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Abstract

We show that, consistently, there can be maximal subtrees of $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\mathrm{fin}$ of arbitrary regular uncountable size below the size of the continuum $\mathfrak{c}$. We also show that there are no maximal subtrees of $\mathcal{P}(\omega)/\mathrm{fin}$ with countable levels. Our results answer several questions of Campero-Arena, Cancino, Hrušák, and Miranda-Perea [CCHM].

1 Introduction

A partial order $(\mathcal{T}, \leq)$ is called a *tree* if it has a largest element $\mathbf{1}$ and for every $t \in \mathcal{T}$, the set of *predecessors* of t in $\mathcal{T}$, $\mathrm{pred}_\mathcal{T}(t) = \{s \in \mathcal{T} : s \geq t\}$ is well-ordered by the reverse order of $\leq$. For each ordinal α, the *α-th level* of $\mathcal{T}$ is given by $\mathrm{Lev}_\alpha(\mathcal{T}) = \{t \in \mathcal{T} : \mathrm{pred}_\mathcal{T}(t)$ has order type $\alpha\}$. The *height* of $\mathcal{T}$, $\mathrm{ht}(\mathcal{T})$, is the least ordinal α such that $\mathrm{Lev}_\alpha(\mathcal{T})$ is empty. The *width* of $\mathcal{T}$ is the cardinal $\sup\{|\mathrm{Lev}_\alpha(\mathcal{T})| : \alpha < \mathrm{ht}(\mathcal{T})\}$. Instead of saying $\mathcal{T}$ has width (at most) κ we may sometimes just say $\mathcal{T}$ has levels of size $\leq \kappa$. Let $(\mathbb{P}, \leq)$ be a partial order with largest element $\mathbf{1}$. $\mathcal{T} \subseteq \mathbb{P}$ is a *subtree* of $(\mathbb{P}, \leq)$ (or, a *tree* in $\mathbb{P}$) if $\mathbf{1} \in \mathcal{T}$ and $(\mathcal{T}, \leq \restriction (\mathcal{T} \times \mathcal{T}))$ is a tree in the above sense. Note that incomparable (equivalently, incompatible) elements of $\mathcal{T}$ are not necessarily incompatible in $\mathbb{P}$; that is, for $s, t \in \mathcal{T}$ with $s \not\leq t$ and $t \not\leq s$ there may exist $r \in \mathbb{P}$ with $r \leq s, t$ (of course, such r cannot belong to $\mathcal{T}$).

Trees are ordered by *end-extension*, that is, $\mathcal{S} \leq \mathcal{T}$ if $\mathcal{S} \subseteq \mathcal{T}$ and $\mathrm{pred}_\mathcal{T}(s) = \mathrm{pred}_\mathcal{S}(s)$ for every $s \in \mathcal{S}$. By Zorn's Lemma, *maximal trees*, that is, trees without proper end-extensions, exist in a given partial order. It is easy to see that $\mathcal{T} \subseteq \mathbb{P}$ is maximal iff for every $p \in \mathbb{P}$

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- either there is $q \in \mathcal{T}$ with $q \leq p$,
- or there are incomparable elements $q, r \in \mathcal{T}$ with $p \leq q, r$.

See [Mo, Proposition 17.11].

We will consider maximal trees for the case when $\mathbb{P}$ is either $\mathcal{P}(\omega) \setminus \{\emptyset\}$, ordered by inclusion, or $(\mathcal{P}(\omega)/\mathrm{fin}) \setminus \{\emptyset\}$, ordered by inclusion mod finite. For simplicity, we will call the former trees in $\mathcal{P}(\omega)$ and the latter trees in $\mathcal{P}(\omega)/\mathrm{fin}$ or, more correctly, in $([\omega]^\omega, \subseteq^*)$. Recall that for $A, B \in [\omega]^\omega$, $A \subseteq^* B$ iff $A \setminus B$ is finite. Monk [Mo, Proposition 17.9] observed that there are always maximal trees in $\mathcal{P}(\omega)$ of size ω and $\mathfrak{c}$, and in $\mathcal{P}(\omega)/\mathrm{fin}$ of size $\mathfrak{c}$, and asked whether there can consistently be maximal trees of other sizes [Mo, Problems 156 and 157]. These questions were solved by Campero-Arena, Cancino, Hrušák, and Miranda-Perea who proved that it is consistent that the continuum hypothesis CH fails and there is a maximal tree of height and width ω_1 in $\mathcal{P}(\omega)/\mathrm{fin}$ [CCHM, Theorem 3.2] and a tree of height ω and width ω_1 which is maximal as a subtree of both $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\mathrm{fin}$ [CCHM, Theorem 4.1]. More explicitly, the existence of such trees follows from one of the parametrized diamond principles of [MHD], and it is well-known that this principle is consistent with $\neg$CH. Define the *tree number* $\mathfrak{tr}$ as the least size of a maximal tree in $\mathcal{P}(\omega)/\mathrm{fin}$ and recall that the *reaping number* $\mathfrak{r}$ (see [Bl, Definition 3.6]) is the least size of a family $\mathcal{A} \subseteq [\omega]^\omega$ such that for all $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that either $A \cap B$ is finite or $A \subseteq^* B$. It is easy to see that $\omega_1 \leq \mathfrak{r} \leq \mathfrak{tr} \leq \mathfrak{c}$ [CCHM, p. 81], and by the mentioned result $\omega_1 = \mathfrak{r} = \mathfrak{tr} < \mathfrak{c}$ is consistent while the consistency of $\omega_1 < \mathfrak{r} = \mathfrak{tr} = \mathfrak{c}$ was established by showing it holds under Martin's axiom MA [Mo, Theorem 17.14 and Corollary 17.15]. This left open the question of whether $\mathfrak{tr}$ can consistently be strictly in between ω_1 and $\mathfrak{c}$ [CCHM, Question 5.3].

We answer this question in the affirmative by proving that for arbitrary regular uncountable κ, maximal trees in $\mathcal{P}(\omega)/\mathrm{fin}$ of height and width κ can be added generically to a model with large continuum (Theorem 5 in Section 3 below). Furthermore, we show that, consistently, we may simultaneously adjoin maximal trees of different sizes (Theorem 6), thus making the *tree spectrum* $\mathsf{Spec}_{\mathrm{tree}} = \{\kappa :$ there is a maximal tree in $\mathcal{P}(\omega)/\mathrm{fin}$ of size $\kappa\}$ large and answering [CCHM, Question 5.4]. By modifying the construction, we also obtain consistently trees of width κ and height ω which are maximal in both $\mathcal{P}(\omega)/\mathrm{fin}$ and $\mathcal{P}(\omega)$, for arbitrary regular uncountable κ (Theorem 11 in Section 4). Again, this construction can be extended to get large spectrum (Theorem 12).

In all such constructions of maximal trees in $\mathcal{P}(\omega)/\mathrm{fin}$, the width is at least the cofinality of the height, and we do not know whether there can consistently be a maximal tree of regular height whose width is smaller than its height (Question 8). However, we prove in ZFC that there are no maximal trees in $\mathcal{P}(\omega)/\mathrm{fin}$ of countable width, thus answering [CCHM, Question 5.2] (see Theorem 2 in Section 2).

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2 Trees with countable levels

A set $A \in [\omega]^\omega$ is a *branching node* in a tree $\mathcal{T}$ if there are incomparable $B, C \in \mathcal{T}$ such that $\mathrm{pred}_{\mathcal{T}}(B) = \mathrm{pred}_{\mathcal{T}}(C)$ and A is the $\subseteq^*$-smallest node of $\mathrm{pred}_{\mathcal{T}}(B)$. $b \subseteq \mathcal{T}$ is a *maximal branch* if b is a maximal linearly ordered subset of $\mathcal{T}$.

If $\mathcal{F} \subseteq [\omega]^\omega$ has the *finite intersection property*, that is, $\bigcap F$ is infinite for every finite $F \subseteq \mathcal{F}$, a set $C \in [\omega]^\omega$ is called a *pseudointersection* of $\mathcal{F}$ if $C \subseteq^* A$ for all $A \in \mathcal{F}$.

Lemma 1. *Assume $\mathcal{T}$ is a tree with countable levels and $b = \{A_\alpha : \alpha < \gamma\}$ is a maximal branch in $\mathcal{T}$ with $\mathrm{cf}(\gamma) > \omega$ such that only countably many nodes A_α of b are branching nodes. Then $\mathcal{T}$ cannot be maximal.*

Proof. Assume $\mathcal{T}$ is maximal. By assumption, for some $\alpha_0 < \gamma$, no branching occurs in b after A_{α_0}. Also, by assumption, the set $\mathcal{B}$ of all $B \in \mathcal{T}$ such that B is an immediate successor of some A_α but $B \notin b$ must be countable. For each $\alpha > \alpha_0$ consider the set $C_\alpha := A_{\alpha_0} \setminus A_\alpha$. By maximality, there must be a set $B_\alpha \in \mathcal{B}$ such that $C_\alpha \subseteq^* B_\alpha$ (otherwise we could add C_α to $\mathcal{T}$). By countability of $\mathcal{B}$ and by $cf(\gamma) > \omega$, we see that there is a single $B \in \mathcal{B}$ such that for all $\alpha > \alpha_0$, $C_\alpha \subseteq^* B$. On the other hand, $A_{\alpha_0} \not\subseteq^* B$. In particular $A_{\alpha_0} \setminus B$ is a pseudointersection of the A_α (which cannot be added to the tree). Using a standard diagonal argument, we can construct a set C such that

- $C \subseteq A_{\alpha_0}$,
- $A_{\alpha_0} \setminus B \not\subseteq^* C$, and
- $C \not\subseteq^* B'$ for all $B' \in \mathcal{B}$.

Now, it is easy to see that C can be added to $\mathcal{T}$: by the third clause, the only predecessors of C in $\mathcal{T}$ are in b. By the second clause, no A_α is almost contained in C. Thus we obtain a contradiction. □

Theorem 2. *There are no maximal trees with countable levels in $\mathcal{P}(\omega)/\mathrm{fin}$.*

Proof. Assume $\mathcal{T}$ were such a tree. Let $b = \{A_\alpha : \alpha < \gamma\}$ be a maximal branch such that the length γ of b is minimal. If $cf(\gamma) \neq \omega_1$, then, because of minimality and the countable levels, there can only be countably many branching nodes in b. More explicitly, if there are ω_1 many branching nodes in b, they are bounded in b because of $cf(\gamma) \neq \omega_1$, say by A_α; the minimality of γ then allows us to build ω_1 many branches branching off from b so that the α-th level is uncountable, a contradiction. In particular, the set $\mathcal{B}$ of all $B \in \mathcal{T}$ such that B is an immediate successor of some element of b yet $B \notin b$ must be countable. If $\gamma = \delta + 1$ is a successor, a standard diagonal argument yields a $C \subseteq A_\delta$ with $A_\delta \not\subseteq^* C$ such that $C \not\subseteq^* B$ for all $B \in \mathcal{B}$. Similarly, if γ has countable cofinality, we obtain $C \subseteq^* A_\delta$ for all $\delta < \gamma$ such that $C \not\subseteq^* B$ for all $B \in \mathcal{B}$. In either case, C can be added to $\mathcal{T}$, showing that $\mathcal{T}$ is not maximal. If $\mathrm{cf}(\gamma) > \omega_1$, we immediately obtain a contradiction by the previous lemma.

So assume $cf(\gamma) = \omega_1$. By the previous lemma, using again minimality and countable levels, we see that there is a cofinal subset of order type ω_1 of branching nodes in b. Furthermore, all but countably many of the branches branching off from b must have length exactly γ: they cannot be shorter by minimality, and not longer by countable levels. In particular, we may find a branching node $A_{\langle\rangle} = A_{\alpha_0} \in b$ such that a branch b' branching off from b in $A_{\langle\rangle}$ has length γ. Applying this argument again to both b and b', we find branching nodes $A_{\langle 0\rangle} \in b$ and $A_{\langle 1\rangle} \in b'$ above $A_{\langle\rangle}$. Let $\alpha_1 > \alpha_0$, $\alpha_1 < \gamma$, be such that the level of $A_{\langle 0\rangle}$ and $A_{\langle 1\rangle}$ is below α_1. Iterating this procedure, we construct nodes A_s, $s \in 2^{<\omega}$, in $\mathcal{T}$ such that A_s and A_t are incomparable for incomparable s and t, and $A_t \subseteq^* A_s$ for t extending s. Furthermore, the level of all A_s, $s \in 2^n$, is below $\alpha_n < \gamma$, and the α_n form a strictly increasing sequence of ordinals. Let $\alpha_\omega = \bigcup_n \alpha_n$. Clearly $\alpha_\omega < \gamma$. Thus, by minimality, for each $f \in 2^\omega$ there is $A_f \in \mathcal{T}$ with $A_f \subseteq^* A_{f\restriction n}$ for all n on level α_ω. In particular, the level α_ω of $\mathcal{T}$ has size $\mathfrak{c}$, a contradiction. □

3 Forcing: matrix trees

Recall that two sets $A, B \in [\omega]^\omega$ are *almost disjoint* if $A \cap B$ is finite. $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint family* (*a.d. family*, for short) if any two distinct members of $\mathcal{A}$ are almost disjoint.

Let $\mathcal{F}$ be a filter on ω containing all cofinite sets. *Mathias forcing* with $\mathcal{F}$, written $\mathbb{M}(\mathcal{F})$, consists of all pairs (s, A) such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, and $\max(s) < \min(A)$. $\mathbb{M}(\mathcal{F})$ is ordered by stipulating that $(t, B) \leq (s, A)$ if $s \subseteq t \subseteq s \cup A$ and $B \subseteq A$. It is well-known and easy to see that $\mathbb{M}(\mathcal{F})$ is a σ-centered forcing which generically adds a pseudointersection X of $\mathcal{F}$ such that X has infinite intersection with all $\mathcal{F}$-positive sets of the ground model. Here $C \in [\omega]^\omega$ is *$\mathcal{F}$-positive* if $C \cap A$ is infinite for all $A \in \mathcal{F}$.

Definition 3. Let γ be an ordinal. Say that a tree $\mathcal{T} = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\}$ in $\mathcal{P}(\omega)/\mathrm{fin}$ is a *matrix tree* if

(i) for $\alpha \leq \gamma$, $\{A_\alpha^\beta : \beta \leq \gamma\}$ is the α-th level of $\mathcal{T}$,

(ii) for $\beta \leq \gamma$ and $\alpha < \alpha' \leq \gamma$, $A_{\alpha'}^\beta \subseteq^* A_\alpha^\beta$,

(iii) for finite $D \subseteq \gamma + 1$ and $\beta \notin D$, $A_0^\beta \setminus \bigcup_{\beta' \in D} A_0^{\beta'}$ is infinite,

(iv) for $\alpha > 0$, $\{A_\alpha^\beta : \beta \leq \gamma\}$ is an a.d. family, and

(v) for $\beta \neq \beta'$, A_γ^β and $A_0^{\beta'}$ are almost disjoint.

Lemma 4. (Extension Lemma) *Assume $\mathcal{T} = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\}$ is a matrix tree. Then there is a ccc forcing end-extending $\mathcal{T}$ to a matrix tree $\mathcal{T}' = \{A_\alpha^\beta : \alpha, \beta \leq \gamma + 2\}$ such that no $C \in [\omega]^\omega$ from the ground model can be added to $\mathcal{T}'$.*

Proof. Let $\mathcal{F}$ be a maximal filter with the property that for all $F \in \mathcal{F}$ and all $\beta \leq \gamma$, $F \cap A_\gamma^\beta$ is infinite. Force with the product $\mathbb{M}(\mathcal{F}) \times \mathbb{M}(\mathcal{F})$. Let X_0 and X_1 be the two generic subsets of ω. We let $A_{\gamma+1}^\beta = X_0 \cap X_1 \cap A_\gamma^\beta$. Clearly this set is infinite by genericity. Choose $A_{\gamma+2}^\beta \subseteq A_{\gamma+1}^\beta$ arbitrarily. We also let $A_0^{\gamma+1} = \omega \setminus X_0$ and $A_0^{\gamma+2} = \omega \setminus X_1$. Then clearly $A_{\gamma+1}^\beta$ and $A_0^{\beta'}$ are disjoint for $\beta \leq \gamma$ and $\beta' \in \{\gamma+1, \gamma+2\}$. A straightforward genericity argument shows that clause (iii) is still satisfied. Thus we can easily add sets $A_1^{\gamma+1} \subseteq A_0^{\gamma+1}$ and $A_1^{\gamma+2} \subseteq A_0^{\gamma+2}$ by ccc forcing such that A_0^β and $A_1^{\beta'}$ are almost disjoint for $\beta' \in \{\gamma+1, \gamma+2\}$ and any $\beta \neq \beta'$. Finally let $\{A_\alpha^{\beta'} : 1 < \alpha \leq \gamma + 2\}$ be decreasing chains below $A_1^{\beta'}$ for $\beta' \in \{\gamma+1, \gamma+2\}$. It follows now that properties (iv) and (v) in the definition of matrix tree hold. Also, $\mathcal{T}'$ is indeed a tree.

Let $C \in [\omega]^\omega$. If $F \cap A_\gamma^\beta \subseteq^* C$ for some $\beta \leq \gamma$ and some $F \in \mathcal{F}$, then $A_{\gamma+1}^\beta \subseteq^* C$, and C cannot be added to $\mathcal{T}'$. So assume this is not the case, that is, $(F \cap A_\gamma^\beta) \setminus C$ is infinite for all $\beta \leq \gamma$ and $F \in \mathcal{F}$. Then $\omega \setminus C \in \mathcal{F}$ by the maximality of $\mathcal{F}$. Hence $X_0 \cup X_1 \subseteq^* \omega \setminus C$ and $C \subseteq^* (\omega \setminus X_0) \cap (\omega \setminus X_1) = A_0^{\gamma+1} \cap A_0^{\gamma+2}$ and, again, C cannot be added to $\mathcal{T}'$. This completes the proof of the lemma. $\square$

Recall that $\mathsf{cov}(\mathcal{M})$ is the least size of a family of meager sets covering the real line. It is well-known that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{r}$ [Bl, Theorem 5.19] and that adding Cohen reals increases $\mathsf{cov}(\mathcal{M})$ [Bl, Subsection 11.3].

Theorem 5. *Let $\kappa \leq \lambda$ be regular uncountable cardinals with $\lambda^\omega = \lambda$. There is a ccc generic extension with $\mathfrak{tr} = \kappa$ and $\mathfrak{c} = \lambda$.*

Proof. First add λ Cohen reals. Then perform a finite support iteration $\langle \mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \kappa \rangle$ of ccc forcing. Let V_γ denote the intermediate model. If γ is an even ordinal, the model $V_{\gamma+1}$ will contain a matrix tree $\mathcal{T}_\gamma = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\}$ such that

- for $\gamma < \delta$, $\mathcal{T}_\delta$ end-extends $\mathcal{T}_\gamma$,
- if $\gamma = \delta + 2$, then no C from V_γ can be added to the tree $\mathcal{T}_\gamma$.

If γ is an odd ordinal, $\dot{\mathbb{Q}}_\gamma$ is the trivial forcing. If $\gamma = \delta + 2$ is even, $\dot{\mathbb{Q}}_\gamma$ is the forcing from the preceding lemma applied to the tree $\mathcal{T}_\delta \in V_{\delta+1} \subseteq V_\gamma$. If γ is a limit ordinal, define $\dot{\mathbb{Q}}_\gamma$ as follows: let $\mathcal{T}_{<\gamma} = \bigcup_{\delta<\gamma} \mathcal{T}_\delta = \{A_\alpha^\beta : \alpha, \beta < \gamma\}$. First add pseudointersections A_γ^β to the decreasing chains $\{A_\alpha^\beta : \alpha < \gamma\}$ for $\beta < \gamma$ (if $cf(\gamma) = \omega$, they can be constructed outright, otherwise they can be forced by ccc forcing). Next add a set A_0^γ almost disjoint from A_0^β, $\beta < \gamma$, by ccc forcing. This can be done by (iii) and will preserve (iii) in Definition 3. Finally let $\{A_\alpha^\gamma : 0 < \alpha \leq \gamma\}$ be a decreasing chain below A_0^γ. Put $\mathcal{T}_\gamma = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\}$.

Then (iv) and (v) in Definition 3 clearly hold as well. This completes the definition of the iteration.

Clearly $\mathcal{T}_\kappa = \bigcup_{\gamma<\kappa} \mathcal{T}_\gamma$ is a maximal tree of size κ by Lemma 4. Therefore $\mathfrak{tr} \leq \kappa$. On the other hand, $\mathfrak{tr} \geq \mathfrak{r} \geq \mathsf{cov}(\mathcal{M}) \geq \kappa$ because of the Cohen reals added in limit stages of the iteration (see [Go, Example 2]). $\square$

Note that the tree $\mathcal{T}_\kappa$ constructed in this proof has height and width κ.

Theorem 6. *Let C be a set of regular uncountable cardinals. There is a ccc generic extension such that for all $\lambda \in C$, there is a maximal tree in $\mathcal{P}(\omega)/\mathrm{fin}$ of size λ.*

Proof. Let $\kappa = \min C$. For $\lambda \in C$ with $\lambda > \kappa$, let $\epsilon_\lambda = \lambda \cdot \kappa$ be the ordinal product. Make a finite support iteration $\langle \mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \kappa \rangle$ of ccc forcing such that

- if $\gamma = \delta + 2$ is even, then $\dot{\mathbb{Q}}_\gamma$ is defined exactly as in the proof of the previous theorem and end-extends the matrix tree $\mathcal{T}_\delta = \{A_\alpha^\beta : \alpha, \beta \leq \delta\} \in V_{\delta+1} \subseteq V_\gamma$ to the matrix tree $\mathcal{T}_\gamma = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\} \in V_{\gamma+1}$,

- if $\gamma = \delta + 1$ is odd, then, for each $\lambda \in C \setminus \{\kappa\}$, $\dot{\mathbb{Q}}_\gamma$ end-extends a matrix tree $\mathcal{T}_\delta^\lambda = \{A_\alpha^\beta : \alpha, \beta \leq \lambda \cdot \delta\} \in V_\gamma$ to a matrix tree $\mathcal{T}_{\delta+2}^\lambda = \{A_\alpha^\beta : \alpha, \beta \leq \lambda \cdot (\delta + 2)\} \in V_{\gamma+1}$ using a finite-support product indexed by $\lambda \in C \setminus \{\kappa\}$ of finite-support iterations of length $\lambda \cdot 2$ for each $\lambda \in C \setminus \{\kappa\}$ as in the proof of the previous theorem,

- if γ is limit, $\dot{\mathbb{Q}}_\gamma$ end-extends $\mathcal{T}_{<\gamma}$ to the matrix tree $\mathcal{T}_\gamma$ as in the proof of the previous theorem and also end-extends the $\mathcal{T}_{<\gamma}^\lambda := \{A_\alpha^\beta : \alpha, \beta < \lambda \cdot \gamma\} \in V_\gamma$ to matrix trees $\mathcal{T}_\gamma^\lambda = \{A_\alpha^\beta : \alpha, \beta \leq \lambda \cdot \gamma\} \in V_{\gamma+1}$.

In the final extension, let $\mathcal{T}_\kappa = \bigcup_{\gamma<\kappa} \mathcal{T}_\gamma$ and $\mathcal{T}_\lambda = \bigcup_{\gamma<\kappa} \mathcal{T}_\gamma^\lambda$ for $\lambda \in C \setminus \{\kappa\}$. By construction, all these trees $\mathcal{T}_\lambda$ are maximal trees, and their respective size is λ. $\square$

We do not know whether there is a way to control the λ for which a maximal tree of size λ is added in this proof.

Question 7. *Let C be a set of regular cardinals (possibly satisfying some additional condition). Is there a ccc forcing extension in which there is a maximal tree of size λ iff $\lambda \in C$?*

Notice that for $\lambda > \kappa := \min C$, the trees in the previous proof all have width $\lambda = |\lambda \cdot \kappa|$ and height $\lambda \cdot \kappa$. In particular, by pruning the branches while keeping maximality, we easily see that we can obtain maximal trees of width λ and height κ as well. Therefore, we see that all maximal trees $\mathcal{T}$ of regular height constructed so far either have width $|\mathcal{T}|$ and height ω (see Theorem 11 below or [CCHM, Theorem 4.1]) or width and height $|\mathcal{T}|$ or width $|\mathcal{T}|$ and height some uncountable regular cardinal below $|\mathcal{T}|$. We do not know whether there can be a maximal tree whose width is smaller than the cofinality of its height:

Question 8. *Is it consistent that there is a maximal tree with levels of size ω_1 and height ω_2 (with all branches of length ω_2)?*

4 Forcing: wide-branching trees

Definition 9. Let γ be an ordinal. Say that a tree $\mathcal{T} = \{A_s : s \in \gamma^{<\omega}\}$ in $\mathcal{P}(\omega)$ is a *wide-branching tree* if

(i) for all n, $\{A_s : s \in \gamma^n\}$ is the n-th level of $\mathcal{T}$,

(ii) for $s \subseteq t$ in $\gamma^{<\omega}$, $A_t \subseteq A_s$,

(iii) for finite $D \subseteq \gamma$ and $\beta \notin D$, $A_{\langle\beta\rangle} \setminus \bigcup_{\alpha\in D} A_{\langle\alpha\rangle}$ is infinite,

(iv) for $n \geq 2$, $\{A_s : s \in \gamma^n\}$ is an a.d. family, and

(v) for all $\alpha \leq \beta < \gamma$ and $s \in \gamma^{\geq 2}$, if $s(0) \neq \alpha$ and $\beta \in \mathrm{ran}(s)$ then A_s and $A_{\langle\alpha\rangle}$ are almost disjoint.

Lemma 10. (Extension Lemma) *Let $\gamma \geq \omega$ be a limit ordinal. Assume $\mathcal{T} = \{A_s : s \in \gamma^{<\omega}\}$ is a wide-branching tree. Then there is a ccc forcing end-extending $\mathcal{T}$ to a wide-branching tree $\mathcal{T}' = \{A_s : s \in (\gamma+\omega)^{<\omega}\}$ such that for every $C \in [\omega]^\omega$ from the ground model, either $A_s \subseteq C$ for some $s \in (\gamma+\omega)^{<\omega}$ or $C \subseteq A_{\langle\gamma+n\rangle} \cap A_{\langle\gamma+n+1\rangle}$ for some $n \in \omega$.*

Proof. Let $\mathcal{F}$ be a maximal filter such that $F \cap A_s$ is infinite for all $s \in \gamma^{<\omega}$ and all $F \in \mathcal{F}$. Force with the finite support product $\mathbb{M}(\mathcal{F})^\omega$ of countably many copies of $\mathbb{M}(\mathcal{F})$. Let $(X_n : n \in \omega)$ be the generic sequence. Put $X := \{m : m \in X_n \text{ for all } n \leq m\}$. By genericity, X is an infinite pseudointersection of the X_n. For each $s \in \gamma^{<\omega} \setminus \{\langle\rangle\}$, let $B_s = A_s \cap X$. Note that for all finite $D, E \subseteq \gamma$ and all $F \in \mathcal{F}$ such that $s(0) \notin E$, $F \cap A_s \setminus (\bigcup_{\beta\in D} A_{s\hat{}\langle\beta\rangle} \cup \bigcup_{\alpha\in E} A_{\langle\alpha\rangle})$ is infinite. (To see this, take $\delta > \max E$, $\delta \notin D$. Then $A_{s\hat{}\langle\delta\rangle}$ is almost disjoint from $\bigcup_{\beta\in D} A_{s\hat{}\langle\beta\rangle} \cup \bigcup_{\alpha\in E} A_{\langle\alpha\rangle}$ by (iv) and (v), and $F \cap A_{s\hat{}\langle\delta\rangle}$ is infinite.) Thus, by genericity, for all finite $D, E \subseteq \gamma$ with $s(0) \notin E$, $B_s \setminus (\bigcup_{\beta\in D} A_{s\hat{}\langle\beta\rangle} \cup \bigcup_{\alpha\in E} A_{\langle\alpha\rangle})$ is infinite. In particular, by a further ccc forcing, we can add pairwise disjoint sets $A_{s\hat{}\langle\gamma+n\rangle}$, $n \in \omega$, contained in B_s such that all of them are almost disjoint from all $A_{s\hat{}\langle\beta\rangle}$, $\beta < \gamma$, and all $A_{\langle\alpha\rangle}$, $\alpha < \gamma$, with $s(0) \neq \alpha$. This means that clauses (iv) and (v) still hold for these sets. In particular, they can be added to the tree (that is, they are neither above an element of the tree, nor below two incomparable elements of the tree). Furthermore, for any $s \in \gamma^{<\omega} \setminus \{\langle\rangle\}$, any $n \in \omega$, and any $t \in (\gamma+\omega)^{<\omega}$, we can now build sets $A_{s\hat{}\langle\gamma+n\rangle\hat{}t}$ contained in $A_{s\hat{}\langle\gamma+n\rangle}$ such that all the clauses are still satisfied.

Next, let $A_{\langle\gamma+n\rangle} = \omega \setminus X_n$ for $n \in \omega$. These sets are almost disjoint from any A_s with $s(0) < \gamma$ and $\gamma + n$ belonging to $\mathrm{ran}(s)$ for some $n \in \omega$. In particular, property (v) is preserved. Also, property (iii) still holds by genericity. Hence an additional ccc forcing adds sets $B_{\langle\gamma+n\rangle} \subseteq A_{\langle\gamma+n\rangle}$ such that $B_{\langle\gamma+n\rangle}$ and $A_{\langle\beta\rangle}$ are almost disjoint for $n \in \omega$ and any $\beta \neq \gamma + n$ with $\beta < \gamma + \omega$. Let $\{A_{\langle\gamma+n\rangle\hat{}\langle\alpha\rangle} : \alpha < \gamma + \omega\}$ be an a.d. family below $B_{\langle\gamma+n\rangle}$ for $n \in \omega$. More generally, for any $n \in \omega$ and any $t \in (\gamma+\omega)^{<\omega}$, we can build sets $A_{\langle\gamma+n\rangle\hat{}t}$ contained in $A_{\langle\gamma+n\rangle}$ such that all the clauses are still satisfied. This completes the definition of $\mathcal{T}'$, and it is clear $\mathcal{T}'$ is a wide-branching tree.

Let $C \in [\omega]^\omega$. If $F \cap A_s \subseteq^* C$ for some $F \in \mathcal{F}$ and $s \in \gamma^{<\omega}$, then $B_s \subseteq^* C$. In particular, for some $n \in \omega$, $A_{s\hat{}\langle\gamma+n\rangle} \subseteq C$. Hence, assume that $(F \cap A_s) \setminus C$ is infinite for all $F \in \mathcal{F}$ and $s \in \gamma^{<\omega}$. Then $\omega \setminus C \in \mathcal{F}$ by maximality of $\mathcal{F}$. Therefore $X_n \subseteq^* \omega \setminus C$ for all $n \in \omega$ and, by genericity, there is in fact an $n \in \omega$ such that $X_n \cup X_{n+1} \subseteq \omega \setminus C$. Therefore, $C \subseteq A_{\langle\gamma+n\rangle} \cap A_{\langle\gamma+n+1\rangle}$ as required. $\square$

Theorem 11. *Let $\kappa \leq \lambda$ be regular uncountable cardinals with $\lambda^\omega = \lambda$. There is a ccc generic extension with $\mathfrak{tr} = \kappa$, $\mathfrak{c} = \lambda$, and, additionally, there is a maximal tree in $\mathcal{P}(\omega)$ of size κ.*

Proof. Add λ Cohen reals and then make a finite support iteration $\langle \mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \kappa\rangle$ of ccc forcing as in the proof of Theorem 5. Let V_γ be the intermediate model. If γ is a limit ordinal, the model V_γ will contain a wide-branching tree $\mathcal{T}_\gamma = \{A_s : s \in \gamma^{<\omega}\}$ such that

- for $\gamma < \delta$, $\mathcal{T}_\delta$ end-extends $\mathcal{T}_\gamma$,
- for $C \in [\omega]^\omega \cap V_\gamma$ either $A_s \subseteq C$ for some $s \in (\gamma+\omega)^{<\omega}$ or $C \subseteq A_{\langle\gamma+n\rangle} \cap A_{\langle\gamma+n+1\rangle}$ for some $n \in \omega$ (in the model $V_{\gamma+1}$ which contains the tree $\mathcal{T}_{\gamma+\omega}$).

If γ is a successor ordinal, $\dot{\mathbb{Q}}_\gamma$ is the trivial forcing. If γ is a limit ordinal, $\dot{\mathbb{Q}}_\gamma$ is the forcing from the preceding lemma applied to the tree $\mathcal{T}_\gamma \in V_\gamma$ and yielding the tree $\mathcal{T}_{\gamma+\omega} \in V_{\gamma+1}$. Here $\mathcal{T}_\gamma$ is obtained as follows: if $\gamma = \delta+\omega$, then $\mathcal{T}_\gamma \in V_{\delta+1}$ has been constructed earlier; if γ is a limit of limits, then $\mathcal{T}_\gamma = \bigcup_{\delta<\gamma} \mathcal{T}_\delta$.

Clearly, $\mathcal{T}_\kappa = \bigcup_{\gamma<\kappa} \mathcal{T}_\gamma$ is a maximal tree in $\mathcal{P}(\omega)$ of size κ by Lemma 10 which is additionally maximal in $\mathcal{P}(\omega)/\text{fin}$. $\mathfrak{tr} = \kappa$ follows as in the proof of Theorem 5. $\square$

As with Theorem 6, the previous result can be extended to yield big spectrum for the size of maximal trees in $\mathcal{P}(\omega)$.

Theorem 12. *Let C be a set of regular uncountable cardinals. There is a ccc generic extension such that for all $\lambda \in C$, there is a maximal tree in $\mathcal{P}(\omega)$ of size λ which is additionally maximal in $\mathcal{P}(\omega)/\text{fin}$.*

Proof. Combine the proofs of Theorems 6 and 11. $\square$

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