In memoriam: James Earl Baumgartner (1943–2011)

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Abstract

James Earl Baumgartner (March 23, 1943 – December 28, 2011) came of age mathematically during the emergence of forcing as a fundamental technique of set theory, and his seminal research changed the way set theory is done. He made fundamental contributions to the development of forcing, to our understanding of uncountable orders, to the partition calculus, and to large cardinals and their ideals. He promulgated the use of logic such as absoluteness and elementary submodels to solve problems in set theory, he applied his knowledge of set theory to a variety of areas in collaboration with other mathematicians, and he encouraged a community of mathematicians with engaging survey talks, enthusiastic discussions of open problems, and friendly mathematical conversations.

1 Overview of Baumgartner's Life

James E. Baumgartner was born on March 23, 1943 in Wichita, Kansas. His high school days included tennis, football, and leading roles in school plays. In 1960 he entered the California Institute of Technology, but stayed only two years, moving to the University of California, Berkeley in 1962, in part because it was co-educational. There he met and married his wife Yolanda. He continued his interest in drama and mathematics as an undergraduate, earned his A.B. in mathematics in 1964, and continued study as a graduate

student. Baumgartner [9, page 2] dated his interest in set theory to the four week long 1967 UCLA Summer Institute on Axiomatic Set Theory.¹ The mathematics for his dissertation was completed in spring 1969, and Baumgartner became a John Wesley Young Instructor at Dartmouth College in fall 1969. His adviser, Robert Vaught, required Baumgartner to include additional details in his dissertation, so he did not earn his doctorate until 1970.

In Fall 1970, he was a Participating Scholar in the New York Academy of Sciences Scholar-in-Residence Program under the sponsorship of Paul Erdős. In Fall 1971, his status at Dartmouth shifted from John Wesley Young Instructor to Assistant Professor but he spent the academic year 1971-1972 at the California Institute of Technology as a Visiting Assistant Professor and regularly attended the UCLA Logic Colloquia on Fridays. Winter and spring quarters of 1975 were spent at the University of California, Berkeley as a Research Associate. He was tenured and promoted to Associate Professor in 1976, promoted to Professor in 1980, became the first John G. Kemeny Professor of Mathematics in 1983. Baumgartner, along with Donald A. Martin, and Saharon Shelah, organized the 1983 American Mathematical Society Summer Research Conference on Axiomatic Set Theory in Boulder, Colorado, and they edited the proceedings [56]. This event was the first large meeting devoted entirely to set theory since the 1967 Summer School held at UCLA. Baumgartner spent a stint as Chair of the department from 1995-1998. He was honored with the Baumgartner Fest in 2003 at which a number of his students spoke. Slowed by the multiple sclerosis diagnosed in 1982, he retired with emeritus status in 2004. He died in 2011 under the care of his loving wife, Yolanda.

Baumgartner had ten doctoral students at Dartmouth listed below with academic affiliations for those who have one: Robert Beaudoin (1985), Stefan Bilaniuk (1989), Trent University, Denis Devlin (1980), Claudia Henrion (1985), Albin Jones (1999), Jean Larson (1972), University of Florida, Thomas Leathrum (1993), Jacksonville State University, Alabama, Tadatoshi Miyamoto (1988), Nanzan University of Nagoya, Alan Taylor (1975), Union College, Stanley Wagon (1975), Macalester College.

Charles K. Landraitis, who is affiliated with Boston College, was a set theory student at Dartmouth College graduating in 1975, and was often included

¹John W. Addison, Jr., who directed Baumgartner's earliest research, made it possible for Baumgartner to attend the 1967 UCLA Summer School.

in group activities for the Baumgartner group of students, even though his adviser was Victor Harnik of Haifa University, who visited Dartmouth College. Baumgartner also advised Peter Dordal who graduated in 1982 from Harvard University and is now in computer science at Loyola University in Chicago.

Baumgartner enjoyed working with a number of mathematicians on post-doctoral positions or visiting positions at Dartmouth, most as John Wesley Young Instructors. These included Uri Abraham, Alessandro Andretta, Jörg Brendle, Elizabeth Theta Brown, James Cummings, Frantisek Franck, Jean-Pierre Levinski, George McNulty, Lee Stanley, Claude Sureson, Stevo Todorcevic, Robert Van Wesep, and Jindrich Zapletal.

2 A personal note

I interacted most with Baumgartner as a graduate student. He arrived at Dartmouth College my second year in graduate school, and I think I met him in my oral qualifying exam where he clarified a question I was being asked enabling me to answer it successfully.

I took his course in set theory starting that fall and decided to ask to work with him. Alas, I was the second to ask, and since the first to ask quickly switched to someone else, at Baumgartner's suggestion, we informally started reading together.

Baumgartner had learned a lot from fellow students in graduate school, so encouraged me to bring an undergraduate into our fall 1970 conversations, enabling me to have a peer with whom to talk. My fellow student took a term of independent study with Baumgartner that fall, but after one semester, continued without credit since he had spent more time than he felt he could afford on it.

In spring 1971 I came up with a short proof of $\omega^{\omega} \to (\omega^{\omega}, n)^2$ which became the cornerstone of my thesis even though C.C. Chang had already proved it for n=3 and Eric Milner showed how to generalize his proof for all n. I had difficulty explaining the proof to Baumgartner, so week after week, he would tell me he did not yet understand, that he was sure I would be able to explain it to him, and he would cheerfully ask me to come back next week to try again.

Once he understood my proof, Baumgartner arranged support for me to attend the 1971 Summer School in Cambridge, organized by Adrian Mathias,

where I met for the first time a very large number of mathematicians that I have continued to see. Baumgartner sent me with a paper of his to hand deliver to Hajnal, guaranteeing a meeting with Erdős, Hajnal and Milner.

My final year in graduate school was spent as a visiting graduate student at UCLA where Baumgartner shared his UCLA office with me 1971-72, while he spent most of his time at Cal Tech.

It was always wonderful to visit the Baumgartners in Hanover and at many conferences over the years. In October 2003, Arthur Apter and Marcia Groszek organized a Baumgartner Fest in honor of his 60th birthday. It was a wonderful conference with many people speaking on mathematics of interest to Baumgartner. At a party at his house during this conference, Baumgartner passed around the framed conference photo from the 1967 UCLA Summer Institute on Axiomatic Set Theory that was the beginning of his interest in set theory. It was a time to reflect on all the meetings we had enjoyed in between the 1967 UCLA Summer Institute and the Baumgartner Fest.

3 Baumgartner's mathematical work

We now turn to the mathematical context in which Baumgartner worked and a discussion of a selected works mainly by date of publication. Jech's book [115] has been generally followed for definitions and notation. Kanamori's book [124] has been an invaluable resource for both mathematics and history.

3.1 Mathematical context and graduate school days

Baumgartner [33, 462] described the mathematical scene in the years just prior to his time as a set theory graduate student:

Once upon a time, not so very long ago, logicians hardly ever wrote anything down. Wonderful results were being obtained almost weekly, and no one wanted to miss out on the next theorem by spending the time to write up the last one. Fortunately there was a Center where these results were collected and organized, but even for the graduate students at the Center life was hard. They had no textbooks for elementary courses, and for advanced courses they were forced to rely on handwritten proof outlines, which were usually illegible and incomplete; handwritten seminar notes, which were usually wrong; and Ph.D. dissertations, which

were usually out of date. Nevertheless, they prospered. Now the Center I have in mind was Berkeley and the time was the early and middle 1960's, ...

In the early and middle 1960's aspects of set theory were developing in concert: forcing,² large cardinals, combinatorial set theory and interactions with model theory.

Forcing was introduced by Paul Cohen in 1963, and it was quickly applied by Easton to the question of the size of powers of regular cardinals in his 1964 thesis [78]. Robert Solovay [174], [175] proved the consistency of ZF with every set of reals being Lebesgue measurable by 1964, but only published the result in 1970. Early lecture series around the world included Prikry's January 1964 lecture in Mostowski's seminar in Warsaw, Levy's course on forcing in 1964, [153, 161] and lectures by Jensen at the University of Bonn in 1965-66 [118].

Large cardinal concepts date back to Hausdorff [112] (weakly inaccessible), Mahlo (Mahlo cardinals³), Banach [5] and Ulam [195] (measurable cardinals⁴.), Erdős and Tarski [85], [91] (weakly compact cardinals). In 1964-1964, H. Jerome Keisler and Alfred Tarski [126] made a systematic study of weakly compact, measurable and strongly compact cardinals. Supercompact cardinals⁵ were introduced by Solovay and William Reinhardt [177] no later than 1966-67 (see [141, page 186]). Another strand of large cardinal properties came out of generalization of partition properties, e.g. Ramsey cardinals⁶ introduced in 1962 by Erdős and Hajnal [88].

Models are foundational for set theory, since in forcing one starts with a model and extends it to get a new model, as Cohen extended Gödel's

²Forcing is a technique for adjoining a generic object to a given model of set theory so that the properties of the generic object and hence the extension of the original model generated from the generic object are determined by the construction and the original model.

 $^{^{3}\}kappa$ is a Mahlo cardinal if the set of regular cardinals below it is stationary.

 $^{^4\}kappa$ is measurable if it has a a non-principle ultrafilter

⁵A cardinal κ is supercompact if and only for every $\lambda \geq \kappa$, there is a normal fine ultrafilter U on $\mathcal{P}_{\kappa}(\lambda)$. Alternatively, a cardinal κ is γ -supercompact for $\gamma \geq \kappa$, if and only if there is an elementary embedding $j: V \to M$ such that κ is the critical point of j, $\gamma < j(\kappa)$, and the model M contains all of its γ -size subsets, and is supercompact if and only if it is γ -supercompact for all $\gamma \geq \kappa$.

⁶A cardinal κ is Ramsey if for every coloring by f of its finite subsets with two colors, there is a subset $H \subseteq \kappa$ of cardinality κ such that for each positive $n < \omega$, all the n-element subsets of H receive the same color.

Constructible Universe to a model in which the Continuum Hypothesis fails. Alfred Tarski and his students developed model theory in the 1950's and 1960's and his students, C.C. Chang and Keisler [73] envisioned in 1963 their textbook on model theory, which would be based on lecture notes, with a significant revision after the 1967 UCLA set theory meeting, but did not appear until 1973. Model theoretic techniques were applied in varied ways to set theory, including the use of absoluteness to transfer results from one model to another as done by Jack Silver⁷ in his 1966 thesis [171].

Since forcing employed partial orders (e.g. Cohen reals), Boolean algebras (e.g. random reals), and trees (e.g. Sacks forcing), they also became combinatorial objects of study in addition to the graphs and hypergraphs of the partition calculus. Notions of largeness included closed unbounded subsets, stationary subsets.

Baumgartner was quickly brought up to speed on current topics in set theory at the four week long 1967 UCLA Summer Institute on Axiomatic Set Theory. Scott⁸ and Joseph Shoenfield [167] gave ten lectures each on forcing. Sacks spoke on the perfect set forcing or tree forcing named for him. Many of the other topics that came up at the meeting and in the two volume proceedings [159], [117] are related to Baumgartner's published work. A variety of large cardinals were discussed including measurable cardinals, real-valued measurable cardinals, and supercompact cardinals, as well as reflection principles. Other topics included λ -saturated ideals, extensions of Lebesgue measure, the partition calculus, Kurepa's Hypothesis, and Chang's Conjecture.

In [153, 161] Gregory Moore, based on an interview with Baumgartner in 1980, reported that "At Berkeley a group of young graduate students (including Baumgartner, Laver, and Mitchell) organized their own seminar — with no faculty invited."

In the acknowledgments section of his thesis, Baumgartner [9, 2] Baumgartner asserted that his "greatest mathematical debt is to the work of Paul Cohen, and to the work of Robert Solovay and others in making it understandable." Baumgartner credited Jack Silver with his "initiation into

⁷Silver received his doctorate from the University of California Berkeley in 1966, and joined the faculty there shortly after with Karel Prikry graduating in 1968 as his first student.

⁸Dana Scott presented the Boolean approach and was expected to submit a paper on it with Solovay to the proceedings of the conference (see [167]) but did not do so.

the techniques of forcing proofs," and noted that "most of the problems treated here were suggested to me by Fred Galvin,⁹ Richard Laver,¹⁰ William Mitchell,¹¹ and Jack Silver, and conversations with them have resulted in the improvement of many proofs and the extension of many results." All of those mentioned attended the 1967 Summer School.

3.2 On Suslin's Question

In 1970, Baumgartner, Jerome Malitz, and William Reinhardt showed, that if the usual axioms of set theory are consistent, then so is a positive answer to Mikhail Suslin's Question of 1920 [181], rephrased below in modern terminology:

Must every complete, dense in itself, linear order without endpoints for which every pairwise disjoint set of intervals is countable be a copy of the real line?

Baumgartner, Malitz and Reinhardt built on work by Đuro Kurepa¹² [134] who conducted the first systematic investigation of uncountable trees, introducing the partition tree of a linear order, the linearization of a tree, Suslin, Aronszajn, and Kurepa trees¹³ He showed the equivalence of the existence of a Suslin tree to a negative answer to Suslin's question.

In their 1970 paper, Baumgartner, Malitz and Reinhardt¹⁴ proved the existence of a forcing extension in which $2^{\aleph_0} = \aleph_2$ and all Aronszajn trees are embeddable in the rationals, ¹⁵ where a tree $(T, <_T)$ embeds in the rationals

 $^{^9\}mathrm{Galvin}$ held pre- and post-doctoral positions at University of California Berkeley during 1965-1968

¹⁰Laver [138] earned his doctorate from University of California Berkeley in 1969.

¹¹Mitchell [148] received his doctorate in 1970.

 $^{^{12}}$ The masterful survey trees and linear orders by Stevo Todorcevic [190] includes an excellent introduction to Kurepa's work.

¹³These are all trees of cardinality ω_1 and height ω_1 : a *Suslin tree* has no uncountable branch and no uncountable antichain, where an *antichain* in a partial order is a set whose elements are pairwise incomparable; an *Aronszajn tree* has no uncountable chain and countable levels; and a *Kurepa tree* has countable levels and more than \aleph_1 branches.

¹⁴Baumgartner (using Martin's Axiom in his thesis) and the team of Jerome Malitz and William Reinhardt independently proved these results.

 $^{^{15}}$ Kurepa [135] constructed the first Aronszajn tree with an embedding into the rational numbers.

if there is a function $f: T \to \mathbb{Q}$ such that $s <_T t$ implies f(s) < f(t). In such a case we say $(T, <_T)$ is *special*.

The heart of the argument is an elegant proof of the countable chain condition of the forcing.

To see how this result is connected to Suslin's Question, note that if (T, <) is Aronszajn tree with an embedding $f: T \to \mathbb{Q}$, then (T, <) is a union of countably many antichains, 16 since for each rational r, $f^{-1}\{r\}$ is an antichain, and it follows that $(T, <_T)$ has an uncountable antichain, since T is uncountable. Thus if an Aronszajn tree is special, it fails to be a Suslin tree. By definition, Suslin trees are Aronszajn trees with no uncountable antichain, so in the Baumgartner-Malitz-Reinhard extension, there are no Suslin trees giving the consistency relative to ZFC of a positive answer to Suslin's Question. At the end of the three author paper there were three questions, the last of which was asked by Baumgartner: Is it consistent with ZFC $+ 2^{\aleph_0} = \aleph_1$ to assume that every Aronszajn tree is embeddable in the rationals?

Baumgartner, Malitz, Reinhardt were preceded by Robert Solovay and Tennenbaum [178] who found their proof in June 1965 that a positive answer to Suslin's Question was relatively consistent with the usual axioms of set theory. Independently, Thomas Jech [114] and Stanley Tennenbaum [186] used forcing to show the consistency relative to the usual axioms of set theory of a negative answer to the famous question by Suslin, so a positive answer is independent of the usual axioms of set theory.

3.3 Generalized Ramsey Theory

Next we turn Baumgartner's work in the partition calculus which grew out of generalizations of Ramsey's Theorem of 1930. Frank P. Ramsey [156] proved that for any partition of the n-element subsets of an infinite set A, there is an infinite subset $H \subseteq A$, all of whose n-element sets lie in the same cell of the partition. To present this in modern notation, we introduce the arrow notation of Richard Rado [83]: for any cardinal κ , for ordinals $\langle \alpha_i \mid i < \kappa$ and β , and any $r \in \omega$ the partition property

$$\beta \to (\alpha_i)_{\kappa}^r$$

 $^{^{16}}$ Both being a countable union of antichains and having an embedding into \mathbb{Q} have been used as the definition of special. See [115, Exercise 9.9] for the equivalence.

is the statement that for any partition $f:[\beta]^r \to \kappa$, there is an $i < \kappa$ and a subset $A \subseteq \beta$ of order type α_i (in symbols, $\operatorname{otp}(A) = \alpha_i$) homogeneous for the partition, that is, all r-tuples from A are in the same cell, i.e. f is constant on $[A]^r$. In such a case we often call f a coloring, β a resource and each α_i a goal. If all the α_i s are equal, say to α , we abbreviate the notation to $\beta \to (\alpha)^r_{\kappa}$. With this notation in hand, Ramsey's Theorem is the statement that for all $k < \omega$, $\omega \to (\omega)^n_k$. In 1930 Sierpiński proved $\omega_1 \to (\omega_1)^2_2$. One of the many equivalent definitions of " κ is a weakly compact cardinal" is that $\kappa \to (\kappa)^2_2$.

In 1973, Baumgartner and András Hajnal [47] solved the $\rho = 0$ case of Problem 10 and all of Problem 10A of the paper by Erdős and Hajnal [81]¹⁷ by proving that for all countable ordinals α and finite k, $\omega_1 \to (\alpha)_k^2$.

According to Hajnal, work on what is known as the Baumgartner-Hajnal Theorem [47] started in 1970. Hajnal [108] learned about Martin's Axiom from István Juhász in Budapest late in 1970. He decided to try it out on the Erdős problem $\omega_1 \to (\alpha, \alpha)^2$ and was delighted to discover it worked.¹⁸

Shortly thereafter he attended the International Congress of Mathematicians in Nice September 1-10, 1970, where he went around telling people, including Solovay of his proof, but there was little interest in the result. Then he contacted Fred Galvin in Budapest and Galvin suggested getting in touch with Baumgartner. Later Galvin wrote to Hajnal that Baumgartner said that he could prove the theorem outright because there was an argument in Silver's thesis [171] (see also [172] cited in their paper) that could be used to eliminate any appeal to Martin's Axiom by absoluteness.

There was a meeting organized by the New York Academy of Sciences, and Erdős, Hajnal and Baumgartner met there and talked about the result. Each told the other what they knew. It took quite awhile for them to be convinced that both parts were right. Then Baumgartner wrote it up. They wrote an initial technical report [47] published in April 1971 and submitted their final report the same year but it did not appear until 1973.

To set the Baumgartner-Hajnal Theorem for ω_1 in context, note that in 1933, Sierpiński [169] proved the analog of Ramsey's Theorem fails for ω_1 : $\omega_1 \nrightarrow (\omega_1)_2^2$. In 1942, Erdős [79] proved an early positive result in the partition calculus for uncountable cardinals when he proved a graph theoretic

¹⁷This paper was based on the lecture by Erdős at the 1967 Summer School at UCLA. ¹⁸Hajnal was not the first to use Martin's Axiom this way. In a personal conversation, Laver told me that he used Martin's Axiom to prove a partition relation equivalent to $\omega_1 \to (\omega_1, (\omega:\omega_1))^2$ early in his post doc at Bristol 1969-1971.

equivalence of $\kappa^{\kappa+} \to (\kappa^+)^2_{\kappa}$; a similar theorem was implicit in work of Kurepa [136] from 1939.

In 1956, Erdős and Rado [84] published the first systematic treatment of the partition calculus. They proved that for any finite n, $\omega_1 \to (\omega + n)_2^2$ and $\omega_1 \to (\omega + 1, \omega_1)_2^2$. They further proved that for any finite positive n, m, and k, and any uncountable order type φ for which neither ω_1 nor its reverse, ω_1^* , embeds in φ , the following partition relations hold:

$$\varphi \to (\omega + n, \omega \cdot n)^2, \quad \varphi \to (\omega + n)_3^2, \quad \varphi \to (\omega + 1)_k^2.$$

In 1960, Hajnal proved that for $n < \omega$, $\omega_1 \to (\omega \cdot 2, \omega \cdot n)$ and that under CH, $\omega_1 \nrightarrow (\omega + 2, \omega_1)^2$. He further proved that for any uncountable order type φ for which neither ω_1 nor its reverse, ω_1^* , embeds in φ , and any finite positive n, countable α , and for η the order type of the rationals, the following partition relations hold:

$$\varphi \to (\alpha \vee \alpha^*, \eta)^2$$
, and $\varphi \to (\omega \cdot n, \alpha)^2$,

where the goal written $\alpha \vee \alpha^*$ is met if there is a homogeneous set for that color isomorphic to one of α and α^* .

Galvin (unpublished) proved no later than May 1970¹⁹ that $\varphi \to (\omega)^1_{\omega}$ implies $\varphi \to (\omega, \omega + 1)^2$. Galvin then revised the conjectures by Erdős and Rado in Problems 10, 10A, 11 for ω_1 , λ , the order type of the set of real numbers, and order types which embed neither ω_1 nor its reverse ω_1^* to a conjecture for order types φ for which $\varphi \to (\omega)^1_{\omega}$, i.e. order types with the property that for every partition into countably many sets, there is one which includes an increasing sequence.

The full Baumgartner-Hajnal Theorem asserts the revised conjecture is true: for any order type φ , if $\varphi \to (\omega)^1_{\omega}$, then for all $\alpha < \omega_1$ and $k < \omega$,

$$\varphi \to (\alpha)_k^2$$
.

Its metamathematical proof took a result proved with additional assumptions and then showed the result is absolute, that is, its truth in a model ZFC with additional assumptions implies that it is a consequence of ZFC. This paper introduced this method to a wide audience.

One of the key lemmas in the proof is the preservation result which says that if φ is an order type such that $\varphi \to (\omega)^1_{\omega}$, then $V^{\mathbb{P}} \models \varphi \to (\omega)^1_{\omega}$ for every ccc forcing notion \mathbb{P} .

¹⁹See [80] for the timing.

Subsequently Galvin [105] gave a proof of the Baumgartner-Hajnal Theorem that was purely combinatorial. Confirming a conjecture of Galvin, Todorcevic [191] extended the Baumgartner-Hajnal Theorem to the class of all partially ordered sets by proving that for every partial order P, if $P \to (\omega)^1_{\omega}$, then $P \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k < \omega$. Note that in this context the analogous absoluteness result that $P \to (\omega)^1_{\omega}$ is preserved by ccc forcing is no longer true, so [191] used a different argument. In 1983, Todorcevic [188] through forcing showed the consistency of a partition relation considerably stronger than the Baumgartner-Hajnal Theorem: 20 $\omega_1 \to (\omega_1, \alpha)^2$ for all countable ordinals α .

In 1991, Prikry and Milner [147] proved $\omega_1 \to (\omega \cdot 2 + 1, 4)^3$ by first showing that the partition relation holds in a model of Todorcevic in which both Martins Axiom and $\omega_1 \to (\omega \cdot 2 + 1)^2$ hold, and then using the approach taken by Baumgartner and Hajnal to show the consistency result is actually a ZFC theorem.

3.4 Basis Problem for Uncountable Order Types

A set of order types B is a basis for a family \mathcal{F} of linear orders if $B \subseteq \mathcal{F}$ and for all $\varphi \in \mathcal{F}$ there is some $\psi \in B$ with ψ embeddable in φ , in symbols $\psi \leq \varphi$. For example, $B_{\aleph_0} = \{\omega, \omega^*\}$ is a two-element basis for countably infinite linear orders, and $B_{\mathbb{Q}} = \{\eta\}$ is a one-element basis for all infinite linear orders that are dense in themselves, since η , the order type of the rationals is embeddable in every infinite dense in itself order type.

A set A of real numbers is \aleph_1 -dense if it has cardinality \aleph_1 and between any two elements of A there are exactly \aleph_1 members of A. In 1973 Baumgartner [10], [11] published his best known theorem on order: it is relatively consistent with ZFC that $2^{\aleph_0} = \aleph_2$ and all \aleph_1 -dense sets of reals are isomorphic. A key idea of the proof is to start with a model of the Continuum Hypothesis, build an iteration of countable chain condition (ccc) forcings that introduces the necessary order isomorphisms between pairs of \aleph_1 -dense sets. The CH is preserved at each intermediate stage and is used to prove that the forcings adding isomorphisms satisfy the ccc. At the end of the iteration, the Continuum Hypothesis no longer holds, but its presence in the intermediate stages was sufficient.

²⁰Todorcevic [188] reported the work on the paper was done during the academic year 1980-1981 when he was visiting Dartmouth College.

Another approach to Baumgartner's result (using the PFA context) is given in Chapter 8 of Todorcevic's book [193], and yet another approach to Baumgartner's result are generalizations given in [1].

In 1980, Shelah [161] proved the consistent existence of a linear order universal in power \aleph_1 with the continuum equal to $\mathfrak{c} = \aleph_2$, Shelah compared and contrasted his use of oracle forcing in the proof with that of Baumgartner in his proof that all \aleph_1 -dense sets of reals can be isomorphic.

Baumgartner used his forcing result to show the consistency of ZFC + "the class of real types has a one element basis." He also asked if it is consistent for all \aleph_2 -dense sets of reals to be isomorphic, and the question was answered in the positive by Itay Neeman (email of March 5, 2016).

Baumgartner noted his proof can be extended to add Martin's Axiom to the conclusion, and asked if "all \aleph_1 -dense sets of reals are isomorphic" follows from Martin's Axiom $+ 2^{\aleph_0} > \aleph_1$. Uri Abraham and Saharon Shelah gave a negative answer in 1981, and Abraham, Matatyahu Rubin, and Shelah [1] showed that it is relatively consistent with ZFC that $2^{\aleph_0} > \aleph_2$ and all \aleph_1 -dense sets of reals are isomorphic and proved that if all \aleph_1 -dense sets of reals are isomorphic, then $2^{\aleph_0} < 2^{\aleph_1}$.

In 1976 Baumgartner [22] considered the problem of finding a nice basis for the class Φ of all uncountable order types which cannot be represented as the union of countably many well-orderings. He set $\Phi_1 = \{\omega_1^*\}$, let Φ_2 be the uncountable order types embeddable in the reals, and let Φ_3 be the uncountable order types which do not embed a subset of type ω_1 nor a subset of type ω_1^* nor an uncountable subset of the real numbers. He called the type types in Φ_3 Specker types.²¹ Finally he let Φ_4 be the uncountable order types φ such that every uncountable subtype $\psi \leq \varphi$ contains an uncountable well-ordering but φ cannot be represented as the union of countably many well-orders. Then every element of Φ embeds some element of $\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$, and the question of a basis for Φ can be subdivided into finding bases for the components. Consequently, a basis for uncountable order types can be obtained by adding ω_1 and its reverse, ω_1^* to a basis for $\Phi_2 \cup \Phi_3$, since every element of Φ_4 embeds ω_1 . Baumgartner answered Galvin's question of whether $\Phi_1 \cup \Phi_2 \cup \Phi_3$ formed a basis for Φ by showing that Φ_4 is non-empty.

 $^{^{21}}$ In [90, page 443] Erdős and Rado conjectured that there were no uncountable linear order types which did not embed ω_1 , nor embed the reverse, ω_1^* , nor embed an uncountable subset of the real numbers, but they included a footnote that Specker had disproved the conjecture.

Most of the paper is devoted to developing a structure theory for elements of Φ_4 , where stationary sets play a significant role. In particular, in Corollary 7.9, he proved that if ZFC + "there exists a weakly compact cardinal" is consistent, then so is ZFC + "for every stationary subset $C \subseteq \omega_2$, if $cf(\alpha) = \omega$ for all $\alpha \in C$, then there is an $\alpha < \omega_2$ such that $C \cap \alpha$ is stationary in α ." In modern language, the latter statement translates to every stationary subset of $\omega_2 \cap \operatorname{cof}(\omega)$ reflects, ²² where $\operatorname{cof}(\omega)$ is the collection of ordinals of cofinality ω . In [22] Baumgartner (as reworked later by Shelah and perhaps others) showed that if one collapses a supercompact cardinal to ω_1 via a standard Lévy collapse, then in the resulting forcing extension, for every regular cardinal $\kappa \geq \omega_2$, and every stationary set $S \subseteq \kappa \cap \operatorname{cof}(\omega)$, there is a $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in γ . Menachem Magidor [142, page 756] asserted that Baumgartner's proof actually showed under the hypothesis of Corollary 7.9, that ZFC + "every pair of stationary subsets of $\omega_2 \cap \operatorname{cof}(\omega)$ has a common point of reflection." Magidor also showed the equiconsistency of this statement with the hypothesis of Corollary 7.9.

In his paper on uncountable order types, Baumgartner also assembles the ingredients for a proof that every element of Φ_3 is the linearization of an Aronszajn tree, that is an Aronszajn line, so Φ_3 and the collection of Aronszajn lines coincide.²³

At the end of the paper Baumgartner [22] asked in Problem 5(i) if ZFC + " Φ_3 (Aronszajn lines) has a finite basis" is consistent and in Problem 5(ii) if ZFC + " $\Phi_2 \cup \Phi_3$ (real types and Aronszajn lines) has a finite basis" is consistent. We will return this question in the section on forcing for all.

3.5 Disjoint Refinements

In 1975, Baumgartner, Hajnal and Attila Máté [50] gave a partial answer to a question of Fodor by giving a condition on the non-stationary ideal NS_{ω_1} which guarantees any ω_1 -sequence of stationary subsets of ω_1 can be resolved

²²In 1985 Leo Harrington and Shelah [110] showed that ZFC + "the existence of a Mahlo cardinal" is equiconsistent with ZFC + "every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ ω reflects."

²³Todorcevic [190] described this result as part of the folklore of the subject, noting that a large part of it was proved by Kurepa [134, page 127-9], and further referring the interested reader to the Erdős-Rado paper [90] which only has their conjecture and the note that Specker refuted it, and to a survey paper by R. Ricabarra. Proofs of this fact can be found in Baumgartner's 1982 survey article [27] on order and Todorcevic's 1984 survey article [190] on trees and linear orders.

into an antichain in $\mathcal{P}(\omega_1)/NS_{\omega_1}$, as described in the following theorem.

Theorem 3.1 (Baumgartner, Hajnal, Máté). Assume I is a normal ideal on ω_1 such that given any $\langle X_{\alpha} \in \mathcal{P}(\omega_1) \setminus I \mid \alpha < \omega_1 \rangle$, there exists $X \in \mathcal{P}(X_0) \setminus I$ such that $X_{\alpha} \setminus X \notin I$ for each $\alpha \geq 1$.²⁴ Then, given $\langle S_{\alpha} \in \mathcal{P}(\omega_1) \setminus I \mid \alpha < \omega_1 \rangle$, there exist $\langle A_{\alpha} \in \mathcal{P}(\omega_1) \setminus I \mid \alpha < \omega_1 \rangle$ such that each $A_{\alpha} \subseteq S_{\alpha}$ and the A_{α} 's are disjoint.

In 2000, Paul Larson [137] used the above theorem in his article Separating Stationary Reflection Principles to show that Todorcevic's Strong Reflection Property (SRP) implies $SR_{\omega_1}^*$, the strongest strengthening of Stationary Reflection (SR) that Larson considered.

In 2014, Monroe Eskew [92] showed that the Baumgartner-Hajnal-Máté result cannot be lifted to ideals in general on larger \aleph_n s when he proved that if ZFC + "there is an almost huge cardinal" is consistent, then for n > 1, so is ZFC + GCH + "there is a normal \aleph_n -complete ideal \mathcal{I} on \aleph_n and a sequence of \aleph_n many \mathcal{I} -positive sets which has no disjoint refinement" is consistent.

3.6 Almost Disjoint Families

In 1976 Baumgartner [21] extended the work [168], [183], [184] of Sierpiński and Tarski²⁵ to give a complete solution to their questions under the assumption of the Generalized Continuum Hypothesis (GCH) after connecting questions about families of almost disjoint sets with his dense set problem of finding out, for a given cardinal κ , which cardinals λ support a linear order with a dense subset of power κ .²⁶ William Mitchell [150] also worked on the dense set problem, and Baumgartner documented the interconnections of their results. Baumgartner developed extensions of an observation of Shelah that permit transfer of a result about the existence of a set S of power λ with a dense subset U of power κ to the statement obtained by replacing λ by λ^{ρ} and κ by κ^{ρ} , and drew conclusions from the extensions about

This condition holds if Y_{ω_1} fails to be ω_1 -dense, where \mathcal{I} is ω_1 -dense on ω_1 if there is a family $D \subseteq \mathcal{P}(\omega_1) \setminus \mathcal{I}$ of power ω_1 such that for any $X \in \mathcal{P} \setminus I$, there is a $Y \in D$ with $Y \setminus X \in \mathcal{I}$.

²⁵They assumed GCH.

²⁶Baumgartner formulated the dense set problem by generalizing a question of Malitz [143] in a proof that the Hanf number for complete $L_{\omega_1,\omega}$ sentences is \beth_{ω_1} . Baumgartner [12] eliminated the use by Malitz of GCH by applying a combinatorial fact due to Hausdorff.

the existence of various families of almost disjoint sets. He constructed a broad range of forcing extensions modeling a variety of answers to the almost disjoint sets questions and the dense sets problems using along the way the Erdős-Rado Theorem for partitions with many parts, Jensen's \Diamond_{κ} , and Easton forcing. In particular, he proved that it is consistent with ZFC that $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1+1}$ and there is no family of 2^{\aleph_1} pairwise almost-disjoint subsets of \aleph_1 .

As a special case of a more general theorem, Sierpiński [168], using the Generalized Continuum Hypothesis, proved that for any infinite set A, there is a family of size \aleph_1 of strongly almost disjoint subsets of A. In 1934, Sierpiński labeled this proposition P_{11} , and showed it is equivalent to the Continuum Hypothesis in his book [170]. Baumgartner [21, pages 424, 428] proved consistency with ZFC and independence from ZFC of the existence of a strongly almost disjoint family of uncountable subsets of ω_1 . He started with an almost disjoint family \mathcal{F} of \aleph_2 many uncountable subsets of ω_1 . He started with an almost disjoint family as $\langle F_{\alpha} \mid \alpha < \omega_2$, and let \mathbb{P} be the set of all finite partial functions $f: \omega_2 \to [\omega_1]^{\omega}$ such that for all $\alpha \in \text{dom}(f)$, f(x) is a finite subset of F_{α} . He let $f \leq g$ if and only if (a) $\text{dom}(g) \subseteq \text{dom}(f)$; (b) for all α in the domain of g, $g(\alpha) \subseteq f(\alpha)$, and (c) for all $\alpha \neq \beta$ in the domain of g, $f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta)$. In the forcing extension $\mathfrak{M}[G]$, the sets $G_{\alpha} = \bigcup \{f(\alpha) \mid f \in G\}$ are considered to be obtained by thinning out the F_{α} , and they form the strongly almost disjoint family in the extension.

Baumgartner used results and adapted techniques used in the study of almost disjoint families to prove theorems about polarized partitions in the final section of his paper. For cardinals κ , λ , μ , ν , and ρ , the polarized partition relation $\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \mu \\ \nu \end{pmatrix}_{\rho}$ holds if and only if for all $f: \kappa \times \lambda \rightarrow \rho$ there are $A \subseteq \kappa$ with $\text{otp}(A) = \mu$ and $B \subseteq \lambda$ with $\text{otp}(B) = \nu$ such that f is constant on $A \times B$. We also consider the variant where the subscript is $< \rho$ for coloring maps whose range has cardinality $< \rho$, and the variant $\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \mu & \sigma \\ \nu & \tau \end{pmatrix}$ when there are only two color classes and different goals for the different colors. This relation was introduced in [84] and studied in [82]. Baumgartner results include the following where $\mathfrak{c} = 2^{\aleph_0}$:

1.
$$\begin{pmatrix} \mathfrak{c} \\ \aleph_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{c} \\ \alpha \end{pmatrix}$$
 for all $\alpha < \omega_1$, but $\begin{pmatrix} \mathfrak{c} \\ \aleph_1 \end{pmatrix} \nrightarrow \begin{pmatrix} 2 \\ \aleph_0 \end{pmatrix}_{\omega}$.

²⁷Such a family can be constructed by transfinite recursion.

2.
$$\begin{pmatrix} \mathfrak{c} \\ \aleph_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{c} & \mathfrak{c} \\ \aleph_1 & \alpha \end{pmatrix}$$
 for all $\alpha < \omega_1$, but $\begin{pmatrix} \mathfrak{c} \\ \aleph_1 \end{pmatrix} \nrightarrow \begin{pmatrix} \aleph_1 & \aleph_1 \\ \aleph_0 & \aleph_0 \end{pmatrix}_{\omega}$.

Almost disjoint families have been and continue to be used to construct interesting examples. Haim Gaifman and Specker [103] showed in 1964 that if $\kappa^{<\kappa} = \kappa$, then there are 2^{κ^+} many different types of normal κ^+ -Aronszajn trees by using a family of almost disjoint sets in their construction.

In 2005, using results under GCH of Sierpiński and Tarski cited above, Lorenz Halbeisen [109] proved the consistency with ZFC that for all cardinals κ , every infinite dimensional Banach space of cardinality κ admits 2^{κ} pairwise almost disjoint normalized Hamel bases. By way of contrast, using a result of Baumgartner [21, Theorem 5.6(b)], Halbeisen proved the consistency with ZFC that $2^{\kappa} \leq \kappa^{++}$ and no infinite dimensional Banach space of cardinality κ admits κ^{++} pairwise almost disjoint normalized Hamel bases.

In 2006, J. Donald Monk [152] revisited and extended Baumgartner's work on families of almost disjoint sets with a focus on the sizes of maximal families.

Cristina Brech and Piotr Koszmider [71] used the product of Baumgartner's \mathbb{P} with the standard σ -closed and ω_2 -cc forcing for adding ω_3 subsets of ω_1 with countable conditions in their construction of a forcing extension in which there is no universal Banach space of density the continuum.

3.7 Translating stationary to closed unbounded

In 1976, Baumgartner, Harrington and Eugene Kleinberg [52] were able to add a closed unbounded set as a subset of a stationary set $A \subseteq \omega_1$ by forcing with closed countable subsets of A whose order type is a successor ordinal. This technique is useful for translating problems about stationary sets into ones about closed unbounded sets, and closed unbounded sets provide the ladder for recursive constructions and inductive proofs.

To start, Baumgartner, Harrington and Kleinberg recalled the well-known theorem that for any regular uncountable cardinal κ , the intersection of fewer than κ many closed unbounded sets is closed unbounded (we will abbreviate "closed unbounded" to club.) Then they observed that \mathcal{F}_{κ} , the family of all subsets of κ that have a club set as a subset, is a κ -additive non-principal filter on κ .

Next they considered the possibility that \mathcal{F}_{κ} is an ultrafilter. If $\kappa > \omega_1$,

then \mathcal{F}_{κ} cannot be an ultrafilter.²⁸ If $\kappa = \aleph_1$ and \mathcal{F} is an ultrafilter, then \aleph_1 is a measurable cardinal. Solovay [176] had shown \aleph_1 being measurable is relatively consistent with the usual axioms of set theory (ZF) together with the Axiom of Determinacy in his model in which all sets of reals are Lebesgue measurable where the Axiom of Choice is not true. Baumgartner, Harrington, and Kleinberg then proved that if one starts with a model \mathcal{M} of the usual axioms of set theory (ZF) together with the Axiom of Choice and a set A in \mathcal{M} such that $\mathcal{M} \models \text{``} A \subseteq \aleph_1$ is not disjoint from any closed unbounded subset of \mathcal{M} ," then there is a generic extension \mathcal{N} with the same reals as \mathcal{M} such that $\mathcal{N} \models \text{``} A$ contains a closed unbounded subset of \aleph_1 ."

Baumgartner, Harrington, and Kleinberg described their "shooting a club through a stationary set" result as an extension of the theorem of Harvey Friedman [100], who proved that every stationary subset of \aleph_1 contains arbitrary long countable closed sets.

In 1978, Abraham and Shelah [2, page 647-8, Theorem 4], building on work by Jonathan Stavi,²⁹ proved that if $\kappa = \mu^+$, $\mu^{<\mu} = \mu$, and S is a fat stationary set,³⁰ then there is a partial order such that forcing with it introduces a club subset of S, does not collapse any cardinals, and does not add new subsets of size $< \mu$.

In Section 6 of his *Handbook of Set Theory* chapter, James Cummings [75] used this forcing of Baumgartner, Harrington and Kleinberg to show that in general (ω_1, ∞) -distributivity is weaker than $< \omega_1$ -strategic closure.

3.8 Cardinal Arithmetic Constraints

In 1976 Baumgartner and Karel Prikry [57] published their elementary proof³¹ of the remarkable result of Jack Silver [173] that if the Generalized Continuum Hypothesis holds below a cardinal κ of uncountable cofinality, then it holds at κ . Silver's paper in the proceedings of the International Congress of Mathematicians of 1974 used metamathematical arguments and his four page proof omitted many details. Let us note that Silver's result has deep

²⁸For regular $\kappa > \omega_1$, consider the set A of ordinals $\alpha < \kappa$ of cofinality ω_1 . Neither it nor its complement can contain a closed unbounded subset of κ .

²⁹Stavi was cited for handwritten notes from 1975 on Adding a closed unbounded set.

 $^{^{30}}$ A stationary set $S \subseteq \kappa$ is called *fat* if and only if for every closed unbounded set $C \subseteq \kappa$, $S \cap C$ contains closed sets of ordinals of arbitrarily large order-types below κ . The terminology is from [94].

³¹Ronald Jensen (unpublished) also had an alternative proof.

roots: Hilbert put the Continuum Problem first on his famous list of problems of 1900. Felix Hausdorff [111, page 133] speculatively used the possible generalization of the continuum hypothesis to larger \aleph_{ν} s in his analysis of order types generalizing the rationals to larger cardinalities. Alfred Tarski [182, page 10] used the phrase generalized continuum hypothesis (hypothèse généralisée du continu) in 1925. As a service to the wider mathematical community, Baumgartner and Prikry [58] then wrote an article for the American Mathematical Monthly on the special case $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$ that used only König's Theorem (the sum of an indexed family of cardinals is less that the product of the indexed family), the Regressive Function Theorem and basic facts about cardinal arithmetic.

3.9 Filters, ideals, and partition relations

Cardinality was the initial notion of largeness for homogeneous sets in the systematic study of the partition calculus by Erdős and Rado [84]. Other notions considered early were having a large order type, e.g. subsets of the reals order isomorphic to the set of reals and being in a κ complete ultrafilter for a large cardinal κ .

The central point for Baumgartner in his two papers on ineffable cardinals was that "many 'large cardinal' properties are better viewed as properties of normal ideals than as properties of cardinals alone," He used a variety of partition relations, inaccessibility and indescribability in his characterizations of the normal ideals associated with a variety of mild large cardinals, that is, ones below a measurable cardinal. He also was able to calibrate the strength of the large cardinal needed for the partition relations in question.

In his 1975 paper on ineffable cardinals Baumgartner [17] analyzed large subsets of ineffable, almost ineffable, and subtle cardinals. These cardinals had been introduced by Jensen and Kunen [120] in their analysis of combinatorial principles that hold in L.

Suppose $A \subseteq \kappa$. Recall a function $f: A \to \kappa$ is regressive if $f(\alpha) < \alpha$ for all $\alpha > 0$; and $f: [A]^n \to \kappa$ is regressive if $f(\vec{a}) < \min(\vec{a})$ for all $\vec{a} \in [A]^n$. In a modern definition, a regular cardinal κ is (1) ineffable; (2) weakly ineffable, (3) subtle respectively if and only if for every regressive function $f: \kappa \to \mathcal{P}(\kappa)$,

1. (ineffable) there is a set $A \subseteq \kappa$ such that the set $\{\alpha < \kappa \mid A \cap \alpha = f(\alpha)\}$ is stationary;

- 2. (weakly ineffable) there is a set $A \subseteq \kappa$ such that the set $\{\alpha < \kappa \mid A \cap \alpha = f(\alpha)\}$ has cardinality κ ;
- 3. (subtle) for every closed unbounded subset $C \subseteq \kappa$ there are $\alpha < \beta \in C$ with $A_{\alpha} = A_{\beta} \cap \alpha$.

Kunen [120] proved that a cardinal κ is ineffable if and only if it satisfies the partition relation $\kappa \to (\text{stationary})_2^2$, where one asks for a stationary homogeneous set rather than one of cardinality κ . He also proved that ineffable cardinals were Π_2^1 -indescribable, and Kunen and Jensen located the least ineffable cardinal above the least cardinal which is Π_m^n -indescribable for all $m, n < \omega$ and above the least cardinal cardinal λ such that $\lambda \to (\omega)_2^{<\omega}$. They showed weakly ineffable cardinals were Π_1^1 -indescribable.

Baumgartner refined subtle, weakly ineffable, and ineffable to n-subtle, n-weakly ineffable, and n-ineffable. Extend the notion of regressive to functions whose domain is a subset $A \subseteq \kappa$ and whose range is a subset of $\mathcal{P}(\kappa)$ as follows: a function $f: A \to \mathcal{P}(\kappa)$ is regressive if $f(\alpha) \subseteq \alpha$ for all $\alpha \in A$ with $\alpha > 0$; and $f: [A]^n \to \kappa$ is regressive if $f(\vec{a}) \subseteq \min(\vec{a})$ for all $\vec{a} \in [A]^n$ with $\min(\vec{a}) > 0$. Call a set $H \subseteq A$ set-homogeneous for a regressive function $f: [\kappa]^n \to \mathcal{P}(\kappa)$ if and only $H \subseteq A$ and for all $\vec{a}, \vec{c} \in [A]^n$, if $\min(\vec{a}) \leq \min(\vec{c})$, then $f(\vec{a}) = f(\vec{c}) \cap \min(\vec{a})$. The concept extends in the natural way for regressive functions from A to $\mathcal{P}(\kappa)$.

A cardinal κ is *n*-subtle if and only if for every regressive function f: $[\kappa]^n \to \mathcal{P}(\kappa)$ and every closed unbounded set $C \subseteq \kappa$, there is a set $H \in [C]^{n+1}$ homogeneous for f. Also $A \subseteq \kappa$ is *n*-ineffable (*n*-weakly ineffable) if and only if every regressive function $f: [A]^n \to \mathcal{P}(\kappa)$ has a homogeneous set which is stationary in κ (of power κ). For any of these cardinal properties, he spoke of a subset $A \subseteq \kappa$ as having the corresponding homogeneity property, if every suitable regressive function had the corresponding homogeneity property.

Baumgartner proved that for each of the notions of largeness for a cardinal κ the corresponding set of small (i.e. not large) sets forms a κ -complete normal ideal on κ .

He used partition properties to characterize the various notions of largeness. He had equivalences for a subset $A \subseteq \kappa$ being subtle, and being *n*-weakly ineffable using both regular and regressive partition relations. For example, the following are equivalent for a subset A of a regular cardinal κ :

- 1. A is n-ineffable.
- 2. $A \to (\text{stationary set})_2^{n+1}$.

- 3. $A \to (\text{stationary set}, \kappa)_2^{n+2}$.
- 4. $A \to (\text{stationary set}, n+3)_2^{n+2}$.

Thus $\kappa \to (\text{stationary set})_2^m$ does not imply $\kappa \to (\text{stationary set})_2^{m+1}$, which stands in contrast to the fact that $\kappa \to (\kappa)_2^n$ implies $\kappa \to (\kappa)_2^n$ for all positive integers n.

Harvey Friedman [101] has adapted Baumgartner's approach to n-subtle cardinals for his program to develop "natural" propositions of finite mathematics whose consistency requires use of large cardinals. Pierre Matet [146] has used it to prove a partition property for $\mathcal{P}_{\kappa}(\lambda)$.

In 1977, Baumgartner [23] extended the association of normal ideals with large cardinals to weakly compact cardinals and Ramsey cardinals, using the notions of α -Erdős cardinals³² and canonical sequences. Qi Feng [93] extended Baumgartner's work in his thesis, and Ian Sharpe and Philip Welch [160] used Baumgartner's canonical sequences and Feng's work to develop an α -weakly Erdős hierarchy as part of their study of implications for inner models of strengthenings of Chang's conjecture. They also modeled the proofs of some of their lemmas on proofs from [37].

In 1977 Baumgartner, Taylor and Wagon [66] used Mahlo's operation M to define a family of ideals they called M-ideals and to define the notion of a cardinal κ being greatly Mahlo, and then proved that a cardinal was greatly Mahlo if and only if it bears an M-ideal. A very satisfying instance of equiconsistency of a combinatorial principle with the existence of a large cardinal was proved in 2011 by John Krueger and Ernest Schimmerling [131]. They showed that the existence of a greatly Mahlo cardinal is equiconsistent with the existence of a regular uncountable cardinal κ such that no stationary subset of κ^+ consisting of ordinals of cofinality κ carries a partial square.³³

 $^{^{32}}$ A cardinal κ is α -Erdős if it is regular and for every regressive function $f: [\kappa]^{<\omega} \to \kappa$ and every closed unbounded set $C \subseteq \kappa$, there is $A \subseteq C$ of order type α which is homogeneous for f, i.e. for every positive n, f is constant on the n-element subsets of A.

³³Partial square sequences were introduced as a weaking of square sequences by Shelah (see [164]). Suppose $\nu < \kappa^+$ is regular and $A \subseteq \kappa^+ \cap \operatorname{cof}(\nu)$. Then A carries a partial square if there is a sequence $\langle c_{\alpha} \mid \alpha \in A \rangle$ such that (a) each c_{α} is a closed unbounded subset of α of order type ν and whenever c_{α} and c_{β} share a common limit point γ , then $c_{\alpha} \cap \gamma = c_{\beta} \cap \gamma$.

3.10 Saturated ideals

Recall that an ideal \mathcal{I} on a cardinal κ is λ -saturated³⁴ if and only if every pairwise \mathcal{I} -almost disjoint collection $F \subseteq \mathcal{I}^+$ of \mathcal{I} -positive sets is of cardinality less than λ . Solovay had shown that if a regular cardinal κ has a nontrivial normal κ -complete λ -saturated ideal for some $\lambda < \kappa$, then κ is measurable in an inner model. In 1970, Kunen [132] extended Solovay's result by showing the result was true for regular cardinals κ carrying a non-trivial κ -complete κ^+ -saturated ideal. Kunen noted that it was unknown whether ω_1 can bear an ω_2 -saturated ideal. He showed that if a successor cardinal κ has a non-trivial κ -complete κ^+ -saturated ideal then 0^{\dagger} exists.

In 1972, Kunen [133] proved that if ZFC + "there exists a huge cardinal" is consistent, then so is ZFC + "there is an ω_2 -saturated ideal on ω_1 ." In his review of the history of the problem, he noted that for an uncountable cardinal κ , the larger the λ , the weaker the property of being λ -saturated ideal on κ . The existence of an ω -saturated ideal on κ was equivalent to κ being measurable; the existence of a κ^+ saturated ideal on κ implied κ is measurable in an inner model, and the existence of a $(2^{\kappa})^+$ -saturated ideal on κ was provable in ZFC. He focused on λ with $\kappa \leq \lambda \leq 2^{\kappa}$. Kunen pointed out that arguments of Ulam [195] showed that if κ is a successor cardinal, then there can be no λ -saturated ideal on κ with $\omega < \lambda \leq \kappa$. He remarked (see [133, p. 72]) that using the techniques of his 1970 paper [132], from an ω_2 -saturated ideal on ω_1 one gets consistency of "inner models with several measurable cardinals."

In 1974-1975, Alan Taylor³⁵ and Stanley Wagon were both in Berkeley for nine months and the Baumgartners spent the winter and spring quarters there. Taylor became interested in Wagon's work on saturation of ideals. At the end of 1975, Baumgartner, Taylor and Wagon [66] submitted their paper On splitting stationary subsets of large cardinals which appeared in 1977. They looked at saturation properties of ideals, especially nonstationary ideals. In 1972 Kunen [133] had shown the consistency of an ω_2 -saturated ideal on ω_1 relative to the existence of a huge cardinal, but only partial results were available on when or if the non-stationary ideal on κ could be κ^+ -saturated.

Baumgartner, Taylor and Wagon [66] showed that given a normal ideal

³⁴Tarski [185] introduced λ -saturation of ideals in 1945.

³⁵Based on email of May 8, 2016 from Alan Taylor.

 \mathcal{I} on κ , \mathcal{I} is κ^+ -saturated if and only if the ideals $\mathcal{I}|A$ generated by I and $\kappa \setminus A$ for $A \in \mathcal{P}(\kappa) \setminus \mathcal{I}$ are the only normal ideals that extend \mathcal{I} . Thus the non-stationary ideal NS_{κ} is κ^+ -saturated if and only if all normal ideals on κ have the form $\mathrm{NS}_{\kappa}|A$ for some $A \subseteq \kappa$. As a corollary, they showed that if \mathcal{I} is a normal κ^+ -saturated ideal on κ and \mathcal{J} is a normal extension of \mathcal{I} , then \mathcal{J} is also κ^+ -saturated. It follows that if NS_{κ} is κ^+ -saturated, then every normal non-trivial ideal on κ is κ^+ -saturated. As a corollary, they showed that if κ is greatly Mahlo, then the nonstationary ideal on κ , NS_{κ} is not κ^+ -saturated. Gitik and Shelah [107] proved that ω_1 is the only uncountable cardinal for which NS_{κ} can be κ^+ -saturated.

Foreman [95] highlighted the power of non-stationary ideals and their restrictions to selected stationary sets when he showed that the consistency of ZFC together with his strengthening of the classical Chang conjectures to the principle of Strong Chang Reflection³⁶ for (ω_{n+3}, ω_n) implies the consistency of ZFC together with the existence of a huge cardinal in a model of the form $L[A^*, \check{I}]$ where \check{I} is the dual of the appropriate nonstationary ideal. Foreman used the proposition below to show that the set A^* was absolutely definable.

Proposition (Baumgartner): Let $M, N \prec H(\theta)$.

If $\sup(M \cap \omega_{n+2}) = \sup(N \cap \omega_{n+2}) \in \operatorname{cof}(>\omega), \ N \cap \omega_{n+1} = M \cap \omega_{n+1}$ and $\sup(N \cap \omega_{n+1}) \in \operatorname{cof}(>\omega)$, then $M \cap \omega_{n+2} = N \cap \omega_{n+2}$.

In 1982, Baumgartner and Taylor [64], [65] published a two part paper on saturation properties where they pioneered the study of conditions on a forcing which preserved the saturation property of ideals in the extension. In the first part, given a cardinal λ , Baumgartner and Taylor concentrated on questions about which properties of an ideal \mathcal{I} and a partial order \mathbb{P} guarantee the λ -saturatedness of the ideal $\overline{\mathcal{I}}$ generated by \mathcal{I} in the generic extension by \mathbb{P} . They focus on instances that do not call for the use of large cardinals. For example they prove that if the forcing has the σ -finite chain condition,³⁷ then one can conclude that in M[G] every ideal on ω_1 is ω_2 -generated, and hence, by a result earlier in the paper, is ω_3 -saturated. They asked whether under ccc forcing, the converse that all ω_3 -saturated ideals are ω_2 -generated. Baumgartner and Taylor used a ccc forcing GH which is a variant of one by Galvin and Hajnal and showed that in the extension, there is an ideal on

 $^{^{36}}$ We omit the definition of this principle but note that it includes second order reflection requirements and that Foreman has shown it is consistent from a 2-huge cardinal.

³⁷A partial order P satisfies the σ -finite chain condition if there is a function $f: P \to \omega$ such that for all $n < \omega$, every pairwise incompatible subset of $f^{-1}(\{n\})$ is finite.

 ω_1 that is not ω_3 -saturated. In Corollary 3.5 they show that it is relatively consistent with ZFC that 2^{ω} is large and the nonstationary ideal \mathcal{I} on $[\omega_2]^{\omega}$ is not 2^{ω} -saturated, but there is a stationary set $S \subseteq [\omega_2]^{\omega}$ such that $\mathcal{I}|S$ is ω_4 -saturated.

In part 2 Baumgartner and Taylor [64] continue the study of preservation under forcing, especially ccc forcing, of saturation properties of countably complete ideals such as ω_2 -saturation or precipitousness.³⁸ They formulate equivalences of the ω_2 -saturation of a countably complete ideal \mathcal{I} on ω_1 being preserved under ccc forcing in terms of a generalized version of Chang's conjecture and a weakening of Kurepa's Hypothesis. They showed that given an ω_2 -saturated ideal \mathcal{I} , in any forcing extension by a σ -finite chain condition forcing, the ideal $\overline{\mathcal{I}}$ induced by \mathcal{I} is ω_2 -saturated on ω_1 . They also showed that after forcing with the partial order for adding a closed unbounded subset of ω_1 with finite conditions (see [29, page 926]), there are no ω_2 -saturated countably complete ideals on ω_1 in the extension. They revisited the variant of the Galvin-Hajnal partial ordering GH used to provide a consistent counterexample to all ω_2 -generated countably complete ideals on ω_1 being ω_3 -saturated, and showed that the ω_2 -saturation of any ideal on ω_1 is preserved when forcing with GH.

Baumgartner and Taylor called an ideal \mathcal{I} presaturated if it is both precipitous and ω_2 -preserving i.e. $\Vdash_{\mathcal{P}(\omega_1)/\mathcal{I}}$ " $\check{\omega}_2$ is a cardinal"). After developing basic properties of presaturated ideals, they prove that (a) any ω_2 -preserving ideal on ω_1 is a weak p-point; and (b) if there is a presaturated ideal on ω_1 , then there is a normal presaturated ideal on ω_1 .

A countably complete ideal \mathcal{I} on ω_1 is *strong* if and only if it is precipitous and $\Vdash_{\mathcal{P}(\omega_1)/\mathcal{I}} j(\omega_1^V) = \omega_2^V$. Baumgartner and Taylor observed that the argument given by Kunen [133] that the consistency of existence of a non-trivial countably complete ω_2 -saturated ideal on ω_1 implies consistency of existence of several measurable cardinals, he only used the fact that the ideal was strong. Every ω_2 -saturated ideal is presaturated, and every presaturated ideal is strong.

In section 5, Baumgartner and Taylor used a technical condition on a forcing \mathbb{P} , being \mathcal{I} -regular, ³⁹ to prove that if an ideal \mathcal{I} in the ground model

³⁸Precipitous ideals were introduced by Jech and Prikry [116]. If I is an ideal on $\mathcal{P}(\kappa)$, then $\mathcal{P}(\kappa)/I$ is a notion of forcing which adds an ultrafilter G extending the filter dual to I, and the ideal I is said to be *precipitous* if $\kappa \Vdash_{\mathcal{P}(I)/I} V^{\kappa}/G$ is wellfounded.

³⁹All ccc forcings and the forcing to add a closed unbounded subset of ω_1 with finite conditions are $\mathcal{I}S$ -regular.

has one of the properties (a) precipitous, (b) strong, (c) presaturated, (d) ω_2 -saturated, then, in the extension by \mathcal{P} , there is a set A which is positive for the ideal $\overline{\mathcal{I}}$ generated from \mathcal{I} such that $\overline{\mathcal{I}}|A$ is has the corresponding property in the extension modulo the following constraints: (a) precipitous (no additional constraint), (b) strong if P does not collapse ω_2 , (c) presaturated if in the extension $J(\mathcal{P})$ does not collapse ω_2^V , (d) ω_2 -saturated if in the extension $j(\mathcal{P})$ is a ccc forcing). Moreover, if \mathbb{P} is a ccc forcing then $A = \omega_1$ for (a), (b), and (c).

In Theorem 5.10 they prove that the consistency of ZFC + "there is a presaturated ideal on ω_1 " implies the consistency of ZFC + "there is a presaturated ideal on ω_1 but no ω_2 -saturated ideals on ω_1 " and prove that consistency of ZFC + "there is a precipitous ideal on ω_1 " implies the consistency of ZFC + "there is a precipitous ideal on ω_1 but no strong ideals on ω_1 " (so also no presaturated ideals on ω_1).

We now review a few of the questions from the final section of the paper. In Question 6.1, Baumgartner and Taylor [65] asked if the ω_2 -saturation of a countably complete ideal on ω_1 is preserved under ccc forcing and in Question 6.2, they asked if ω_1 can carry a countable complete ω_2 -saturated ideal which satisfies Chang's conjecture. They had proved for an ω_2 -saturated ideal \mathcal{I} , that \mathcal{I} satisfies Chang's conjecture if and only if for every ccc partial ordering \mathbb{P} , $\Vdash_{\mathbb{P}}$ " \mathcal{I} generates an ω_2 -saturated ideal". Foreman, Magidor and Shelah [99, p. 24, Corollary 17] proved that if Martin's Maximum (MM) holds, then the NS $_{\omega_1}$ is ω_2 -saturated and there is no ccc forcing which destroys its saturation, so NS $_{\omega_1}$ satisfies Chang's conjecture, giving a model in which the answers to Questions 6.1 and 6.2 are yes. Donder and Levinski (unpublished) gave a model in which the answer is no to Question 6.1. Boban Velickovic [196] gave another negative answer to Question 6.1 in a forcing extension of a model of MM.

In Question 6.5, Baumgartner and Taylor asked if every ω_2 -preserving countably complete ideal on ω_1 is precipitous. John Krueger [129, page 844, Corollary 6] gave a positive answer for normal ideals on ω_1 under the cardinality constraint $2^{\omega_1} \leq \omega_3$ when he proved that if κ is regular and $2^{\kappa} \leq \kappa^{++}$, then the properties of κ^+ -preserving and presaturated⁴⁰ are equivalent for normal ideals.

⁴⁰What Baumgartner and Taylor called presaturated, Krueger called weakly presaturated, under the hypothesis of this theorem, Kruegar proved the Baumgartner-Taylor version and his version were equivalent.

In Question 6.10, Baumgartner and Taylor asked if the consistency of the existence of a strong ideal on ω_1 is equivalent to the consistency of the existence of a normal strong ideal. In 2010, Gitik [106, page 196, Proposition 3.1] gave a positive answer.

In Question 6.11, Baumgartner and Taylor asked if the consistency of the existence of a strong ideal on ω_1 is equivalent to the consistency of the existence of an ω_2 -saturated ideal on ω_1 . Consider the following four statements:

- (a) ZFC + "there exists a Woodin cardinal."
- (b) ZFC + "there exists an ω_2 -saturated ideal on ω_1 ."
- (c) ZFC + "there exists a presaturated ideal on ω_1 ."
- (d) ZFC + "there exists a strong ideal on ω_1 ."

That the consistency of (a) implies the consistency of (b) was shown in a series of papers each building on the previous which dropped the large cardinal needed from a supercompact in 1983^{41} , to a Shelah cardinal in 1984^{42} to a Woodin cardinal after its invention in 1984^{43} and no later than 1985^{44} . Baumgartner and Taylor [65] observed that an ω_2 -saturated ideal is presaturated and strong, so the consistency of each statement implies the consistency of the next on the list. John Steel and Jensen [122] building on Steel [179] proved the consistency of (c) implies the consistency of (a), and Benjamin Claverie and Ralf Schindler [74, Section 6] proved the consistency of (d) implies the consistency of (a). Hence all four statements are equiconsistent and Question 6.11 is answered positively.

3.11 Iterated forcing and Axiom A

In 1976 Richard Laver [139] introduced the modern form of iterated countable support forcing in his celebrated paper on the consistency of the Borel Conjecture. Other early countable support interations include a term forcing of Mitchell [148], [149] and the forcing in Jensen's consistency proof of

⁴¹See [99] for the proof. The timing is from a private communication from Foreman, November 18, 2016.

⁴²See [166] for the proof and [3] for the timing of the major breakthrough.

⁴³See [180] for the timing.

 $^{^{44}}$ See [163] for an announcement of the result and its timing, and [165] for a proof

Suslin's Hypothesis with the Continuum Hypothesis [77] which appeared in 1974.

In 1979 Baumgartner and Laver [55] developed a countable support iterated Sacks forcing, and used it to prove the consistency of ZFC + $2^{\aleph_0} = \aleph_2$ with every selective ultrafilter⁴⁵ being \aleph_1 -generated, solving Erdő-Hajnal Problem 26 in [81]. They also used their forcing to give a new proof of the result of Mitchell [150] that it is consistent that there are no ω_2 -Aronszajn trees.

Baumgartner's invention of Axiom A forcing was a critical point in the development of generalizations of Martin's axiom. A partial order (\mathbb{P}, \leq) satisfies $Axiom\ A$ if and only if there is a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings of \mathbb{P} such that $p \leq_0 q$ implies $p \leq q$, for every $n, p \leq_{n+1} q$ implies $p \leq_n q$, and the following conditions hold:

- 1. if $\langle p_n \in \mathbb{P} : n < \omega \rangle$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 \cdots \geq_{n-1} p_n \geq_n \ldots$, then there is a $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all n;
- 2. for every $p \in \mathbb{P}$, for every n and for every ordinal name $\dot{\alpha}$, there exist a $q \leq_n p$ and a countable set B such that $q \Vdash \dot{\alpha} \in B$.

The class of Axiom A forcings includes countable chain condition forcings, countably closed forcings, Sacks (perfect set) forcing, Prikry forcing, and Mathias forcing. It is both a generalization of ccc and σ -closed forcing. Baumgartner proved the consistency of a forcing axiom generalizing Martin's axiom to cover all Axiom A forcings, and in the summer of 1978 he included the proof in a series of lectures on iterated forcing in the three week long Summer School in Set Theory in Cambridge, England organized by Harrington, Magidor and Mathias. Baumgartner's expository paper [28] growing out of these lectures was aimed at individuals with basic knowledge of forcing, and has been a popular introduction to the subject for graduate students for many years. ⁴⁶ Baumgartner indicated that the approach he took to iterated forcing was "strongly influenced by Laver's paper [11]" [28, page 2]⁴⁷ and thanked Laver and Shelah for conversations and correspondence.

 $^{^{45}}$ Selective ultrafilters are also known as Ramsey ultrafilters and as Rudin-Keisler minimal ultrafilters.

⁴⁶James Cummings [75, page 7]) describes Section 7 of his chapter *Iterated Forcing* and *Elementary Embeddings* for the *Handbook of Set Theory* "as essentially following the approach of Baumgartner's survey."

⁴⁷The paper of Laver [11] is his Borel Conjecture paper [139]

In 2005 Tetsuya Ishiu [113] proved that a poset is forcing equivalent to a poset satisfying Axiom A if and only if it is α -proper for every $\alpha < \omega_1$. A notion of forcing is *proper* if for all regular uncountable cardinals λ , the forcing preserves stationary subsets of $[\lambda]^{\omega}$. Properness was developed by Shelah starting in 1978 and first appeared in print in 1980 (see [161]). Being α -proper is a natural strengthening by Shelah of being proper (see Shelah's book [162]).

3.12 Proper forcing and the Proper Forcing Axiom

Axiom A forcing was in important influence in the development of the Proper Forcing Axiom (PFA), which further extended the class of applicable forcings to include proper forcing. Early in 1979 (see [29, 926]), Baumgartner formulated the Proper Forcing Axiom, which can be briefly described as the extension of Martin's Axiom to include proper forcing as stated below:

Proper Forcing Axiom (PFA): If \mathbb{P} is a proper forcing and \mathcal{D} is a collection of at most \aleph_1 dense sets, then there is a filter $G \subseteq \mathbb{P}$ which meets every element of \mathcal{D} .

Baumgartner used a Laver Diamond to prove that if ZFC together with the existence of a supercompact cardinal. is consistent, then so is ZFC + $2^{\aleph_0} = \aleph_2 + \text{PFA}$. A cardinal κ is λ -supercompact if there is an elementary embedding $j: V \to M$ so that the critical point of j is κ , $j(\kappa) > \lambda$, and M contains all its λ sequences. A cardinal κ is supercompact if it is λ -supercompact for all $\lambda \geq \kappa$. Alternatively, κ is supercompact if for all A of power at least κ , there is a normal measure on $[A]^{<\lambda}$. A Laver diamond [140] for a supercompact cardinal κ is a function $f: \kappa \to V_{\kappa}$ such that for every $x \in V_{\kappa}$ and every $\lambda \geq |TC(x)|$, there is a supercompact ultrafilter U_{λ} on $[\lambda]^{<\kappa}$ such that $(j_{\lambda}f)(\kappa) = x$. The Laver Diamond was used to organize the critical iteration.

In 1983, Baumgartner's expository paper on iterated forcing, which was based his lectures at the 1978 Cambridge Summer School, appeared in the proceedings of that conference. Devlin published his *Yorkshireman's Guide to Proper Forcing* [76] in the same proceedings in which he gave a proof of the Proper Forcing Axiom. Devlin, who was not at the Cambridge summer school, was encouraged to write the article independently by Baumgartner, Rudi Göbel, and Todorcevic. He based his article, which started out as personal notes, on the following materials:

- Notes written by Shelah in Berkeley in 1978 when he was giving lectures on proper forcing on material that eventually appeared in Chapters III, IV, V of the first edition of *Proper Forcing*; and
- Notes written by Juris Steprans based on the ten lectures given by Baumgartner at the SETTOP Meeting in July and August, 1980 in Toronto.

In 1984, the Handbook of Set-Theoretic Topology was published and became an important reference for set theorists and set-theoretic topologists. Baumgartner [29] wrote an extensive article, Applications of the Proper Forcing Axiom starting from the definitions but requiring knowledge of forcing. He gave an example of a forcing that was just barely proper, namely the forcing P (see [29, page 926]) to add a club to ω_1 with finite conditions. Conditions in P are finite functions from ω_1 into ω_1 approximating an enumeration of a closed unbounded set, and conditions are ordered by reverse inclusion. This construction was generalized by Todorcevic in [189] in which he developed his seminal method for building proper partial orders using models as side conditions. Building on Baumgartner's elegant approach to adding a club to ω_1 , Friedman [102], Mitchell [151], and Neeman [155] all developed forcings with finite conditions to add a club to ω_2 . Inspired by the forcings of Friedman and Mitchell, John Krueger defined adequate sets and S-adequate $sets^{48}$ and developed a type of forcing for adding interesting combinatorial objects with finite conditions using S-adequate sets of models as side conditions. In [130] he used the approach to add a closed, unbounded set to a given fat stationary set.

Baumgartner gave proofs from PFA of a number of statements known to be consistent some of which are listed below.

- 1. Theorem 6.9: PFA implies all \aleph_1 -dense sets of reals are isomorphic.
- 2. Theorem 7.2: PFA implies there are no \aleph_2 -Aronszajn trees.
- 3. Theorem 7.10: PFA implies every tree of height ω_1 and cardinality \aleph_1 is essentially special, and therefore weak Kurepa's Hypothesis (wKH) is false.
- 4. Theorem 7.12: PFA implies that \square_{ω_1} is false.

⁴⁸We omit these definitions for brevity.

In Theorem 7.13 Baumgartner proved that PFA implies $\Diamond(E)$ for every stationary subset of $\{\alpha < \omega_2 : \text{cf } \alpha = \omega_1\}$.

Theorem 6.9 above is useful for the Basis Problem for uncountable linear orders. As noted earlier, Baumgartner [11]) proved all \aleph_1 -dense sets of reals are isomorphic relative to ZFC + $2^{\aleph_0} = \aleph_2$. In 2006, Justin Moore [154] used PFA to show there is a two element basis for the collection of Aronszajn lines, namely a Countryman type and its reverse answering question 5.1(i) of [22] Since the forcing Baumgartner [11], [29, Theorem 6.9] used in the consistency of a one element basis for the class of real types is proper, it can be combined with the consistency of a two element basis for the class Aronszajn lines relative to PFA to get a positive answer relative to PFA to Question 5(ii) of Baumgartner [22], and with the addition of ω_1 and ω_1^* , we obtain the consistency that the class of uncountable orderings has a five element basis relative to PFA.

With regard to Theorem 7.2, Silver (see [150]) using a model of Mitchell proved that the non-existence of an \aleph_2 -Aronszajn tree is equiconsistent with the existence of a weakly compact cardinal.

With regard to Theorem 7.10, Mitchell [150] proved the failure of weak Kurepa's hypothesis is equiconsistent with the existence of an inaccessible cardinal over ZFC. Baumgartner [28] and independently and earlier Todorcevic [187] proved ZFC + MA + \neg wKH is consistent relative to the existence of an inaccessible cardinal, and Todorcevic gave consequences in his paper.

Theorem 7.12 was improved by Todorcevic [189] who proved that PFA implies \square_{κ} fails for all uncountable cardinals. Note that the failure of \square_{κ} , for a regular uncountable cardinal κ , is equiconsistent with the existence of a Mahlo cardinal. In the early 1970s, Solovay⁴⁹ proved that if $\lambda > \kappa$ is a Mahlo cardinal, then in an extension by the Lévy collapse $\operatorname{Coll}(\kappa, <\lambda)$, $\lambda = \kappa^+$ and $\neg \square_{\kappa}$. Jensen [119] proved that if κ^+ is not Mahlo in L, then \square_{κ} holds.

Baumgartner [29, Section 8] introduced a strengthening of PFA which he called PFA⁺ and under its assumption proved two theorems on stationary-set reflection and pointed out that that such results imply the consistency of many measurable cardinals (see [125]).

In 2009, Jensen, Schimmerling, Schindler and Steel [121] used the results of Todorcevic [189], that PFA implies $2^{\aleph_0} = \aleph_2$ and \square_{κ} fails for all uncountable cardinals, together with core model theory to show that PFA implies

⁴⁹For timing of the result, see Math Review MR2833150 (2012g:03134) by Kanamori; for the attribution see [131], [115, page 547].

there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals, and indiscernibles for such a model.

In 2011, Matteo Viale and Christoph Weiß [198] proved that if one can force PFA with a proper forcing that collapses a large cardinal κ to ω_2 and satisfies the κ -covering and κ -approximation properties, then κ is supercompact. These papers of Jensen, Schimmerling, Schindler and Steel and of Viale and Weiss suggest that Baumgartner's use of a supercompact cardinal in obtaining the consistency of PFA is likely necessary.

3.13 Chromatic number of graphs

Let us call a coloring of the vertices of a graph good if no pair joined by an edge have the same color. an edge in the graph have the same color. The chromatic number of a graph is the smallest number of colors for which there is a good coloring. In 1984 Baumgartner [30] proved that If ZFC is consistent, then so is ZFC + GCH + "there is a graph of cardinality \aleph_2 and chromatic number \aleph_2 such that every subgraph of cardinality $< \aleph_2$ has chromatic number $\le \aleph_0$ " providing a consistent negative answer to a question from 1961 of Erdős and Hajnal [87, page 118] (this quote has been mildly rephrased with modern notation):

Let there be given a graph G of power \aleph_2 . Suppose that every subgraph G_1 of G of cardinality at most \aleph_1 has chromatic number not greater than \aleph_0 . Is it then true that the chromatic number of G is not greater than \aleph_0 ?

The question was reiterated in print by Erdős and Hajnal in 1966 in [89, pages 92-93] in a paper dedicated to the to the 60th birthdays of well-known Hungarian mathematians Rózsa Péter and László Kalmár both born in 1905. In their 1968 paper emerging from a 1966 conference, Erdős and Hajnal gave a negative answer under CH when they proved there is a graph on $(2^{\aleph_0})^+$ many vertices whose chromatic number is at least \aleph_1 and all of whose subgraphs of smaller cardinality have chromatic number at most \aleph_0 . The statement of the theorem was immediately followed by two questions: (A) Does there exist a graph of power $\omega_{\omega+1}$ and uncountable chromatic number all of whose smaller subgraphs have countable chromatic number? (B) Does there exist a graph of power and chromatic number ω_2 all of whose smaller subgraphs have countable chromatic number. These appear as Problem 41(A) and 41(B) in the problem paper [81] growing out of the presentation at the 1967 UCLA

summer school and were reiterated in 1975 in [86, page 415] and in 1973 and Galvin [104] reframed one question by asking if every graph of chromatic number \aleph_2 has a subgraph of chromatic number \aleph_1 . Thus Baumgartner gave a positive answer to problem 41(B) and a negative answer to Galvin's problem. In 1988, Komjath⁵⁰ [127] provided a different positive answer to problem 41(B) since in his forcing extension $2^{\aleph_0} = \aleph_3$. Also in 1988, Foreman and Laver [97] proved the relative consistency of the opposite conclusions, assuming the existence of a huge cardinal to construct a forcing extension in which ZFC + GCH hold and every graph of power \aleph_2 and chromatic number \aleph_2 has a subgraph of size and chromatic number \aleph_1 . In 1997 Todorcevic [194] constructed in ZFC a graph of power 2^{\aleph_2} of uncountable chromatic number with no subgraph of power and chromatic number \aleph_1 . Recent work on the construction of graphs of large power and uncountable chromatic number all of whose smaller subgraphs have countable chromatic number includes the use of $\square_{\lambda} + 2^{\lambda} = \lambda^{+}$ for an uncountable cardinal λ by Assaf Rinot [158] to get graphs of size λ^+ of chromatic number of any desired value $\kappa \leq \lambda$.

3.14 A thin very-tall superatomic Boolean algebra

In 1987 Baumgartner and Shelah published their proof of the consistent existence of a thin very-tall superatomic Boolean algebra, where a Boolean algebra is superatomic if and only all of its homomorphic images are atomic. Their collaboration came about as follows. Baumgartner circulated a preprint which included his proof of this result by a two step forcing, a countably closed forcing and a ccc forcing. Baumgartner used a function f_* with special properties in his construction of the second forcing to guarantee it was ccc. Fleissner found an error in the proof and it was later discovered that no function exists with the special properties Baumgartner had envisioned. Shelah came up with a different set of properties Δ for a function f_* and a different countably closed forcing to make the whole construction work. The combined forcing of Baumgartner and Shelah [59] has proven useful in other settings.

In 2001, Juan Carlos Martínez [144] generalized the result to show that Con(ZFC) implies Con(ZFC + "for all $\alpha < \omega_3$, there is a superatomic Boolean algebra of width ω and height α ").

⁵⁰In 2002, Komjath [128] made a systematic study of the set of of chromatic numbers realized by subgraphs of a given graph.

In 2013, Boban Velickovic and Giorgio Venturi [197] used Neeman's method of forcing with generalized side conditions with two types of models and finite support to give a new proof of the Baumgartner-Shelah result.

Recent research in the area (see [145]) has turned to proofs of existence of cardinal sequences of locally compact scattered (Hausdorff) spaces (LCS), and these results can be translated into results about superatomic Boolean algebras.

3.15 Closed unbounded sets

In 1991, Baumgartner [37] published the paper on the structure of closed unbounded subsets of $[\lambda]^{<\kappa}$, and the stationary sets associated with them, with a focus on $\lambda = \kappa^{+n}$. Given cardinals $\kappa < \lambda$, he introduced a family of sets $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$ parameterized for some $n < \omega$ by a sequence of regular cardinals $\kappa_0, \kappa_1, \ldots, \kappa_n$ all smaller than κ , and he proved that they are stationary sets. His stated goal was to find closed unbounded sets C so that each $x \in C \cap S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$ was determined as much as possible by the sequence $\langle \sup(x \cap \kappa_0), \ldots, \sup(x \cap \kappa_n) \rangle$. He used these sets and their intersections to prove limitations on the size of intersections of closed unbounded sets with these sets $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$. For example, for κ regular and $\lambda = \kappa^{+n}$, Baumgartner proved that if all the κ_i 's are regular, $\kappa_i = 0$ for some i > 0, and $\lambda = \kappa^{+n}$, then the intersection of every closed unbounded set C with $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$ has cardinality at least λ^{ω} .

He also introduced two combinatorial principles which were useful in pinning down the intersections of closed unbounded sets with stationary sets of the form $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$. Then he used various types of \square -sequences, the weakening of Erdős cardinals to remarkable cardinals, and notions of reverse-Easton-like forcings to prove consistency with and independence from ZFC of the combinatorial principles.

For a cardinal κ he wrote $\square(\kappa)^{51}$ to indicate that there is $\langle C_{\alpha} \mid \alpha < \kappa, \kappa \rangle$ singular limit such that for each singular limit $\alpha < \kappa, C_{\alpha}$ is closed and unbounded subset of α of order type $< \alpha$; and if $\beta < \alpha$ is a limit point of C_{α} , then $C_{\beta} = \beta \cap C_{\alpha}$ and indicated that $\square(\kappa^{+})$ was equivalent to \square_{κ} .

⁵¹ Currently $\Box(\kappa)$ denotes a sequence $\langle C_{\alpha} \mid \alpha < \kappa \rangle$ such that C_{α} for limit α is a closed unbounded subset of α ; if β is a limit point of C_{α} , then $\beta \cap C\alpha = C_{\beta}$; and there is no threading, i.e. no closed unbounded set $C \subseteq \kappa$ such that $C \cap \alpha = C_{\alpha}$ for all limit points C. This variant of Jensen's \Box_{λ} principle is due to Todorcevic [192]. See [157, page 298] for context.

In 1980 it was shown in [70] that if V = L, then $\square(\kappa)$ holds for all κ . In Theorem 6.6 Baumgartner proved that if κ is γ -Erdős for some $\gamma < \kappa$, then there is a forcing extension in which κ remains γ -Erdős and both $\square(\kappa)$ and $\square(\{\alpha < \kappa \mid \alpha \text{ is regular}\})$ hold.

Foreman and Magidor [98, page 66, Corollary 2.11(b)] generalized the function $S(\kappa, \lambda; \kappa_0, \kappa_1, \kappa_2, \kappa_3)$ by replacing λ with the collection $H(\lambda)$ of sets hereditarily of cardinality $< \lambda$). While Baumgartner used the sets $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$ to control sizes of clubs and their intersections with the stationary set, Foreman and Magidor use their version to construct interesting stationary sets, e.g. there is a non-reflecting stationary subset $C \subseteq S(\omega_2, H(\lambda); \omega_2, \omega_3; \omega_1, \omega)$.

3.16 Revisiting partition relations

A key result of Erdős and Rado was the following theorem [84, pages 467-8].

Positive Stepping Up Lemma (modern form): For all infinite cardinals κ , all γ with $2 \leq \gamma < \kappa$, all finite r, and all cardinals $\langle \alpha_{\nu} : \nu < \gamma \rangle$, if $\kappa \to (\alpha_{\nu})^{r}_{\nu < \gamma}$, then $(2^{<\kappa})^{+} \to (\alpha_{\nu} + 1)^{r+1}_{\nu < \gamma}$

Let $\exp_n(\kappa)$ denote *n*-times iterated exponentiation, that is, $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$. With this notation and the cardinal arithmetic above, we state below the modern version of their Theorem 39, obtained using the Positive Stepping Up Lemma repeatedly and starting from the clear fact that $\exp_0(\kappa) = \kappa \to (\kappa)^{\gamma}$ for $\gamma < \operatorname{cf}(\kappa)$.

Erdős-Rado Theorem (modern form): For every infinite cardinal κ , every finite $r \geq 2$, and all $\gamma < \operatorname{cf}(\kappa)$, $(\exp_r(2^{<\kappa}))^+ \to (\kappa + (r-1))_{\gamma}^r$.

In 1993, Baumgartner, Hajnal and Todorcevic [51] published their extensions of the Erdős-Rado Theorem as follows.

Theorem 3.2 (Balanced Baumgartner-Hajnal-Todorcevic Theorem). Suppose κ is regular and uncountable. For all $\ell < \omega$ and all ordinals ξ with $2^{\xi} < \kappa$, $(2^{<\kappa})^+ \to (\kappa + \xi)_{\ell}^2$.

Theorem 3.3 (Unbalanced Baumgartner-Hajnal-Todorcevic Theorem). Suppose κ is regular and uncountable. For all $\ell, n < \omega$,

$$(2^{<\kappa})^+ \to (\kappa^{\omega+2} + 1, \kappa + n)^2.$$

In 2001, Baumgartner and Hajnal [49] published a polarized version of the Erdős-Rado Theorem for pairs. Baumgartner and Hajnal proved that for every cardinal κ ,

$$\begin{pmatrix} (2^{<\kappa})^{++} \\ (2^{<\kappa})^{+} \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}_{< cf \kappa} .$$

If κ is weakly compact, then this can be improved to $\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \rightarrow \begin{pmatrix} \kappa \\ \kappa+1 \end{pmatrix}_{<\kappa}$

In 2003, Matthew Foreman and Hajnal [96] used techniques from [51] and ideals coming from elementary submodels in their proof that if $\kappa^{<\kappa}$ and κ carries a κ -dense ideal, then $\kappa^+ \to (\kappa^2 + 1, \alpha)^2$ for all $\alpha < \kappa^+$. A proper, non-principal ideal \mathcal{J} on κ is λ -dense if the Boolean algebra $\mathcal{P}(\kappa)/\mathcal{J}$ has a dense set of size λ . Equivalently, \mathcal{J} is λ -dense if there is a family \mathcal{D} of λ many \mathcal{J} -positive sets so that every \mathcal{J} -positive set is \mathcal{J} -almost included in some element of \mathcal{D} and any two different elements of \mathcal{D} have intersection in \mathcal{J} .

Moreover, if κ is a measurable cardinal, then there is a rather large ordinal $\Omega < \kappa^+$ such that for all $n < \omega$ and $\alpha < \Omega$, $\kappa^+ \to (\alpha)_n^2$. They give multiple definitions of Ω and use them to show that Ω is rather large. Specifically, they observe that it follows from the definitions that L_{Ω} is a model of ZFC; they show that the statement " $\alpha < \Omega$ " is upwards absolute; and for U a normal ultrafilter on κ , they show that the least ordinal ν such that $L_{\nu}[U] \cap \kappa^{<\kappa} = L[U] \cap \kappa^{<\kappa}$ is a lower bound.

In 2006, Albin Jones [123] extended the weakly compact polarized partition relation of Baumgartner and Hajnal [49] to specify the order type of sets chosen to be homogeneous.

In 2014, Ari Brodsky gave a generalization of the Balanced Baumgartner-Hajnal-Todorcevic Theorem to all partially ordered sets. More specifically, Brodsky [72] proved that if P is a partial order such that $P \to (2^{<\kappa})_{2^{<\kappa}}^1$ for some uncountable regular cardinal κ , and if $\ell < \omega$ and ξ is an ordinal such that $2^{|\xi|} < \kappa$, then $P \to (\kappa + \xi)_{\ell}^2$.

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