_{i+1}$. If we are in the former case for some $i < k$, then the condition $d = (F, X_{i+1}, S)$ is an extension of c such that $\#(d) < \#(c)$. Otherwise, the condition $d = (F \cup \{x\}, X_k, S)$ is the desired extension of c. $\square$

We now prove that $\mathcal{C}$ has no $G \oplus C$-computable c.b-enum for every set G yielded by a sufficiently generic filter.

Lemma 2.6 For every condition $c = (F, X, S)$, every $k \in \mathbb{N}$ and every Turing functional Γ, there is an extension d of c such that either $\#(d) < \#(c)$, or d forces $\Gamma^{G \oplus C}$ not to be a valid k-enum of $\mathcal{C}$.

Proof. Suppose that $S = \{g_0, \dots, g_{m-1}\}$. For ease of notation, whenever $\Gamma^{G\oplus C}(n)$ halts, we will interpret $\Gamma^{G\oplus C}(n)$ as a finite set D_n of size k such that $|\sigma| = n$ for every $\sigma \in D_n$. For every $n \in \mathbb{N}$, let $C_n = \{\sigma \in 2^n : \mathcal{C} \cap [\sigma] \neq \emptyset\}$. For every set $D \subseteq 2^n$, let $\mathcal{F}_D$ be the the $\Pi^{0,X\oplus C}_1$ class of all m-tuples of functions $\langle h_0, \dots, h_{m-1}\rangle$ over $\mathbb{N} \to \mathbb{N}$ dominated by b, such that $h_i(y) \geq h_i(x)$ for each $x \in F$, $y \in X$ and $i < m$, and such that $\Gamma^{(F\cup E)\oplus C}(n) \uparrow$ or $\Gamma^{(F\cup E)\oplus C}(n) \cap D \neq \emptyset$ for every set $E \subset X$ non-decreasing for every h_i simultaneously. Finally, for each $n \in \mathbb{N}$, let $P_n = \{D \subseteq 2^n : \mathcal{F}_D \neq \emptyset\}$. We have three outcomes.

In the first case, $C_n \not\in P_n$ for some $n \in \mathbb{N}$. In other words, $\mathcal{F}_{C_n} = \emptyset$. In particular, $\langle g_0, \dots, g_{m-1}\rangle \not\in \mathcal{F}_{C_n}$, so there is a finite set $E \subseteq X$ non-decreasing for every g_i simultaneously, such that $\Gamma^{(F\cup E)\oplus C}(n) \cap \mathcal{C} = \emptyset$. Apply strong c.b-enum avoidance of the infinite pigeonhole principle as in Lemma 2.5 to obtain an infinite set $Y \subseteq X$ such that either $d = (F, Y, S)$ is an extension of c satisfying $\#(d) < \#(c)$, or $d = (F \cup E, Y, S)$ is a valid extension of c forcing $[\Gamma^{G\oplus C}(n)] \cap \mathcal{C} = \emptyset$.

In the second case, there is some $n \in \mathbb{N}$ such that for every k-partition $\mathcal{V}_0, \dots, \mathcal{V}_{k-1}$ of P_n, there is some $i < k$ such that $\bigcap \mathcal{V}_i = \emptyset$. For each $D \in P_n$, pick some $\langle h^D_0, \dots, h^D_{m-1}\rangle \in \mathcal{F}_D$. The condition $d = (F, X, T)$, where $T = S \cup \bigcup_{D\in P_n} \{h^D_0, \dots, h^D_{m-1}\}$, is a valid extension of c forcing $\Gamma^{G\oplus C}(n) \uparrow$. To see that, suppose that $\Gamma^{G\oplus C}(n) \downarrow= \{\sigma_0, \dots, \sigma_{k-1}\}$, and let $\mathcal{V}_i = \{D \in P_n : \sigma_i \in D\}$. We claim that $\mathcal{V}_0, \dots, \mathcal{V}_{k-1}$ forms a k-partition of P_n. Indeed, for any $D \in P_n$, since G satisfies d, $G \setminus F$ is non-decreasing for $h^D_0, \dots, h^D_{m-1}$, so $\{\sigma_0, \dots, \sigma_{k-1}\} \cap D \neq \emptyset$ and $D \in \mathcal{V}_i$ for some $i < k$. But then there is some $i < k$ such that $\bigcap \mathcal{V}_i = \emptyset$, contradicting $\sigma_i \in \bigcap \mathcal{V}_i$.

In the last case, for every $n \in \mathbb{N}$, $C_n \in P_n$ and there is a k-partition $\mathcal{V}_0, \dots, \mathcal{V}_{k-1}$ of P_n such that $\bigcap \mathcal{V}_i \neq \emptyset$ for each $i < k$. In this case, we claim that $\mathcal{C}$ admits an $X \oplus C$-computable k-enum, contradicting our assumption. First note that the set P_n is $X \oplus C$-co-c.e. uniformly in n. Therefore, given $n \in \mathbb{N}$, we can $X \oplus C$-computably find a stage s and a k-partition $\mathcal{V}_0, \dots, \mathcal{V}_{k-1}$ of $P_{n,s}$ such that $\bigcap \mathcal{V}_i \neq \emptyset$ for each $i < k$. Let D_n be the set obtained by picking a σ in each $\bigcap \mathcal{V}_i$. The set D_n has size k, and $\mathcal{C} \cap [D_n] \neq \emptyset$ since $C_n \in P_{n,s}$. The sequence $D_0, D_1, \dots$ is an $X \oplus C$-computable k-enum of $\mathcal{C}$. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega, \{f\})$, where $c_s = (F_s, X_s, S_s)$. The filter $\mathcal{F}$ yields a unique set $G = \bigcup_s F_s$. By Lemma 2.5, the set G is infinite, and by definition of the extension relation, G is non-decreasing for f. By Lemma 2.6, $G \oplus C$ computes no c.b-enum of $\mathcal{C}$. This completes the proof of Theorem 2.4. □

Corollary 2.7 CNS does not imply WWKL over RCA_0.

Proof. Let T be a computable tree of positive measure whose paths are Martin-Löf randoms. By Liu [11], $[T]$ has no computable c.b-enum. Iterate Theorem 2.4 to build a model $\mathcal{M}$ of CNS such that $[T]$ has no X-computable c.b-enum for any $X \in \mathcal{M}$. In particular, $T \in \mathcal{M}$, but there is no path through T in $\mathcal{M}$, so $\mathcal{M} \not\models \mathsf{WWKL}$. $\square$

The statement CNS enjoys two important properties. First, any infinite subset of a non-decreasing sequence is itself non-decreasing. Second, for any function $f : \mathbb{N} \to \mathbb{N}$ and any infinite set $X \subseteq \mathbb{N}$, one can find an infinite non-decreasing subsequence $Y \subseteq X$. These two features are shared with a whole family of statements coming from Ramsey theory. Given a set X and some $n \in \mathbb{N}$, let $[X]^n$ denote the set of all unordered n-tuples over X. Recall that Ramsey's theorem for n-tuples and k colors (RT^n_k) asserts the existence, for every coloring $f : [\mathbb{N}]^n \to k$, of an infinite *homogeneous set*, that is, a set $H \subseteq \mathbb{N}$ such that $[H]^n$ is monochromatic. SRT^2_k is the restriction of RT^2_k to stable colorings. Any stable coloring $f : [\mathbb{N}]^2 \to k$ can be seen as the Δ^0_2 approximation of the computably bounded function $\tilde{f} : \mathbb{N} \to \mathbb{N}$ defined by $\tilde{f}(x) = \lim_s f(x, s)$. Moreover, any infinite non-decreasing subsequence for $\tilde{f}$ is, up to finite changes, homogeneous for $\tilde{f}$, and can be $f \oplus H$-computably thinned out to obtain an infinite f-homogeneous set. By Chong, Lempp and Yang [3], this argument can be formalized in RCA_0, therefore CNS implies SRT^2_k over RCA_0 for every standard $k \in \mathbb{N}$. We will prove in the next section that the converse does not hold. For now, we show that LNS does not imply CNS over RCA_0 using the notion of preservation of hyperimmunity.

A function $f : \mathbb{N} \to \mathbb{N}$ is *hyperimmune* if it is not dominated by any computable function. An infinite set is hyperimmune if its principal function is hyperimmune, where the *principal function* of a set $X = \{x_0 < x_1 < \dots\}$ is defined by $p_X(n) = x_n$. A problem P *admits preservation of hyperimmunity* if for each set C, each countable collection of C-hyperimmune sets $A_0, A_1, \dots$, and each P-instance $X \leq_T Z$, there exists a solution Y to X such that the A's are $Y \oplus C$-hyperimmune. The author proved [12] that weak statements such as the stable version of the ascending descending principle (SADS) do not admit preservation of hyperimmunity, while the Erdős-Moser theorem (EM) does. We shall use this notion to separate LNS from SADS over RCA_0. In particular, this will separate LNS from CNS since CNS implies SRT^2_2, which itself implies SADS over RCA_0 (see [8]).

We have seen that for every computable instance $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of LNS, its limit function $\tilde{f}$ is computably bounded, and that this bounding feature is sufficient to obtain cone avoidance. We will now exploit another property enjoyed by $\tilde{f}$ to prove that LNS admits preservation of hyperimmunity. A function $g : \mathbb{N} \to \mathbb{N}$ is *eventually increasing* if each $y \in \mathbb{N}$, the preimage of $\{y\}$ by g is finite. For every computable instance $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of LNS with no computable solution, its limit function $\tilde{f}$ must be eventually increasing, otherwise the set $\{x : \tilde{f}(x, s) = y\}$ would be an infinite, computable non-decreasing subsequence for $\tilde{f}$ for the least y with infinitely many predecessors by $\tilde{f}$. Let ICNS be the restriction of CNS to eventually increasing functions. We will now prove that ICNS, and therefore LNS, admits preservation of hyperimmunity.

Theorem 2.8 Fix a set C and a countable sequence $A_0, A_1, \dots$ of C-hyperimmune sets. For every C-computably bounded, eventually increasing function $f : \mathbb{N} \to$

$\mathbb{N}$, there is an infinite non-decreasing subsequence G such that the A's are $G \oplus C$-hyperimmune.

Proof. Let $b : \mathbb{N} \to \mathbb{N}$ be a C-computable function bounding f. As usual, assume that there is no infinite set G non-decreasing for f such that the A's are $G \oplus C$-hyperimmune, otherwise we are done. We will build the set G by a variant of Mathias forcing.

A condition is a tuple (F, X) where F is a finite set of integers non-decreasing for f, X is an infinite set such that $\max F < \min X$, $f(x) \leq f(y)$ for every $x \in F$, $y \in X$, and the A's are $X \oplus C$-hyperimmune. The notions of condition extension and of set satisfaction inherit from Mathias forcing. We again prove that every sufficiently generic filter for this notion of forcing yields an infinite set.

Lemma 2.9 For every condition $c = (F, X)$, there is an extension $d = (E, Y)$ such that $|E| > |F|$.

Proof. Pick any $x \in X$ and let $Y = \{y \in X : y > x \wedge f(y) \geq f(x)\}$. The set Y is obtained from X by removing finitely many elements since f is eventually increasing, so the A's are $Y \oplus C$-hyperimmune. The condition $(F \cup \{x\}, Y)$ is the desired extension of c. $\square$

Next, we prove that every sufficiently generic filter yields a set G such that the A's are $G \oplus C$-hyperimmune.

Lemma 2.10 For every condition $c = (F, X)$, every Turing functional Γ and every $i \in \mathbb{N}$, there is an extension forcing $\Gamma^{G \oplus C}$ not to dominate p_{A_i}.

Proof. Let h be the partial $X \oplus C$-computable function which on input x searches for a finite set of integers U such that for every function $g : \mathbb{N} \to \mathbb{N}$ bounded by b, there is a finite set $E \subseteq X$ non-decreasing for g such that $\Phi_e^{(F \cup E) \oplus C}(x) \downarrow \in U$. If such a set U is found, $h(x) = \max U$, otherwise $h(x) \uparrow$. We have two cases.

Case 1: h is total. By $X \oplus C$-hyperimmunity of A_i, there is some x such that $h(x) < p_{A_i}(x)$. Let U be the finite set witnessing $h(x) \downarrow$. In particular, taking $g = f$, there is a finite set $E \subseteq X$ non-decreasing for f such that $\Phi_e^{(F \cup E) \oplus C}(x) \downarrow \in U$. By removing finitely many elements from X, we obtain a set Y such that $(F \cup E, Y)$ is a valid extension of c forcing $\Phi_e^{G \oplus C}(x) \downarrow < p_{A_i}(x)$.

Case 2: there is some x such that $h(x) \uparrow$. Let $\mathcal{C}$ be the $\Pi_1^{0, X \oplus C}$ class of functions $g : \mathbb{N} \to \mathbb{N}$ bounded by b such that for every finite set $E \subseteq X$ non-decreasing for g, $\Phi_e^{(F \cup E) \oplus C}(x) \uparrow$. By compactness, $\mathcal{C} \neq \emptyset$, so by preservation of hyperimmunity of WKL, there exists some $g \in \mathcal{C}$ such that the A's are $g \oplus X \oplus C$-hyperimmune. We can $g \oplus X$-computably thin out the set X to obtain an infinite set $Y \subseteq X$ non-decreasing for g. The condition (F, Y) is an extension of c forcing $\Phi_e^{G \oplus C}(x) \uparrow$. $\square$

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \omega)$, where $c_s = (F_s, X_s)$. The filter $\mathcal{F}$ yields a unique set $G = \bigcup_s F_s$. By Lemma 2.9, the set G is infinite, and by definition of a condition, G is non-decreasing for f. By Lemma 2.10, the A's are $G \oplus C$-hyperimmune. This completes the proof of Theorem 2.8. $\square$

Corollary 2.11 $\mathsf{ICNS} \wedge \mathsf{EM} \wedge \mathsf{WKL}$ does not imply SADS over RCA_0.

Proof. By Theorem 2.8, by [12] and by the hyperimmune-free basis theorem [10], ICNS, EM and WKL admit preservation of hyperimmunity, while SADS does not. One can therefore build an ω-model of $\mathsf{ICNS} \wedge \mathsf{EM} \wedge \mathsf{WKL}$ in which SADS does not hold. □

3. The strength of non-decreasing subsequences

We continue our study of the strength of the non-decreasing statements by considering their ability to compute functions not dominated by some classes of functions. Since stable Ramsey's theorem for pairs is computably reducible to CNS, there is a computable instance of CNS whose solutions are all of hyperimmune degree. We now prove that the same property holds for LNS.

Theorem 3.1 There is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $f(x, s+1) \leq f(x, s)$ for every $x, s \in \mathbb{N}$ and such that every infinite limit non-decreasing subsequence for f is hyperimmune.

Proof. We will construct the function f so that p_H is hyperimmune for every infinite limit non-decreasing subsequence H for f. We want to satisfy the following requirements for every $e \in \mathbb{N}$.

> $\mathcal{R}_e$: If Φ_e is total and increasing, then $\Phi_e(x_0) \downarrow= x_1$ and $\Phi_e(x_1) \downarrow= x_2$ for some $x_0 < x_1 < x_2 \in \mathbb{N}$ such that $\lim_s f(x, s) > \lim_s f(y, s)$ for each $x \in [x_0, x_1)$ and $y \in [x_1, x_2)$.

Indeed, given an infinite limit non-decreasing subsequence H for f let Φ_e be any computable increasing function. By $\mathcal{R}_e$, either $H \cap [x_0, x_1) = \emptyset$ or $H \cap [x_1, x_2) = \emptyset$. In the former case, $p_H(x_0) \geq x_1 = \Phi_e(x_0)$, while in the latter case $p_H(x_1) \geq x_2 = \Phi_e(x_1)$.

The overall construction is a finite injury priority argument. The local strategy for $\mathcal{R}_e$ *requires attention* at stage s if $\Phi_e(x_0) \downarrow= x_1$ and $\Phi_e(x_1) \downarrow= x_2$ for some $x_0 < x_1 < x_2 < s$ such that $f(x, s) > e$ for each $x \in [x_0, x_2)$ and such that no value in $[x_0, x_2)$ is restrained by a strategy of higher priority. The strategy for $\mathcal{R}_e$ commits $f(x, t)$ to be equal to e for every $x \in [x_1, x_2)$ and any $t \geq s$. It then puts restraint on every value in $[x_0, x_2)$ and is declared *satisfied*. If at a later stage, some strategy of higher priority restrains some value in $[x_0, x_2)$, then the strategy for $\mathcal{R}_e$ is *injured* and starts over, releasing all its restraint.

The global construction works as follows. At stage 0, f is the empty function. Suppose that at stage s, the function f is defined over $[0, s)^2$. If some strategy requires attention, then pick the one of highest priority and run it. In any case, set $f(x, s) = e$ for every strategy $\mathcal{R}_e$ which has committed such an assignment. Then set $f(x, s) = f(x, s-1)$ for every $x < s$ which has not been assigned yet, and $f(s, t) = s$ for every $t \leq s$. Then go to the next stage. This finishes the construction. We now turn to the verification.

First notice that each strategy acts finitely often, and therefore that each strategy is injured finitely many times. Moreover, notice that $f(x, s+1) \leq f(x, s)$ for every $x, s \in \mathbb{N}$ since when $f(x, s+1) \neq f(x, s)$, this is caused by a strategy which made its value decrease. We claim that each strategy $\mathcal{R}_e$ is eventually satisfied. To see that, let Φ_e be a total increasing function and let $s_0 > e$ be a stage after which no strategy of higher priority ever acts. By construction, $f(x,s) > e$ for every $x, s \geq s_0$. Therefore at some later stage s_1, there will be some $x_0 < x_1 < x_2 < s_1$ such that $\Phi_e(x_0) \downarrow = x_1$ and $\Phi_e(x_1) \downarrow = x_2$. In particular, $f(x,s) > e$ for each $x \in [x_0, x_2)$, so the strategy for $\mathcal{R}_e$ will require attention and will be satisfied since no strategy of higher priority acts. This completes the verification. $\square$

We will now prove that $\mathsf{RT}^2_2 \wedge \mathsf{WKL}$ does not imply CNS over RCA_0 using the notion of hypersurjectivity. A formula $\varphi(U)$, where U is a finite coded set parameter, is *essential* if for every $x \in \mathbb{N}$, there is some finite set $A > x$ such that $\varphi(A)$ holds. Given a set C and an infinite set $L \subseteq \mathbb{N}$, a function $f : \mathbb{N} \to \mathbb{N}$ is *C-hypersurjective for L* if for every essential $\Sigma^{0,C}_1$ formula $\varphi(U)$ and every $y \in L$, $f(A) = \{y\}$ for some finite set A such that $\varphi(A)$ holds. We say that f is C-hypersurjective if it is C-hypersurjective for some infinite set $L \subseteq \mathbb{N}$. A problem P *admits preservation of hypersurjectivity* if for each set C, each function $f : \mathbb{N} \to \mathbb{N}$ which is C-hypersurjective, and each P-instance $X \leq_T C$, there exists a solution Y to X and such that f is $Y \oplus C$-hypersurjective.

Theorem 3.2 CNS does not admit preservation of hypersurjectivity.

Proof. We will build a Δ^0_2 function $f : \mathbb{N} \to \mathbb{N}$ hypersurjective for $\mathbb{N}$, such that $f(x) \leq x$ for every $x \in \mathbb{N}$. We first claim that for every infinite set H non-decreasing for f and every infinite set $L \subseteq \mathbb{N}$, f is not H-hypersurjective for L. Therefore, f is a computable instance of CNS whose solutions do not preserve its own hypersurjectivity.

Suppose for the sake of contradiction that f is H-hypersurjective for some infinite set $L \subseteq \mathbb{N}$. Let y be the first element of L. We have two cases. First, suppose that $f(x) \leq y$ for every $x \in H$. Let $\varphi(U)$ be the $\Sigma^{0,H}_1$ formula which holds if U is a non-empty subset of H. The formula $\varphi(U)$ is essential since H is infinite. However, let z be the second element of L. There is no finite set A such that $f(A) = \{z\}$ and $\varphi(A)$, otherwise there would be some $x \in A \subseteq H$ such that $f(x) = z > y$. This contradicts H-hypersurjectivity of f for L. Second, suppose that there is some $x \in H$ such that $f(x) > y$. Let $\psi(U)$ be the $\Sigma^{0,H}_1$ formula which holds if U is a non-empty subset of $H \setminus [0,x]$. The formula $\psi(U)$ is again essential since H is infinite. However, if there is a finite set A such that $f(A) = \{y\}$ and $\psi(A)$ holds, then there is some $z \in A \subseteq H \setminus [0,x]$ such that $f(z) = y$. In particular, $x < z$ and $f(x) > f(z)$ which contradicts the fact that H is non-decreasing for f. Therefore f is not H-hypersurjective.

We now build the function $f : \mathbb{N} \to \mathbb{N}$ by the finite extension method in a Δ^0_2 construction. Fix an enumeration $\varphi_0(U), \varphi_1(U), \dots$ of all Σ^0_1 formulas. Start at stage 0 with the empty function f. Suppose that at stage $s = \langle y, e \rangle$, the function f is defined over some domain $[0, m)$. Decide in $\emptyset'$ whether there is a finite set $A \geq m$ such that $\varphi_e(A)$ holds. If so, set $f(x) = y$ for every $x \in [m, \max A]$, otherwise set $f(m) = 0$. In both case, go to the next stage. This completes the construction. $\square$

Before proving that RT^2_2 admits preservation of hypersurjectivity, we first need to prove that so does WKL for any fixed $L \subseteq \mathbb{N}$. Indeed, the latter will be used in the proof of the former. The proof of the following theorem is a slight modification of the proof of Theorem 14 in [12].

Theorem 3.3 WKL admits preservation of hypersurjectivity for any fixed L.

Proof. Fix a set C, let $f : \mathbb{N} \to \mathbb{N}$ be a function C-hypersurjective for L, and let $T \subseteq 2^{<\omega}$ be a C-computable infinite binary tree. We construct an infinite decreasing sequence of computable subtrees $T = T_0 \supseteq T_1 \supseteq \dots$ such that for every path P through $\bigcap_s T_s$, f is $P \oplus C$-hypersurjective for L. Note that the intersection $\bigcap_s T_s$ is non-empty since the T's are infinite trees. More precisely, if we interpret s as a tuple $\langle y, \varphi \rangle$ where $y \in L$ and $\varphi(G, U)$ is a $\Sigma^{0,C}_1$ formula, we want to satisfy the following requirement.

> $\mathcal{R}_s$: For every path P through T_{s+1}, either $\varphi(P, U)$ is not essential, or $\varphi(P, A)$ holds for some finite set A such that $f(A) = \{y\}$.

At stage $s = \langle y, \varphi \rangle$, given some infinite, computable binary tree T_s, define the $\Sigma^{0,C}_1$ formula

$$\psi(U) = (\exists n)(\forall \tau \in T_s \cap 2^n)(\exists \tilde{A} \subseteq U)\varphi(\tau, \tilde{A})$$

We have two cases. In the first case, $\psi(U)$ is not essential with some witness t. By compactness, the following set is an infinite C-computable subtree of T_s:

$$T_{s+1} = \{\tau \in T_s : (\forall A > t)\neg\varphi(\tau, A)\}$$

The tree T_{s+1} has been defined so that $\varphi(P, U)$ is not essential for every $P \in [T_{s+1}]$. In the second case, $\psi(U)$ is essential. By C-hypersurjectivity of f for L, there is a finite set A such that $\psi(A)$ holds and $f(A) = \{y\}$. We claim that for every path $P \in [T_s]$, $\varphi(P, \tilde{A})$ holds for some set $\tilde{A}$ such that $f(\tilde{A}) = \{y\}$. Fix some path $P \in [T_s]$. Unfolding the definition of $\psi(A)$, there is some n such that $\varphi(P \upharpoonright n, \tilde{A})$ holds for some set $\tilde{A} \subseteq A$. By continuity, $\varphi(P, \tilde{A})$ holds. Moreover, $f(\tilde{A}) = \{y\}$ since $f(A) = \{y\}$. Set $T_{s+1} = T_s$ and go to the next stage. This completes the proof of Theorem 3.3. □

We are now ready to prove that Ramsey's theorem for pairs admits preservation of hypersurjectivity.

Theorem 3.4 RT^2_2 admits preservation of hypersurjectivity.

Proof. Let C be a set and $g : \mathbb{N} \to \mathbb{N}$ be a function C-hypersurjective for some infinite set $L \subseteq \mathbb{N}$. Fix a C-computable coloring $f : [\mathbb{N}]^2 \to 2$. As usual, assume that there is no infinite f-homogeneous set H such that g is $H \oplus C$-hypersurjective, otherwise we are done. We will build two infinite sets G_0, G_1, f-homogeneous for color 0 and 1, respectively, and such that g is either $G_0 \oplus C$-hypersurjective, or $G_1 \oplus C$-hypersurjective.

We will use forcing conditions (F_0, F_1, X), where F_0 and F_1 are finite sets of integers, X is an infinite set such that $\max(F_0, F_1) < \min X$ and for every $i < 2$ and every $x \in X$, $F_i \cup \{x\}$ is f-homogeneous for color i. We furthermore impose that g is $X \oplus C$-hypersurjective for L. A condition $d = (E_0, E_1, Y)$ extends $c = (F_0, F_1, X)$ if (E_i, Y) Mathias extends (F_i, X) for each $i < 2$. A pair of sets G_0, G_1 satisfies a condition (F_0, F_1, X) if for each $i < 2$, G_i if f-homogeneous for color i and satisfies the Mathias condition (F_i, X). Again, we start by proving that every infinite filter yields two infinite sets.

Lemma 3.5 For every condition $c = (F_0, F_1, X)$ and every $i < 2$, there is an extension $d = (E_0, E_1, X)$ of c such that $|E_i| > |F_i|$.

Proof. For every $x \in X$, let $S_x = \{y \in X : y > x \wedge f(x, y) = i\}$. If S_x is finite for every $x \in X$, then one can $X \oplus C$-computably thin out the set X to obtain an infinite set f-homogeneous for color $1-i$, contradicting our assumption. Therefore, there is some $x \in X$ such that S_x is infinite. The condition (E_0, E_1, S_x) where $E_i = F_i \cup \{x\}$ and $E_{1-i} = F_{1-i}$ is the desired extension of c. □

Fix an enumeration $\varphi_0(G, U), \varphi_1(G, U), \dots$ of all $\Sigma^{0,C}_1$ formulas. We will now ensure the following disjunctive requirements for each $e_0, e_1 \in \mathbb{N}$ and $y \in L$.

$$\mathcal{R}_{e_0,e_1,y} : \mathcal{R}^{G_0}_{e_0,y} \vee \mathcal{R}^{G_1}_{e_1,y}$$

where $\mathcal{R}^G_{e,y}$ is the statement "If $\varphi_e(G, U)$ is essential, then $\varphi(G, A)$ holds for some finite set A such that $g(A) = \{y\}$". If all the disjunctive requirements are satisfied for some pair of sets G_0, G_1, then there will be a 2-partition $L_0 \cup L_1 = L$ such that g is $G_i \oplus C$-hypersurjective for L_i for each $i < 2$. Among L_0 and L_1, at least one must be infinite. We will then pick the corresponding G_i. We say that a condition c forces $\mathcal{R}_{e_0,e_1,y}$ if it holds for every pair of sets G_0, G_1 satisfying c.

Lemma 3.6 For every condition $c = (F_0, F_1, X)$, every pair of indices $e_0, e_1 \in \mathbb{N}$ and every $y \in L$, there is an extension d of c forcing $\mathcal{R}_{e_0,e_1,y}$.

Proof. Let $\psi(U)$ be the $\Sigma^{0,X\oplus C}_1$ formula which holds if there is a finite set $H \subseteq X$ such that for every 2-partition $H_0 \cup H_1 = H$, there is some $i < 2$, some finite set $U_i \subseteq U$ and some set $E \subseteq H_i$ f-homogeneous for color i such that $\varphi_{e_i}(F_i \cup E, U_i)$ holds. We have two cases.

Case 1: the formula $\psi(U)$ is not essential, with witness $x \in \mathbb{N}$. Let $\mathcal{C}$ be the $\Pi^{0,X\oplus C}_1$ class of all sets $H_0 \oplus H_1$ such that $H_0 \cup H_1 = X$ and for every $i < 2$, every finite set $U_i > x$ and every finite set $E \subseteq H_i$ f-homogeneous for color i, $\varphi_{e_i}(F_i \cup E, U_i)$ does not hold. Since there is no finite set $U > x$ such that $\varphi(U)$ holds, then by a compactness argument $\mathcal{C}$ is non-empty. By preservation of hypersurjectivity of WKL for L (Theorem 3.3), there is some $H_0 \oplus H_1 \in \mathcal{C}$ such that g is $H_0 \oplus H_1 \oplus X \oplus C$-hypersurjective for L. Let $i < 2$ be such that H_i is infinite. The condition (F_0, F_1, H_i) is an extension of c forcing $\mathcal{R}^{G_i}_{e_i,y}$, hence $\mathcal{R}_{e_0,e_1,y}$.

Case 2: the formula $\psi(U)$ is essential. By $X \oplus C$-hypersurjectivity of g for L, $\psi(A)$ holds for some finite set A such that $g(A) = \{y\}$. Let $H \subseteq X$ be the finite set witnessing that $\psi(A)$ holds. Every $z \in X$ induces a 2-partition $H_0 \cup H_1 = H$ defined by $H_i = \{x \in H : f(x, z) = i\}$. Since there are finitely many 2-partitions of H, there is a 2-partition $H_0 \cup H_1 = H$ such that the set

$$Y = \{z : X : z > \max H \wedge (\forall i < 2)(\forall x \in H_i) f(x, z) = i\}$$

is infinite. In particular, there is some $i < 2$ and some set $E \subseteq H_i$ f-homogeneous for color i such that $\varphi_{e_i}(F_i \cup E, A_i)$ holds for some set $A_i \subseteq A$. The condition (E_0, E_1, Y) defined by $E_i = F_i \cup E$ and $E_{1-i} = F_{1-i}$ is an extension of c forcing $\mathcal{R}^{G_i}_{e_i,y}$, hence $\mathcal{R}_{e_0,e_1,y}$. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter containing $(\emptyset, \emptyset, \omega)$, where $c_s = (F_{0,s}, F_{1,s}, X_s)$. The filter $\mathcal{F}$ yields a unique pair of sets $G_0 = \bigcup_s F_{0,s}$

and $G_1 = \bigcup_s F_{1,s}$. By Lemma 3.5, both G_0 and G_1 are infinite. By Lemma 3.6, there is some $i < 2$ and some infinite set $L_i \subseteq L$ such that g is $G_i \oplus C$-hypersurjective for L. This completes the proof. □

Corollary 3.7 $\mathsf{RT}^2_2 \wedge \mathsf{WKL}$ does not imply CNS over RCA_0.

4. Low$_2$ non-decreasing subsequences

Cholak, Jockusch and Slaman [2] proved that every computable instance of RT^2_2 admits a low$_2$ solution. In this section, we prove that the same property holds for CNS. Given two sets X and A, an integer $e \in \mathbb{N}$ is a *$\Delta^{0,X}_2$ index* of A if $\Phi^{X'}_e = A$. Similarly, e is an *X-jump index* of A if $\Phi^X_e = A'$. A function $f : \mathbb{N} \to \{0,1\}$ is *X-dnc$_2$* if $f(e) \neq \Phi^X_e(e)$ for every e. The following theorem is obtained by looking at the uniformity of the first jump control of Cholak, Jockusch and Slaman [2].

Theorem 4.1 There are two computable functions $h_0, h_1 : \mathbb{N} \to \mathbb{N}$ such that for every set C and every C'-dnc$_2$ function f, if e is the $\Delta^{0,C}_2$ index of a set A, then either $h_0(e)$ is an f-jump index of $Y_0 \oplus C$, where Y_0 is an infinite subset of A, or $h_1(e)$ is an f-jump index of $Y_1 \oplus C$, where Y_1 is an infinite subset of $\overline{A}$.

Proof. Fix a set C and let f be a C'-dnc$_2$ function and e be an index of a $\Delta^{0,C}_2$ set A. We will describe an f-computable construction of a set G such that for every pair $e_0, e_1 \in \mathbb{N}$, either $((G \cap A) \oplus C)'(e_0)$, or $((G \cap \overline{A}) \oplus C)'(e_1)$ is decided. We work with Mathias conditions (F, X) where X is low over C. An *index* of such a condition $c = (F, X)$ is a code $\langle F, i\rangle$ such that $\Phi^{C'}_i = (X \oplus C)'$.

To simplify our notation, we let $A_0 = A$ and $A_1 = \overline{A}$. Given some $i < 2$ and some $e_i \in \mathbb{N}$, a condition $c = (F, X)$ *decides* $((G \cap A_i) \oplus C)'(e_i)$ if either $\Phi^{(F \cap A_i) \oplus C}_{e_i}(e_i)\downarrow$, or $\Phi^{((F \cap A_i) \cup H) \oplus C}_{e_i}(e_i)\uparrow$ for every set $H \subseteq X$. Note that H is not necessarily included in A_i. This detail will be used in Lemma 4.3.

Lemma 4.2 (Lemma 4.6 in [2]) Given a condition $c = (F, X)$ and a pair of indices $e_0, e_1 \in \mathbb{N}$, there is an extension d of c deciding either $((G \cap A) \oplus C)'(e_0)$, or $((G \cap \overline{A}) \oplus C)'(e_1)$. Furthermore, a index of d may be f-computably computed from e_0, e_1 and an index of c, and one can f-computably decide which case applies.

Using Lemma 4.2, build an infinite f-computable decreasing sequence of conditions $(\emptyset, \omega) = (F_0, X_0) \geq (F_1, X_1) \geq \dots$ such that for every $s = \langle e_0, e_1\rangle$, (F_{s+1}, X_{s+1}) decides either $((G \cap A) \oplus C)'(e_0)$, or $((G \cap \overline{A}) \oplus C)'(e_1)$. Unlike the original construction [2], we do not interleave requirements to ensure that both $G \cap A$ and $G \cap \overline{A}$ are infinite, and indeed, it is not possible since we cannot uniformly decide whether there is a low solution or not. Thankfully, the infinity requirements are already ensured by the decision process, as shows the following lemma.

Lemma 4.3 If $G \cap A_i$ is finite, then for some $e_i \in \mathbb{N}$, there is no stage s at which (F_{s+1}, X_{s+1}) decides $((G \cap A_i) \oplus C)'(e_i)$.

Proof. Let $k = |G \cap A_i|$, and let $e_i \in \mathbb{N}$ be such that for every set H, $\Phi_{e_i}^{H \oplus C}(e_i) \downarrow$ if and only if H contains at least $k+1$ elements. Suppose there is a stage s at which (F_{s+1}, X_{s+1}) decides $((G \cap A_i) \oplus C)'(e_i)$. By definition, $\Phi_{e_i}^{(F \cap A_i) \oplus C}(e_i) \downarrow$, or $\Phi_{e_i}^{((F \cap A_i) \cup H) \oplus C}(e_i) \uparrow$ for every set $H \subseteq X$. The former does not hold since $|F \cap A_i| = k$, and neither does the latter since $\Phi_{e_i}^{((F \cap A_i) \cup H) \oplus C}(e_i) \downarrow$ for any infinite set $H \subseteq X$. □

For each $i < 2$, let $h_i(e)$ be the Turing index of the f-algorithm which on input e_i, f-computably runs the construction until it finds some stage s at which (F_{s+1}, X_{s+1}) decides $((G \cap A) \oplus C)'(e_i)$. If such stage is found, the algorithm outputs the answer, otherwise it does not terminate. We claim that one of the two following holds:

(a) $h_0(e)$ is an f-jump index of $(G \cap A) \oplus C$ and $G \cap A$ is infinite;
(b) $h_1(e)$ is an f-jump index of $(G \cap \overline{A}) \oplus C$ and $G \cap \overline{A}$ is infinite.

If case (a) does not hold, then either $G \cap A$ is finite, or the algorithm of $h_0(e)$ is not total, and by Lemma 4.3, the former implies the latter. Moreover, if the algorithm of $h_0(e)$ is not total, then by the usual pairing argument, the algorithm of $h_1(e)$ is total, and by the contrapositive of Lemma 4.3, $G \cap \overline{A}$ is infinite, so case (b) holds. This completes the proof. □

We are now ready to prove the main theorem of the section. In what follows, we write $P \gg X$ to say that P is of PA degree relative to X.

Theorem 4.4 Fix a set C and a set $P \gg C'$. For every C-computable instance of CNS, there is an infinite non-decreasing subsequence G such that $(G \oplus C)' \leq_T P$.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a $\Delta_2^{0,C}$ function C-computably bounded by some function $b : \mathbb{N} \to \mathbb{N}$. Let $h_0, h_1, \dots$ be a uniformly P-computable sequence of functions such that $h_0 \equiv_T C'$, and h_{i+1} is h_i-dnc$_2$ for every $i \in \mathbb{N}$. Assume that there is no infinite set G over which f is constant, and such that $(G \oplus C)' \leq_T P$, otherwise we are done. We will build our set G by a P-computable construction using variants of Mathias conditions.

An h_i-*condition* is a tuple (F, X, S) where (F, X) is a Mathias condition, $g(x) \leq g(y)$ for every $x \in F$, $y \in X$ and $g \in S \cup \{f\}$, and S is a finite collection of functions bounded by b, such that $(X \oplus S \oplus C)' \leq_T h_i$. An h_j-condition $d = (E, Y, T)$ extends an h_i-condition $c = (F, X, S)$ if $j \geq i$, $E \supseteq F$, $Y \subseteq X$, $T \supseteq S$ and $E \setminus F$ is a non-decreasing subset of X for every $g \in S \cup \{f\}$. A set G satisfies (F, X, S) if $F \subseteq G \subseteq F \cup X$ and $G \setminus F$ is non-decreasing for every $g \in S \cup \{f\}$.

An h_i-*index* of (F, X, S) is a code $\langle i, F, e \rangle$ such that $\Phi_e^{h_i} = (X \oplus S \oplus C)'$. Given an h_i-condition $c = (F, X, S)$, we let $\#(c)$ be the number of functions $g \in S$ such that g is not constant over X. Note that an h_{i+1}-index of c can be P-computed from an h_i-index of c, and that $\#(c)$ can be h_i-computed from an h_i-index of c.

Lemma 4.5 For every $n \in \mathbb{N}$ and every h_n-condition $c = (F, X, S)$, there is an h_{n+1}-extension $d = (E, Y, S)$ of c such that either $\#(d) < \#(c)$, or $|E| > |F|$. Furthermore, an h_{n+1}-index of d may be P-computed from an h_n-index of c.

Proof. Pick any $x \in X$. Since $h_{n+1} \gg (X \oplus S \oplus C)'$, one can h_{n+1}-computably decide if there is some $g \in S$ and some $u < g(x)$ such that the set $Y_0 = \{y \in X : g(y) = u\}$ is infinite, or whether the set $Y_1 = \{y \in X : (\forall g \in S) g(y) \geq g(x)\}$ is infinite. In the first case, the h_{n+1}-condition $d = (F, Y_0, S)$ is an extension of c such that $\#(d) < \#(c)$. In the second case, let $A = \{y \in Y_1 : f(y) \geq f(x)\}$. By Theorem 4.1, either we obtain a h_{n+1}-jump index of $Z_0 \oplus X \oplus S \oplus C$, where Z_0 is an infinite subset of A, or an h_{n+1}-jump index of $Z_1 \oplus X \oplus S \oplus C$, where Z_1 is an infinite subset of $Y_1 \setminus A$. The second case cannot happen since otherwise, f would be of bounded range over Z_1, and by further applications of Theorem 4.1, one would obtain an infinite set H over which f is constant and such that $(H \oplus C)' \leq_T P$, contradicting our initial assumption. We therefore obtain an infinite set $Z_0 \subseteq A$ such that $h_{n+1} \geq_T (Z_0 \oplus S \oplus C)'$. The h_{n+1}-condition $d = (F \cup \{x\}, Z_0, S)$ is an extension of c satisfying the desired property. $\square$

An h_i-condition $c = (F, X)$ *decides* $(G \oplus C)'(e)$ if either $\Phi_e^{F \oplus C}(e) \downarrow$, or $\Phi_e^{(F \cup H) \oplus C}(e) \uparrow$ for every set $H \subseteq X$.

Lemma 4.6 For every $n \in \mathbb{N}$, every h_n-condition $c = (F, X, S)$ and every $e \in \mathbb{N}$, there is an h_{n+1}-extension d of c such that either $\#(d) < \#(c)$, or d decides $(G \oplus C)'(e)$. Furthermore, an h_{n+1}-index of d may be P-computed from an h_n-index of c, and one can P-computably decide which case applies.

Proof. Let $\mathcal{C}$ be the $\Pi_1^{0, X \oplus S \oplus C}$ class of all functions $p : \mathbb{N} \to \mathbb{N}$ bounded by b, such that $p(x) \leq p(y)$ for every $x \in F$ and $y \in X$, and for every finite set $E \subseteq X$ non-decreasing for every $g \in S \cup \{p\}$ simultaneously, $\Phi^{(F \cup E) \oplus C}(e) \uparrow$. Since $h_n \geq_T (X \oplus S \oplus C)'$, one can h_n-decide whether $\mathcal{C}$ is empty or not.

If $\mathcal{C}$ is empty, then in particular $f \notin \mathcal{C}$. Unfolding the definition, there is a finite set $E \subseteq X$ non-decreasing for every $g \in S \cup \{f\}$, such that $\Phi^{(F \cup E) \oplus C)}(e) \downarrow$. As in Lemma 4.5, one can h_{n+1}-computably decide whether there is some $g \in S$ and some $u < \max\{g(x) : x \in E\}$ such that the set $Y_0 = \{y \in X : g(y) = u\}$ is infinite, or whether the set $Y_1 = \{y \in X : (\forall g \in S)(\forall x \in E) g(y) \geq g(x)\}$ is infinite. In the first case, the h_{n+1}-condition $d = (F, Y_0, S)$ is an extension of c such that $\#(d) < \#(c)$. In the second case, let $A = \{y \in Y_1 : (\forall x \in E) f(y) \geq f(x)\}$. Still by the same argument as in Lemma 4.5, one can P-computably find an infinite set $Z_0 \subseteq A$ such that $h_{n+1} \geq_T (Z_0 \oplus S \oplus C)'$. The h_{n+1}-condition $(F \cup E, Z_0, S)$ is an extension of c forcing $(G \oplus C)'(e) = 1$.

If $\mathcal{C} \neq \emptyset$, then by the relativized low basis theorem [10], one can h_n-computably pick some $g \in \mathcal{C}$ such that $h_n \geq_T (g \oplus X \oplus S \oplus C)'$. The h_{n+1}-condition $(F, X, S \cup \{g\})$ is an extension of c forcing $(G \oplus C)'(e) = 0$. $\square$

Using Lemma 4.5 and Lemma 4.6, build an infinite P-computable decreasing sequence of tuples $(\emptyset, \omega, \emptyset) = (F_0, X_0, S_0) \geq (F_1, X_1, S_1) \geq \dots$ such that for every $s \in \mathbb{N}$,

(i) (F_s, X_s, S_s) is an h_n-condition for some $n \in \mathbb{N}$
(ii) $|F_s| \geq s$
(iii) (F_s, X_s, S_s) decides $(G \oplus C)'(e)$

The set $G = \bigcup_s F_s$ is an infinite non-decreasing subsequence for f such that $(G \oplus C)' \leq_T P$. This completes the proof of Theorem 4.4. $\square$

Corollary 4.7 Every computable instance of CNS admits a low_2 solution.

Proof. Let f be a computable instance of CNS. By the relativized low basis theorem [10], there is a set $P \gg \emptyset'$ such that $P' \leq_T \emptyset''$. By Theorem 4.4, there is an infinite non-decreasing subsequence G for f, such that $G' \leq_T P$. In particular, $G'' \leq_T P' \leq_T \emptyset''$, so G is low_2. □

5. Summary and open questions

In this section, we summarize the known relations between CNS, LNS, and existing principles in reverse mathematics. We also state some remaining open questions. In Figure 5, a plain arrow from P to Q means that P implies Q over RCA_0, while a dotted arrow stands for an open implication. Here, ACA stands for the Arithmetic Comprehension Axiom [13], DNC and HYP assert the existence, for every set X, of an X-dnc function and an X-hyperimmune set, respectively. HYP is also known as OPT, standing for Omitting Partial Types [9].

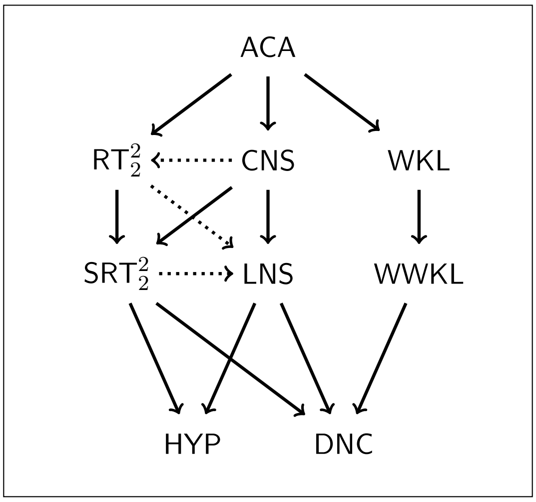


Figure 1. Non-decreasing subsequences in reverse mathematics

We wonder about the two remaining open implications between Ramsey's theorem for pairs and the non-decreasing sequence statements.

Question 5.1 Does CNS imply RT^2_2 over RCA_0?

Question 5.2 Does RT^2_2 imply LNS over RCA_0?

The same questions hold for ω-models and over computable reducibility.

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