# OD elements of countable OD sets in the Solovay model 

Vladimir Kanovei*

September 9, 2018


#### Abstract

It is true in the Solovay model that every countable ordinal-definable set of sets of reals contains only ordinal-definable elements.


## 1 Introduction

It is known that the existence of a non-empty OD (ordinal-definable) set of reals $X$ with no OD element is consistent with ZFC; the set of all non-constructible reals gives a transparent example in many generic models.

Can such a set $X$ be countable?
This question was initiated and discussed at the Mathoverflow website ${ }^{1}$ and at FOM ${ }^{2}$. In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support countable product $\mathbb{P}^{<\omega}$ (see [2]) of the Jensen "minimal $\Pi_{2}^{1}$ real singleton forcing" $\mathbb{P}$ defined in [4] (see also Section 28A of [3]). We proved in [5] that indeed, in a $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$, the set of all reals $\mathbb{P}$-generic over $\mathbf{L}$ is a countable $\Pi_{2}^{1}$ set with no OD elements. Moreover there is a modification $\mathbb{P}^{\prime}$ of $\mathbb{P}$ such that it is true in a $\mathbb{P}^{\prime}$-generic extension of $\mathbf{L}$ that there is a $\Pi_{2}^{1} \mathrm{E}_{0}$-equivalence class containing no OD reals, 7 .

On the other hand, one may ask do countable non-empty OD sets without OD elements exist in such a more typical generic extension as the Solovay model? We partially answer this question in the negative.

[^0]Theorem 1.1. It is true in the Solovay model that every non-empty OD countable or finite set $\mathscr{X}$ of sets of reals necessarily contains an $O D$ element, and hence, in fact, consists of $O D$ elements.

The Solovay model here is a model of ZFC defined in [8] in which all projective (and generally all ROD, real-ordinal definable) sets of reals are Lebesgue measurable. The case, when $\mathscr{X}$ is a (non-empty OD countable) set of reals in this theorem, is well known and is implicitly contained in the proof of the perfect set property by Solovay [8]. Hovever the proofs known for this particular case of sets of reals (as, e.g., in [9] or [6]) do not work even for sets $\mathscr{X} \subseteq \mathscr{P}\left(\omega^{\omega}\right)$ (as in the theorem). In this paper, we present the proof of Theorem 1.1.

## 2 Notation

We consider the constructible universe $\mathbf{L}$ as the ground model by default. Suppose that $\Omega$ is an inaccessible cardinal.

Blanket assumption 2.1. By a generic set we'll always mean a filter, that is, both pairwise compatible in itself and containing all weaker conditions.

Definition 2.2. We represent the Levy - Solovay forcing associated with $\Omega$ is the set LS of all partial maps $p: \operatorname{dom} p \rightarrow \Omega$ such that $\operatorname{dom} p \subseteq \Omega \times \omega$ is a finite set and $p(\alpha, n)<\alpha$ whenever $\langle\alpha, n\rangle \in \operatorname{dom} p$. Let $|p|=\{\alpha: \exists n(\langle\alpha, n\rangle \in \operatorname{dom} p)\}$.

If $\gamma \leq \Omega$ then $\mathbf{L S}{ }_{\gamma}=\{p \in \mathbf{L S}:|p| \subseteq \gamma\}$; in particular $\mathbf{L S}_{\Omega}=\mathbf{L S}$.
If $p \in \mathbf{L S}$ and $\alpha<\Omega$ then the $\alpha$-component $p_{\alpha}$ of $p$ is a map defined on the set $\operatorname{dom} p_{\alpha}=\{n:\langle\alpha, n\rangle \in \operatorname{dom} p\} \subseteq \omega$ by $p_{\alpha}(n)=p(\alpha, n)$.

If $G \subseteq \mathbf{L S}$ is an $\mathbf{L S}$-generic set over $\mathbf{L}$ then $\mathbf{L}[G]$ is the Solovay model, to which Theorem 1.1 refers. The next lemma will be important below.

Lemma 2.3 (reduction to ROD). It is true in the Solovay model that if $\mathscr{X}$ is a non-empty $O D$ countable set and $X \in \mathscr{X}$ is $R O D$ then $X$ is $O D$.

Thus somewhat surprisingly, it turns out that it suffices to prove the existence of a ROD (real-ordinal definable) element $X \in \mathscr{X}$ in Theorem 1.1.

Proof. Arguing in the Solovay model, assume that

$$
X=X_{p_{0}}=\left\{x: \varphi\left(x, p_{0}\right)\right\},
$$

where $\varphi$ is a formula with a real parameter $p_{0} \in \omega^{\omega}$ and hidden ordinal parameters. The set $P=\left\{p \in \omega^{\omega}: X_{p} \in \mathscr{X}\right\}$ is OD and contains $p_{0}$, and the equivalence relation, $p \mathrm{E} q$ iff $X_{p}=X_{q}$ on $P$, is OD as well, and E has at most
countably many equivalence classes in $P$. However it is known that, in the Solovay model, if an OD equivalence relation on $\omega^{\omega}$ has at most countably many equivalence classes then all its equivalence classes are OD, [6, 9]. In particular $\left[p_{0}\right]_{\mathrm{E}}$ is OD, and hence the set $X=X_{p_{0}}=\left\{x: \exists p \in\left[p_{0}\right]_{\mathrm{E}} \varphi\left(x, p_{0}\right)\right\}$ is OD.

Definition 2.4 (ramified names). We'll use the ordinary ramified system of LS-names for differens sets in $\mathbf{L}[G]$, so that $U \llbracket G \rrbracket$ will be the $G$-interpretation of a name $U$ (basically, any set) defined by $\in$-rank induction by

$$
U \llbracket G \rrbracket=\{u \llbracket G \rrbracket: \exists p \in G(\langle p, u\rangle \in U)\} .
$$

Then, if $G \subseteq \mathbf{L S}$ is generic over $\mathbf{L}$ then $\mathbf{L}[G]=\{U \llbracket G \rrbracket: U \in \mathbf{L}\}$.
Each set $x \in \mathbf{L}$ has a canonical $\mathbf{L S}$-name $\check{x} \in \mathbf{L}$, such that $\check{x}[G]=x$ for any generic set $G \subseteq \mathbf{L S}$. Yet following common practice we shall identify $\check{x}$ with $x$ itself whenever possible.

Definition 2.5 (simple names). To somewhat simplify notation, we'll make use of a simpler system of names particularly for subsets of $\mathbf{L S}$. Let $\mathbf{N}=\mathscr{P}(\mathbf{L S} \times$ $\mathbf{L S})$, and if $t \in \mathbf{N}$ and $G \subseteq \mathbf{L S}$ then $t[G]=\{q: \exists p \in G(\langle p, q\rangle \in t)\} \subseteq \mathbf{L S}$.

Thus $\mathbf{N}$ consists of all $\mathbf{L S}$-names for subsets of $\mathbf{L S}$.
If $\gamma<\Omega$ then let $\mathbf{N}_{\gamma}=\mathscr{P}\left(\left(\mathbf{L} \mathbf{S}_{\gamma}\right) \times\left(\mathbf{L} \mathbf{S}_{\gamma}\right)\right)$, so that any $t \in \mathbf{N}_{\gamma}$ is a $\mathbf{L} \mathbf{S}_{\gamma^{-}}$ name for a subset of $\mathbf{L} \mathbf{S}_{\gamma}$.

The name $\underline{G}=\{\langle p, p\rangle: p \in \mathbf{L S}\}$ belongs to $\mathbf{N}$, and $\underline{G}[G]=G$.

## 3 Double names

In many cases below, we'll consider pairs of $\mathbf{L S}$-generic sets $G, G^{\prime} \subseteq \mathbf{L S}$ over $\mathbf{L}$, such that $\mathbf{L}[G]=\mathbf{L}\left[G^{\prime}\right]$; note that this is not a $(\mathbf{L S} \times \mathbf{L S})$-generic pair! Similar pairs will be considered for the forcing notions $\mathbf{L S} \boldsymbol{S}_{\gamma}(\gamma<\Omega)$ instead of $\mathbf{L S}$. The next definition introduces a useful tool related to such pairs.

Definition 3.1. In $\mathbf{L}$, if $\gamma \leq \Omega$ then any pair $a=\left\langle t_{\text {lef }}^{a}, t_{\text {rig }}^{a}\right\rangle$ of names $t_{\text {lef }}^{a}, t_{\text {rig }}^{a} \in \mathbf{N}_{\gamma}$ will be called a double-name. Let $\mathbf{D} \mathbf{N}_{\gamma}$ consist of all doublenames $a=\left\langle t_{\text {lef }}^{a}, t_{\text {rig }}^{a}\right\rangle$ such that $t_{\text {lef }}^{a} \neq \varnothing, t_{\text {rig }}^{a} \neq \varnothing$, and
(1) if $p \in \operatorname{dom} t_{\text {lef }}^{a}$ then $p \mathbf{L} \mathbf{S}_{\gamma}$-forces: (a) $t_{\text {lef }}^{a}[\underline{G}]$ is $\mathbf{L} \mathbf{S}_{\gamma}$-generic, and
(b) $\underline{G}=t_{\text {rig }}^{a}\left[t_{\text {lef }}^{a}[\underline{G}]\right]$;
(2) if $p \in \operatorname{dom} t_{\text {rig }}^{a}$ then $p \mathbf{L} \mathbf{S}_{\gamma}$-forces: (a) $t_{\text {rig }}^{a}[\underline{G}]$ is $\mathbf{L} \mathbf{S}_{\gamma}$-generic, and
(b) $\underline{G}=t_{\text {lef }}^{a}\left[t_{\text {rig }}^{a}[\underline{G}]\right]$.

Define $\mathbf{D N}=\bigcup_{\gamma<\Omega} \mathbf{D} \mathbf{N}_{\gamma}$; this is different from $\mathbf{D} \mathbf{N}_{\Omega}$. It follows from (1) or (2) that for any $a \in \mathbf{D N}$ there is a unique $\gamma=|a|<\Omega$ such that $a \in \mathbf{D} \mathbf{N}_{\gamma}$.

Note that all sets $\mathbf{N}_{\gamma}$ and $\mathbf{D} \mathbf{N}_{\gamma}$ belong to $\mathbf{L}$.
Lemma 3.2. Assume that $\gamma \leq \Omega$ and $a \in \mathbf{D N}_{\gamma}$. Then:
(i) if $G_{\text {lef }} \subseteq \mathbf{L} \mathbf{S}_{\gamma}$ is an $\mathbf{L} \mathbf{S}_{\gamma}$-generic set and $G_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$ then $G_{\text {rig }}=$ $t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]$ is $\mathbf{L} \mathbf{S}_{\gamma}$-generic, $G_{\text {rig }} \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$, and $G_{\text {lef }}=t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$;
(ii) if $G_{\text {rig }} \subseteq \mathbf{L} \mathbf{S}_{\gamma}$ is $\mathbf{L S}_{\gamma}$-generic and $G_{\text {rig }} \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$ then $G_{\text {lef }}=$ $t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$ is $\mathbf{L} \mathbf{S}_{\gamma}$-generic, $G_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing, G_{\text {rig }}=t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]$.

Thus each $a \in \mathbf{D} \mathbf{N}_{\gamma}$ induces a bijection between all $\mathbf{L} \mathbf{S}_{\gamma}$-generic sets $G \subseteq \mathbf{L S}_{\gamma}$ satisfying $G \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$ and those satisfying $G \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$.

Corollary 3.3. If $\gamma \leq \Omega, a \in \mathbf{D N}_{\gamma},\langle q, p\rangle \in t_{\text {rig }}^{a}$, and $q \subseteq q^{\prime} \in \mathbf{L} \mathbf{S}_{\gamma}$ then there is a condition $p^{\prime} \in \mathbf{L} \mathbf{S}_{\gamma}$ compatible with $p$ and such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in t_{\text {lef }}^{a}$.

Proof. Let $G_{\text {rig }} \subseteq \mathbf{L S}_{\gamma}$ be a generic set containing $q^{\prime}$, hence containing $q$ as well. Then $G_{\text {lef }}=t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$ is a $\mathbf{L} \mathbf{S}_{\gamma}$-generic set containing $p$, and $G_{\text {rig }}=$ $t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]$ by Lemmaio. As $q^{\prime} \in G_{\text {rig }}$, there is a condition $p^{\prime} \in G_{\text {lef }}$ such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in t_{\text {lef }}^{a}$. As $p$ also belongs to $G_{\text {lef }}, p, p^{\prime}$ are compatible.

## 4 Full, regular, equivalent names

Recall that a set $D \subseteq \mathbf{L S}_{\gamma}$ is dense if for any $p \in \mathbf{L} \mathbf{S}_{\gamma}$ there is $q \in D$ with $p \subseteq q$, and is open if $\left(p \in D \wedge p \subseteq q \in \mathbf{L} \mathbf{S}_{\gamma}\right) \Longrightarrow q \in D$.

Definition 4.1. Let $\gamma \leq \Omega$. A name $t \in \mathbf{N}_{\gamma}$ is full if the set dom $t$ is dense in $\mathbf{L} \mathbf{S}_{\gamma}$. A double-name $a \in \mathbf{D} \mathbf{N}_{\gamma}$ is full if such are the names $t_{\text {lef }}^{a}$ and $t_{\text {rig }}^{a}$.

A name $t \in \mathbf{N}_{\gamma}$ is regular, if the following holds: if $p, q \in \mathbf{L} \mathbf{S}_{\gamma}$ and $p \mathbf{L} \mathbf{S}_{\gamma^{-}}$ forces $q \in t[\underline{G}]$ then $\langle p, q\rangle \in t$. In particular, in this case, if $\langle p, q\rangle \in t$ and $p \subseteq p^{\prime} \in \mathbf{L} \mathbf{S}_{\gamma}$ then $\left\langle p^{\prime}, q\right\rangle \in t$, too. A double-name $a \in \mathbf{D} \mathbf{N}_{\gamma}$ is regular, if so are both components $t_{\text {lef }}^{a}$ and $t_{\text {rig }}^{a}$. Define the regular hull

$$
{ }^{\mathrm{rh}} t=\left\{\langle p, q\rangle \in \mathbf{L} \mathbf{S}_{\gamma} \times \mathbf{L} \mathbf{S}_{\gamma}: p \mathbf{L S}_{\gamma} \text {-forces } q \in t[\underline{G}]\right\}
$$

of any $t \in \mathbf{N}_{\gamma}$. If $a \in \mathbf{D} \mathbf{N}_{\gamma}$ then let ${ }^{\text {rh }} a=\left\langle{ }^{\mathrm{rh}} t_{\text {lef }}^{a},{ }^{\mathrm{rh}} t_{\text {rig }}^{a}\right\rangle$.
Lemma 4.2. Assume that $\gamma \leq \Omega$ and $a \in \mathbf{D N}_{\gamma}$ is full. Then $\operatorname{ran} t_{\text {lef }}^{a}=$ $\operatorname{ran} t_{\mathrm{rig}}^{a}=\mathbf{L} \mathbf{S}_{\gamma}$, and if $G \subseteq \mathbf{L S}_{\gamma}$ is $\mathbf{L S}_{\gamma}$-generic then so are $t_{\text {lef }}^{a}[G]$ and $t_{\mathrm{rig}}^{a}[G]$.

Proof. To prove the genericity claim note that if say dom $t_{1 \text { ef }}^{a}$ is dense then any generic set $G \subseteq \mathbf{L S}_{\gamma}$ intersects dom $t_{\text {lef }}^{a}$, then use Lemma 3.2. To prove the first claim, let $q \in \mathbf{L} \mathbf{S}_{\gamma}$. Consider a generic set $G_{\text {rig }} \subseteq \mathbf{L} \mathbf{S}_{\gamma}$ containing $q$. Then $G \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$, see above. It follows that $G_{\text {lef }}=t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$ is generic and $G_{\text {rig }}=t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]$ by Lemma 3.2. But $q \in G_{\text {rig }}$, hence $q \in \operatorname{ran} t_{\text {lef }}^{c}$.

Definition 4.3. Names $s, t \in \mathbf{N}_{\gamma}$ are equivalent if $s[G]=t[G]$ for any generic set $G \subseteq \mathbf{L} \mathbf{S}_{\gamma}$, or equivalently, if any $p \in \mathbf{L} \mathbf{S}_{\gamma} \mathbf{L S}_{\gamma}$-forces $s[\underline{G}]=t[\underline{G}]$. Double-names $a, b \in \mathbf{D N}_{\gamma}$ are equivalent if $t_{\text {lef }}^{b}, t_{\text {rig }}^{b}$ are equivalent to resp. $t_{\text {lef }}^{a}, t_{\text {rig }}^{a}$.

Lemma 4.4. Assume that $\gamma \leq \Omega$. Then:
(i) if $t \in \mathbf{N}_{\gamma}$ then ${ }^{\text {rh }} t$ is regular and equivalent to $t$;
(ii) if $a \in \mathbf{D N}_{\gamma}$ then ${ }^{\text {rh }} a \in \mathbf{D N}_{\gamma}, a \leqslant{ }^{\mathrm{rh}} a$, and ${ }^{\text {rh }} a$ is equivalent to $a$ therefore the set $\mathbf{D N}_{\gamma}^{\mathrm{reg}}=\left\{b \in \mathbf{D N}_{\gamma}: b\right.$ is regular $\}$ is dense in $\mathbf{D N}_{\gamma}$;
(iii) if $a, b \in \mathbf{D N}_{\gamma}$ then $a$ is equivalent to $b$ iff ${ }^{\mathrm{rh}} a={ }^{\mathrm{rh}} b$.

Proof. (i)] To establish the equivalence, assume that $G \subseteq \mathbf{L} \mathbf{S}_{\gamma}$ is generic and $q \in{ }^{\mathrm{rh}} t[G]$. Then there is $p \in G$ such that $\langle p, q\rangle \in{ }^{\mathrm{rh}} t$. By definition $p \mathbf{L S}_{\gamma^{-}}$ forces $q \in t[\underline{G}]$. But then $q \in t[G]$, as required. To establish the regularity, assume that $p, q \in \mathbf{L} \mathbf{S}_{\gamma}$, and $p \mathbf{L} \mathbf{S}_{\gamma}$-forces $q \in{ }^{\mathrm{rh}} t[\underline{G}]$ - therefore $p \mathbf{L} \mathbf{S}_{\gamma}$-forces $q \in t[\underline{G}]$ by the equivalence already proved. Then by definition $\langle p, q\rangle \in{ }^{\mathrm{rh}} t$.
(ii) follows from (i). The direction $\Longleftarrow$ in (iii) immediately follows from (ii). To prove the opposite direction, it suffices to show that if names $s, t \in \mathbf{N}_{\gamma}$ are equivalent then ${ }^{\mathrm{rh}} s={ }^{\mathrm{rh}} t$. Assume that $\langle p, q\rangle \in{ }^{\mathrm{rh}} s$. By definition $p \mathbf{L S}_{\gamma^{-}}$ forces $q \in s[\underline{G}]$. Then, as $s, t$ are equivalent, $p$ also forces $q \in t[\underline{G}]$. It follows that $\langle p, q\rangle \in{ }^{-\mathrm{rh}} t$, as required.

Example 4.5. If $\gamma<\Omega$ then let $t_{\gamma}=\left\{\langle p, q\rangle: p, q \in \mathbf{L S}_{\gamma} \wedge q \subseteq p\right\}$ and $\mathbf{i d}[\gamma]=\left\langle t_{\gamma}, t_{\gamma}\right\rangle$. Then $\mathbf{i d}[\gamma] \in \mathbf{D N}_{\gamma}$ is a full regular double-name and $t_{\text {lef }}^{\mathrm{id}[\gamma]}[G]=$ $t_{\text {lef }}^{\text {id }[\gamma]}[G]=G$ for any $\mathbf{L} \mathbf{S}_{\gamma}$-generic set $G \subseteq \mathbf{L} \mathbf{S}_{\gamma}$ : the identity name.

## 5 Double-name representation theorem

The next theorem shows that the double-name tool adequately represents the case of a pair of $\mathbf{L S}$-generic sets $G, G^{\prime} \subseteq \mathbf{L S}$ such that $\mathbf{L}[G]=\mathbf{L}\left[G^{\prime}\right]$.

Theorem 5.1. Assume that $\gamma \leq \Omega, G_{\text {lef }}, G_{\text {rig }} \subseteq \mathbf{L S} \boldsymbol{L}_{\gamma}$ are $\mathbf{L S}_{\gamma}$-generic sets over $\mathbf{L}$, and $\mathbf{L}\left[G_{1 \text { ef }}\right]=\mathbf{L}\left[G_{\text {rig }}\right]$. Then there is a full regular double-name $c \in \mathbf{D N}_{\gamma}$ such that $G_{\text {rig }}=t_{\text {lef }}^{c}\left[G_{\text {lef }}\right], G_{\text {lef }}=t_{\text {rig }}^{c}\left[G_{\text {rig }}\right]$, and $t_{\text {lef }}^{c}=t_{\text {rig }}^{c}$.

Proof. If $G_{\text {lef }}=G_{\text {rig }}$ then it suffices to define $c$ by $t_{\text {lef }}^{c}=t_{\text {rig }}^{c}=\mathbf{i d}[\gamma]$. Therefore assume that $G_{\text {lef }} \neq G_{\text {rig }}$. Then there exist conditions $p_{\text {lef }} \in G_{\text {lef }}$ and $p_{\text {rig }} \in G_{\text {rig }}$ incompatible in $\mathbf{L} \mathbf{S}_{\gamma}$. By a basic forcing theorem, there exist names $s_{\text {lef }}, s_{\text {rig }} \in \mathbf{N}_{\gamma}$ such that $G_{\text {rig }}=s_{\text {lef }}\left[G_{\text {lef }}\right], G_{\text {lef }}=s_{\text {rig }}\left[G_{\text {rig }}\right]$, and every condition $p \in \operatorname{dom} s_{\text {lef }}$ satisfies $p_{\text {lef }} \subseteq p$ while every condition $q \in \operatorname{dom} s_{\text {rig }}$ satisfies $p_{\text {rig }} \subseteq q$. It is not true immediately that $\left\langle s_{\text {lef }}, s_{\text {rig }}\right\rangle \in \mathbf{D N}_{\gamma}$; we need to somewhat modify the names by shrinking.

We can wlog assume that $s_{\text {lef }}$ and $s_{\text {rig }}$ are regular; as otherwise we can replace them by resp. ${ }^{\text {rh }} s_{\text {lef }}$ and ${ }^{\mathrm{rh}} s_{\text {rig }}$ and use Lemma 4.4(i).

Define $a=\left\langle t_{\text {lef }}^{a}, t_{\text {rig }}^{a}\right\rangle$, where $t_{\text {lef }}^{a}$ consists of all pairs $\langle p, q\rangle \in s_{\text {lef }}$ such that
$p \mathbf{L S}_{\gamma}$-forces that $s_{\text {lef }}[\underline{G}]$ is $\mathbf{L S}_{\boldsymbol{\gamma}}$-generic and $\underline{G}=s_{\text {rig }}\left[s_{1 \text { ef }}[\underline{G}]\right]$,
and $t_{\text {rig }}^{a}$ consists of all pairs $\langle q, p\rangle \in s_{\text {rig }}$ such that
$q \mathbf{L S}_{\gamma}$-forces that $s_{\text {rig }}[\underline{G}]$ is $\mathbf{L S}_{\gamma}$-generic and $\underline{G}=s_{\text {lef }}\left[s_{\text {rig }}[\underline{G}]\right] ;$
then $\varnothing \neq t_{\text {lef }}^{a} \subseteq s_{\text {lef }}$ and $\varnothing \neq t_{\text {rig }}^{a} \subseteq s_{\text {rig }}$.
We claim that $a \in \mathbf{D N}_{\gamma}$, and still $G_{\text {rig }}=t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]$ and $G_{\text {lef }}=t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$.
Lemma 5.2. If $H_{\text {lef }}$ is an $\mathbf{L S}_{\gamma}$-generic set and $H_{\text {lef }} \cap \operatorname{dom} t_{1 \text { ef }}^{a} \neq \varnothing$ then $t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]=s_{\text {lef }}\left[H_{\text {lef }}\right]$. Similarly if $H_{\text {rig }}$ is an $\mathbf{L S}_{\gamma}$-generic set and $H_{\text {rig }} \cap$ $\operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$ then $t_{\text {rig }}^{a}\left[H_{\text {rig }}\right]=s_{\text {lef }}\left[H_{\text {rig }}\right]$.
Proof (lemma). By construction $t_{\text {lef }}^{a}\left[H_{\text {lef }}\right] \subseteq=s_{\text {lef }}\left[H_{\text {lef }}\right]$. Consider any $q \in$ $s_{\text {lef }}\left[H_{\text {lef }}\right]$, so that there is $p \in H_{\text {lef }}$ with $\langle p, q\rangle \in s_{\text {lef }}$. On the other hand, as $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$, there is a condition $p^{\prime} \in H_{\text {lef }}$ with $p \subseteq p^{\prime}$ which $\mathbf{L S}_{\gamma^{-}}$ forces that $s_{\text {lef }}[\underline{G}]$ is $\mathbf{L S}_{\gamma}$-generic and $\underline{G}=s_{\text {rig }}\left[s_{1 \text { ef }}[\underline{G}]\right]$. Then $\left\langle p^{\prime}, q\right\rangle \in t_{1 \text { ef }}^{a}$ by the regularity assumption, and we have $q \in t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]$.
$\square$ (Lemma)
Now to check 3.1](1) for $a$ let $H_{\text {lef }}$ be an $\mathbf{L S}_{\boldsymbol{\gamma}}$-generic set and $H_{\text {lef }} \cap$ $\operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$. Then $t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]=s_{\text {lef }}\left[H_{\text {lef }}\right]$ by the lemma. Therefore $H_{\text {rig }}=$ $t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]$ is $\mathbf{L S}_{\gamma}$-generic and $H_{\text {lef }}=s_{\text {rig }}\left[H_{\text {rig }}\right]$ by the definition of $t_{\text {lef }}^{a}$. Thus $s_{\mathrm{rig}}\left[H_{\mathrm{rig}}\right]$ is generic and $s_{\text {lef }}\left[s_{\text {rig }}\left[H_{\text {rig }}\right]\right]=H_{\text {rig }}$ by construction. This is forced by some $q \in H_{\text {rig }}$. On the other hand, as $H_{\text {lef }}=s_{\text {rig }}\left[H_{\text {rig }}\right] \neq \varnothing$, there exists some $q^{\prime} \in H_{\text {rig }} \cap \operatorname{dom} s_{\text {rig }}$. We can assume that $q^{\prime} \subseteq q$. Then $q \in \operatorname{dom} s_{\text {rig }}$, too, by the regularity assumption, and hence $q \in \operatorname{dom} t_{\text {rig }}^{a}$, and $H_{\text {rig }} \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$. We conclude that $t_{\text {rig }}^{a}\left[H_{\text {rig }}\right]=s_{\text {rig }}\left[H_{\text {rig }}\right]=H_{\text {lef }}$, by the lemma. Finally $t_{\text {rig }}^{a}\left[t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]\right]=H_{\text {lef }}$; this ends the verification of 3.1](1) for $a$.

Thus $a \in \mathbf{D N}_{\gamma}$. In addition, by the choice of $s_{\text {lef }}$ and $s_{\text {rig }}$, some $p \in G_{\text {lef }}$ forces that " $s_{\text {lef }}[\underline{G}]$ is generic and $\underline{G}=s_{\text {rig }}\left[s_{\text {lef }}[G]\right]$ ". Then $p \in \operatorname{dom} s_{\text {lef }}$, $p \in \operatorname{dom} t_{\text {lef }}^{a}, G_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$, and $t_{\text {lef }}^{a}\left[G_{\text {lef }}\right]=s_{\text {lef }}\left[G_{\text {lef }}\right]=G_{\text {rig }}$, as above. Similarly we have $G_{\text {lef }}=t_{\text {rig }}^{a}\left[G_{\text {rig }}\right]$.

To fix the regularity condition of the theorem, let $b={ }^{\mathrm{rh}} a$; then still $b \in \mathbf{D N}_{\gamma}$, $G_{\text {rig }}=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right], G_{\text {lef }}=t_{\text {rig }}^{b}\left[G_{\text {rig }}\right]$, and $b$ is regular, by Lemma 4.4.

It is not necessarily true, of course, that sets $\operatorname{dom} t_{\text {1ef }}^{b}$ and $\operatorname{dom} t_{\text {rig }}^{b}$ are dense. To fix this shortcoming, we define

$$
W=\left\{p \in \mathbf{L S}_{\gamma}: \forall q \in \operatorname{dom} t_{\text {lef }}^{b} \cup \operatorname{dom} t_{\mathrm{rig}}^{b}(p \text { is incompatible with } q)\right\}
$$

and let $c=\left\langle t_{\text {lef }}^{c}, t_{\text {rig }}^{c}\right\rangle$, where $t_{\text {lef }}^{c}=t_{\text {rig }}^{c}=t_{\text {lef }}^{b} \cup t_{\text {rig }}^{b} \cup\{\langle p, q\rangle: p \in W \wedge q \subseteq p\}$.
The set $\operatorname{dom} t_{\text {lef }}^{c}=\operatorname{dom} t_{\text {rig }}^{c}=\operatorname{dom} t_{\text {lef }}^{b} \cup \operatorname{dom} t_{\text {rig }}^{b} \cup W$ is dense in $\mathbf{L} \mathbf{S}_{\gamma}$ by construction. We claim that $c \in \mathbf{D N}_{\gamma}$. Indeed let $H_{\text {lef }} \subseteq \mathbf{L S}_{\gamma}$ be an $\mathbf{L S}_{\gamma^{-}}$ generic set. Then $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{c} \neq \varnothing$. But $\operatorname{dom} t_{\text {lef }}^{c}=\operatorname{dom} t_{\text {lef }}^{b} \cup \operatorname{dom} t_{\text {rig }}^{b} \cup W$.

Case 1: $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{b} \neq \varnothing$. Then $H_{\text {lef }} \cap \operatorname{dom} t_{\text {rig }}^{b}=\varnothing$ since if $p^{\prime} \in$ $\operatorname{dom} t_{\text {lef }}^{b}$ and $q^{\prime} \in \operatorname{dom} t_{\text {lef }}^{b}$ then $p^{\prime}, q^{\prime}$ are incompatible by the original choice of $p_{\text {lef }}, p_{\text {rig }}$. We also have $H_{\text {lef }} \cap W=\varnothing$ by obvious reasons. It follows that $t_{\text {lef }}^{c}\left[H_{\text {lef }}\right]=t_{\text {lef }}^{b}\left[H_{\text {lef }}\right]$, and hence $H_{\text {rig }}=t_{\text {lef }}^{c}\left[H_{\text {lef }}\right]$ is an $\mathbf{L S}_{\gamma}$-generic set and $H_{\text {lef }}=t_{\text {rig }}^{b}\left[H_{\text {rig }}\right]$, because $b \in \mathbf{D N}_{\gamma}$. In particular $H_{\text {rig }} \cap \operatorname{dom} t_{\text {lef }}^{b} \neq \varnothing$, so that $t_{\text {rig }}^{c}\left[H_{\text {rig }}\right]=t_{\text {rig }}^{b}\left[H_{\text {rig }}\right]$, as above.

Case 2: $H_{\text {lef }} \cap \operatorname{dom} t_{\text {rig }}^{b} \neq \varnothing$, similar.
Case 3: $H_{\text {lef }} \cap W \neq \varnothing$. Then $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{b}=H_{\text {lef }} \cap \operatorname{dom} t_{\text {rig }}^{b}=\varnothing$ as above. It follows that $t_{\text {lef }}^{c}\left[H_{\text {lef }}\right]=t_{\text {rig }}^{c}\left[H_{\text {lef }}\right]=H_{\text {lef }}$.

Thus indeed $c \in \mathbf{D N}_{\gamma}, t_{\text {lef }}^{c}=t_{\text {rig }}^{c}$, the set dom $t_{\text {lef }}^{c}=\operatorname{dom} t_{\text {rig }}^{c}$ is open dense in $\mathbf{L} \mathbf{S}_{\gamma}$, and the arguments above (Case 1) also imply that $G_{\text {rig }}=t_{\text {lef }}^{c}\left[G_{\text {lef }}\right]$, $G_{\text {lef }}=t_{\text {rig }}^{c}\left[G_{\text {rig }}\right]$. Moreover, $c$ inherits the regularity of $b$.

## 6 Extensions

Definition 6.1 (extension). Suppose that $a, b$ are double-names. We say that $b$ extends $a$, in symbol $a \leqslant b$, if just $t_{\text {lef }}^{a} \subseteq t_{\text {lef }}^{b}$ and $t_{\text {rig }}^{a} \subseteq t_{\text {rig }}^{b}$.

Lemma 6.2 (in $\mathbf{L}$ ). If $\beta<\gamma \leq \Omega$ and $a \in \mathbf{D N}_{\beta}$, then there is a double-name $b \in \mathbf{D N}_{\gamma}$ which extends $a$.

Proof. Let $t_{\text {lef }}^{b}$ consist of all pairs $\langle p \cup r, q \cup r\rangle$, where $\langle p, q\rangle \in t_{\text {lef }}^{a}$ and $r$ is a condition in $\mathbf{L S}{ }_{\gamma}$ satisfying $|r| \subseteq \gamma \backslash \beta$; let $t_{\text {rig }}^{b}$ be defined the same way.

This can be explained as follows. Suppose that $G_{\text {lef }} \subseteq \mathbf{L S} \boldsymbol{S}_{\gamma}$ is a $\mathbf{L S} \mathbf{S}_{\gamma}$-generic set containing $p_{\text {lef }}$. Then the factors $G_{\text {lef }}^{\prime}=G_{\text {lef }} \cap \mathbf{L} \mathbf{S}_{\beta}$ and $G_{\text {lef }}^{\prime \prime}=G_{\text {lef }} \cap$ $\mathbf{L S}_{\gamma \backslash \beta}$ are resp. $\mathbf{L S}_{\beta}$-generic and $\mathbf{L S} \mathbf{S}_{\gamma \backslash \beta}$-generic, and $G_{\text {lef }}$ can be identified with $G_{1 \text { ef }}^{\prime} \times G_{1 \text { ef }}^{\prime \prime}$ by the product forcing theorem. Then by definition the set $G_{\text {rig }}=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right]$ has the form $G_{\text {rig }}^{\prime} \times G_{\text {rig }}^{\prime \prime}$, where $G_{\text {rig }}^{\prime}=t_{\text {lef }}^{a}\left[G_{\text {lef }}^{\prime}\right]$ while simply $G_{\text {rig }}^{\prime \prime}=G_{\text {lef }}^{\prime \prime}$. The genericity of $G_{\text {rig }}$ easily follows.

Definition 6.3 (restriction). Let $\alpha<\beta \leq \Omega$. If $t \in \mathbf{L S}_{\beta}$ then define $t \upharpoonright \alpha=$ $t \cap\left(\mathbf{L S}_{\alpha} \times \mathbf{L S}_{\alpha}\right) ; t \upharpoonright \alpha \in \mathbf{N}_{\alpha}$. If $a \in \mathbf{D N}_{\beta}$, then let $a \upharpoonright \alpha=\left\langle t_{\text {lef }}^{a} \upharpoonright \alpha, t_{\text {rig }}^{a} \upharpoonright \alpha\right\rangle$.

It is not asserted that always $a \upharpoonright \alpha \in \mathbf{D N}_{\alpha}$ !
Lemma 6.4. If, in $\mathbf{L}, \alpha<\beta \leq \Omega, a \in \mathbf{D N}_{\alpha}, b \in \mathbf{D N}_{\beta}$, and $a \leqslant b$, then
(i) if $G_{\text {lef }} \subseteq \mathbf{L S}_{\beta}$ is an $\mathbf{L S}_{\beta}$-generic set then (a) $H_{\text {lef }}=G_{\text {lef }} \cap \mathbf{L S}_{\alpha}$ is $\mathbf{L S}_{\alpha}$ generic, and (b) if $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$ then $t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right] \cap \mathbf{L S} \mathbf{S}_{\alpha}$;
(ii) if $G_{\text {rig }} \subseteq \mathbf{L S} \mathbf{S}_{\beta}$ is an $\mathbf{L S}_{\beta}$-generic set then (a) $H_{\text {rig }}=G_{\mathrm{rig}} \cap \mathbf{L S _ { \alpha }}$ is $\mathbf{L S}_{\alpha^{-}}$ generic, and (b) if $G_{\text {rig }} \cap \operatorname{dom} t_{\text {rig }}^{a} \neq \varnothing$ then $t_{\text {rig }}^{a}\left[H_{\text {rig }}\right]=t_{\text {rig }}^{b}\left[G_{\text {rig }}\right] \cap \mathbf{L S} \mathbf{S}_{\alpha}$;
(iii) $c=b \upharpoonright \alpha$ belongs to $\mathbf{D N}_{\alpha}$ and $a \leqslant c \leqslant b$.

Proof. (i)(a) That $H_{\text {lef }}$ is generic holds by the product forcing theorem.
(i)(b) If $H_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$ then $G_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{b} \neq \varnothing$, and hence the sets $G_{\text {rig }}=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right]$ and $H_{\text {rig }}=t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]$ are generic sets in resp. $\mathbf{L S} \boldsymbol{S}_{\beta}$ and $\mathbf{L S} \mathbf{S}_{\alpha}$ by Lemma [3.2, and $H_{\text {rig }} \subseteq G_{\text {rig }}$ since $a \leqslant b$. Therefore $H_{\text {rig }} \subseteq H_{\text {rig }}^{\prime}=$ $G_{\text {rig }} \cap \mathbf{L S}_{\alpha}$. However $H_{\text {rig }}^{\prime}$ is $\mathbf{L S} \mathbf{S}_{\alpha}$-generic by the product forcing. Thus both $H_{\text {rig }} \subseteq H_{\text {rig }}^{\prime}$ are generic sets, hence easily $H_{\text {rig }}=H_{\text {rig }}^{\prime}$ as required.
(iii) To check 3.1](1) (a) for some $p \in \operatorname{dom} t_{\text {lef }}^{c}$, consider any $\mathbf{L S}_{\alpha}$-generic set $H_{\text {lef }} \subseteq \mathbf{L} \mathbf{S}_{\alpha}$ containing $p$ and extend it to a $\mathbf{L S} \boldsymbol{\beta}_{\beta}$-generic set $G_{\text {lef }} \subseteq \mathbf{L} \mathbf{S}_{\alpha}$ so that $H_{\text {lef }}=G_{\text {lef }} \cap \mathbf{L} \mathbf{S}_{\alpha}$. The (generic by Lemma (3.2) sets $H_{\text {rig }}=t_{\text {lef }}^{a}\left[H_{\text {lef }}\right]$ and $G_{\text {rig }}=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right]$ satisfy $H_{\text {rig }}=G_{\text {rig }} \cap \mathbf{L S} \beta$ by (i). On the other hand $H_{\text {rig }} \subseteq t_{\text {lef }}^{c}\left[H_{\text {lef }}\right] \subseteq G_{\text {rig }} \cap \mathbf{L S}_{\beta}$, hence $t_{\text {lef }}^{c}\left[H_{\text {lef }}\right]=H_{\text {rig }}$ is generic, as required. The verification of 3.1)(1) (b) also is very simple.

Lemma 6.5. In $\mathbf{L}$, assume that $\alpha<\beta \leq \Omega$. Then:
(i) if $s \in \mathbf{N}_{\alpha}, t \in \mathbf{N}_{\beta}$, and $s \subseteq t$, then ${ }^{\mathrm{rh}} s \subseteq{ }^{\mathrm{rh}} t$;
(ii) therefore if $a \in \mathbf{D N}_{\alpha}, b \in \mathbf{D N}_{\beta}$, and $a \leqslant b$, then ${ }^{\mathrm{rh}} a \leqslant{ }^{\mathrm{rh}} b$;
(iii) if $b \in \mathbf{D N}_{\beta}$ is regular and $a=b \upharpoonright \alpha \in \mathbf{D N}_{\alpha}$ then $a$ is regular, too.

Proof. (i) Suppose that $\left\langle p^{\prime}, q\right\rangle \in{ }^{\mathrm{rh}} s$, i.e., $p^{\prime}, q \in \mathbf{L S}_{\alpha}$ and there is a condition $p \subseteq p^{\prime}$ which $\mathbf{L S}_{\alpha}$-forces that $q \in t[\underline{G}]$. Prove that $p$ also $\mathbf{L S}_{\beta}$-forces $q \in t[\underline{G}]$. Let a set $G_{\text {lef }} \subseteq \mathbf{L} \mathbf{S}_{\beta}$ be a set $\mathbf{L} \mathbf{S}_{\beta}$-generic over $\mathbf{L}$ and containing $p$; prove that $q \in G_{\text {rig }}=t\left[G_{\text {lef }}\right]$. The set $H_{\text {lef }}=G_{\text {lef }} \upharpoonright \mathbf{L S}_{\alpha}$ is $\mathbf{L S}_{\alpha}$-generic by Lemma 6.4 and still $p \in H_{\text {lef }}$, hence $q \in s\left[H_{\text {lef }}\right] \subseteq t\left[G_{\text {lef }}\right]=G_{\text {rig }}$, as required.
(iii) Assume that $p, q, p^{\prime} \in \mathbf{L S}_{\alpha}, p \subseteq p^{\prime}$ and $p \mathbf{L S}_{\alpha}$-forces $q \in t_{1 \mathrm{ef}}^{a}[\underline{G}]$; we have to prove that $\left\langle p^{\prime}, q\right\rangle \in t_{\text {lef }}^{a}$. As $a=b \upharpoonright \alpha$, it suffices to show that $\left\langle p^{\prime}, q\right\rangle \in t_{\text {lef }}^{b}$. The same argument based on Lemma 6.4 shows that $p$ also $\mathbf{L S}_{\beta^{-}}$ forces $q \in t_{1 \text { ef }}^{a}[\underline{G}]$. Therefore $\left\langle p^{\prime}, q\right\rangle \in t_{\text {lef }}^{b}$ since $b$ is regular.

## 7 Increasing sequences

Suppose that a set $\Gamma \subseteq \mathbf{D N}$ is pairwise $\leqslant-$ compatible. Then define the doublename $A=\bigvee \Gamma$ by $t_{\text {lef }}^{A}=\bigcup_{a \in \Gamma} t_{\text {lef }}^{a}, t_{\text {rig }}^{A}=\bigcup_{a \in \Gamma} t_{\text {rig }}^{a}$.

Lemma 7.1 (in $\mathbf{L}$ ). (i) If $\lambda<\Omega$ is a limit ordinal and $\left\{a_{\xi}\right\}_{\xi<\lambda}$ is $a \leqslant-$ increasing sequence in $\mathbf{D N}$ then $A=\bigvee\left\{a_{\xi}: \xi<\lambda\right\}$ belongs to $\mathbf{D N}$;
(ii) therefore the set $\mathbf{D N}=\bigcup_{\gamma<\Omega} \mathbf{D N}_{\gamma}$ is $\Omega$-closed in the sense of $\leqslant$;
(iii) if $\left\{a_{\xi}\right\}_{\xi<\Omega}$ is a strictly $\leqslant$-increasing sequence in $\mathbf{D N}$ then the doublename $A=\bigvee\left\{a_{\xi}: \xi<\lambda\right\}$ belongs to $\mathbf{D N}_{\Omega}$.

Proof. (i) Suppose that $\left\{\gamma_{\xi}\right\}_{\xi<\lambda}$ is a strictly increasing sequence of ordinals $\gamma_{\xi}<\Omega$, and double-names $a_{\xi}=\left\langle t_{\text {1ef }}^{\xi}, t_{\text {rig }}^{\xi}\right\rangle \in \mathbf{D N}_{\gamma_{\xi}}$ form a strictly $\leqslant$-increasing sequence: if $\xi<\eta<\lambda$ then $t_{\text {1ef }}^{\xi} \subseteq t_{\text {lef }}^{\eta}$ and $t_{\text {rig }}^{\xi} \subseteq t_{\text {rig }}^{\eta}$. Let $t_{\text {lef }}^{A}=\bigcup_{\xi<\lambda} t_{\text {lef }}^{\xi}$, $t_{\text {rig }}^{A}=\bigcup_{\xi<\lambda} t_{\text {rig }}^{\xi}$, and $\gamma=\sup _{\xi<\lambda} \gamma_{\xi}$. We claim that $A=\left\langle t_{\text {lef }}^{A}, t_{\text {rig }}^{A}\right\rangle \in \mathbf{D N}_{\gamma}$.

Let's verify [3.1](1)] Assume that $G_{\text {lef }} \subseteq \mathbf{L S}_{\gamma}$ is a generic set containing some $p \in \operatorname{dom} t_{\text {lef }}^{A}$; we have to prove that $G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]$ is $\mathbf{L} \mathbf{S}_{\gamma^{-}}$-generic and $G_{\text {lef }}=t_{\text {rig }}^{A}\left[G_{\text {rig }}\right]$. Note first of all that each set $G_{\text {lef }}^{\xi}=G_{\text {lef }} \cap \mathbf{L S} \mathbf{\gamma}_{\xi}, \xi<\lambda$, is $\mathbf{L} \mathbf{S}_{\gamma_{\xi}}$-generic by the product forcing theorem, and $p$ belongs to some dom $t_{\text {lef }}^{a_{\zeta}}$, $\zeta<\Omega$. We can assume that $\zeta=0$ (otherwise simply cut all double-names $a_{\xi}$, $\xi<\zeta)$. Then $p \in \operatorname{dom} t_{1 \text { ef }}^{0}$, therefore $p \in \operatorname{dom} t_{\text {ef }}^{\xi}$ for all $\xi<\Omega$. It follows that each set $G_{\text {rig }}^{\xi}=t_{\text {1ef }}^{\xi}\left[G_{\text {ief }}^{\xi}\right] \subseteq \mathbf{L S}_{\gamma_{\xi}}$ is $\mathbf{L S}_{\gamma_{\xi}}$-generic, $G_{\text {rig }}^{\xi} \cap \operatorname{dom} t_{\text {rig }}^{\xi} \neq \varnothing$, and $G_{\text {1ef }}^{\xi}=t_{\text {rig }}^{\xi}\left[G_{\text {rig }}^{\xi}\right]$, by Lemma 3.2. And as $G_{\text {rig }}=\bigcup_{\xi<\lambda} G_{\mathrm{rig}}^{\xi}$, we conclude that at least $G_{\text {rig }}$ is a filter in $\mathbf{L S} \boldsymbol{S}_{\gamma}$ and $G_{\text {lef }}=t_{\text {rig }}^{A}\left[G_{\text {rig }}\right]$, that is, 3.1](1)(b).

To continue with 3.1](1) (a), we prove the $\mathbf{L S} \mathbf{S}_{\gamma}$-genericity of $G_{\text {rig }}$.
Let $D \subseteq \mathbf{L S} \mathbf{S}_{\gamma}$ be a dense subset of $\mathbf{L} \mathbf{S}_{\gamma}$, in $\mathbf{L}$. Assume towards the contrary that $G_{\text {rig }} \cap D=\varnothing$. Then there is a condition $p \in G_{\text {lef }}$ which $\mathbf{L S}_{\gamma}$-forces that $t_{\text {lef }}^{A}[\underline{G}] \cap D=\varnothing$. Then $p \in G_{1 \text { ef }}^{\xi}$ for some $\xi<\lambda$, and there is a condition $q \in G_{\text {rig }}^{\xi}$ which puts $p$ in $G_{\text {lef }}^{\xi}=t_{\text {rig }}^{\xi}\left[G_{\text {rig }}^{\xi}\right]$ in the sense that $\langle q, p\rangle \in t_{\text {rig }}^{\xi}$. As $D$ is dense, there is some $q^{\prime} \in D$ with $q \subseteq q^{\prime}$. Then $q^{\prime}$ belongs to some $\mathbf{L} \mathbf{S}_{\gamma_{\eta}}$, $\xi<\eta<\lambda$. By Corollary [3.3, there is a condition $p^{\prime} \in \mathbf{L} \mathbf{S}_{\gamma_{\eta}}$, compatible with $p$ and such that $\left\langle p^{\prime}, q^{\prime}\right\rangle \in t_{1 \text { ef }}^{\eta}$. Then $p^{\prime} \mathbf{L S}_{\gamma^{-}}$-forces $q^{\prime} \in t_{1 \text { ef }}^{\eta}[\underline{G}] \cap D$, while $p$, a compatible condition, forces the opposite, which is a contradiction.
(iii) Pretty similar argument.

Corollary $\mathbf{7 . 2}$ (in L). Assume that $c \in \mathbf{D N}_{\Omega}$. Then
(i) the set $\Xi=\left\{\gamma<\Omega: c \upharpoonright \gamma \in \mathbf{D N}_{\gamma}\right\}$ is a club in $\Omega$;
(ii) if $c$ is full (Definition 4.1) then $\Xi^{\prime}=\{\gamma \in \Xi: c \upharpoonright \gamma$ is full $\}$ is a club;
(iii) if $\Xi^{\prime \prime}=\{\gamma \in \Xi: c \upharpoonright \gamma$ is regular $\}$ is unbounded in $\Omega$ then $\Xi^{\prime \prime}=\Xi$.

Proof. (i) That $\Xi$ is closed follows from Lemma 7.1](i). To prove that $\Xi$ is unbounded, let $\alpha<\Omega$ and find a larger ordinal $\beta \in \Xi$.

Recall that to decide a sentence $\Phi$ means to force $\Phi$ or to force $\neg \Phi$.
By basic forcing theorems, if $p \in \mathbf{L S}$ then the set

$$
D_{p}=\left\{p \in \mathbf{L S}: p \text { decides } q \in t_{1 \mathrm{ef}}^{c}[\underline{G}] \text { and decides } q \in t_{\mathrm{rig}}^{c}[\underline{G}]\right\}
$$

is dense in $\mathbf{L S}$, therefore by the ccc property of $\mathbf{L S}$ there is an ordinal $\beta$, $\alpha<\beta<\Omega$, such that $D_{p}$ is dense in $\mathbf{L S} \mathbf{S}_{\beta}$ for all $p \in \mathbf{L S} \mathbf{S}_{\beta}$. Then $\beta \in \Xi$.
(ii) easily follows from (i). To prove (iii) apply Lemma 6.5) (iii),

## 8 Superpositions

Assume that $\gamma \leq \Omega$ and $a, c \in \mathbf{D N}_{\gamma}$. Define

$$
\begin{aligned}
t_{\text {lef }}^{a \cdot c} & =\left\{\left\langle p^{\prime}, q\right\rangle \in \mathbf{L} \mathbf{S}_{\gamma} \times \mathbf{L} \mathbf{S}_{\gamma}: \exists p \in \mathbf{L S}_{\gamma}\left(\left\langle p^{\prime}, p\right\rangle \in t_{\text {lef }}^{c} \wedge\langle p, q\rangle \in t_{\text {lef }}^{a}\right)\right\}, \\
t_{\mathrm{rig}}^{a \cdot c} & =\left\{\left\langle q, p^{\prime}\right\rangle \in \mathbf{L} \mathbf{S}_{\gamma} \times \mathbf{L} \mathbf{S}_{\gamma}: \exists p \in \mathbf{L S}_{\gamma}\left(\langle q, p\rangle \in t_{\mathrm{rig}}^{a} \wedge\left\langle p, p^{\prime}\right\rangle \in t_{\mathrm{rig}}^{c}\right)\right\} .
\end{aligned}
$$

and $a \cdot c=\left\langle t_{\text {lef }}^{a \cdot c}, t_{\text {rig }}^{a \cdot c}\right\rangle$.
Lemma 8.1. If $\gamma \leq \Omega, a, c \in \mathbf{D N}_{\gamma}$, and $G \subseteq \mathbf{L S}_{\gamma}$, then $t_{\text {lef }}^{a \cdot c}[G]=t_{\text {lef }}^{a}\left[t_{\text {lef }}^{c}[G]\right]$ and $t_{\text {rig }}^{a \cdot c}[G]=t_{\text {rig }}^{c}\left[t_{\text {rig }}^{a}[G]\right]$.
Proof. Assume that $q \in t_{1 \text { ef }}^{a \cdot c}[G]$. Then there is a pair $\left\langle p^{\prime}, q\right\rangle \in t_{1 \text { ef }}^{a \cdot c}$ with $p^{\prime} \in G$. By definition there is a condition $p$ such that $\left\langle p^{\prime}, p\right\rangle \in t_{\text {lef }}^{c}$ and $\langle p, q\rangle \in t_{\text {lef }}^{a}$. Then $p \in t_{\text {lef }}^{c}[G]$ and hence $q \in t_{\text {lef }}^{a}\left[t_{\text {lef }}^{c}[G]\right]$. To prove the converse assume that $q \in t_{1 \text { ef }}^{a}\left[t_{\text {lef }}^{c}[G]\right]$. Then there is a pair $\langle p, q\rangle \in t_{1 \text { ef }}^{a}$ with $p \in t_{\text {lef }}^{c}[G]$, and further there is a pair $\left\langle p^{\prime}, p\right\rangle \in t_{1 \mathrm{ef}}^{c}$ with $p^{\prime} \in G$. Then $p$ witnesses that $\left\langle p^{\prime}, q\right\rangle \in t_{\text {lef }}^{a \cdot c}$, and hence $q \in t_{1 \text { ef }}^{a \cdot c}[G]$.

Corollary 8.2. Assume that $\gamma<\Omega$ and $a, b, c \in \mathbf{L S} \mathbf{S}_{\gamma}$. If $a, b$ are equivalent (in the sense of Definition 4.3) then so are $a \cdot c$ and $b \cdot c$.
Lemma 8.3. If $\gamma \leq \Omega$ and $a, c \in \mathbf{D N}_{\gamma}$ then the following are equivalent:
(1) $\operatorname{ran} t_{\text {lef }}^{c} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$, (2) $r a n t_{\text {rig }}^{a} \cap \operatorname{dom} t_{\text {rig }}^{c} \neq \varnothing$, (3) $a \cdot c \in \mathbf{D N}_{\gamma}$.

Proof. Let ran $t_{\text {lef }}^{c} \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$. To prove (3) consider an $\mathbf{L S}_{\gamma}$-generic set $G^{\prime} \subseteq \mathbf{L S}_{\gamma}$, and let $p^{\prime} \in G^{\prime} \cap \operatorname{dom} t_{\text {lef }}^{a \cdot c}$. Then $p^{\prime} \in \operatorname{dom} t_{\text {lef }}^{c}$, hence $G=t_{\text {lef }}^{b}\left[G^{\prime}\right]$ is an $\mathbf{L S}_{\gamma}$-generic set by Lemma 3.2, As $p^{\prime} \in \operatorname{dom} t_{\text {lef }}^{a \cdot b}, G \cap \operatorname{dom} t_{\text {lef }}^{a} \neq \varnothing$. It follows that $H=t_{1 \text { ef }}^{a}[G]$ is an $\mathbf{L S}_{\gamma^{\prime}}$-generic set. Finally $H=t_{\text {lef }}^{a \cdot c}\left[G^{\prime}\right]$ by Lemma 8.1.

This argument also proves that $G^{\prime}=t_{r i g}^{a \cdot c}[H]$. Thus $(1) \Longrightarrow(3)$.
That $(3) \Longrightarrow(1)$ is obvious.

Corollary 8.4. If $\gamma \leq \Omega, a, c \in \mathbf{D N}_{\gamma}$, and $c$ is full (in the sense of Definition (4.1) then $a \cdot c \in \mathbf{D N}_{\gamma}$.
Proof. By Lemma 4.2, ran $t_{\text {lef }}^{c}=\operatorname{ran} t_{\text {rig }}^{c}=\mathbf{L S} \boldsymbol{S}_{\gamma}$. Now use Lemma 8.3,
Thus if $c \in \mathbf{D N}_{\gamma}$ is a full double-name then $a \mapsto a \cdot c$ is a map $\mathbf{D N}_{\gamma} \rightarrow$ $\mathbf{D N}_{\gamma}$. In this case, consider the inverse double-name $c^{-1}=\left\langle t_{\text {rig }}^{c}, t_{\text {lef }}^{c}\right\rangle$, let $a \in \mathbf{D N}_{\gamma}$, and compare $a$ with $a^{\prime}=a \cdot c \cdot c^{-1}$. On the one hand, we have $t_{1 \text { ef }}^{a^{\prime}}[G]=t_{1 \text { ef }}^{a}\left[t_{\text {lef }}^{c}\left[t_{1 \text { ef }}^{c^{-1}}[G]\right]\right]$ for any $\mathbf{L S}_{\gamma^{-}}$-generic set $G$ by Lemma 8.1. It follows that $t_{\text {lef }}^{a^{\prime}}[G]=t_{\text {lef }}^{a}\left[t_{\text {lef }}^{c}\left[t_{\text {rig }}^{c}[G]\right]\right]=t_{\text {lef }}^{a}[G]$ since the successive action of $t_{\text {lef }}^{c}$ and $t_{\text {rig }}^{c}$ is the identity by Lemma 3.2, Similarly $t_{\text {rig }}^{a^{\prime}}[G]=t_{\text {rig }}^{a}[G]$. Therefore $a$ and $a^{\prime}$ are equivalent, and hence ${ }^{\mathrm{rh}} a={ }^{\mathrm{rh}} a^{\prime}$ by Lemma 4.4, but generally speaking we cannot assert that straightforwardly $a=a^{\prime}$.

To fix this problem, define the modified action $a * c={ }^{\mathrm{rh}}(a \cdot c)$.
Lemma 8.5. Let $\gamma<\Omega$ and let $c \in \mathbf{D N}_{\gamma}$ be a full double-name. If $a \in \mathbf{D N}_{\gamma}$ is regular (that is, $a={ }^{\mathrm{rh}} a$ ) then $b=a * c \in \mathbf{D N}_{\gamma}, b$ is regular, and $a=b * c^{-1}$.

Proof. That $b \in \mathbf{D N}_{\gamma}$ follows from Corollary 8.4. The regularity holds by Lemma 4.4. To prove $a=b * c^{-1}$, note that both $a$ and $b * c^{-1}$ are regular double-names, and hence it suffices, by Lemma 4.4, to prove that $a$ and $b * c^{-1}$ are equivalent. However, still by Lemma 4.4, $b * c^{-1}$ is equivalent to $b \cdot c^{-1}$, and $b=a * c$ is equivalent to $a \cdot c$, hence overall $b * c^{-1}$ is equivalent to $a \cdot c \cdot c^{-1}$ by Corollary 8.2. Finally $a$ is equivalent to $a \cdot c \cdot c^{-1}$, see above.

Lemma 8.6. Assume that $\gamma<\delta \leq \Omega, c \in \mathbf{D N}_{\gamma}$ and $d \in \mathbf{D N}_{\delta}$ are full double-names, $c=d \upharpoonright \gamma$, and $a \in \mathbf{D N}_{\gamma}, b \in \mathbf{D N}_{\delta}$. Then
(i) if $a \leqslant b$ then $a \cdot c \leqslant b \cdot d$;
(ii) if $a, b$ are regular then $a \leqslant b$ iff $a * c \leqslant b * d$.

Proof. (i) is clear since $a \cdot c$ is monotone on both $a$ and $c$. As for (ii), the implication $\Longrightarrow$ holds by (i) and Lemma 6.5 while to prove the inverse make use of Lemma 8.5.

## 9 Generic double-names and product forcing

By Lemma 7.1, we can consider the set $\mathbf{D N}=\bigcup_{\gamma<\Omega} \mathbf{D} \mathbf{N}_{\gamma}$ ordered by $\leqslant$ as an $\Omega$-closed forcing notion in $\mathbf{L}$ ( $\leqslant$-bigger double-names are stronger conditions). Suppose that $\Gamma \subseteq \mathbf{D N}$ is a $\mathbf{D N}$-generic set over $\mathbf{L}$. Then a double-name $A=$ $\bigvee \Gamma \in \mathbf{L}[\Gamma]$ can be defined as in Section 7, we call such double-names $A=\bigvee \Gamma$ generic over $\mathbf{L}$ (together with the background generic sets $\Gamma$ ).

Let $\underline{\Gamma}$ and $\underline{A}$ be canonical DN-names of resp. $\Gamma$ and $A=\bigvee \Gamma$.

Remark 9.1. As $\mathbf{L}$ is our default ground model unless otherwise specified, the sets $\Gamma$ and $A=\bigvee \Gamma$ do not belong to $\mathbf{L}$, however all reals and generally all sets $x \subseteq \gamma<\Omega$ in $\mathbf{L}[\Gamma]$ belong to $\mathbf{L}$ by Lemma [7.1. It follows that the definition of $\mathbf{D} \mathbf{N}_{\gamma}(\gamma<\Omega)$ in $\mathbf{L}$ is absolute for $\mathbf{L}[\Gamma]$. That is, if $a \in \mathbf{D} \mathbf{N}_{\gamma}$ in $\mathbf{L}$ then it is true in $\mathbf{L}[\Gamma]$ that $a \in \mathbf{D N}_{\gamma}$. And conversely, if $a \in \mathbf{L}[\Gamma]$ and it is true in $\mathbf{L}[\Gamma]$ that $a \in \mathbf{D N}_{\gamma}$ then $a \in \mathbf{L}$ and it is true in $\mathbf{L}$ that $a \in \mathbf{D N}_{\gamma}$.

Corollary 9.2. Assume that $\Gamma$ is $\mathbf{D N}$-generic over $\mathbf{L}$ and $A=\bigvee \Gamma$. Then
(i) it holds in $\mathbf{L}[\Gamma]$ that $A$ belongs to $\mathbf{D N}_{\Omega}$;
(ii) if $G_{\text {lef }}$ is $\mathbf{L S}$-generic over $\mathbf{L}[\Gamma]$, and $G_{\text {lef }} \cap \operatorname{dom} t_{\text {lef }}^{A} \neq \varnothing$, then $G_{\text {rig }}$ is $\mathbf{L S}$-generic over $\mathbf{L}[\Gamma]$ and $G_{\text {lef }}=t_{\text {rig }}^{A}\left[G_{\text {rig }}\right]$;
(iii) if $a \in \mathbf{D N}, a \subseteq A$, and $\gamma=|a|$ then $A \upharpoonright \gamma \in \mathbf{D N}_{\gamma} \cap \Gamma$ and $a \leqslant A \upharpoonright \gamma \leqslant A$.

Proof. (i) Remark 9.1 allows simply to refer to Lemma 7.1 .
(ii) Make use of Lemma 3.2.
(iii) To prove that $a^{\prime}=A \upharpoonright \gamma \in \mathbf{D N}_{\gamma}$ and $a \leqslant a^{\prime} \leqslant A$ refer to Lemma 6.4)(iii), To prove that $a^{\prime} \in \Gamma$ note that by Lemma 7.1 there is some $c \in \Gamma$ which decides each $b \in \mathbf{L} \mathbf{S}_{\gamma}$ to belong or not to belong to $\Gamma$; then $a^{\prime} \subseteq c$.

## 10 The first ingredient

Generic double-names and forcing with $\mathbf{L S} \times \mathbf{D N}$ enable us to carry out the first main step towards Theorem 1.1.

In $\mathbf{L}$, let $\mathbf{H} \Omega$ be the set of all sets $x$ such that the transitive closure $\operatorname{TC}(x)$ has cardinality $\operatorname{card}(\operatorname{TC}(x))<\Omega$ strictly.

Blanket assumption 10.1. Thus suppose that $G_{0} \subseteq \mathbf{L S}$ is a LS-generic set over $\mathbf{L}$, let $\mathscr{X} \in \mathbf{L}\left[G_{0}\right]$, and it is true in $\mathbf{L}\left[G_{0}\right]$ that $\mathscr{X}$ is a countable OD non-empty set of sets of reals. There is a formula $\varphi(\cdot, \pi)$ with some $\pi \in$ Ord as the only parameter, such that it is true in $\mathbf{L}\left[G_{0}\right]$ that $\mathscr{X}$ is the only set $x$ satisfying $\varphi(x, \pi)$.

There is a sequence $u=\left\{U_{n}\right\}_{n \in \omega} \in \mathbf{L}$ of names $U_{n} \in \mathbf{L}$, such that $\mathscr{X}=$ $u \llbracket G_{0} \rrbracket:=\left\{U_{n} \llbracket G_{0} \rrbracket: n \in \omega\right\}$. Each $U_{n}$ can be assumed to be an LS-name of a set of reals, that is, in $\mathbf{L}, U_{n} \subseteq \mathbf{L S} \times \mathbb{T}$, where $T$ is the set of all LS-names for reals. Furthermore, according to the $\Omega$-cc property of the forcing LS, each LSname for a real can be assumed to be a set in $\mathbf{H} \Omega$. Therefore we shall wlog assume that $U_{n} \subseteq \mathbf{H} \Omega$ for all $n$.

Anyway there is a condition $\bar{p} \in G_{0}$ which $\mathbf{L S}$-forces over $\mathbf{L}$ that "u $[\underline{G} \rrbracket$ is the only set $x$ satisfying $\varphi(x, \pi)$, and $« \llbracket \underline{G} \rrbracket$ is a set of sets of reals". Let $\bar{\gamma}<\Omega$ be the least ordinal satisfying $\bar{p} \in \mathbf{L S}_{\bar{\gamma}}$.

Let a $\bar{p}$-pair be any pair $\langle p, a\rangle \in \mathbf{L S} \times \mathbf{D N}$ such that $\bar{p} \subseteq p \in \operatorname{dom} t_{\text {lef }}^{a}$ and $p$ $\mathbf{L S}_{\gamma}$-forces that $\bar{p} \in t_{\text {lef }}^{a}[\underline{G}]$, where $\gamma=|a|$.

Remark 10.2. Let $\bar{a}=\mathbf{i d}[\bar{\gamma}]$. Then $\langle\bar{p}, \bar{a}\rangle$ is a $\bar{p}$-pair; $\bar{p} \mathbf{L} \mathbf{S}_{\bar{\gamma}}$-forces that $t_{\text {lef }}^{\bar{a}}[\underline{G}]=\underline{G}$.

Lemma 10.3. Let $\langle p, a\rangle \in \mathbf{L S} \times \mathbf{D N}$ be a $\bar{p}$-pair, $q \in \mathbf{L S}, b \in \mathbf{D N}, p \subseteq q$, $a \leqslant b$. There is a double-name $c \in \mathbf{D N}$ such that $b \leqslant c$ and $\langle q, c\rangle$ is a $\bar{p}$-pair.
Proof. If $q \in \mathbf{L S}_{\gamma}$, where $\gamma=|b|$, then to define $c$ add to $t_{\text {lef }}^{b}$ all pairs $\langle q, r\rangle$ such that already $\langle p, r\rangle \in b$. We claim that $\langle q, c\rangle$ is a $\bar{p}$-pair. Indeed if $G_{\text {lef }} \subseteq \mathbf{L S}_{\gamma}$ is generic then easily $(*) t_{\text {lef }}^{c}\left[G_{\text {lef }}\right]=t_{\text {lef }}^{b}\left[G_{\text {lef }}\right]$, hence $c \in \mathbf{D N}_{\gamma}$. Further $\bar{p} \subseteq p \subseteq q \in \operatorname{dom} t_{\text {lef }}^{c}$ by construction. Finally $q \mathbf{L S} \mathbf{S}_{\gamma}$-forces that $\bar{p} \in t_{\text {lef }}^{a}[\underline{G}]$ because so does $p$, and we can replace $t_{\text {lef }}^{a}$ by $t_{\text {lef }}^{c}$ since $a \subseteq b \subseteq c$.

If $q \notin \mathbf{L} \mathbf{S}_{\gamma}$ then still $q \in \mathbf{L} \mathbf{S}_{\delta}$ for some $\delta, \gamma<\delta<\Omega$. Use Lemma 6.2 to get a double-name $b^{\prime} \in \mathbf{D N}_{\delta}$ with $b \leqslant b^{\prime}$, and argue as in the first case.

Theorem 10.4. Suppose that $G_{\text {lef }} \times \Gamma$ is a $\mathbf{L S} \times \mathbf{D N}$-generic set over $\mathbf{L}$, $A=\bigvee \Gamma$, and $\langle p, a\rangle \in G_{\text {lef }} \times \Gamma$ is a $\bar{p}$-pair. Then
(i) $p, \bar{p} \in G_{\text {lef }}, \bar{p} \in G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]$, and $G_{\text {rig }}$ is $\mathbf{L S}$-generic over $\mathbf{L}[\Gamma]$;
(ii) $u \llbracket G_{\text {lef }} \rrbracket=u \llbracket G_{\text {rig }} \rrbracket-$ in other words, any $\bar{p}$-pair $\langle p, a\rangle(\mathbf{L S} \times \mathbf{D N})$-forces $u \llbracket \underline{G} \rrbracket=u \llbracket t_{l_{\mathrm{ef}}}^{\underline{G}}[\underline{G}] \rrbracket$ over $\mathbf{L}$.

Proof. (i) To prove the genericity apply Corollary 9.2 ,
To prove (ii) suppose otherwise. Then there is a pair $\langle q, b\rangle$ in $\mathbf{L S} \times \mathbf{D N}$ with $p \subseteq q, a \leqslant b$, which $(\mathbf{L S} \times \mathbf{D N})$-forces $u \llbracket \underline{G} \rrbracket \neq u \llbracket t \frac{A}{l_{\mathrm{ef}}[\underline{G}] \rrbracket \text {, that is }}$
$(\dagger)$ if $G_{\text {lef }} \times \Gamma$ is a $(\mathbf{L S} \times \mathbf{D N})$-generic set over $\mathbf{L}$ containing $\langle q, b\rangle, A=\bigvee \Gamma$,


Let $\mathcal{L} \in \mathbf{L}$ be an elementary submodel of a large model, such that $\mathbf{H} \Omega \subseteq \mathcal{L}$, $\Omega$ and $\pi$ belong to $\mathcal{L}, \operatorname{card}(\mathcal{L})=\Omega$ in $\mathbf{L}$, and $\mathcal{L}$ is an elementary submodel of $\mathbf{L}$ v.r.t. all $\Sigma_{100}$ formulas. Let $\mathcal{L}^{\prime} \in \mathbf{L}$ be the Mostowski collapse of $\mathcal{L}$; still $\operatorname{card}\left(\mathcal{L}^{\prime}\right)=\Omega$ in $\mathbf{L}$. Note that $\mathcal{L}^{\prime}$ is a transitive model of Zermelo with choice, and the collapse $\operatorname{map} \phi: \mathcal{L} \xrightarrow{\text { onto }} \mathcal{L}^{\prime}$ is the identity on $\mathbf{H} \Omega$, hence even on $\mathscr{P}(\mathbf{H} \Omega) \cap \mathcal{L}$. In particular, $\phi(\Omega)=\Omega, \phi(u)=u, \phi\left(U_{n}\right)=U_{n}$ for all $n$, $\phi(\mathbf{L S})=\mathbf{L S}, \phi(\mathbf{D N})=\mathbf{D N}, \mathbf{H} \Omega \subseteq \mathcal{L}^{\prime}$, and even $\mathscr{P}(\mathbf{H} \Omega) \cap \mathcal{L} \subseteq \mathcal{L}^{\prime}$.

By the elementary submodel property, $\langle q, b\rangle$ still $(\mathbf{L S} \times \mathbf{D N})$-forces over $\mathcal{L}^{\prime}$ that $u \llbracket \underline{G} \rrbracket \neq u \llbracket t t_{\underline{\text { lef }}}[\underline{G}] \rrbracket$ - that is
$(\ddagger)$ if $G_{\text {lef }} \times \Gamma$ is a $(\mathbf{L S} \times \mathbf{D N})$-generic set over $\mathcal{L}^{\prime}$ containing $\langle q, b\rangle, A=\bigvee \Gamma$, and $G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]$, then $\llbracket \llbracket G_{\text {lef }} \rrbracket \neq u \llbracket G_{\text {rig }} \rrbracket$.

To infer a contradiction, note that since $\operatorname{card}\left(\mathcal{L}^{\prime}\right)=\Omega$ in $\mathbf{L}$, by Lemma 7.1] there exists a set $\Gamma \in \mathbf{L}, \mathbf{D N}$-generic over $\mathcal{L}^{\prime}$ and containing $b$, hence containing $a$ as well. We underline that $\Gamma \in \mathbf{L}$, and then $A=\bigvee \Gamma$ belongs to $\mathbf{L}$, too. Let $G_{\text {lef }} \subseteq \mathbf{L S}$ be a set $\mathbf{L S}$-generic over $\mathbf{L}$, hence over $\mathcal{L}^{\prime}[\Gamma]$ as well, and containing $q$, and then containing $p$. Then the set $G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]$ is $\mathbf{L S}$-generic over $\mathbf{L}$ and over $\mathcal{L}^{\prime}[\Gamma]$ by Lemma 3.2, and in addition, u $\llbracket G_{\text {lef }} \rrbracket \neq u \llbracket G_{\text {rig }} \rrbracket$ by ( $\ddagger$ ),

Recall that $\langle p, a\rangle$ also belongs to $G_{\text {lef }} \times A$. Therefore $\bar{p} \in G_{\text {lef }} \cap G_{\text {rig }}$ by (i). Thus $G_{\text {lef }}$ and $G_{\text {rig }}$ are LS-generic sets over $\mathbf{L}$ and both contain $\bar{p}$, $u\left[G_{\text {lef }}\right]$ is the only set $x$ satisfying $\varphi(x, \pi)$ in $\mathbf{L}\left[G_{1 \text { ef }}\right]$ while $u\left[G_{\text {rig }}\right]$ is the only set $x$ satisfying $\varphi(x, \pi)$ in $\mathbf{L}\left[G_{\text {rig }}\right]$. However $\mathbf{L}\left[G_{\text {lef }}\right]=\mathbf{L}\left[G_{\text {rig }}\right]$ (because $G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right], G_{\text {lef }}=t_{\text {rig }}^{A}\left[G_{\text {rig }}\right]$, and $\left.A \in \mathbf{L}\right)$, while on the other hand $u \llbracket G_{\text {lef }} \rrbracket \neq u \llbracket G_{\text {rig }} \rrbracket$, which is a contradiction.

## 11 Stabilizing pairs and second ingredient

Let a stabilizing $\bar{p}$-pair be any $\bar{p}$-pair $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ which, for some $n$, $(\mathbf{L S} \times \mathbf{D N})$-forces $U_{0} \llbracket \underline{G} \rrbracket=U_{n} \llbracket t \frac{A}{1 \mathrm{ef}}[\underline{G}] \rrbracket$ over $\mathbf{L}$.

Corollary 11.1. If $G_{\text {lef }}$ is an $\mathbf{L S}$-generic set over $\mathbf{L}$ containing $\bar{p}$, then there is a stabilizing $\bar{p}$-pair $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ with $\hat{p} \in G_{\text {1ef }}$.

Proof. Let $\bar{a}=\mathbf{i d}[\bar{\gamma}]$, see Remark 10.2, Let $\Gamma \subseteq \mathbf{D N}$ be a set $\mathbf{D N}$-generic over $\mathbf{L}\left[G_{\text {lef }}\right]$ and containing $\bar{a}$, so that $G_{\text {lef }} \times \Gamma$ is $(\mathbf{L S} \times \mathbf{D N})$-generic. Let $A=\bigvee \Gamma$. Then the set $G_{\text {rig }}=t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]$ satisfies $u \llbracket G_{\text {lef }} \rrbracket=u \llbracket G_{\text {rig }} \rrbracket$ by Theorem 10.4, Therefore there is a number $n \in \omega$ such that $U_{0} \llbracket G_{\text {lef }} \rrbracket=U_{n} \llbracket G_{\text {rig }} \rrbracket$. Then there is a stronger pair $\langle\hat{p}, \hat{a}\rangle \in G_{\text {lef }} \times \Gamma(\bar{p} \subseteq \hat{p}$ and $\bar{a} \leqslant \hat{a})$ which ( $\mathbf{L S} \times \mathbf{D N}$ )-forces $U_{0} \llbracket \underline{G} \rrbracket=U_{n} \llbracket t \frac{A}{1 \mathrm{ef}}[\underline{G} \rrbracket \rrbracket$. We can assume that $\langle\hat{p}, \hat{a}\rangle$ is a $\bar{p}$-pair, by Lemma 10.3 ,
Proposition 11.2. Let $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ be a stabilizing $\bar{p}$-pair. Assume that $G_{\text {lef }} \times \Gamma, G_{\text {lef }}^{\prime} \times \Gamma^{\prime}$ are sets $(\mathbf{L S} \times \mathbf{D N})$-generic over $\mathbf{L}$ and containing $\langle\hat{p}, \hat{a}\rangle$, $A=\bigvee \Gamma, A^{\prime}=\bigvee \Gamma^{\prime}$, and $t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]=t_{\text {lef }}^{A^{\prime}}\left[G_{\text {lef }}^{\prime}\right]$. Then $U_{0} \llbracket G_{\text {lef }} \rrbracket=U_{0} \llbracket G_{\text {lef }}^{\prime} \rrbracket$.
Proof. By definition, $U_{0} \llbracket G_{\text {lef }} \rrbracket=U_{n} \llbracket t_{\text {lef }}^{A}\left[G_{\text {lef }}\right] \rrbracket$ and $U_{0} \llbracket G_{\text {1ef }}^{\prime} \rrbracket=U_{n} \llbracket t_{\text {lef }}^{A^{\prime}}\left[G_{1 \text { ef }}^{\prime} \rrbracket \rrbracket\right.$ for one and the same $n$.

The second ingredient in the proof of Theorem 1.1 will be the following:
Theorem 11.3. Assume that $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ is a stabilizing $\bar{p}$-pair, $\hat{\gamma}<\Omega$, $\hat{a} \in \mathbf{D N}_{\hat{\gamma}}, \hat{p} \in \mathbf{L S}_{\hat{\gamma}}, G_{\text {lef }}, G_{\text {lef }}^{\prime} \subseteq \mathbf{L S}$ are $\mathbf{L S}$-generic sets over $\mathbf{L}$ containing $\hat{p}$, $G_{\text {lef }} \cap \mathbf{L S}_{\hat{\gamma}}=G_{\text {lef }}^{\prime} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}$, and $\mathbf{L}\left[G_{\text {lef }}\right]=\mathbf{L}\left[G_{\text {lef }}^{\prime}\right]$. Then $U_{0} \llbracket G_{\text {lef }} \rrbracket=U_{0} \llbracket G_{1 \mathrm{ef}}^{\prime} \rrbracket$.

Let's show how this implies Theorem 1.1. The proof of Theorem 11.3 itself will follow in the next sections.

Proof (Theorem 1.1 from Theorem 11.3). We argue in the assumptions and notation of 10.1. Let $G_{\text {lef }}=G_{0}$, so that $\bar{p} \in G_{\text {lef }}$ by 10.1. Then by Corollary 11.1, there is a stabilizing $\bar{p}$-pair $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ such that $\hat{p} \in G_{\text {lef }}$. Pick $\hat{\gamma}<\Omega$ such that $\hat{a} \in \mathbf{D N}_{\hat{\gamma}}$ and $\hat{p} \in \mathbf{L} \mathbf{S}_{\hat{\gamma}}$. Consider, in $\mathbf{L}\left[G_{\text {lef }}\right]$, the set $\mathscr{G}$ of all sets $G \subseteq \mathbf{L S}, \mathbf{L S}$-generic over $\mathbf{L}$ and satisfying $\mathbf{L}[G]=\mathbf{L}\left[G_{1 \text { ef }}\right], \hat{p} \in G$, and $G \cap \mathbf{L S}_{\hat{\gamma}}=G_{\text {lef }} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}$. In particular $G_{\text {lef }} \in \mathscr{G}$. The only essential parameter of the definition of $\mathscr{G}$ which is not immediately OD - is $G_{\text {lef }} \cap \mathbf{L S} \boldsymbol{\gamma}_{\hat{\gamma}}$. However $G_{\text {lef }} \cap \mathbf{L S} \mathbf{S}_{\hat{\gamma}}$ itself, as basically any subset of any $\mathbf{L} \mathbf{S}_{\gamma}, \gamma<\Omega$, is ROD in the Solovay model. We conclude that $\mathscr{G}$ is ROD in $\mathbf{L}\left[G_{\text {lef }}\right]$.

On the other hand, suppose that $G \in \mathscr{G}$. Then $U_{0} \llbracket G_{1 \text { ef }} \rrbracket=U_{0} \llbracket G \rrbracket$ by Theorem 11.3 . Therefore the set $U_{0} \llbracket G_{1 \text { ef }} \rrbracket$ can be defined as $U_{0} \llbracket G \rrbracket$ for some / every $G \in \mathscr{G}$. This witnesses that $U_{0} \llbracket G_{\text {lef }} \rrbracket$ is ROD in $\mathbf{L}\left[G_{\text {lef }}\right]$, because so is $\mathscr{G}$ by the above. Thus the set $\mathscr{X}=u \llbracket G_{\text {lef }} \rrbracket$ contains a ROD element. It follows that $\mathscr{X}$ contains an OD element, by Lemma 2.3, as required.
(Thm 1.1 mod Thm 11.3)

## 12 Final

Here we prove Theorem 11.3 and finally prove Theorem 1.1. We argue in the assumptions and notation of Theorem 11.3. That is,
(1) $\langle\hat{p}, \hat{a}\rangle \in \mathbf{L S} \times \mathbf{D N}$ is a stabilizing $\bar{p}$-pair, $\hat{\gamma}<\Omega, \hat{a} \in \mathbf{D N}_{\hat{\gamma}}, \hat{p} \in \mathbf{L} \mathbf{S}_{\hat{\gamma}}$, the sets $G_{\text {lef }}, G_{\text {lef }}^{\prime} \subseteq \mathbf{L S}$ are $\mathbf{L S}$-generic over $\mathbf{L}$ and both contain $\hat{p}$, and in addition $G_{\text {lef }} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}=G_{\text {lef }}^{\prime} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}, \mathbf{L}\left[G_{\text {lef }}\right]=\mathbf{L}\left[G_{\text {lef }}^{\prime}\right]$.

In this assumption, we have to prove that $U_{0} \llbracket G_{\text {lef }} \rrbracket=U_{0} \llbracket G_{1 \text { ef }}^{\prime} \rrbracket$. Working towards this goal, our plan will be to find:
$(*)$ sets $\Gamma, \Gamma^{\prime} \subseteq \mathbf{D N}, \quad \mathbf{D N}$-generic over $\mathbf{L}\left[G_{\text {lef }}\right]=\mathbf{L}\left[G_{\text {lef }}^{\prime}\right]$, containing $\hat{a}$, and satisfying $t_{\text {lef }}^{A}\left[G_{\text {lef }}\right]=t_{\text {lef }}^{A^{\prime}}\left[G_{\text {lef }}^{\prime}\right]$, where $A=\bigvee \Gamma$ and $A^{\prime}=\bigvee \Gamma^{\prime}$;
then the products $G_{\text {lef }} \times \Gamma$ and $G_{\text {lef }}^{\prime} \times \Gamma^{\prime}$ will be $(\mathbf{L S} \times \mathbf{D N})$-generic over $\mathbf{L}$ and containing $\langle\hat{p}, \hat{a}\rangle$, so that $U_{0} \llbracket G_{1 \text { ef }} \rrbracket=U_{0} \llbracket G_{\text {1ef }}^{\prime} \rrbracket$ follows by Proposition 11.2 , accomplishing the proof of Theorem 11.3,

By Theorem 5.1 there is a double-name $C \in \mathbf{D N}_{\Omega}$ in $\mathbf{L}$, such that
(2) $C$ is full, $t_{\text {lef }}^{C}=t_{\text {rig }}^{C}, G_{\text {lef }}=t_{\text {lef }}^{C}\left[G_{1 \mathrm{ef}}^{\prime}\right]$, and $G_{\text {lef }}^{\prime}=t_{\text {rig }}^{C}\left[G_{\text {lef }}\right]$.

As $G_{\text {lef }} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}=G_{\text {lef }}^{\prime} \cap \mathbf{L} \mathbf{S}_{\hat{\gamma}}$, we can further assume that
(3) the restricted double-name $C \upharpoonright \hat{\gamma}$ coincides with $\mathbf{i d}[\hat{\gamma}]$ of Example 4.5, so that $C \upharpoonright \hat{\gamma} \in \mathbf{L} \mathbf{S}_{\hat{\gamma}}$ is full and regular, and $t_{\text {lef }}^{C \mid \hat{\gamma}}[G]=t_{\text {rig }}^{C \mid \hat{\gamma}}[G]=G$ for all $G$.

Let $\Gamma$ be any set $\Gamma \subseteq \mathbf{D N}$ with $\hat{a} \in \Gamma$, DN-generic over $\mathbf{L}\left[G_{\text {lef }}\right]$. Then $A=\bigvee \Gamma \in \mathbf{D N}_{\Omega}$ in $\mathbf{L}[\Gamma]$ by Corollary 9.2 , and $\bar{p} \subseteq \hat{p} \in \operatorname{dom} A$ since $\hat{a} \in \Gamma$.

Corollary 12.1. (i) The set $X=\left\{\gamma<\Omega: A \upharpoonright \gamma \in \mathbf{D N}_{\gamma}\right\} \in \mathbf{L}[\Gamma]$ is a club in $\Omega$, and if $\gamma \in X$ then $A \upharpoonright \gamma$ is regular;
(ii) the set $Y=\left\{\gamma<\Omega: C \upharpoonright \gamma \in \mathbf{D N}_{\gamma}\right.$ and $C \upharpoonright \gamma$ is full $\} \in \mathbf{L}$ is a club in $\Omega$;
(iii) therefore $Z=\{\gamma \in X \cap Y: \hat{\gamma} \leq \gamma\}$ is a club, and in addition $\hat{\gamma} \in Z$.

Proof. To prove (i) and (ii) apply Corollary 7.2, the unboundedness condition in 7.2(iii) follows from the genericity of $\Gamma$ and the density of the set of all regular double-names $a \in \mathbf{D N}$ by Lemma 4.4)(ii).

Claim $\hat{\gamma} \in Z$ in (iii) follows from (3).
Now suppose that $\gamma \in Y$, hence $C \upharpoonright \gamma \in \mathbf{D N}_{\gamma}$ is full. Let $a \in \mathbf{D N}_{\gamma}$ be regular. Define $a * C=a *(C \upharpoonright \gamma)$ (see Section [8).

Lemma 12.2. The map $a \mapsto a * C$ is $a \leqslant$-preserving bijection of the set $\mathbf{D N}_{\text {reg }}^{Y}=\{a \in \mathbf{D N}: a$ is regular $\wedge|a| \in Y\}$ onto itself, satisfying $a * C * C=a$.

Proof. If $a \in \mathbf{D N}_{\text {reg }}^{Y}$ and $\gamma=|a|$ then $a * C=a *(C \upharpoonright \gamma)$ belongs to $\mathbf{D N}_{\gamma}$ and is regular by Lemma 8.5, hence $a * C \in \mathbf{D N}_{\text {reg }}^{Y}$. If $\delta>\gamma$ is a bigger ordinal still in $Y$, and $b \in \mathbf{D N}_{\text {reg }}^{Y}, \delta=|b|$, then $a \leqslant b$ iff $a * C \leqslant b * C$ by Lemma 8.6.(ii), Finally $a * C * C=a$ holds still by Lemma 8.5, because $C^{-1}=C$ (that is, $t_{\text {lef }}^{C}=t_{\text {rig }}^{C}$ ) by (2).

In particular, if $\gamma \in Z$ then $A \upharpoonright \gamma \in \mathbf{D N}_{\text {reg }}^{Y}$, and hence $(A \upharpoonright \gamma) * C \in \mathbf{D N}_{\text {reg }}^{Y}$ is a regular double-name. Thus $\{(A \upharpoonright \gamma) * C\}_{\gamma \in Z} \in \mathbf{L}[\Gamma]$ is a $\leqslant$-increasing sequence of regular double-names. The following is a key fact.

Lemma 12.3. The sequence $\{(A \upharpoonright \gamma) * C\}_{\gamma \in Z}$ is $\mathbf{D N}$-generic over $\mathbf{L}\left[G_{1 e f}\right]=$ $\mathbf{L}\left[G_{\text {lef }}^{\prime}\right]$, in the sense that if a set $D^{\prime} \subseteq \mathbf{D N}, D^{\prime} \in \mathbf{L}\left[G_{\text {lef }}\right]$, is open dense in DN then there is an ordinal $\gamma \in Z$ such that $(A \upharpoonright \gamma) * C \in D^{\prime}$.

Proof. The set $\Delta^{\prime}=D^{\prime} \cap \mathbf{D N}_{\text {reg }}^{Y}$ belongs to $\mathbf{L}\left[G_{\text {lef }}\right]$ and still is dense in $\mathbf{D N}$ by Lemma 4.4(ii). Therefore its $C$-image $\Delta=\left\{a * C: a \in \Delta^{\prime}\right\}$ still belongs to $\mathbf{L}\left[G_{\text {lef }}\right]$ and is dense in $\mathbf{D N}$ by Lemma 12.2, It follows by the genericity of $\Gamma$ that $A \upharpoonright \gamma \in \Delta$ for some $\gamma \in Z$. Then $a=(A \upharpoonright \gamma) * C \in \Delta^{\prime}$, since $a * C=A \upharpoonright \gamma$ by Lemma 12.2 .

Corollary 12.4. The set $\Gamma^{\prime}=\{a \in \mathbf{D N}: \exists \gamma \in Z(a \leqslant(A \upharpoonright \gamma) * C)\}$ is $\mathbf{D N}$ generic over $\mathbf{L}\left[G_{\text {lef }}\right]=\mathbf{L}\left[G_{\text {lef }}^{\prime}\right]$.

Let us check the other intended properties of $\Gamma^{\prime}$ as in (*).
To see that $\hat{a} \in \Gamma^{\prime}$, recall that $\hat{a} \in \Gamma \cap \mathbf{D N}_{\hat{\gamma}}$. It follows by Corollary 9.2](iii) that $\hat{a} \leqslant a=A \upharpoonright \hat{\gamma}$. However $\hat{\gamma} \in Z$ by Corollary 12.1](iii), We conclude that $\hat{a} * C \in \Gamma^{\prime}$. Finally $\hat{a} * C=\hat{a} *(C \upharpoonright \hat{\gamma})=\hat{a}$ since $C \upharpoonright \hat{\gamma}=\mathbf{i d}[\hat{\gamma}]$ by (3). Thus $\hat{a} \in \Gamma^{\prime}$, as required.

Finally prove that $t_{\text {lef }}^{A}\left[G_{1 \mathrm{ef}}\right]=t_{\text {lef }}^{A^{\prime}}\left[G_{1 \mathrm{ef}}^{\prime}\right]$, where $A=\bigvee \Gamma$ and $A^{\prime}=\bigvee \Gamma^{\prime}$. It suffices to show that if $\gamma \in Z$ then

$$
\begin{equation*}
t_{1 \mathrm{ef}}^{A \upharpoonright \gamma}\left[G_{\mathrm{lef}} \cap \mathbf{L S}_{\gamma}\right]=t_{\mathrm{lef}}^{A^{\prime} \uparrow \gamma}\left[G_{\mathrm{lef}}^{\prime} \cap \mathbf{L S}_{\gamma}\right] . \tag{5}
\end{equation*}
$$

However by construction $A^{\prime} \upharpoonright \gamma=(A \upharpoonright \gamma) * C=(A \upharpoonright \gamma) *(C \upharpoonright \gamma)$, and on the other hand $t_{\text {lef }}^{(A \upharpoonright \gamma) *(C \upharpoonright \gamma)}[G]=t_{\text {lef }}^{A \upharpoonright \gamma}\left[t_{\text {lef }}^{C \upharpoonleft \gamma}[G]\right]$ for all $G$ by Lemma 8.1, therefore (5) is equivalent to

$$
t_{\mathrm{lef}}^{A \mid \gamma}\left[G_{\mathrm{lef}} \cap \mathbf{L} \mathbf{S}_{\gamma}\right]=t_{\text {lef }}^{A \mid \gamma}\left[t_{\mathrm{lef}}^{C \upharpoonleft \gamma}\left[G_{\mathrm{lef}}^{\prime} \cap \mathbf{L S}_{\gamma}\right]\right],
$$

which obviously follows from

$$
G_{\text {lef }} \cap \mathbf{L S}_{\gamma}=t_{\text {lef }}^{C \upharpoonleft \gamma}\left[G_{\text {lef }}^{\prime} \cap \mathbf{L S}_{\gamma}\right],
$$

and this is a corollary of the equality $G_{1 \mathrm{ef}}=t_{\text {lef }}^{C}\left[G_{1 \mathrm{ef}}^{\prime}\right]$ in (2) by Lemma 6.4](i).(b).
(Theorem 11.3)
This also completes the proof of Theorem 1.1 (see the end of Section 11).
$\square$ (Theorem 1.1)

## 13 Conclusive remarks

Question 13.1. Is Theorem 1.1 true for arbitrary sets $\mathscr{X}$, not necessarily sets of reals? In this general case, the proof given above fails in the proof of Theorem 10.4, since it is not true anymore that $U_{n} \subseteq \mathbf{H} \Omega$ and $\phi\left(U_{n}\right)=U_{n}$.

It follows from Theorem 1.1 that, in the Solovay model, any OD set $\mathscr{X}$ of sets of reals containing non-OD elements is uncountable. If moreover $\mathscr{X}$ is a set of reals then in fact $\mathscr{X}$ contains a perfect subset and hence has cardinality $\mathfrak{c}$ by a profound theorem in [8]. Does this stronger result reasonably generalize to sets of sets of reals and more complex sets?

Conjecture 13.2. It is true in the Solovay model that if $\mathscr{X}$ is an OD set then
(I) if $\mathscr{X}$ contains only OD elements then it is OD-wellorderable;
(II) if $\mathscr{X}$ contains only ROD elements, among them at leat one non-OD element, then $\mathscr{X}$ includes a ROD-image of the continuum $2^{\omega}$;
(III) if $\mathscr{X}$ contains a non-ROD element then $\mathscr{X}$ has cardinality $\geq 2^{c}$.

The set of all $\mathbf{L S}$-generic sets over $\mathbf{L}$ is a less trivial example of a set of type (III) in the Solovay model.

A proof of (III) would be an alternative (and perhaps simpler) proof of Theorem 1.1 of this paper.

It remains to note that Caicedo and Ketchersid [1] obtained a somewhat similar trichotomy result in in a strong determinacy assumption.

## References

[1] Andrés Eduardo Caicedo and Richard Ketchersid. A trichotomy theorem in natural models of $\mathrm{AD}^{+}$. In Set theory and its applications. Annual Boise extravaganza in set theory, Boise, ID, USA, 1995-2010, pages 227-258. Providence, RI: American Mathematical Society (AMS), 2011.
[2] Ali Enayat. On the Leibniz-Mycielski axiom in set theory. Fundam. Math., 181(3):215-231, 2004.
[3] Thomas Jech. Set theory. Berlin: Springer, the third millennium revised and expanded edition, 2003.
[4] Ronald Jensen. Definable sets of minimal degree. Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, pp. 122-128, 1970.
[5] V. Kanovei and V. Lyubetsky. A countable definable set of reals containing no definable elements. ArXiv e-prints, 1408.3901, August 2014.
[6] Vladimir Kanovei. An Ulm-type classification theorem for equivalence relations in Solovay model. J. Symb. Log., 62(4):1333-1351, 1997.
[7] Vladimir Kanovei and Vassily Lyubetsky. A definable $E_{0}$ class containing no definable elements. Arch. Math. Logic, 54(5-6):711-723, 2015.
[8] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. Ann. Math. (2), 92:1-56, 1970.
[9] Jacques Stern. On Lusin's restricted continuum problem. Ann. Math. (2), 120:7-37, 1984.

## Index

double-name, 3
$C, 15$
equivalent, 4
full, 4
inverse, $c^{-1}, 11$
regular, 4
extends, 7
name
equivalent, 4
full, 4
id $[\gamma], 5$
regular, 4
pair
$\bar{p}$-pair, 12
stabilizing, 14
regular hull, 4
restriction
$a \upharpoonright \alpha, 7$
$t \upharpoonright \alpha, 7$
set
dense, 4
open, 4
Solovay model, 2
superposition
$a \cdot c, 10$
$a * c, 11$
A, 16
$a \upharpoonright \alpha, 7$
$a \cdot c, 10$
$a * c, 11$
$\hat{a}, 15$
$|a|, 3$
$A^{\prime}, 17$
A, 11
C, 15
$c^{-1}, 11$
DN, 3
$\mathbf{D N}_{\gamma}, 3$
$\mathbf{D N}_{\text {reg }}^{Y}, 16$
$G_{0}, 12$
$\Gamma, 16$
$\bar{\gamma}, 12$
V $\Gamma, 8$
$\hat{\gamma}, 15$
$\Gamma^{\prime}, 16$
$\underline{\Gamma}, 11$
$G_{\text {lef }}, 15$
$G_{1 \text { ef }}^{\prime}, 15$
$\underline{G}, 3$
$\mathbf{i d}[\gamma], 5$
LS, 2
$\mathbf{L S}_{\gamma}, 2$
$\mathbf{N}, 3$
$\mathbf{N}_{\gamma}, 3$
$\bar{p}, 12$
$\bar{p}$-pair, 12
stabilizing, 14
$\hat{p}, 15$
$\varphi(\cdot, \pi), 12$
$|p|, 2$
${ }^{\text {rh }} t, 4$
$t_{1 \mathrm{ef}}^{a \cdot c}, 10$
$t_{\text {rig }}^{a \cdot c}, 10$
$t_{\text {lef }}^{a}, 3$
$t \upharpoonright \alpha, 7$
$t_{\text {rig }}^{a}, 3$
$t[G], 3$
${ }^{\text {rh }} t, 4$
u, 12
$U \llbracket G \rrbracket, 3$
$U_{n}, 12$
$\mathscr{X}, 12$
$X, Y, Z, 16$
V $\Gamma, 8$
$\leqslant, 7$
separate index for Section 12 see below
A, 16
a, 15
$A^{\prime}, 17$

$$
\begin{aligned}
& \mathbf{D N}_{\text {reg }}^{Y}, 16 \\
& \Gamma, 16 \\
& \hat{\gamma}, 15 \\
& \Gamma^{\prime}, 16 \\
& G_{\text {lef }}, 15 \\
& G_{\text {lef }}^{\prime}, 15 \\
& \hat{p}, 15 \\
& X, Y, Z, 16
\end{aligned}
$$


[^0]:    *IITP RAS and MIIT, Moscow, Russia, kanovei@googlemail. com - contact author.
    ${ }^{1}$ Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.
    ${ }^{2}$ FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html

