# OD elements of countable OD sets in the Solovay model

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#### Abstract

It is true in the Solovay model that every countable ordinal-definable set of sets of reals contains only ordinal-definable elements.

#### Introduction

It is known that the existence of a non-empty OD (ordinal-definable) set of reals X with no OD element is consistent with **ZFC**; the set of all non-constructible reals gives a transparent example in many generic models.

Can such a set X be countable?

This question was initiated and discussed at the *Mathoverflow* website <sup>1</sup> and at FOM<sup>2</sup>. In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support countable product  $\mathbb{P}^{<\omega}$  (see [2]) of the Jensen "minimal  $\Pi_2^1$  real singleton forcing"  $\mathbb P$  defined in [4] (see also Section 28A of [3]). We proved in [5] that indeed, in a  $\mathbb{P}^{<\omega}$ -generic extension of **L**, the set of all reals  $\mathbb{P}$ -generic over **L** is a countable  $\Pi_2^1$  set with no OD elements. Moreover there is a modification  $\mathbb{P}'$  of  $\mathbb{P}$  such that it is true in a  $\mathbb{P}'$ -generic extension of **L** that there is a  $\Pi_2^1$  E<sub>0</sub>-equivalence class containing no OD reals, [7].

On the other hand, one may ask do countable non-empty OD sets without OD elements exist in such a more typical generic extension as the Solovay model? We partially answer this question in the negative.

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Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.
 FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html

**Theorem 1.1.** It is true in the Solovay model that every non-empty OD countable or finite set  $\mathscr{X}$  of sets of reals necessarily contains an OD element, and hence, in fact, consists of OD elements.

The Solovay model here is a model of **ZFC** defined in [8] in which all projective (and generally all ROD, real-ordinal definable) sets of reals are Lebesgue measurable. The case, when  $\mathscr{X}$  is a (non-empty OD countable) **set of reals** in this theorem, is well known and is implicitly contained in the proof of the perfect set property by Solovay [8]. Hovever the proofs known for this particular case of sets of reals (as, e.g., in [9] or [6]) do not work even for sets  $\mathscr{X} \subseteq \mathscr{P}(\omega^{\omega})$  (as in the theorem). In this paper, we present the proof of Theorem 1.1.

#### 2 Notation

We consider the constructible universe  $\mathbf{L}$  as the ground model by default. Suppose that  $\Omega$  is an inaccessible cardinal.

**Blanket assumption 2.1.** By a *generic set* we'll always mean a *filter*, that is, both pairwise compatible in itself and containing all weaker conditions.

**Definition 2.2.** We represent the Levy – Solovay forcing associated with  $\Omega$  is the set **LS** of all partial maps  $p : \operatorname{dom} p \to \Omega$  such that  $\operatorname{dom} p \subseteq \Omega \times \omega$  is a finite set and  $p(\alpha, n) < \alpha$  whenever  $\langle \alpha, n \rangle \in \operatorname{dom} p$ . Let  $|p| = \{\alpha : \exists n \ (\langle \alpha, n \rangle \in \operatorname{dom} p)\}$ .

If  $\gamma \leq \Omega$  then  $LS_{\gamma} = \{ p \in LS : |p| \subseteq \gamma \}$ ; in particular  $LS_{\Omega} = LS$ .

If  $p \in \mathbf{LS}$  and  $\alpha < \Omega$  then the  $\alpha$ -component  $p_{\alpha}$  of p is a map defined on the set  $\operatorname{dom} p_{\alpha} = \{n : \langle \alpha, n \rangle \in \operatorname{dom} p\} \subseteq \omega$  by  $p_{\alpha}(n) = p(\alpha, n)$ .

If  $G \subseteq \mathbf{LS}$  is an  $\mathbf{LS}$ -generic set over  $\mathbf{L}$  then  $\mathbf{L}[G]$  is the Solovay model, to which Theorem 1.1 refers. The next lemma will be important below.

**Lemma 2.3** (reduction to ROD). It is true in the Solovay model that if  $\mathscr X$  is a non-empty OD countable set and  $X \in \mathscr X$  is ROD then X is OD.

Thus somewhat surprisingly, it turns out that it suffices to prove the existence of a ROD (real-ordinal definable) element  $X \in \mathcal{X}$  in Theorem 1.1.

**Proof.** Arguing in the Solovay model, assume that

$$X = X_{p_0} = \{x : \varphi(x, p_0)\},\$$

where  $\varphi$  is a formula with a real parameter  $p_0 \in \omega^{\omega}$  and hidden ordinal parameters. The set  $P = \{p \in \omega^{\omega} : X_p \in \mathcal{X}\}$  is OD and contains  $p_0$ , and the equivalence relation,  $p \to q$  iff  $X_p = X_q$  on P, is OD as well, and  $\to q$  has at most

countably many equivalence classes in P. However it is known that, in the Solovay model, if an OD equivalence relation on  $\omega^{\omega}$  has at most countably many equivalence classes then all its equivalence classes are OD, [6, 9]. In particular  $[p_0]_{\mathsf{E}}$  is OD, and hence the set  $X = X_{p_0} = \{x : \exists p \in [p_0]_{\mathsf{E}} \varphi(x, p_0)\}$  is OD.  $\square$ 

**Definition 2.4** (ramified names). We'll use the ordinary ramified system of **LS**-names for differens sets in  $\mathbf{L}[G]$ , so that U[G] will be the *G*-interpretation of a name U (basically, any set) defined by  $\in$ -rank induction by

$$U[\![G]\!] = \left\{u[\![G]\!] : \exists \, p \in G \, (\langle p, u \rangle \in U)\right\}.$$

Then, if  $G \subseteq \mathbf{LS}$  is generic over  $\mathbf{L}$  then  $\mathbf{L}[G] = \{U[G] : U \in \mathbf{L}\}.$ 

Each set  $x \in \mathbf{L}$  has a canonical **LS**-name  $\check{x} \in \mathbf{L}$ , such that  $\check{x}[G] = x$  for any generic set  $G \subseteq \mathbf{LS}$ . Yet following common practice we shall identify  $\check{x}$  with x itself whenever possible.

**Definition 2.5** (simple names). To somewhat simplify notation, we'll make use of a simpler system of names particularly for subsets of **LS**. Let  $\mathbf{N} = \mathscr{P}(\mathbf{LS} \times \mathbf{LS})$ , and if  $t \in \mathbf{N}$  and  $G \subseteq \mathbf{LS}$  then  $t[G] = \{q : \exists p \in G(\langle p, q \rangle \in t)\} \subseteq \mathbf{LS}$ .

Thus N consists of all LS-names for subsets of LS.

If  $\gamma < \Omega$  then let  $\mathbf{N}_{\gamma} = \mathscr{P}((\mathbf{LS}_{\gamma}) \times (\mathbf{LS}_{\gamma}))$ , so that any  $t \in \mathbf{N}_{\gamma}$  is a  $\mathbf{LS}_{\gamma}$ -name for a subset of  $\mathbf{LS}_{\gamma}$ .

The name  $\underline{G} = \{ \langle p, p \rangle : p \in \mathbf{LS} \}$  belongs to  $\mathbf{N}$ , and  $\underline{G}[G] = G$ .

#### 3 Double names

In many cases below, we'll consider pairs of **LS**-generic sets  $G, G' \subseteq \mathbf{LS}$  over **L**, such that  $\mathbf{L}[G] = \mathbf{L}[G']$ ; note that this is not a  $(\mathbf{LS} \times \mathbf{LS})$ -generic pair! Similar pairs will be considered for the forcing notions  $\mathbf{LS}_{\gamma}$  ( $\gamma < \Omega$ ) instead of **LS**. The next definition introduces a useful tool related to such pairs.

**Definition 3.1.** In **L**, if  $\gamma \leq \Omega$  then any pair  $a = \langle t_{\tt lef}^a, t_{\tt rig}^a \rangle$  of names  $t_{\tt lef}^a, t_{\tt rig}^a \in \mathbf{N}_{\gamma}$  will be called a *double-name*. Let  $\mathbf{DN}_{\gamma}$  consist of all double-names  $a = \langle t_{\tt lef}^a, t_{\tt rig}^a \rangle$  such that  $t_{\tt lef}^a \neq \varnothing$ ,  $t_{\tt rig}^a \neq \varnothing$ , and

- (1) if  $p \in \text{dom } t_{\texttt{lef}}^a$  then  $p \ \textbf{LS}_{\gamma}$ -forces: (a)  $t_{\texttt{lef}}^a[\underline{G}]$  is  $\textbf{LS}_{\gamma}$ -generic, and
  - $\text{(b) }\underline{G}=t_{\mathrm{rig}}^{a}[t_{\mathrm{lef}}^{a}[\underline{G}]];$
- (2) if  $p \in \text{dom}\,t^a_{ t rig}$  then p  $\mathbf{LS}_{\gamma}$ -forces: (a)  $t^a_{ t rig}[\underline{G}]$  is  $\mathbf{LS}_{\gamma}$ -generic, and
  - (b)  $\underline{G} = t_{\texttt{lef}}^a[t_{\texttt{rig}}^a[\underline{G}]].$

Define  $\mathbf{DN} = \bigcup_{\gamma < \Omega} \mathbf{DN}_{\gamma}$ ; this is different from  $\mathbf{DN}_{\Omega}$ . It follows from (1) or (2) that for any  $a \in \mathbf{DN}$  there is a unique  $\gamma = |a| < \Omega$  such that  $a \in \mathbf{DN}_{\gamma}$ .

Note that all sets  $N_{\gamma}$  and  $DN_{\gamma}$  belong to L.

**Lemma 3.2.** Assume that  $\gamma \leq \Omega$  and  $a \in \mathbf{DN}_{\gamma}$ . Then:

- (i) if  $G_{\mathsf{lef}} \subseteq \mathbf{LS}_{\gamma}$  is an  $\mathbf{LS}_{\gamma}$ -generic set and  $G_{\mathsf{lef}} \cap \mathsf{dom} \, t_{\mathsf{lef}}^a \neq \emptyset$  then  $G_{\mathsf{rig}} = t_{\mathsf{lef}}^a[G_{\mathsf{lef}}]$  is  $\mathbf{LS}_{\gamma}$ -generic,  $G_{\mathsf{rig}} \cap \mathsf{dom} \, t_{\mathsf{rig}}^a \neq \emptyset$ , and  $G_{\mathsf{lef}} = t_{\mathsf{rig}}^a[G_{\mathsf{rig}}]$ ;
- (ii) if  $G_{\text{rig}} \subseteq \mathbf{LS}_{\gamma}$  is  $\mathbf{LS}_{\gamma}$ -generic and  $G_{\text{rig}} \cap \text{dom} \, t_{\text{rig}}^a \neq \varnothing$  then  $G_{\text{lef}} = t_{\text{rig}}^a[G_{\text{rig}}]$  is  $\mathbf{LS}_{\gamma}$ -generic,  $G_{\text{lef}} \cap \text{dom} \, t_{\text{lef}}^a \neq \varnothing$ ,  $G_{\text{rig}} = t_{\text{lef}}^a[G_{\text{lef}}]$ .

Thus each  $a \in \mathbf{DN}_{\gamma}$  induces a bijection between all  $\mathbf{LS}_{\gamma}$ -generic sets  $G \subseteq \mathbf{LS}_{\gamma}$  satisfying  $G \cap \operatorname{dom} t_{\mathtt{lef}}^a \neq \emptyset$  and those satisfying  $G \cap \operatorname{dom} t_{\mathtt{rig}}^a \neq \emptyset$ .

Corollary 3.3. If  $\gamma \leq \Omega$ ,  $a \in \mathbf{DN}_{\gamma}$ ,  $\langle q, p \rangle \in t^a_{rig}$ , and  $q \subseteq q' \in \mathbf{LS}_{\gamma}$  then there is a condition  $p' \in \mathbf{LS}_{\gamma}$  compatible with p and such that  $\langle p', q' \rangle \in t^a_{lef}$ .

**Proof.** Let  $G_{\text{rig}} \subseteq \mathbf{LS}_{\gamma}$  be a generic set containing q', hence containing q as well. Then  $G_{\text{lef}} = t_{\text{rig}}^a[G_{\text{rig}}]$  is a  $\mathbf{LS}_{\gamma}$ -generic set containing p, and  $G_{\text{rig}} = t_{\text{lef}}^a[G_{\text{lef}}]$  by Lemmaio. As  $q' \in G_{\text{rig}}$ , there is a condition  $p' \in G_{\text{lef}}$  such that  $\langle p', q' \rangle \in t_{\text{lef}}^a$ . As p also belongs to  $G_{\text{lef}}$ , p, p' are compatible.

# 4 Full, regular, equivalent names

Recall that a set  $D \subseteq \mathbf{LS}_{\gamma}$  is *dense* if for any  $p \in \mathbf{LS}_{\gamma}$  there is  $q \in D$  with  $p \subseteq q$ , and is *open* if  $(p \in D \land p \subseteq q \in \mathbf{LS}_{\gamma}) \Longrightarrow q \in D$ .

**Definition 4.1.** Let  $\gamma \leq \Omega$ . A name  $t \in \mathbf{N}_{\gamma}$  is *full* if the set dom t is dense in  $\mathbf{LS}_{\gamma}$ . A double-name  $a \in \mathbf{DN}_{\gamma}$  is *full* if such are the names  $t_{\mathtt{lef}}^{a}$  and  $t_{\mathtt{rig}}^{a}$ .

A name  $t \in \mathbf{N}_{\gamma}$  is regular, if the following holds: if  $p, q \in \mathbf{LS}_{\gamma}$  and  $p \in \mathbf{LS}_{\gamma}$  forces  $q \in t[\underline{G}]$  then  $\langle p, q \rangle \in t$ . In particular, in this case, if  $\langle p, q \rangle \in t$  and  $p \subseteq p' \in \mathbf{LS}_{\gamma}$  then  $\langle p', q \rangle \in t$ , too. A double-name  $a \in \mathbf{DN}_{\gamma}$  is regular, if so are both components  $t_{\mathtt{lef}}^a$  and  $t_{\mathtt{rig}}^a$ . Define the regular hull

$$^{\mathsf{rh}}t = \{\langle p, q \rangle \in \mathbf{LS}_{\gamma} \times \mathbf{LS}_{\gamma} : p \ \mathbf{LS}_{\gamma} \text{-forces } q \in t[\underline{G}] \}.$$

of any  $t \in \mathbf{N}_{\gamma}$ . If  $a \in \mathbf{DN}_{\gamma}$  then let  ${}^{\mathsf{rh}}a = \langle {}^{\mathsf{rh}}t^{a}_{\mathsf{lef}}, {}^{\mathsf{rh}}t^{a}_{\mathsf{rig}} \rangle$ .

**Lemma 4.2.** Assume that  $\gamma \leq \Omega$  and  $a \in \mathbf{DN}_{\gamma}$  is full. Then  $\operatorname{ran} t_{\mathtt{lef}}^a = \operatorname{ran} t_{\mathtt{rig}}^a = \mathbf{LS}_{\gamma}$ , and if  $G \subseteq \mathbf{LS}_{\gamma}$  is  $\mathbf{LS}_{\gamma}$ -generic then so are  $t_{\mathtt{lef}}^a[G]$  and  $t_{\mathtt{rig}}^a[G]$ .

**Proof.** To prove the genericity claim note that if say  $\operatorname{dom} t_{\mathtt{lef}}^a$  is dense then any generic set  $G \subseteq \mathbf{LS}_{\gamma}$  intersects  $\operatorname{dom} t_{\mathtt{lef}}^a$ , then use Lemma 3.2. To prove the first claim, let  $q \in \mathbf{LS}_{\gamma}$ . Consider a generic set  $G_{\mathtt{rig}} \subseteq \mathbf{LS}_{\gamma}$  containing q. Then  $G \cap \operatorname{dom} t_{\mathtt{rig}}^a \neq \varnothing$ , see above. It follows that  $G_{\mathtt{lef}} = t_{\mathtt{rig}}^a[G_{\mathtt{rig}}]$  is generic and  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^a[G_{\mathtt{lef}}]$  by Lemma 3.2. But  $q \in G_{\mathtt{rig}}$ , hence  $q \in \mathtt{ran} t_{\mathtt{lef}}^c$ .

**Definition 4.3.** Names  $s, t \in \mathbf{N}_{\gamma}$  are equivalent if s[G] = t[G] for any generic set  $G \subseteq \mathbf{LS}_{\gamma}$ , or equivalently, if any  $p \in \mathbf{LS}_{\gamma}$   $\mathbf{LS}_{\gamma}$ -forces  $s[\underline{G}] = t[\underline{G}]$ . Double-names  $a, b \in \mathbf{DN}_{\gamma}$  are equivalent if  $t_{\mathtt{lef}}^b$ ,  $t_{\mathtt{rig}}^b$  are equivalent to resp.  $t_{\mathtt{lef}}^a$ ,  $t_{\mathtt{rig}}^a$ .

**Lemma 4.4.** Assume that  $\gamma \leq \Omega$ . Then:

- (i) if  $t \in \mathbf{N}_{\gamma}$  then  $^{\mathsf{rh}}t$  is regular and equivalent to t;
- (ii) if  $a \in \mathbf{DN}_{\gamma}$  then  ${}^{\mathsf{rh}}a \in \mathbf{DN}_{\gamma}$ ,  $a \leqslant {}^{\mathsf{rh}}a$ , and  ${}^{\mathsf{rh}}a$  is equivalent to a—
  therefore the set  $\mathbf{DN}_{\gamma}^{\mathsf{reg}} = \{b \in \mathbf{DN}_{\gamma} : b \text{ is regular}\}$  is dense in  $\mathbf{DN}_{\gamma}$ ;
- (iii) if  $a, b \in \mathbf{DN}_{\gamma}$  then a is equivalent to b iff  ${}^{\mathsf{rh}}a = {}^{\mathsf{rh}}b$ .
- **Proof.** (i) To establish the equivalence, assume that  $G \subseteq \mathbf{LS}_{\gamma}$  is generic and  $q \in {}^{\mathsf{rh}}t[G]$ . Then there is  $p \in G$  such that  $\langle p,q \rangle \in {}^{\mathsf{rh}}t$ . By definition  $p \ \mathbf{LS}_{\gamma}$ -forces  $q \in t[\underline{G}]$ . But then  $q \in t[G]$ , as required. To establish the regularity, assume that  $p, q \in \mathbf{LS}_{\gamma}$ , and  $p \ \mathbf{LS}_{\gamma}$ -forces  $q \in {}^{\mathsf{rh}}t[\underline{G}]$  therefore  $p \ \mathbf{LS}_{\gamma}$ -forces  $q \in t[\underline{G}]$  by the equivalence already proved. Then by definition  $\langle p,q \rangle \in {}^{\mathsf{rh}}t$ .
- (ii) follows from (i). The direction  $\Leftarrow$  in (iii) immediately follows from (ii). To prove the opposite direction, it suffices to show that if names  $s, t \in \mathbf{N}_{\gamma}$  are equivalent then  ${}^{\mathsf{rh}}s = {}^{\mathsf{rh}}t$ . Assume that  $\langle p, q \rangle \in {}^{\mathsf{rh}}s$ . By definition  $p \mathbf{LS}_{\gamma}$ -forces  $q \in s[\underline{G}]$ . Then, as s, t are equivalent, p also forces  $q \in t[\underline{G}]$ . It follows that  $\langle p, q \rangle \in {}^{\mathsf{rh}}t$ , as required.

**Example 4.5.** If  $\gamma < \Omega$  then let  $t_{\gamma} = \{\langle p,q \rangle : p,q \in \mathbf{LS}_{\gamma} \land q \subseteq p\}$  and  $\mathbf{id}[\gamma] = \langle t_{\gamma}, t_{\gamma} \rangle$ . Then  $\mathbf{id}[\gamma] \in \mathbf{DN}_{\gamma}$  is a full regular double-name and  $t_{\mathtt{lef}}^{\mathtt{id}[\gamma]}[G] = t_{\mathtt{lef}}^{\mathtt{id}[\gamma]}[G] = G$  for any  $\mathbf{LS}_{\gamma}$ -generic set  $G \subseteq \mathbf{LS}_{\gamma}$ : the *identity* name.

#### 5 Double-name representation theorem

The next theorem shows that the double-name tool adequately represents the case of a pair of **LS**-generic sets  $G, G' \subseteq \mathbf{LS}$  such that  $\mathbf{L}[G] = \mathbf{L}[G']$ .

**Theorem 5.1.** Assume that  $\gamma \leq \Omega$ ,  $G_{\mathsf{lef}}, G_{\mathsf{rig}} \subseteq \mathbf{LS}_{\gamma}$  are  $\mathbf{LS}_{\gamma}$ -generic sets over  $\mathbf{L}$ , and  $\mathbf{L}[G_{\mathsf{lef}}] = \mathbf{L}[G_{\mathsf{rig}}]$ . Then there is a full regular double-name  $c \in \mathbf{DN}_{\gamma}$  such that  $G_{\mathsf{rig}} = t^c_{\mathsf{lef}}[G_{\mathsf{lef}}]$ ,  $G_{\mathsf{lef}} = t^c_{\mathsf{rig}}[G_{\mathsf{rig}}]$ , and  $t^c_{\mathsf{lef}} = t^c_{\mathsf{rig}}$ .

**Proof.** If  $G_{\mathsf{lef}} = G_{\mathsf{rig}}$  then it suffices to define c by  $t_{\mathsf{lef}}^c = t_{\mathsf{rig}}^c = \mathsf{id}[\gamma]$ . Therefore assume that  $G_{\mathsf{lef}} \neq G_{\mathsf{rig}}$ . Then there exist conditions  $p_{\mathsf{lef}} \in G_{\mathsf{lef}}$  and  $p_{\mathsf{rig}} \in G_{\mathsf{rig}}$  incompatible in  $\mathsf{LS}_{\gamma}$ . By a basic forcing theorem, there exist names  $s_{\mathsf{lef}}, s_{\mathsf{rig}} \in \mathbb{N}_{\gamma}$  such that  $G_{\mathsf{rig}} = s_{\mathsf{lef}}[G_{\mathsf{lef}}], \ G_{\mathsf{lef}} = s_{\mathsf{rig}}[G_{\mathsf{rig}}],$  and every condition  $p \in \mathsf{dom}\, s_{\mathsf{lef}}$  satisfies  $p_{\mathsf{lef}} \subseteq p$  while every condition  $q \in \mathsf{dom}\, s_{\mathsf{rig}}$  satisfies  $p_{\mathsf{rig}} \subseteq q$ . It is not true immediately that  $\langle s_{\mathsf{lef}}, s_{\mathsf{rig}} \rangle \in \mathsf{DN}_{\gamma}$ ; we need to somewhat modify the names by shrinking.

We can wlog assume that  $s_{lef}$  and  $s_{rig}$  are regular; as otherwise we can replace them by resp.  ${}^{rh}s_{lef}$  and  ${}^{rh}s_{rig}$  and use Lemma 4.4(i).

Define  $a = \langle t_{\texttt{lef}}^a, t_{\texttt{rig}}^a \rangle$ , where  $t_{\texttt{lef}}^a$  consists of all pairs  $\langle p, q \rangle \in s_{\texttt{lef}}$  such that

p LS<sub> $\gamma$ </sub>-forces that  $s_{\text{lef}}[\underline{G}]$  is LS<sub> $\gamma$ </sub>-generic and  $\underline{G} = s_{\text{rig}}[s_{\text{lef}}[\underline{G}]]$ ,

and  $t_{rig}^a$  consists of all pairs  $\langle q, p \rangle \in s_{rig}$  such that

q LS<sub> $\gamma$ </sub>-forces that  $s_{\text{rig}}[\underline{G}]$  is LS<sub> $\gamma$ </sub>-generic and  $\underline{G} = s_{\text{lef}}[s_{\text{rig}}[\underline{G}]];$ 

then  $\varnothing \neq t_{\texttt{lef}}^a \subseteq s_{\texttt{lef}}$  and  $\varnothing \neq t_{\texttt{rig}}^a \subseteq s_{\texttt{rig}}$ .

We claim that  $a \in \mathbf{DN}_{\gamma}$ , and still  $G_{rig} = t_{lef}^a[G_{lef}]$  and  $G_{lef} = t_{rig}^a[G_{rig}]$ .

**Lemma 5.2.** If  $H_{\text{lef}}$  is an  $\mathbf{LS}_{\gamma}$ -generic set and  $H_{\text{lef}} \cap \text{dom}\, t_{\text{lef}}^a \neq \varnothing$  then  $t_{\text{lef}}^a[H_{\text{lef}}] = s_{\text{lef}}[H_{\text{lef}}]$ . Similarly if  $H_{\text{rig}}$  is an  $\mathbf{LS}_{\gamma}$ -generic set and  $H_{\text{rig}} \cap \text{dom}\, t_{\text{rig}}^a \neq \varnothing$  then  $t_{\text{rig}}^a[H_{\text{rig}}] = s_{\text{lef}}[H_{\text{rig}}]$ .

**Proof** (lemma). By construction  $t_{\mathtt{lef}}^a[H_{\mathtt{lef}}] \subseteq s_{\mathtt{lef}}[H_{\mathtt{lef}}]$ . Consider any  $q \in s_{\mathtt{lef}}[H_{\mathtt{lef}}]$ , so that there is  $p \in H_{\mathtt{lef}}$  with  $\langle p, q \rangle \in s_{\mathtt{lef}}$ . On the other hand, as  $H_{\mathtt{lef}} \cap \mathsf{dom}\,t_{\mathtt{lef}}^a \neq \varnothing$ , there is a condition  $p' \in H_{\mathtt{lef}}$  with  $p \subseteq p'$  which  $LS_{\gamma}$ -forces that  $s_{\mathtt{lef}}[\underline{G}]$  is  $LS_{\gamma}$ -generic and  $\underline{G} = s_{\mathtt{rig}}[s_{\mathtt{lef}}[\underline{G}]]$ . Then  $\langle p', q \rangle \in t_{\mathtt{lef}}^a$  by the regularity assumption, and we have  $q \in t_{\mathtt{lef}}^a[H_{\mathtt{lef}}]$ .

Now to check 3.1(1) for a let  $H_{\text{lef}}$  be an  $\mathbf{LS}_{\gamma}$ -generic set and  $H_{\text{lef}} \cap \text{dom } t_{\text{lef}}^a \neq \varnothing$ . Then  $t_{\text{lef}}^a[H_{\text{lef}}] = s_{\text{lef}}[H_{\text{lef}}]$  by the lemma. Therefore  $H_{\text{rig}} = t_{\text{lef}}^a[H_{\text{lef}}]$  is  $\mathbf{LS}_{\gamma}$ -generic and  $H_{\text{lef}} = s_{\text{rig}}[H_{\text{rig}}]$  by the definition of  $t_{\text{lef}}^a$ . Thus  $s_{\text{rig}}[H_{\text{rig}}]$  is generic and  $s_{\text{lef}}[s_{\text{rig}}[H_{\text{rig}}]] = H_{\text{rig}}$  by construction. This is forced by some  $q \in H_{\text{rig}}$ . On the other hand, as  $H_{\text{lef}} = s_{\text{rig}}[H_{\text{rig}}] \neq \varnothing$ , there exists some  $q' \in H_{\text{rig}} \cap \text{dom } s_{\text{rig}}$ . We can assume that  $q' \subseteq q$ . Then  $q \in \text{dom } s_{\text{rig}}$ , too, by the regularity assumption, and hence  $q \in \text{dom } t_{\text{rig}}^a$ , and  $H_{\text{rig}} \cap \text{dom } t_{\text{rig}}^a \neq \varnothing$ . We conclude that  $t_{\text{rig}}^a[H_{\text{rig}}] = s_{\text{rig}}[H_{\text{rig}}] = H_{\text{lef}}$ , by the lemma. Finally  $t_{\text{rig}}^a[t_{\text{lef}}^a[H_{\text{lef}}]] = H_{\text{lef}}$ ; this ends the verification of 3.1(1) for a.

Thus  $a \in \mathbf{DN}_{\gamma}$ . In addition, by the choice of  $s_{\mathsf{lef}}$  and  $s_{\mathsf{rig}}$ , some  $p \in G_{\mathsf{lef}}$  forces that " $s_{\mathsf{lef}}[\underline{G}]$  is generic and  $\underline{G} = s_{\mathsf{rig}}[s_{\mathsf{lef}}[G]]$ ". Then  $p \in \mathsf{dom}\, s_{\mathsf{lef}}$ ,  $p \in \mathsf{dom}\, t_{\mathsf{lef}}^a$ ,  $G_{\mathsf{lef}} \cap \mathsf{dom}\, t_{\mathsf{lef}}^a \neq \emptyset$ , and  $t_{\mathsf{lef}}^a[G_{\mathsf{lef}}] = s_{\mathsf{lef}}[G_{\mathsf{lef}}] = G_{\mathsf{rig}}$ , as above. Similarly we have  $G_{\mathsf{lef}} = t_{\mathsf{rig}}^a[G_{\mathsf{rig}}]$ .

To fix the regularity condition of the theorem, let  $b = {}^{\mathsf{rh}}a$ ; then still  $b \in \mathbf{DN}_{\gamma}$ ,  $G_{\mathsf{rig}} = t^b_{\mathsf{lef}}[G_{\mathsf{lef}}]$ ,  $G_{\mathsf{lef}} = t^b_{\mathsf{rig}}[G_{\mathsf{rig}}]$ , and b is regular, by Lemma 4.4.

It is not necessarily true, of course, that sets  $dom t_{lef}^b$  and  $dom t_{rig}^b$  are dense. To fix this shortcoming, we define

$$W = \{p \in \mathbf{LS}_{\gamma} : \forall \, q \in \mathrm{dom} \, t_{\mathtt{lef}}^b \cup \mathrm{dom} \, t_{\mathtt{rig}}^b \, (p \ \mathrm{is \, incompatible \, \, with \, } \, q) \}$$

and let  $c = \langle t_{\mathtt{lef}}^c, t_{\mathtt{rig}}^c \rangle$ , where  $t_{\mathtt{lef}}^c = t_{\mathtt{rig}}^c = t_{\mathtt{lef}}^b \cup t_{\mathtt{rig}}^b \cup \{\langle p, q \rangle : p \in W \land q \subseteq p\}$ . The set  $\mathtt{dom}\, t_{\mathtt{lef}}^c = \mathtt{dom}\, t_{\mathtt{rig}}^c = \mathtt{dom}\, t_{\mathtt{lef}}^b \cup \mathtt{dom}\, t_{\mathtt{rig}}^b \cup W$  is dense in  $\mathbf{LS}_{\gamma}$  by construction. We claim that  $c \in \mathbf{DN}_{\gamma}$ . Indeed let  $H_{\mathtt{lef}} \subseteq \mathbf{LS}_{\gamma}$  be an  $\mathbf{LS}_{\gamma}$ -generic set. Then  $H_{\mathtt{lef}} \cap \mathtt{dom}\, t_{\mathtt{lef}}^c \neq \varnothing$ . But  $\mathtt{dom}\, t_{\mathtt{lef}}^c = \mathtt{dom}\, t_{\mathtt{lef}}^b \cup \mathtt{dom}\, t_{\mathtt{rig}}^b \cup W$ .

Case 1:  $H_{\mathsf{lef}} \cap \mathsf{dom}\, t_{\mathsf{lef}}^b \neq \varnothing$ . Then  $H_{\mathsf{lef}} \cap \mathsf{dom}\, t_{\mathsf{rig}}^b = \varnothing$  since if  $p' \in \mathsf{dom}\, t_{\mathsf{lef}}^b$  and  $q' \in \mathsf{dom}\, t_{\mathsf{lef}}^b$  then p', q' are incompatible by the original choice of  $p_{\mathsf{lef}}, p_{\mathsf{rig}}$ . We also have  $H_{\mathsf{lef}} \cap W = \varnothing$  by obvious reasons. It follows that  $t_{\mathsf{lef}}^c[H_{\mathsf{lef}}] = t_{\mathsf{lef}}^b[H_{\mathsf{lef}}]$ , and hence  $H_{\mathsf{rig}} = t_{\mathsf{lef}}^c[H_{\mathsf{lef}}]$  is an  $\mathsf{LS}_{\gamma}$ -generic set and  $H_{\mathsf{lef}} = t_{\mathsf{rig}}^b[H_{\mathsf{rig}}]$ , because  $b \in \mathsf{DN}_{\gamma}$ . In particular  $H_{\mathsf{rig}} \cap \mathsf{dom}\, t_{\mathsf{lef}}^b \neq \varnothing$ , so that  $t_{\mathsf{rig}}^c[H_{\mathsf{rig}}] = t_{\mathsf{rig}}^b[H_{\mathsf{rig}}]$ , as above.

Case 2:  $H_{lef} \cap dom t_{rig}^b \neq \emptyset$ , similar.

Case 3:  $H_{\tt lef} \cap W \neq \varnothing$ . Then  $H_{\tt lef} \cap \mathsf{dom}\, t^b_{\tt lef} = H_{\tt lef} \cap \mathsf{dom}\, t^b_{\tt rig} = \varnothing$  as above. It follows that  $t^c_{\tt lef}[H_{\tt lef}] = t^c_{\tt rig}[H_{\tt lef}] = H_{\tt lef}$ .

Thus indeed  $c \in \mathbf{DN}_{\gamma}$ ,  $t_{\mathtt{lef}}^c = t_{\mathtt{rig}}^c$ , the set  $\mathtt{dom}\, t_{\mathtt{lef}}^c = \mathtt{dom}\, t_{\mathtt{rig}}^c$  is open dense in  $\mathbf{LS}_{\gamma}$ , and the arguments above (Case 1) also imply that  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^c[G_{\mathtt{lef}}]$ ,  $G_{\mathtt{lef}} = t_{\mathtt{rig}}^c[G_{\mathtt{rig}}]$ . Moreover, c inherits the regularity of b.

# 6 Extensions

**Definition 6.1** (extension). Suppose that a, b are double-names. We say that b extends a, in symbol  $a \leq b$ , if just  $t_{\mathtt{lef}}^a \subseteq t_{\mathtt{lef}}^b$  and  $t_{\mathtt{rig}}^a \subseteq t_{\mathtt{rig}}^b$ .

**Lemma 6.2** (in **L**). If  $\beta < \gamma \leq \Omega$  and  $a \in \mathbf{DN}_{\beta}$ , then there is a double-name  $b \in \mathbf{DN}_{\gamma}$  which extends a.

**Proof.** Let  $t_{\mathtt{lef}}^b$  consist of all pairs  $\langle p \cup r, q \cup r \rangle$ , where  $\langle p, q \rangle \in t_{\mathtt{lef}}^a$  and r is a condition in  $\mathbf{LS}_{\gamma}$  satisfying  $|r| \subseteq \gamma \setminus \beta$ ; let  $t_{\mathtt{rig}}^b$  be defined the same way.

This can be explained as follows. Suppose that  $G_{\text{lef}} \subseteq \mathbf{LS}_{\gamma}$  is a  $\mathbf{LS}_{\gamma}$ -generic set containing  $p_{\text{lef}}$ . Then the factors  $G'_{\text{lef}} = G_{\text{lef}} \cap \mathbf{LS}_{\beta}$  and  $G''_{\text{lef}} = G_{\text{lef}} \cap \mathbf{LS}_{\gamma \setminus \beta}$  are resp.  $\mathbf{LS}_{\beta}$ -generic and  $\mathbf{LS}_{\gamma \setminus \beta}$ -generic, and  $G_{\text{lef}}$  can be identified with  $G'_{\text{lef}} \times G''_{\text{lef}}$  by the product forcing theorem. Then by definition the set  $G_{\text{rig}} = t^b_{\text{lef}}[G_{\text{lef}}]$  has the form  $G'_{\text{rig}} \times G''_{\text{rig}}$ , where  $G'_{\text{rig}} = t^a_{\text{lef}}[G'_{\text{lef}}]$  while simply  $G''_{\text{rig}} = G''_{\text{lef}}$ . The genericity of  $G_{\text{rig}}$  easily follows.

**Definition 6.3** (restriction). Let  $\alpha < \beta \leq \Omega$ . If  $t \in \mathbf{LS}_{\beta}$  then define  $t \upharpoonright \alpha = t \cap (\mathbf{LS}_{\alpha} \times \mathbf{LS}_{\alpha}); \ t \upharpoonright \alpha \in \mathbf{N}_{\alpha}$ . If  $a \in \mathbf{DN}_{\beta}$ , then let  $a \upharpoonright \alpha = \langle t_{\mathtt{lef}}^{a} \upharpoonright \alpha, t_{\mathtt{rig}}^{a} \upharpoonright \alpha \rangle$ .

It is not asserted that always  $a \upharpoonright \alpha \in \mathbf{DN}_{\alpha}!$ 

**Lemma 6.4.** If, in **L**,  $\alpha < \beta \leq \Omega$ ,  $a \in \mathbf{DN}_{\alpha}$ ,  $b \in \mathbf{DN}_{\beta}$ , and  $a \leq b$ , then

- (i) if  $G_{\text{lef}} \subseteq \mathbf{LS}_{\beta}$  is an  $\mathbf{LS}_{\beta}$ -generic set then (a)  $H_{\text{lef}} = G_{\text{lef}} \cap \mathbf{LS}_{\alpha}$  is  $\mathbf{LS}_{\alpha}$ generic, and (b) if  $H_{\text{lef}} \cap \text{dom } t_{\text{lef}}^a \neq \emptyset$  then  $t_{\text{lef}}^a[H_{\text{lef}}] = t_{\text{lef}}^b[G_{\text{lef}}] \cap \mathbf{LS}_{\alpha}$ ;
- (ii) if  $G_{rig} \subseteq \mathbf{LS}_{\beta}$  is an  $\mathbf{LS}_{\beta}$ -generic set then (a)  $H_{rig} = G_{rig} \cap \mathbf{LS}_{\alpha}$  is  $\mathbf{LS}_{\alpha}$ -generic, and (b) if  $G_{rig} \cap \mathsf{dom} \, t^a_{rig} \neq \varnothing$  then  $t^a_{rig}[H_{rig}] = t^b_{rig}[G_{rig}] \cap \mathbf{LS}_{\alpha}$ ;
- (iii)  $c = b \upharpoonright \alpha$  belongs to  $\mathbf{DN}_{\alpha}$  and  $a \leqslant c \leqslant b$ .

**Proof.** (i)(a) That  $H_{lef}$  is generic holds by the product forcing theorem.

- (i)(b) If  $H_{\text{lef}} \cap \text{dom } t_{\text{lef}}^a \neq \emptyset$  then  $G_{\text{lef}} \cap \text{dom } t_{\text{lef}}^b \neq \emptyset$ , and hence the sets  $G_{\text{rig}} = t_{\text{lef}}^b[G_{\text{lef}}]$  and  $H_{\text{rig}} = t_{\text{lef}}^a[H_{\text{lef}}]$  are generic sets in resp.  $\mathbf{LS}_{\beta}$  and  $\mathbf{LS}_{\alpha}$  by Lemma 3.2, and  $H_{\text{rig}} \subseteq G_{\text{rig}}$  since  $a \leq b$ . Therefore  $H_{\text{rig}} \subseteq H'_{\text{rig}} = G_{\text{rig}} \cap \mathbf{LS}_{\alpha}$ . However  $H'_{\text{rig}}$  is  $\mathbf{LS}_{\alpha}$ -generic by the product forcing. Thus both  $H_{\text{rig}} \subseteq H'_{\text{rig}}$  are generic sets, hence easily  $H_{\text{rig}} = H'_{\text{rig}}$  as required.
- (iii) To check 3.1(1)(a) for some  $p \in \text{dom } t_{\text{lef}}^c$ , consider any  $\mathbf{LS}_{\alpha}$ -generic set  $H_{\text{lef}} \subseteq \mathbf{LS}_{\alpha}$  containing p and extend it to a  $\mathbf{LS}_{\beta}$ -generic set  $G_{\text{lef}} \subseteq \mathbf{LS}_{\alpha}$  so that  $H_{\text{lef}} = G_{\text{lef}} \cap \mathbf{LS}_{\alpha}$ . The (generic by Lemma 3.2) sets  $H_{\text{rig}} = t_{\text{lef}}^a[H_{\text{lef}}]$  and  $G_{\text{rig}} = t_{\text{lef}}^b[G_{\text{lef}}]$  satisfy  $H_{\text{rig}} = G_{\text{rig}} \cap \mathbf{LS}_{\beta}$  by (i). On the other hand  $H_{\text{rig}} \subseteq t_{\text{lef}}^c[H_{\text{lef}}] \subseteq G_{\text{rig}} \cap \mathbf{LS}_{\beta}$ , hence  $t_{\text{lef}}^c[H_{\text{lef}}] = H_{\text{rig}}$  is generic, as required. The verification of 3.1(1)(b) also is very simple.

**Lemma 6.5.** In **L**, assume that  $\alpha < \beta \leq \Omega$ . Then:

- (i) if  $s \in \mathbf{N}_{\alpha}$ ,  $t \in \mathbf{N}_{\beta}$ , and  $s \subseteq t$ , then  ${}^{\mathsf{rh}} s \subseteq {}^{\mathsf{rh}} t$ ;
- (ii) therefore if  $a \in \mathbf{DN}_{\alpha}$ ,  $b \in \mathbf{DN}_{\beta}$ , and  $a \leqslant b$ , then  ${}^{\mathsf{rh}}a \leqslant {}^{\mathsf{rh}}b$ ;
- (iii) if  $b \in \mathbf{DN}_{\beta}$  is regular and  $a = b \upharpoonright \alpha \in \mathbf{DN}_{\alpha}$  then a is regular, too.
- **Proof.** (i) Suppose that  $\langle p', q \rangle \in {}^{\mathsf{rh}}s$ , i.e.,  $p', q \in \mathbf{LS}_{\alpha}$  and there is a condition  $p \subseteq p'$  which  $\mathbf{LS}_{\alpha}$ -forces that  $q \in t[\underline{G}]$ . Prove that p also  $\mathbf{LS}_{\beta}$ -forces  $q \in t[\underline{G}]$ . Let a set  $G_{\mathsf{lef}} \subseteq \mathbf{LS}_{\beta}$  be a set  $\mathbf{LS}_{\beta}$ -generic over  $\mathbf{L}$  and containing p; prove that  $q \in G_{\mathsf{rig}} = t[G_{\mathsf{lef}}]$ . The set  $H_{\mathsf{lef}} = G_{\mathsf{lef}} \upharpoonright \mathbf{LS}_{\alpha}$  is  $\mathbf{LS}_{\alpha}$ -generic by Lemma 6.4 and still  $p \in H_{\mathsf{lef}}$ , hence  $q \in s[H_{\mathsf{lef}}] \subseteq t[G_{\mathsf{lef}}] = G_{\mathsf{rig}}$ , as required.
- (iii) Assume that  $p,q,p' \in \mathbf{LS}_{\alpha}$ ,  $p \subseteq p'$  and  $p \mathbf{LS}_{\alpha}$ -forces  $q \in t^a_{\mathtt{lef}}[\underline{G}]$ ; we have to prove that  $\langle p',q \rangle \in t^a_{\mathtt{lef}}$ . As  $a = b \upharpoonright \alpha$ , it suffices to show that  $\langle p',q \rangle \in t^b_{\mathtt{lef}}$ . The same argument based on Lemma 6.4 shows that p also  $\mathbf{LS}_{\beta}$ -forces  $q \in t^a_{\mathtt{lef}}[\underline{G}]$ . Therefore  $\langle p',q \rangle \in t^b_{\mathtt{lef}}$  since b is regular.

#### 7 Increasing sequences

Suppose that a set  $\Gamma \subseteq \mathbf{DN}$  is pairwise  $\leq$ -compatible. Then define the double-name  $A = \bigvee \Gamma$  by  $t_{\mathsf{lef}}^A = \bigcup_{a \in \Gamma} t_{\mathsf{lef}}^a$ ,  $t_{\mathsf{rig}}^A = \bigcup_{a \in \Gamma} t_{\mathsf{rig}}^a$ .

**Lemma 7.1** (in **L**). (i) If  $\lambda < \Omega$  is a limit ordinal and  $\{a_{\xi}\}_{{\xi}<\lambda}$  is a  $\leqslant$ -increasing sequence in **DN** then  $A = \bigvee \{a_{\xi} : {\xi} < \lambda\}$  belongs to **DN**;

- (ii) therefore the set  $\mathbf{DN} = \bigcup_{\gamma < \Omega} \mathbf{DN}_{\gamma}$  is  $\Omega$ -closed in the sense of  $\leq$ ;
- (iii) if  $\{a_{\xi}\}_{{\xi}<\Omega}$  is a strictly  $\leq$ -increasing sequence in **DN** then the double-name  $A = \bigvee \{a_{\xi} : {\xi} < {\lambda}\}$  belongs to  $\mathbf{DN}_{\Omega}$ .

**Proof.** (i) Suppose that  $\{\gamma_{\xi}\}_{\xi<\lambda}$  is a strictly increasing sequence of ordinals  $\gamma_{\xi}<\Omega$ , and double-names  $a_{\xi}=\langle t_{\mathtt{lef}}^{\xi}, t_{\mathtt{rig}}^{\xi}\rangle\in\mathbf{DN}_{\gamma_{\xi}}$  form a strictly  $\leqslant$ -increasing sequence: if  $\xi<\eta<\lambda$  then  $t_{\mathtt{lef}}^{\xi}\subseteq t_{\mathtt{lef}}^{\eta}$  and  $t_{\mathtt{rig}}^{\xi}\subseteq t_{\mathtt{rig}}^{\eta}$ . Let  $t_{\mathtt{lef}}^{A}=\bigcup_{\xi<\lambda}t_{\mathtt{lef}}^{\xi}$ ,  $t_{\mathtt{rig}}^{A}=\bigcup_{\xi<\lambda}t_{\mathtt{rig}}^{\xi}$ , and  $\gamma=\sup_{\xi<\lambda}\gamma_{\xi}$ . We claim that  $A=\langle t_{\mathtt{lef}}^{A}, t_{\mathtt{rig}}^{A}\rangle\in\mathbf{DN}_{\gamma}$ .

Let's verify 3.1(1). Assume that  $G_{\text{lef}} \subseteq \mathbf{LS}_{\gamma}$  is a generic set containing some  $p \in \text{dom } t_{\text{lef}}^A$ ; we have to prove that  $G_{\text{rig}} = t_{\text{lef}}^A[G_{\text{lef}}]$  is  $\mathbf{LS}_{\gamma}$ -generic and  $G_{\text{lef}} = t_{\text{rig}}^A[G_{\text{rig}}]$ . Note first of all that each set  $G_{\text{lef}}^{\xi} = G_{\text{lef}} \cap \mathbf{LS}_{\gamma_{\xi}}$ ,  $\xi < \lambda$ , is  $\mathbf{LS}_{\gamma_{\xi}}$ -generic by the product forcing theorem, and p belongs to some  $\text{dom } t_{\text{lef}}^{a_{\zeta}}$ ,  $\zeta < \Omega$ . We can assume that  $\zeta = 0$  (otherwise simply cut all double-names  $a_{\xi}$ ,  $\xi < \zeta$ ). Then  $p \in \text{dom } t_{\text{lef}}^0$ , therefore  $p \in \text{dom } t_{\text{lef}}^{\xi}$  for all  $\xi < \Omega$ . It follows that each set  $G_{\text{rig}}^{\xi} = t_{\text{lef}}^{\xi}[G_{\text{lef}}^{\xi}] \subseteq \mathbf{LS}_{\gamma_{\xi}}$  is  $\mathbf{LS}_{\gamma_{\xi}}$ -generic,  $G_{\text{rig}}^{\xi} \cap \text{dom } t_{\text{rig}}^{\xi} \neq \emptyset$ , and  $G_{\text{lef}}^{\xi} = t_{\text{rig}}^{\xi}[G_{\text{rig}}^{\xi}]$ , by Lemma 3.2. And as  $G_{\text{rig}} = \bigcup_{\xi < \lambda} G_{\text{rig}}^{\xi}$ , we conclude that at least  $G_{\text{rig}}$  is a filter in  $\mathbf{LS}_{\gamma}$  and  $G_{\text{lef}} = t_{\text{rig}}^{\xi}[G_{\text{rig}}^{\xi}]$ , that is, 3.1(1)(b).

To continue with 3.1(1)(a), we prove the  $\overline{LS}_{\gamma}$ -genericity of  $G_{rig}$ .

Let  $D \subseteq \mathbf{LS}_{\gamma}$  be a dense subset of  $\mathbf{LS}_{\gamma}$ , in  $\mathbf{L}$ . Assume towards the contrary that  $G_{\mathtt{rig}} \cap D = \varnothing$ . Then there is a condition  $p \in G_{\mathtt{lef}}$  which  $\mathbf{LS}_{\gamma}$ -forces that  $t_{\mathtt{lef}}^A[\underline{G}] \cap D = \varnothing$ . Then  $p \in G_{\mathtt{lef}}^{\xi}$  for some  $\xi < \lambda$ , and there is a condition  $q \in G_{\mathtt{rig}}^{\xi}$  which puts p in  $G_{\mathtt{lef}}^{\xi} = t_{\mathtt{rig}}^{\xi}[G_{\mathtt{rig}}^{\xi}]$  in the sense that  $\langle q, p \rangle \in t_{\mathtt{rig}}^{\xi}$ . As D is dense, there is some  $q' \in D$  with  $q \subseteq q'$ . Then q' belongs to some  $\mathbf{LS}_{\gamma_{\eta}}$ ,  $\xi < \eta < \lambda$ . By Corollary 3.3, there is a condition  $p' \in \mathbf{LS}_{\gamma_{\eta}}$ , compatible with p and such that  $\langle p', q' \rangle \in t_{\mathtt{lef}}^{\eta}$ . Then p'  $\mathbf{LS}_{\gamma}$ -forces  $q' \in t_{\mathtt{lef}}^{\eta}[\underline{G}] \cap D$ , while p, a compatible condition, forces the opposite, which is a contradiction.

(iii) Pretty similar argument.

Corollary 7.2 (in L). Assume that  $c \in DN_{\Omega}$ . Then

- (i) the set  $\Xi = \{ \gamma < \Omega : c \upharpoonright \gamma \in \mathbf{DN}_{\gamma} \}$  is a club in  $\Omega$ ;
- (ii) if c is full (Definition 4.1) then  $\Xi' = \{ \gamma \in \Xi : c \mid \gamma \text{ is full} \}$  is a club;

(iii) if  $\Xi'' = \{ \gamma \in \Xi : c \mid \gamma \text{ is regular} \}$  is unbounded in  $\Omega$  then  $\Xi'' = \Xi$ .

**Proof.** (i) That  $\Xi$  is closed follows from Lemma 7.1(i). To prove that  $\Xi$  is unbounded, let  $\alpha < \Omega$  and find a larger ordinal  $\beta \in \Xi$ .

Recall that to decide a sentence  $\Phi$  means to force  $\Phi$  or to force  $\neg \Phi$ . By basic forcing theorems, if  $p \in \mathbf{LS}$  then the set

$$D_p = \{ p \in \mathbf{LS} : p \text{ decides } q \in t_{\mathtt{lef}}^c[\underline{G}] \text{ and decides } q \in t_{\mathtt{rig}}^c[\underline{G}] \}$$

is dense in **LS**, therefore by the ccc property of **LS** there is an ordinal  $\beta$ ,  $\alpha < \beta < \Omega$ , such that  $D_p$  is dense in **LS**<sub>\beta</sub> for all  $p \in \mathbf{LS}_{\beta}$ . Then  $\beta \in \Xi$ .

(ii) easily follows from (i). To prove (iii) apply Lemma 6.5(iii).

#### 8 Superpositions

Assume that  $\gamma \leq \Omega$  and  $a, c \in \mathbf{DN}_{\gamma}$ . Define

$$t_{\mathtt{lef}}^{a \cdot c} = \{ \langle p', q \rangle \in \mathbf{LS}_{\gamma} \times \mathbf{LS}_{\gamma} : \exists \, p \in \mathbf{LS}_{\gamma} \, (\langle p', p \rangle \in t_{\mathtt{lef}}^{c} \wedge \langle p, q \rangle \in t_{\mathtt{lef}}^{a}) \},$$

$$t_{\mathtt{rig}}^{a \cdot c} \ = \ \{ \langle q, p' \rangle \in \mathbf{LS}_{\gamma} \times \mathbf{LS}_{\gamma} : \exists \ p \in \mathbf{LS}_{\gamma} \left( \langle q, p \rangle \in t_{\mathtt{rig}}^{a} \wedge \langle p, p' \rangle \in t_{\mathtt{rig}}^{c} \right) \}.$$

and  $a \cdot c = \langle t_{\texttt{lef}}^{a \cdot c}, t_{\texttt{rig}}^{a \cdot c} \rangle$ .

**Lemma 8.1.** If  $\gamma \leq \Omega$ ,  $a, c \in \mathbf{DN}_{\gamma}$ , and  $G \subseteq \mathbf{LS}_{\gamma}$ , then  $t_{\mathsf{lef}}^{a \cdot c}[G] = t_{\mathsf{lef}}^{a}[t_{\mathsf{lef}}^{c}[G]]$  and  $t_{\mathsf{rig}}^{a \cdot c}[G] = t_{\mathsf{rig}}^{c}[t_{\mathsf{rig}}^{a}[G]]$ .

**Proof.** Assume that  $q \in t_{\mathsf{lef}}^{a \cdot c}[G]$ . Then there is a pair  $\langle p', q \rangle \in t_{\mathsf{lef}}^{a \cdot c}$  with  $p' \in G$ . By definition there is a condition p such that  $\langle p', p \rangle \in t_{\mathsf{lef}}^c$  and  $\langle p, q \rangle \in t_{\mathsf{lef}}^a$ . Then  $p \in t_{\mathsf{lef}}^c[G]$  and hence  $q \in t_{\mathsf{lef}}^a[t_{\mathsf{lef}}^c[G]]$ . To prove the converse assume that  $q \in t_{\mathsf{lef}}^a[t_{\mathsf{lef}}^c[G]]$ . Then there is a pair  $\langle p, q \rangle \in t_{\mathsf{lef}}^a$  with  $p \in t_{\mathsf{lef}}^c[G]$ , and further there is a pair  $\langle p', p \rangle \in t_{\mathsf{lef}}^c$  with  $p' \in G$ . Then p witnesses that  $\langle p', q \rangle \in t_{\mathsf{lef}}^{a \cdot c}$ , and hence  $q \in t_{\mathsf{lef}}^{a \cdot c}[G]$ .

**Corollary 8.2.** Assume that  $\gamma < \Omega$  and  $a, b, c \in \mathbf{LS}_{\gamma}$ . If a, b are equivalent (in the sense of Definition 4.3) then so are  $a \cdot c$  and  $b \cdot c$ .

**Lemma 8.3.** If  $\gamma \leq \Omega$  and  $a, c \in \mathbf{DN}_{\gamma}$  then the following are equivalent:

$$(1) \ \operatorname{ran} t^c_{\operatorname{lef}} \cap \operatorname{dom} t^a_{\operatorname{lef}} \neq \varnothing \,, \ (2) \ \operatorname{ran} t^a_{\operatorname{rig}} \cap \operatorname{dom} t^c_{\operatorname{rig}} \neq \varnothing \,, \ (3) \ a \cdot c \in \mathbf{DN}_\gamma \,.$$

**Proof.** Let  $\operatorname{ran} t_{\mathtt{lef}}^c \cap \operatorname{dom} t_{\mathtt{lef}}^a \neq \varnothing$ . To prove (3) consider an  $\operatorname{LS}_{\gamma}$ -generic set  $G' \subseteq \operatorname{LS}_{\gamma}$ , and let  $p' \in G' \cap \operatorname{dom} t_{\mathtt{lef}}^{a \cdot c}$ . Then  $p' \in \operatorname{dom} t_{\mathtt{lef}}^c$ , hence  $G = t_{\mathtt{lef}}^b[G']$  is an  $\operatorname{LS}_{\gamma}$ -generic set by Lemma 3.2. As  $p' \in \operatorname{dom} t_{\mathtt{lef}}^{a \cdot b}$ ,  $G \cap \operatorname{dom} t_{\mathtt{lef}}^a \neq \varnothing$ . It follows that  $H = t_{\mathtt{lef}}^a[G]$  is an  $\operatorname{LS}_{\gamma}$ -generic set. Finally  $H = t_{\mathtt{lef}}^{a \cdot c}[G']$  by Lemma 8.1.

This argument also proves that  $G' = t_{rig}^{a \cdot c}[H]$ . Thus  $(1) \Longrightarrow (3)$ . That  $(3) \Longrightarrow (1)$  is obvious.

Corollary 8.4. If  $\gamma \leq \Omega$ ,  $a, c \in \mathbf{DN}_{\gamma}$ , and c is full (in the sense of Definition 4.1) then  $a \cdot c \in \mathbf{DN}_{\gamma}$ .

**Proof.** By Lemma 4.2,  $\operatorname{ran} t_{\text{lef}}^c = \operatorname{ran} t_{\text{rig}}^c = \operatorname{LS}_{\gamma}$ . Now use Lemma 8.3.

Thus if  $c \in \mathbf{DN}_{\gamma}$  is a full double-name then  $a \mapsto a \cdot c$  is a map  $\mathbf{DN}_{\gamma} \to \mathbf{DN}_{\gamma}$ . In this case, consider the *inverse* double-name  $c^{-1} = \langle t^c_{\mathbf{rig}}, t^c_{\mathbf{lef}} \rangle$ , let  $a \in \mathbf{DN}_{\gamma}$ , and compare a with  $a' = a \cdot c \cdot c^{-1}$ . On the one hand, we have  $t^{a'}_{\mathsf{lef}}[G] = t^a_{\mathsf{lef}}[t^c_{\mathsf{lef}}[f^{c^{-1}}_{\mathsf{lef}}[G]]]$  for any  $\mathbf{LS}_{\gamma}$ -generic set G by Lemma 8.1. It follows that  $t^{a'}_{\mathsf{lef}}[G] = t^a_{\mathsf{lef}}[t^c_{\mathsf{lef}}[t^c_{\mathsf{rig}}[G]]] = t^a_{\mathsf{lef}}[G]$  since the successive action of  $t^c_{\mathsf{lef}}$  and  $t^c_{\mathsf{rig}}$  is the identity by Lemma 3.2. Similarly  $t^{a'}_{\mathsf{rig}}[G] = t^a_{\mathsf{rig}}[G]$ . Therefore a and a' are equivalent, and hence  ${\mathsf{rh}}a = {\mathsf{rh}}a'$  by Lemma 4.4, but generally speaking we cannot assert that straightforwardly a = a'.

To fix this problem, define the modified action  $a * c = {}^{\mathsf{rh}}(a \cdot c)$ .

**Lemma 8.5.** Let  $\gamma < \Omega$  and let  $c \in \mathbf{DN}_{\gamma}$  be a full double-name. If  $a \in \mathbf{DN}_{\gamma}$  is regular (that is,  $a = {}^{\mathsf{rh}}a$ ) then  $b = a * c \in \mathbf{DN}_{\gamma}$ , b is regular, and  $a = b * c^{-1}$ .

**Proof.** That  $b \in \mathbf{DN}_{\gamma}$  follows from Corollary 8.4. The regularity holds by Lemma 4.4. To prove  $a = b * c^{-1}$ , note that both a and  $b * c^{-1}$  are regular double-names, and hence it suffices, by Lemma 4.4, to prove that a and  $b * c^{-1}$  are equivalent. However, still by Lemma 4.4,  $b * c^{-1}$  is equivalent to  $b \cdot c^{-1}$ , and b = a \* c is equivalent to  $a \cdot c$ , hence overall  $b * c^{-1}$  is equivalent to  $a \cdot c \cdot c^{-1}$  by Corollary 8.2. Finally a is equivalent to  $a \cdot c \cdot c^{-1}$ , see above.

**Lemma 8.6.** Assume that  $\gamma < \delta \leq \Omega$ ,  $c \in \mathbf{DN}_{\gamma}$  and  $d \in \mathbf{DN}_{\delta}$  are full double-names,  $c = d \upharpoonright \gamma$ , and  $a \in \mathbf{DN}_{\gamma}$ ,  $b \in \mathbf{DN}_{\delta}$ . Then

- (i) if  $a \leq b$  then  $a \cdot c \leq b \cdot d$ ;
- (ii) if a, b are regular then  $a \leq b$  iff  $a * c \leq b * d$ .

**Proof.** (i) is clear since  $a \cdot c$  is monotone on both a and c. As for (ii), the implication  $\implies$  holds by (i) and Lemma 6.5 while to prove the inverse make use of Lemma 8.5.

# 9 Generic double-names and product forcing

By Lemma 7.1, we can consider the set  $\mathbf{DN} = \bigcup_{\gamma < \Omega} \mathbf{DN}_{\gamma}$  ordered by  $\leqslant$  as an  $\Omega$ -closed forcing notion in  $\mathbf{L}$  ( $\leqslant$ -bigger double-names are stronger conditions). Suppose that  $\Gamma \subseteq \mathbf{DN}$  is a  $\mathbf{DN}$ -generic set over  $\mathbf{L}$ . Then a double-name  $A = \bigvee \Gamma \in \mathbf{L}[\Gamma]$  can be defined as in Section 7; we call such double-names  $A = \bigvee \Gamma$  generic over  $\mathbf{L}$  (together with the background generic sets  $\Gamma$ ).

Let  $\underline{\Gamma}$  and  $\underline{A}$  be canonical **DN**-names of resp.  $\Gamma$  and  $A = \bigvee \Gamma$ .

Remark 9.1. As  $\mathbf{L}$  is our default ground model unless otherwise specified, the sets  $\Gamma$  and  $A = \bigvee \Gamma$  do not belong to  $\mathbf{L}$ , however all reals and generally all sets  $x \subseteq \gamma < \Omega$  in  $\mathbf{L}[\Gamma]$  belong to  $\mathbf{L}$  by Lemma 7.1. It follows that the definition of  $\mathbf{DN}_{\gamma}$  ( $\gamma < \Omega$ ) in  $\mathbf{L}$  is absolute for  $\mathbf{L}[\Gamma]$ . That is, if  $a \in \mathbf{DN}_{\gamma}$  in  $\mathbf{L}$  then it is true in  $\mathbf{L}[\Gamma]$  that  $a \in \mathbf{DN}_{\gamma}$ . And conversely, if  $a \in \mathbf{L}[\Gamma]$  and it is true in  $\mathbf{L}[\Gamma]$  that  $a \in \mathbf{DN}_{\gamma}$  then  $a \in \mathbf{L}$  and it is true in  $\mathbf{L}$  that  $a \in \mathbf{DN}_{\gamma}$ .

Corollary 9.2. Assume that  $\Gamma$  is DN-generic over L and  $A = \bigvee \Gamma$ . Then

- (i) it holds in  $\mathbf{L}[\Gamma]$  that A belongs to  $\mathbf{DN}_{\Omega}$ ;
- (ii) if  $G_{\mathtt{lef}}$  is  $\mathtt{LS}$ -generic over  $\mathtt{L}[\Gamma]$ , and  $G_{\mathtt{lef}} \cap \mathtt{dom}\, t_{\mathtt{lef}}^A \neq \varnothing$ , then  $G_{\mathtt{rig}}$  is  $\mathtt{LS}$ -generic over  $\mathtt{L}[\Gamma]$  and  $G_{\mathtt{lef}} = t_{\mathtt{rig}}^A[G_{\mathtt{rig}}]$ ;
- (iii) if  $a \in \mathbf{DN}$ ,  $a \subseteq A$ , and  $\gamma = |a|$  then  $A \upharpoonright \gamma \in \mathbf{DN}_{\gamma} \cap \Gamma$  and  $a \leqslant A \upharpoonright \gamma \leqslant A$ .

**Proof.** (i) Remark 9.1 allows simply to refer to Lemma 7.1.

- (ii) Make use of Lemma 3.2.
- (iii) To prove that  $a' = A \upharpoonright \gamma \in \mathbf{DN}_{\gamma}$  and  $a \leqslant a' \leqslant A$  refer to Lemma 6.4(iii). To prove that  $a' \in \Gamma$  note that by Lemma 7.1 there is some  $c \in \Gamma$  which decides each  $b \in \mathbf{LS}_{\gamma}$  to belong or not to belong to  $\Gamma$ ; then  $a' \subseteq c$ .

#### 10 The first ingredient

Generic double-names and forcing with  $\mathbf{LS} \times \mathbf{DN}$  enable us to carry out the first main step towards Theorem 1.1.

In **L**, let  $\mathbf{H}\Omega$  be the set of all sets x such that the transitive closure  $\mathtt{TC}(x)$  has cardinality  $\mathtt{card}(\mathtt{TC}(x)) < \Omega$  strictly.

Blanket assumption 10.1. Thus suppose that  $G_0 \subseteq \mathbf{LS}$  is a  $\mathbf{LS}$ -generic set over  $\mathbf{L}$ , let  $\mathscr{X} \in \mathbf{L}[G_0]$ , and it is true in  $\mathbf{L}[G_0]$  that  $\mathscr{X}$  is a countable OD non-empty set of sets of reals. There is a formula  $\varphi(\cdot,\pi)$  with some  $\pi \in \operatorname{Ord}$  as the only parameter, such that it is true in  $\mathbf{L}[G_0]$  that  $\mathscr{X}$  is the only set x satisfying  $\varphi(x,\pi)$ .

There is a sequence  $\mathbf{u} = \{U_n\}_{n \in \omega} \in \mathbf{L}$  of names  $U_n \in \mathbf{L}$ , such that  $\mathscr{X} = \mathbf{u}[\![G_0]\!] := \{U_n[\![G_0]\!] : n \in \omega\}$ . Each  $U_n$  can be assumed to be an **LS**-name of a set of reals, that is, in  $\mathbf{L}$ ,  $U_n \subseteq \mathbf{LS} \times \mathbb{T}$ , where T is the set of all **LS**-names for reals. Furthermore, according to the  $\Omega$ -cc property of the forcing **LS**, each **LS**-name for a real can be assumed to be a set in  $\mathbf{H}\Omega$ . Therefore we shall wlog assume that  $U_n \subseteq \mathbf{H}\Omega$  for all n.

Anyway there is a condition  $\bar{p} \in G_0$  which **LS**-forces over **L** that " $\mathbb{U}[\underline{G}]$  is the only set x satisfying  $\varphi(x,\pi)$ , and  $\mathbb{U}[\underline{G}]$  is a set of sets of reals". Let  $\bar{\gamma} < \Omega$  be the least ordinal satisfying  $\bar{p} \in \mathbf{LS}_{\bar{\gamma}}$ .

Let a  $\bar{p}$ -pair be any pair  $\langle p, a \rangle \in \mathbf{LS} \times \mathbf{DN}$  such that  $\bar{p} \subseteq p \in \mathsf{dom}\, t_{\mathsf{lef}}^a$  and  $p \in \mathsf{LS}_{\gamma}$ -forces that  $\bar{p} \in t_{\mathsf{lef}}^a[\underline{G}]$ , where  $\gamma = |a|$ .

**Remark 10.2.** Let  $\bar{a} = \mathrm{id}[\bar{\gamma}]$ . Then  $\langle \bar{p}, \bar{a} \rangle$  is a  $\bar{p}$ -pair;  $\bar{p}$  LS $_{\bar{\gamma}}$ -forces that  $t_{\mathrm{lef}}^{\bar{a}}[\underline{G}] = \underline{G}$ .

**Lemma 10.3.** Let  $\langle p, a \rangle \in \mathbf{LS} \times \mathbf{DN}$  be a  $\bar{p}$ -pair,  $q \in \mathbf{LS}$ ,  $b \in \mathbf{DN}$ ,  $p \subseteq q$ ,  $a \leq b$ . There is a double-name  $c \in \mathbf{DN}$  such that  $b \leq c$  and  $\langle q, c \rangle$  is a  $\bar{p}$ -pair.

**Proof.** If  $q \in \mathbf{LS}_{\gamma}$ , where  $\gamma = |b|$ , then to define c add to  $t_{\mathsf{lef}}^b$  all pairs  $\langle q, r \rangle$  such that already  $\langle p, r \rangle \in b$ . We claim that  $\langle q, c \rangle$  is a  $\bar{p}$ -pair. Indeed if  $G_{\mathsf{lef}} \subseteq \mathbf{LS}_{\gamma}$  is generic then easily (\*)  $t_{\mathsf{lef}}^c[G_{\mathsf{lef}}] = t_{\mathsf{lef}}^b[G_{\mathsf{lef}}]$ , hence  $c \in \mathbf{DN}_{\gamma}$ . Further  $\bar{p} \subseteq p \subseteq q \in \mathsf{dom}\,t_{\mathsf{lef}}^c$  by construction. Finally  $q \ \mathbf{LS}_{\gamma}$ -forces that  $\bar{p} \in t_{\mathsf{lef}}^a[\underline{G}]$  because so does p, and we can replace  $t_{\mathsf{lef}}^a$  by  $t_{\mathsf{lef}}^c$  since  $a \subseteq b \subseteq c$ . If  $q \notin \mathbf{LS}_{\gamma}$  then still  $q \in \mathbf{LS}_{\delta}$  for some  $\delta, \gamma < \delta < \Omega$ . Use Lemma 6.2 to get a double-name  $b' \in \mathbf{DN}_{\delta}$  with  $b \leqslant b'$ , and argue as in the first case.

**Theorem 10.4.** Suppose that  $G_{\mathtt{lef}} \times \Gamma$  is a  $\mathtt{LS} \times \mathtt{DN}$ -generic set over  $\mathtt{L}$ ,  $A = \bigvee \Gamma$ , and  $\langle p, a \rangle \in G_{\mathtt{lef}} \times \Gamma$  is a  $\bar{p}$ -pair. Then

- $(\mathrm{i}) \ p, \bar{p} \in G_{\mathtt{lef}} \, , \ \bar{p} \in G_{\mathtt{rig}} = t_{\mathtt{lef}}^A[G_{\mathtt{lef}}] \, , \ and \ G_{\mathtt{rig}} \ is \ \mathbf{LS} \text{-}generic \ over \ } \mathbf{L}[\Gamma] \, ;$
- (ii)  $\mathbb{U}[G_{\mathtt{lef}}] = \mathbb{U}[G_{\mathtt{rig}}] in \text{ other words, any } \bar{p}\text{-pair } \langle p, a \rangle \text{ } (\mathbf{LS} \times \mathbf{DN})\text{-forces}$   $\mathbb{U}[\underline{G}] = \mathbb{U}[t_{\mathtt{lef}}^{\underline{A}}[\underline{G}]] \text{ over } \mathbf{L}.$

**Proof.** (i) To prove the genericity apply Corollary 9.2.

To prove (ii) suppose otherwise. Then there is a pair  $\langle q,b\rangle$  in  $\mathbf{LS}\times\mathbf{DN}$  with  $p\subseteq q,\ a\leqslant b,$  which  $(\mathbf{LS}\times\mathbf{DN})$ -forces  $\mathbb{E}[\underline{G}]\neq\mathbb{E}[\underline{G}]$ , that is

(†) if  $G_{\mathtt{lef}} \times \Gamma$  is a  $(\mathbf{LS} \times \mathbf{DN})$ -generic set over  $\mathbf{L}$  containing  $\langle q, b \rangle$ ,  $A = \bigvee \Gamma$ , and  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^A[G_{\mathtt{lef}}]$ , then  $\mathbb{u}[\![G_{\mathtt{lef}}]\!] \neq \mathbb{u}[\![G_{\mathtt{rig}}]\!]$ .

Let  $\mathcal{L} \in \mathbf{L}$  be an elementary submodel of a large model, such that  $\mathbf{H}\Omega \subseteq \mathcal{L}$ ,  $\Omega$  and  $\pi$  belong to  $\mathcal{L}$ ,  $\operatorname{card}(\mathcal{L}) = \Omega$  in  $\mathbf{L}$ , and  $\mathcal{L}$  is an elementary submodel of  $\mathbf{L}$  v.r.t. all  $\mathcal{L}_{100}$  formulas. Let  $\mathcal{L}' \in \mathbf{L}$  be the Mostowski collapse of  $\mathcal{L}$ ; still  $\operatorname{card}(\mathcal{L}') = \Omega$  in  $\mathbf{L}$ . Note that  $\mathcal{L}'$  is a transitive model of Zermelo with choice, and the collapse map  $\phi : \mathcal{L} \xrightarrow{\operatorname{onto}} \mathcal{L}'$  is the identity on  $\mathbf{H}\Omega$ , hence even on  $\mathscr{P}(\mathbf{H}\Omega) \cap \mathcal{L}$ . In particular,  $\phi(\Omega) = \Omega$ ,  $\phi(\mathbf{u}) = \mathbf{u}$ ,  $\phi(U_n) = U_n$  for all n,  $\phi(\mathbf{LS}) = \mathbf{LS}$ ,  $\phi(\mathbf{DN}) = \mathbf{DN}$ ,  $\mathbf{H}\Omega \subseteq \mathcal{L}'$ , and even  $\mathscr{P}(\mathbf{H}\Omega) \cap \mathcal{L} \subseteq \mathcal{L}'$ .

By the elementary submodel property,  $\langle q, b \rangle$  still  $(\mathbf{LS} \times \mathbf{DN})$ -forces over  $\mathcal{L}'$  that  $\mathbb{U}[\underline{G}] \neq \mathbb{U}[t_{\mathsf{lef}}^{\underline{A}}[\underline{G}]]$  — that is

(‡) if  $G_{\mathtt{lef}} \times \Gamma$  is a  $(\mathbf{LS} \times \mathbf{DN})$ -generic set over  $\mathcal{L}'$  containing  $\langle q, b \rangle$ ,  $A = \bigvee \Gamma$ , and  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^A[G_{\mathtt{lef}}]$ , then  $\mathbb{U}[G_{\mathtt{lef}}] \neq \mathbb{U}[G_{\mathtt{rig}}]$ .

To infer a contradiction, note that since  $\operatorname{card}(\mathcal{L}') = \Omega$  in  $\mathbf{L}$ , by Lemma 7.1 there exists a set  $\Gamma \in \mathbf{L}$ ,  $\mathbf{DN}$ -generic over  $\mathcal{L}'$  and containing b, hence containing a as well. We underline that  $\Gamma \in \mathbf{L}$ , and then  $A = \bigvee \Gamma$  belongs to  $\mathbf{L}$ , too. Let  $G_{\mathsf{lef}} \subseteq \mathbf{LS}$  be a set  $\mathbf{LS}$ -generic over  $\mathbf{L}$ , hence over  $\mathcal{L}'[\Gamma]$  as well, and containing q, and then containing p. Then the set  $G_{\mathsf{rig}} = t_{\mathsf{lef}}^A[G_{\mathsf{lef}}]$  is  $\mathsf{LS}$ -generic over  $\mathsf{L}$  and over  $\mathcal{L}'[\Gamma]$  by Lemma 3.2, and in addition,  $\mathsf{u}[G_{\mathsf{lef}}] \neq \mathsf{u}[G_{\mathsf{rig}}]$  by  $(\ddagger)$ .

Recall that  $\langle p,a \rangle$  also belongs to  $G_{\mathtt{lef}} \times A$ . Therefore  $\bar{p} \in G_{\mathtt{lef}} \cap G_{\mathtt{rig}}$  by (i). Thus  $G_{\mathtt{lef}}$  and  $G_{\mathtt{rig}}$  are **LS**-generic sets over **L** and both contain  $\bar{p}$ ,  $\mathsf{u}[G_{\mathtt{lef}}]$  is the only set x satisfying  $\varphi(x,\pi)$  in  $\mathbf{L}[G_{\mathtt{lef}}]$  while  $\mathsf{u}[G_{\mathtt{rig}}]$  is the only set x satisfying  $\varphi(x,\pi)$  in  $\mathbf{L}[G_{\mathtt{rig}}]$ . However  $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G_{\mathtt{rig}}]$  (because  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^A[G_{\mathtt{lef}}]$ ,  $G_{\mathtt{lef}} = t_{\mathtt{rig}}^A[G_{\mathtt{rig}}]$ , and  $A \in \mathbf{L}$ ), while on the other hand  $\mathsf{u}[G_{\mathtt{lef}}] \neq \mathsf{u}[G_{\mathtt{rig}}]$ , which is a contradiction.

### 11 Stabilizing pairs and second ingredient

Let a stabilizing  $\bar{p}$ -pair be any  $\bar{p}$ -pair  $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$  which, for some n,  $(\mathbf{LS} \times \mathbf{DN})$ -forces  $U_0[\underline{G}] = U_n[t_{1ef}^{\underline{A}}[\underline{G}]]$  over  $\mathbf{L}$ .

Corollary 11.1. If  $G_{\mathtt{lef}}$  is an LS-generic set over L containing  $\bar{p}$ , then there is a stabilizing  $\bar{p}$ -pair  $\langle \hat{p}, \hat{a} \rangle \in \mathtt{LS} \times \mathtt{DN}$  with  $\hat{p} \in G_{\mathtt{lef}}$ .

**Proof.** Let  $\bar{a} = \mathrm{id}[\bar{\gamma}]$ , see Remark 10.2. Let  $\Gamma \subseteq \mathbf{DN}$  be a set  $\mathbf{DN}$ -generic over  $\mathbf{L}[G_{\mathtt{lef}}]$  and containing  $\bar{a}$ , so that  $G_{\mathtt{lef}} \times \Gamma$  is  $(\mathbf{LS} \times \mathbf{DN})$ -generic. Let  $A = \bigvee \Gamma$ . Then the set  $G_{\mathtt{rig}} = t_{\mathtt{lef}}^A[G_{\mathtt{lef}}]$  satisfies  $\mathfrak{u}[G_{\mathtt{lef}}] = \mathfrak{u}[G_{\mathtt{rig}}]$  by Theorem 10.4. Therefore there is a number  $n \in \omega$  such that  $U_0[G_{\mathtt{lef}}] = U_n[G_{\mathtt{rig}}]$ . Then there is a stronger pair  $\langle \hat{p}, \hat{a} \rangle \in G_{\mathtt{lef}} \times \Gamma$   $(\bar{p} \subseteq \hat{p} \text{ and } \bar{a} \leqslant \hat{a})$  which  $(\mathbf{LS} \times \mathbf{DN})$ -forces  $U_0[G] = U_n[t_{\mathtt{lef}}^A[G]]$ . We can assume that  $\langle \hat{p}, \hat{a} \rangle$  is a  $\bar{p}$ -pair, by Lemma 10.3.  $\square$ 

**Proposition 11.2.** Let  $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$  be a stabilizing  $\bar{p}$ -pair. Assume that  $G_{\mathtt{lef}} \times \Gamma$ ,  $G'_{\mathtt{lef}} \times \Gamma'$  are sets  $(\mathbf{LS} \times \mathbf{DN})$ -generic over  $\mathbf{L}$  and containing  $\langle \hat{p}, \hat{a} \rangle$ ,  $A = \bigvee \Gamma$ ,  $A' = \bigvee \Gamma'$ , and  $t_{\mathtt{lef}}^A[G_{\mathtt{lef}}] = t_{\mathtt{lef}}^{A'}[G'_{\mathtt{lef}}]$ . Then  $U_0[G_{\mathtt{lef}}] = U_0[G'_{\mathtt{lef}}]$ .

**Proof.** By definition,  $U_0\llbracket G_{\mathtt{lef}} \rrbracket = U_n\llbracket t_{\mathtt{lef}}^A[G_{\mathtt{lef}}] \rrbracket$  and  $U_0\llbracket G'_{\mathtt{lef}} \rrbracket = U_n\llbracket t_{\mathtt{lef}}^{A'}[G'_{\mathtt{lef}}] \rrbracket$  for one and the same n.

The second ingredient in the proof of Theorem 1.1 will be the following:

Theorem 11.3. Assume that  $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$  is a stabilizing  $\bar{p}$ -pair,  $\hat{\gamma} < \Omega$ ,  $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$ ,  $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$ ,  $G_{\mathsf{lef}}, G'_{\mathsf{lef}} \subseteq \mathbf{LS}$  are  $\mathbf{LS}$ -generic sets over  $\mathbf{L}$  containing  $\hat{p}$ ,  $G_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$ , and  $\mathbf{L}[G_{\mathsf{lef}}] = \mathbf{L}[G'_{\mathsf{lef}}]$ . Then  $U_0[G_{\mathsf{lef}}] = U_0[G'_{\mathsf{lef}}]$ .

Let's show how this implies Theorem 1.1. The proof of Theorem 11.3 itself will follow in the next sections.

**Proof** (Theorem 1.1 from Theorem 11.3). We argue in the assumptions and notation of 10.1. Let  $G_{\mathsf{lef}} = G_0$ , so that  $\bar{p} \in G_{\mathsf{lef}}$  by 10.1. Then by Corollary 11.1, there is a stabilizing  $\bar{p}$ -pair  $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$  such that  $\hat{p} \in G_{\mathsf{lef}}$ . Pick  $\hat{\gamma} < \Omega$  such that  $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$  and  $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$ . Consider, in  $\mathbf{L}[G_{\mathsf{lef}}]$ , the set  $\mathscr{G}$  of all sets  $G \subseteq \mathbf{LS}$ ,  $\mathbf{LS}$ -generic over  $\mathbf{L}$  and satisfying  $\mathbf{L}[G] = \mathbf{L}[G_{\mathsf{lef}}]$ ,  $\hat{p} \in G$ , and  $G \cap \mathbf{LS}_{\hat{\gamma}} = G_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$ . In particular  $G_{\mathsf{lef}} \in \mathscr{G}$ . The only essential parameter of the definition of  $\mathscr{G}$  which is not immediately  $\mathrm{OD}$  — is  $G_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$ . However  $G_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$  itself, as basically any subset of any  $\mathbf{LS}_{\gamma}$ ,  $\gamma < \Omega$ , is ROD in the Solovay model. We conclude that  $\mathscr{G}$  is ROD in  $\mathbf{L}[G_{\mathsf{lef}}]$ .

On the other hand, suppose that  $G \in \mathcal{G}$ . Then  $U_0[\![G_{\mathtt{lef}}]\!] = U_0[\![G]\!]$  by Theorem 11.3. Therefore the set  $U_0[\![G_{\mathtt{lef}}]\!]$  can be defined as  $U_0[\![G]\!]$  for some / every  $G \in \mathcal{G}$ . This witnesses that  $U_0[\![G_{\mathtt{lef}}]\!]$  is ROD in  $\mathbf{L}[G_{\mathtt{lef}}]\!]$ , because so is  $\mathcal{G}$  by the above. Thus the set  $\mathcal{X} = \mathfrak{u}[\![G_{\mathtt{lef}}]\!]$  contains a ROD element. It follows that  $\mathcal{X}$  contains an OD element, by Lemma 2.3, as required.

 $\square$  (Thm 1.1 mod Thm 11.3)

#### 12 Final

Here we prove Theorem 11.3 and finally prove Theorem 1.1. We argue in the assumptions and notation of Theorem 11.3. That is,

(1)  $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$  is a stabilizing  $\bar{p}$ -pair,  $\hat{\gamma} < \Omega$ ,  $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$ ,  $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$ , the sets  $G_{\mathsf{lef}}, G'_{\mathsf{lef}} \subseteq \mathbf{LS}$  are  $\mathbf{LS}$ -generic over  $\mathbf{L}$  and both contain  $\hat{p}$ , and in addition  $G_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{\mathsf{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$ ,  $\mathbf{L}[G_{\mathsf{lef}}] = \mathbf{L}[G'_{\mathsf{lef}}]$ .

In this assumption, we have to prove that  $U_0[G_{lef}] = U_0[G'_{lef}]$ . Working towards this goal, our plan will be to find:

(\*) sets  $\Gamma, \Gamma' \subseteq \mathbf{DN}$ ,  $\mathbf{DN}$ -generic over  $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$ , containing  $\hat{a}$ , and satisfying  $t_{\mathtt{lef}}^A[G_{\mathtt{lef}}] = t_{\mathtt{lef}}^{A'}[G'_{\mathtt{lef}}]$ , where  $A = \bigvee \Gamma$  and  $A' = \bigvee \Gamma'$ ;

then the products  $G_{\mathtt{lef}} \times \Gamma$  and  $G'_{\mathtt{lef}} \times \Gamma'$  will be  $(\mathbf{LS} \times \mathbf{DN})$ -generic over  $\mathbf{L}$  and containing  $\langle \hat{p}, \hat{a} \rangle$ , so that  $U_0[\![G_{\mathtt{lef}}]\!] = U_0[\![G'_{\mathtt{lef}}]\!]$  follows by Proposition 11.2, accomplishing the proof of Theorem 11.3.

By Theorem 5.1 there is a double-name  $C \in \mathbf{DN}_{\Omega}$  in L, such that

(2) 
$$C$$
 is full,  $t_{\texttt{lef}}^C = t_{\texttt{rig}}^C$ ,  $G_{\texttt{lef}} = t_{\texttt{lef}}^C[G'_{\texttt{lef}}]$ , and  $G'_{\texttt{lef}} = t_{\texttt{rig}}^C[G_{\texttt{lef}}]$ .

As  $G_{lef} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{lef} \cap \mathbf{LS}_{\hat{\gamma}}$ , we can further assume that

(3) the restricted double-name  $C \upharpoonright \hat{\gamma}$  coincides with  $\mathbf{id}[\hat{\gamma}]$  of Example 4.5, so that  $C \upharpoonright \hat{\gamma} \in \mathbf{LS}_{\hat{\gamma}}$  is full and regular, and  $t_{\mathtt{lef}}^{C \upharpoonright \hat{\gamma}}[G] = t_{\mathtt{rig}}^{C \upharpoonright \hat{\gamma}}[G] = G$  for all G.

Let  $\Gamma$  be any set  $\Gamma \subseteq \mathbf{DN}$  with  $\hat{a} \in \Gamma$ ,  $\mathbf{DN}$ -generic over  $\mathbf{L}[G_{\mathtt{lef}}]$ . Then  $A = \bigvee \Gamma \in \mathbf{DN}_{\Omega}$  in  $\mathbf{L}[\Gamma]$  by Corollary 9.2, and  $\bar{p} \subseteq \hat{p} \in \mathtt{dom} A$  since  $\hat{a} \in \Gamma$ .

- Corollary 12.1. (i) The set  $X = \{ \gamma < \Omega : A \upharpoonright \gamma \in \mathbf{DN}_{\gamma} \} \in \mathbf{L}[\Gamma]$  is a club in  $\Omega$ , and if  $\gamma \in X$  then  $A \upharpoonright \gamma$  is regular;
  - (ii) the set  $Y = \{ \gamma < \Omega : C \upharpoonright \gamma \in \mathbf{DN}_{\gamma} \text{ and } C \upharpoonright \gamma \text{ is full} \} \in \mathbf{L}$  is a club in  $\Omega$ ;
- (iii) therefore  $Z = \{ \gamma \in X \cap Y : \hat{\gamma} \leq \gamma \}$  is a club, and in addition  $\hat{\gamma} \in Z$ .

**Proof.** To prove (i) and (ii) apply Corollary 7.2; the unboundedness condition in 7.2(iii) follows from the genericity of  $\Gamma$  and the density of the set of all regular double-names  $a \in \mathbf{DN}$  by Lemma 4.4(ii).

Claim 
$$\hat{\gamma} \in Z$$
 in (iii) follows from (3).

Now suppose that  $\gamma \in Y$ , hence  $C \upharpoonright \gamma \in \mathbf{DN}_{\gamma}$  is full. Let  $a \in \mathbf{DN}_{\gamma}$  be regular. Define  $a * C = a * (C \upharpoonright \gamma)$  (see Section 8).

**Lemma 12.2.** The map  $a \mapsto a * C$  is a  $\leqslant$ -preserving bijection of the set  $\mathbf{DN}_{reg}^Y = \{a \in \mathbf{DN} : a \text{ is regular} \land |a| \in Y\}$  onto itself, satisfying a \* C \* C = a.

**Proof.** If  $a \in \mathbf{DN_{reg}}^Y$  and  $\gamma = |a|$  then  $a * C = a * (C \upharpoonright \gamma)$  belongs to  $\mathbf{DN_{\gamma}}$  and is regular by Lemma 8.5, hence  $a * C \in \mathbf{DN_{reg}}^Y$ . If  $\delta > \gamma$  is a bigger ordinal still in Y, and  $b \in \mathbf{DN_{reg}}^Y$ ,  $\delta = |b|$ , then  $a \leq b$  iff  $a * C \leq b * C$  by Lemma 8.6(ii). Finally a \* C \* C = a holds still by Lemma 8.5, because  $C^{-1} = C$  (that is,  $t_{1ef}^C = t_{rig}^C$ ) by (2).

In particular, if  $\gamma \in Z$  then  $A \upharpoonright \gamma \in \mathbf{DN^Y_{reg}}$ , and hence  $(A \upharpoonright \gamma) * C \in \mathbf{DN^Y_{reg}}$  is a regular double-name. Thus  $\{(A \upharpoonright \gamma) * C\}_{\gamma \in Z} \in \mathbf{L}[\Gamma]$  is a  $\leqslant$ -increasing sequence of regular double-names. The following is a key fact.

**Lemma 12.3.** The sequence  $\{(A \upharpoonright \gamma) * C\}_{\gamma \in Z}$  is **DN**-generic over  $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$ , in the sense that if a set  $D' \subseteq \mathbf{DN}$ ,  $D' \in \mathbf{L}[G_{\mathtt{lef}}]$ , is open dense in **DN** then there is an ordinal  $\gamma \in Z$  such that  $(A \upharpoonright \gamma) * C \in D'$ .

**Proof.** The set  $\Delta' = D' \cap \mathbf{DN}_{\mathsf{reg}}^Y$  belongs to  $\mathbf{L}[G_{\mathsf{lef}}]$  and still is dense in  $\mathbf{DN}$  by Lemma 4.4(ii). Therefore its C-image  $\Delta = \{a * C : a \in \Delta'\}$  still belongs to  $\mathbf{L}[G_{\mathsf{lef}}]$  and is dense in  $\mathbf{DN}$  by Lemma 12.2. It follows by the genericity of  $\Gamma$  that  $A \upharpoonright \gamma \in \Delta$  for some  $\gamma \in Z$ . Then  $a = (A \upharpoonright \gamma) * C \in \Delta'$ , since  $a * C = A \upharpoonright \gamma$  by Lemma 12.2.

Corollary 12.4. The set  $\Gamma' = \{a \in \mathbf{DN} : \exists \gamma \in Z (a \leqslant (A \upharpoonright \gamma) * C)\}$  is  $\mathbf{DN}$ -generic over  $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$ .

Let us check the other intended properties of  $\Gamma'$  as in (\*).

To see that  $\hat{a} \in \Gamma'$ , recall that  $\hat{a} \in \Gamma \cap \mathbf{DN}_{\hat{\gamma}}$ . It follows by Corollary 9.2(iii) that  $\hat{a} \leqslant a = A \upharpoonright \hat{\gamma}$ . However  $\hat{\gamma} \in Z$  by Corollary 12.1(iii). We conclude that  $\hat{a} * C \in \Gamma'$ . Finally  $\hat{a} * C = \hat{a} * (C \upharpoonright \hat{\gamma}) = \hat{a}$  since  $C \upharpoonright \hat{\gamma} = \mathbf{id}[\hat{\gamma}]$  by (3). Thus  $\hat{a} \in \Gamma'$ , as required.

Finally prove that  $t_{\mathtt{lef}}^A[G_{\mathtt{lef}}] = t_{\mathtt{lef}}^{A'}[G'_{\mathtt{lef}}]$ , where  $A = \bigvee \Gamma$  and  $A' = \bigvee \Gamma'$ . It suffices to show that if  $\gamma \in Z$  then

$$t_{\mathsf{lef}}^{A \upharpoonright \gamma}[G_{\mathsf{lef}} \cap \mathbf{LS}_{\gamma}] = t_{\mathsf{lef}}^{A' \upharpoonright \gamma}[G'_{\mathsf{lef}} \cap \mathbf{LS}_{\gamma}]. \tag{5}$$

However by construction  $A' \upharpoonright \gamma = (A \upharpoonright \gamma) * C = (A \upharpoonright \gamma) * (C \upharpoonright \gamma)$ , and on the other hand  $t_{\mathtt{lef}}^{(A \upharpoonright \gamma) * (C \upharpoonright \gamma)}[G] = t_{\mathtt{lef}}^{A \upharpoonright \gamma}[t_{\mathtt{lef}}^{C \upharpoonright \gamma}[G]]$  for all G by Lemma 8.1, therefore (5) is equivalent to

$$t_{\mathtt{lef}}^{A \upharpoonright \gamma}[G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}] = t_{\mathtt{lef}}^{A \upharpoonright \gamma}[t_{\mathtt{lef}}^{C \upharpoonright \gamma}[G_{\mathtt{lef}}' \cap \mathbf{LS}_{\gamma}]] \,,$$

which obviously follows from

$$G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma} = t_{\mathtt{lef}}^{C \upharpoonright \gamma} [G'_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}],$$

and this is a corollary of the equality  $G_{\texttt{lef}} = t_{\texttt{lef}}^C[G'_{\texttt{lef}}]$  in (2) by Lemma 6.4(i)(b).  $\Box$  (Theorem 11.3)

This also completes the proof of Theorem 1.1 (see the end of Section 11).

 $\Box$  (Theorem 1.1)

#### 13 Conclusive remarks

Question 13.1. Is Theorem 1.1 true for *arbitrary* sets  $\mathcal{X}$ , not necessarily sets of reals? In this general case, the proof given above fails in the proof of Theorem 10.4, since it is not true anymore that  $U_n \subseteq \mathbf{H}\Omega$  and  $\phi(U_n) = U_n$ .  $\square$ 

It follows from Theorem 1.1 that, in the Solovay model, any OD set  $\mathscr{X}$  of sets of reals containing non-OD elements is uncountable. If moreover  $\mathscr{X}$  is a set of reals then in fact  $\mathscr{X}$  contains a perfect subset and hence has cardinality  $\mathfrak{c}$  by a profound theorem in [8]. Does this stronger result reasonably generalize to sets of sets of reals and more complex sets?

Conjecture 13.2. It is true in the Solovay model that if  $\mathscr X$  is an OD set then

- (I) if  $\mathscr{X}$  contains only OD elements then it is OD-wellorderable;
- (II) if  $\mathscr{X}$  contains only ROD elements, among them at leat one non-OD element, then  $\mathscr{X}$  includes a ROD-image of the continuum  $2^{\omega}$ ;

(III) if  $\mathscr{X}$  contains a non-ROD element then  $\mathscr{X}$  has cardinality  $\geq 2^{\mathfrak{c}}$ .

The set of all **LS**-generic sets over **L** is a less trivial example of a set of type (III) in the Solovay model.  $\Box$ 

A proof of (III) would be an alternative (and perhaps simpler) proof of Theorem 1.1 of this paper.

It remains to note that Caicedo and Ketchersid [1] obtained a somewhat similar trichotomy result in in a strong determinacy assumption.

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