# ON A QUESTION OF SILVER ABOUT GAP-TWO CARDINAL TRANSFER PRINCIPLES

## MOHAMMAD GOLSHANI AND SHAHRAM MOHSENIPOUR

ABSTRACT. Assuming the existence of a Mahlo cardinal, we produce a generic extension of Gödel's constructible universe L, in which the GCH holds and the transfer principles  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$  and  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  fail simultaneously. The result answers a question of Silver from 1971. We also extend our result to higher gaps.

## 1. Introduction

In this paper we study cardinal transfer principles introduced by Vaught [6], [7], and prove some consistency results related to them.

Assume  $\mathcal{L}$  is a first order language which contains a unary predicate U. By a  $(\kappa, \lambda)$ -model for  $\mathcal{L}$ , we mean a model  $\mathcal{M} = (M, U^{\mathcal{M}}, \dots)$ , where  $|M| = \kappa$  and  $|U^{\mathcal{M}}| = \lambda$ , where  $U^{\mathcal{M}}$  is the interpretation of U in  $\mathcal{M}$ . Following Devlin [2], we use the notation

$$(\kappa, \lambda) \to (\kappa', \lambda')$$

to mean the following transfer principle:

For every countable first order language  $\mathcal{L}$  as above, and every first order theory T of  $\mathcal{L}$ , if T has a  $(\kappa, \lambda)$ -model, then it has a  $(\kappa', \lambda')$ -model.

For any natural number  $n \geq 1$ , by the gap-n-cardinal transfer principle we mean the statement

$$\forall \kappa \ \forall \lambda \ (\kappa^{+n}, \kappa) \to (\lambda^{+n}, \lambda).$$

In [5], Silver proved the independence of gap-2-cardinal transfer principle. Starting from an inaccessible cardinal, he was able to produce a model in which the cardinal transfer  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  fails. His proof is simply as follows: By a result of Vaught [7], there

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exists a sentence  $\phi_{KH}$  in a suitable first order language, such that for any infinite cardinal  $\beta$ ,

$$\phi_{KH}$$
 has a  $(\beta^{++}, \beta)$ -model  $\iff$  there exists a  $\beta^{+}$ -Kurepa tree.

Now, starting from an inaccessible cardinal  $\kappa$ , Silver shows that in the generic extension by the Levy collapse  $\operatorname{Col}(\aleph_1, < \kappa)$ , there are no  $\aleph_1$ -Kurepa trees. If we start with V = L, then in the resulting extension, there are  $\aleph_2$ -Kurepa trees, and so the transfer principle  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  fails in it. Similarly if we force with  $\operatorname{Col}(\aleph_2, < \kappa)$ , then in the extension there are no  $\aleph_2$ -Kurepa trees, and we can use it to prove the independence of  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ . The following question was asked by Silver [5].

**Question 1.1.** <sup>1</sup> Is it consistent with GCH that both transfer principles  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  and  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$  fail simultaneously?

**Remark 1.2.** If we drop the GCH assumption from the question, then one can easily answer the above question. Assume  $\kappa$  is an inaccessible cardinals and let G\*H be  $Col(\aleph_1, < \kappa) * Add(\aleph_0, \kappa)$ -generic over L. In the generic extension L[G\*H] there are no  $\aleph_1$ -Kurepa trees (see Devlin [3]) but there exists an  $\aleph_2$ -Kurepa tree, and hence by the remarks above, the transfer principle  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  fails in L[G\*H].

On the other hand L[G\*H] satisfies " $2^{\aleph_0} = 2^{\aleph_1} = \kappa = \aleph_2$ ". Let  $\mathcal{L} = (U, F)$ , where U is a unary predicate symbol and F is a binary predicate symbol. let T be an  $\mathcal{L}$ -theory which says the following:

- (1)  $\forall x, y \ F(x, y) \to U(y)$ . In particular, for each x, F determines a subset  $F_x$  of U, namely,  $F_x = \{y : F(x, y)\}$ .
- (2) For all  $x \neq x', F_x \neq F_y$ .

Then T has an  $(\aleph_2, \aleph_0)$  model but it does not have an  $(\aleph_3, \aleph_1)$ -model (as otherwise we should have  $2^{\aleph_1} \geq \aleph_3$ ). Thus the transfer principle  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$  fails in L[G \* H].

We give an affirmative answer to this question by proving the following theorem:

<sup>&</sup>lt;sup>1</sup>On page 388 of [5], Silver writes "One can also get a GCH model in which  $(\aleph_7, \aleph_5) \to (\aleph_3, \aleph_1)$  fails and a GCH model which  $(\aleph_3, \aleph_1) \to (\aleph_7, \aleph_5)$  fails (though I don't see how to get the  $\to$  both ways to fail simultaneously)".

**Theorem 1.3.** Assume  $\kappa$  is a Mahlo cardinal. Then there is a generic extension of L, the Gödel's constructible universe, in which the GCH holds and the cardinal transfer principles  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$  and  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  fail.

Then we prove a general model theoretic fact, and use it to extend the above result to higher gaps:

**Theorem 1.4.** Assume  $\kappa$  is a Mahlo cardinal. Then there is a generic extension of L in which the GCH holds and for all  $n \geq 2$ , the cardinal transfer principles  $(\aleph_n, \aleph_0) \rightarrow (\aleph_{n+1}, \aleph_1)$  and  $(\aleph_{n+1}, \aleph_1) \rightarrow (\aleph_n, \aleph_0)$  fail.

Remark 1.5. Our proofs can be easily extended to get the following consistency result: assume  $\alpha < \beta$  are regular cardinals and assume there exists a Mahlo cardinal above them. Then in a generic extension of L, the GCH holds and both transfer principles  $(\alpha^{+n}, \alpha) \rightarrow (\beta^{+n}, \beta)$  and  $(\beta^{+n}, \beta) \rightarrow (\alpha^{+n}, \alpha)$  fail.

In Section 2 we prove Theorem 1.3 and in Section 3, we prove Theorem 1.4. In the last section, we discuss the same problem for the case of gap-1.

## 2. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

2.1. On a result of Jensen. In this subsection we state a result of Jensen [4] and mention some of its basic properties which are needed. Let  $\mathcal{L} = \{\in, A, \mathcal{C}\}$ , where A is a unary predicate and  $\mathcal{C}$  is a function symbol. Let  $T_J$  be the following theory in  $\mathcal{L}$ :

" $ZFC^- + GCH + A^+$  is the largest cardinal+ $\mathcal C$  is a  $\square_{A^+}$ -sequence".

By a  $(\kappa, \lambda)$ -model of  $T_J$  we mean a model  $\mathcal{M} = (M, \in^{\mathcal{M}}, A^{\mathcal{M}}, \mathcal{C}^{\mathcal{M}})$  of  $T_J$ , where  $|M| = \kappa$  and  $|A^{\mathcal{M}}| = \lambda$ .

**Theorem 2.1.** (Jensen [4]) Assume  $GCH + \lozenge_{\beta^+}$  holds, where  $\beta$  is a regular cardinal, and suppose  $\kappa > \beta$  is a Mahlo cardinal. Then there is a forcing notion  $\mathbb{P}_{\beta,\kappa}$  such that if K is  $\mathbb{P}_{\beta,\kappa}$ -generic over V, then the following hold in V[K]:

(a) 
$$V[K] \models \text{``GCH''}$$
.

- (b) The principle  $\Diamond_{\beta^+}^+$  holds.
- (c) The theory  $T_J$  does not have any  $(\beta^{++}, \beta)$ -model.

*Proof.* As requested by the referees, we sketch the proof of the theorem, by providing the forcing construction  $\mathbb{P}_{\beta,\kappa}$ , and refer to [4] for details. Let G be  $\operatorname{Col}(\beta^+, < \kappa)$ -generic over V, where

$$\operatorname{Col}(\beta^+,<\kappa)=\{p:\beta^+\times\kappa\to\kappa:|p|\leq\beta\text{ and for all }(\alpha,\lambda)\in\operatorname{dom}(p),\ p(\alpha,\lambda)<\lambda\}$$

is the Levy collapse. The next claim is standard.

Claim 2.2. (a) The forcing  $Col(\beta^+, < \kappa)$  is  $\beta^+$ -closed and  $\kappa$ -c.c.

- (b)  $V[G] \models \text{``}GCH + \lozenge_{\beta^+}\text{''}.$
- (c)  $V[G] \models "\kappa = \beta^{++} \text{ and } \square_{\beta^{++}} \text{ fails}".$

In [4], the following strengthening of Claim 2.2(c) is proved.

Claim 2.3. In V[G], the theory  $T_J$  has no  $(\beta^{++}, \beta)$ -model

From now on we work in V[G]. Let  $S = \langle S_{\alpha} : \alpha < \beta^{+} \rangle$  witness  $\Diamond_{\beta^{+}}$ . For each  $\alpha < \beta^{+}$  let  $d_{\alpha} : \beta \to \alpha$  be an onto function and set  $d = \langle d_{\alpha} : \alpha < \beta^{+} \rangle$ . For  $\alpha < \beta^{+}$  set

$$M_{\alpha} = L_{\gamma_{\alpha}}[\mathcal{S} \upharpoonright \alpha + 1, d \upharpoonright \alpha + 1],$$

where  $\gamma_{\alpha}$  is the least ordinal  $\gamma > \alpha$  such that  $\gamma > \sup_{\nu < \alpha} \gamma_{\nu}$  and

$$L_{\gamma}[S \upharpoonright \alpha + 1, d \upharpoonright \alpha + 1] \models "ZFC^{-}".$$

Define

$$\mathcal{S}^* = \langle S_{\alpha}^* : \alpha < \beta^+ \rangle,$$

where  $S_{\alpha}^* = P(\alpha) \cap M_{\alpha}$ . We find a generic extension of V[G] in which  $\mathcal{S}^*$  is a  $\Diamond_{\beta^+}^+$ -sequence. Let  $A \subseteq \kappa$  be such that  $L_{\kappa}[A] = H(\kappa)$  and define the sequence  $\langle \rho_{\nu} : \nu < \kappa \rangle$  by recursion on  $\nu$  as follows:  $\rho_{\nu}$  is the least ordinal  $\rho > \beta^+$  such that

- $\rho > \sup_{\xi < \nu} \rho_{\xi}$ .
- $\langle M_{\alpha} : \alpha < \beta^{+} \rangle \in L_{\rho}[A].$
- $cf(\rho) = \beta^+$ .

• 
$$L_{\rho}[A] \models "ZFC^- + \forall x, |x| \leq \beta^+$$
.

Set  $\tilde{\rho}_{\nu} = \beta^{+} \cup \sup_{\xi < \nu} \rho_{\xi}$ ,

$$\mathcal{U}_{\nu} = \langle L_{\rho_{\nu}}[A], \in, A \cap \rho_{\nu}, \langle M_{\alpha} : \alpha < \beta^{+} \rangle \rangle,$$

and for  $\nu > 0$  set

$$\tilde{\mathcal{U}}_{\nu} = \bigcup_{\xi < \nu} \tilde{\mathcal{U}}_{\xi} = \langle L_{\tilde{\rho}_{\nu}}[A], \in, A \cap \tilde{\rho}_{\nu}, \langle M_{\alpha} : \alpha < \beta^{+} \rangle \rangle.$$

Then set

$$f_{\nu} = \text{the } \mathcal{U}_{\nu}\text{-least bijection } f: \beta^{+} \leftrightarrow \tilde{\rho}_{\nu}.$$
  $a_{\xi} = \text{the } \xi\text{-th } a \subseteq \beta^{+} \text{ in } L_{\kappa}[A].$   $\tilde{a}_{\nu} = \{(\xi, \mu) : \xi \in a_{f_{\nu}(\mu)}\}.$ 

We are now ready to define the desired forcing notion, that we denote by  $Add(\Diamond_{\beta^+}^+)$ . First we define the forcing notions  $Add(\Diamond_{\beta^+}^+)_{\nu}, \nu < \kappa$ , which are the building blocks of the main forcing construction <sup>2</sup>.

A condition in  $Add(\Diamond_{\beta^+}^+)_{\nu}$  is a subset p of  $\beta^+$  such that

- (1)  $p \subseteq \beta^+$  is closed and bounded.
- (2)  $\alpha \in p \implies \tilde{a}_{\nu} \cap \alpha \in M_{\alpha}$ .

 $Add(\Diamond_{\beta^+}^+)_{\nu}$  is ordered by end extension:

$$p \le q \iff q = p \cap (\max(p) + 1).$$

Let us now define  $Add(\Diamond_{\beta^+}^+)$ . A condition in  $Add(\Diamond_{\beta^+}^+)$  is a function p such that

- (1)  $dom(p) \subseteq \kappa$  and  $|dom(p)| \leq \beta$ .
- (2)  $\forall \nu \in \text{dom}(p), p(\nu) \in Add(\Diamond_{\beta^+}^+)_{\nu}.$
- (3) If  $\nu \in \text{dom}(p)$ , then
  - (a)  $f_{\nu}''[\max(p(\nu))] \subseteq \text{dom}(p)$ .
  - (b) For each  $\xi \in f_{\nu}^{\prime\prime}[\max(p(\nu))], \ \max(p(\xi)) \ge \max(p(\nu)).$
  - (c)  $\alpha \in p(\nu) \implies \tilde{C}_{p,\nu} \cap \alpha \in M_{\alpha}$ , where

$$\tilde{C}_{p,\nu} = \{(\mu,\xi) \in \max(p(\nu)) \times \max(p(\nu)) : \mu \in p(f_{\nu}(\xi))\}.$$

<sup>&</sup>lt;sup>2</sup>In [4], the forcing notion  $Add(\Diamond_{\beta^+}^+)_{\nu}$  is denoted by  $\mathbb{P}^A_{\nu}$  and the forcing notion  $Add(\Diamond_{\beta^+}^+)$  is denoted by  $\mathbb{P}^A$ 

The forcing  $Add(\lozenge_{\beta^+}^+)$  is ordered as follows:  $p \leq q$  if and only if

$$\mathrm{dom}(p)\supseteq\mathrm{dom}(q)\text{ and for all }\nu\in\mathrm{dom}(q), p(\nu)\leq_{Add(\diamondsuit_{\beta^+}^+)_{\nu}}q(\nu).$$

Let H be  $Add(\Diamond_{\beta^+}^+)$ -generic over V[G]. The next claim is proved in [4].

Claim 2.4. (a)  $Add(\Diamond_{\beta^+}^+)$  is  $\beta^+$ -distributive and  $\kappa = \beta^{++}$ -c.c.".

- (b)  $V[G * H] \models \text{``GCH''}.$
- (c)  $S^*$  witnesses that  $\lozenge_{\beta^+}^+$  holds in V[G\*H].
- (d) The theory  $T_J$  does not have a  $(\beta^{++}, \beta)$ -model in V[G \* H].

Then 
$$\mathbb{P}_{\beta,\kappa} = \operatorname{Col}(\beta^+, <\kappa) * Add(\Diamond_{\beta^+}^+)$$
 is as required.

Suppose K = G \* H is  $\mathbb{P}_{\beta,\kappa}$ -generic over V. As  $\Diamond_{\beta^+}^+$  implies the existence of a  $\beta^+$ -Kurepa tree [2], in V[K], we have  $\beta^+$ -Kurepa trees.

2.2. Completing the proof of Theorem 1.3. In this subsection we complete the proof of Theorem 1.3. Thus assume V = L and let  $\kappa$  be a Mahlo cardinal. Let  $\lambda$  be the least inaccessible cardinal. So  $\lambda < \kappa$ . Let G be  $\operatorname{Col}(\aleph_1, < \lambda)$ -generic over L. Then:

**Lemma 2.5.** (a)  $L[G] \models$  "There are no  $\aleph_1$ -Kurepa trees".

- (b)  $L[G] \models$  "GCH holds".
- (c)  $L[G] \models$  "  $\kappa$  is a Mahlo cardinal".

*Proof.* (a) and (b) hold by [5], and (c) is clear, as the forcing  $Col(\aleph_1, < \lambda)$  has size  $< \kappa$ .  $\square$ 

Let K be  $\mathbb{P}^{L[G]}_{\aleph_1,\kappa}$ -generic over L[G]. We show that L[G\*K] is the required model. First note that by Theorem 2.1,

$$L[G * K] \models$$
 "there exists an  $\aleph_2$ -Kurepa tree".

But by Lemma 2.5,  $L[G] \models$  "There are no  $\aleph_1$ -Kurepa trees". On the other hand,  $L[G] \models$  " $\mathbb{P}_{\aleph_1,\kappa}$  is  $\lambda = \aleph_2$ -distributive", in particular

$$L[G*K] \models$$
 "There are no  $\aleph_1$ -Kurepa trees".

It follows that

$$L[G * K] \models$$
 " $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$  fails ".

On the other hand, by Theorem 2.1(b),  $L[G*K] \models \text{``}T_J$  does not have an  $(\aleph_3, \aleph_1)$ -model". We show that  $T_J$  has an  $(\aleph_2, \aleph_0)$ -model in L[G\*K]. First note that  $\aleph_2^{L[G*K]} = \lambda$ , which is inaccessible but not Mahlo in L, so it follows from results of Jensen and Solovay (see [2]) that  $\square_{\aleph_1}$  holds in both L[G] and L[G\*K]. Let  $\mathcal{C} = \langle C_\alpha : \alpha < \lambda, \lim(\alpha) \rangle \in L[G]$  witness this. Consider the model

$$\mathcal{M} = (H(\lambda)^{L[G]}, \in, \aleph_0, \mathcal{C}),$$

where  $\aleph_0$  is considered as the interpretation of A. Then  $\mathcal{M}$  is an  $(\aleph_2, \aleph_0)$ -model of T. So

$$L[G * K] \models$$
 " $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$  fails ".

The theorem follows.

3. A GENERAL MODEL THEORETIC FACT AND THE PROOF OF THEOREM 1.4

In this section we prove a general model theoretic fact, and use it to prove Theorem 1.4.

3.1. A general model theoretic fact. In this subsection we prove the following lemma and consider some of its consequences.

**Lemma 3.1.** Assume  $n \geq 1$ ,  $\mathcal{L}$  is a first order language which contains a unary predicate U, and T is a theory in  $\mathcal{L}$ . Then there are  $\mathcal{L}^+ \supseteq \mathcal{L}$  and a theory  $T^+$  in  $\mathcal{L}^+$ , such that for all infinite cardinals  $\beta$ :

$$T$$
 has a  $(\beta^{+n}, \beta)$ -model  $\iff T^+$  has a  $(\beta^{+n+1}, \beta)$ -model.

Proof. Let  $\mathcal{L}^+ = \mathcal{L} \cup \{<, W_0, \dots, W_n, F_{-1}, F_0, \dots, F_n\}$  where < is a binary predicate symbol,  $W_i$ 's are unary predicate symbols,  $F_{-1}$  is a binary predicate symbol and  $F_i$ 's,  $0 \le i \le n$ , are ternary predicate symbols. Let  $T^+$  consists of the following axioms:

- (1)  $\phi^{W_n}$ , for each  $\phi \in T$ , where  $\phi^{W_n}$  is the relativization of  $\phi$  to  $W_n$ .
- (2) < is a linear ordering of the universe.
- (3) Under <, each  $W_i$  is an initial segment of  $W_{i+1}, i < n$ , and  $W_n$  is an initial segment of the universe (in particular  $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n$ ).
- (4)  $U \subseteq W_n$  (i.e.,  $\forall x(U(x) \to W_n(x))$ ).
- (5)  $F_{-1} \subseteq U \times W_0$  defines a bijection from U onto  $W_0$ .

- (6) For each  $0 \le i < n, F_i \subseteq (W_{i+1} \setminus W_i) \times W_i \times W_{i+1}$  is such that if  $x \in W_{i+1} \setminus W_i$ , then  $\{(y, z) : F_i(x, y, z)\}$  is a bijection from  $W_i$  onto  $\{z \in W_{i+1} : z < x\}$ .
- (7)  $F_n$  is such that if  $x \notin W_n$ , then  $\{(y,z) : F_n(x,y,z)\}$  is a bijection from  $W_n$  onto  $\{z : z < x\}$ .

Now suppose that T has a  $(\beta^{+n}, \beta)$ -model  $\mathcal{M} = (\beta^{+n}, U^{\mathcal{M}}, \dots)$ . Consider the model

$$\mathcal{M}^+ = (\beta^{+n+1}, \mathcal{M}, <, \beta, \dots, \beta^{+n}, f_{-1}, f_0, \dots, f_n),$$

where  $f_{-1}: U^{\mathcal{M}} \leftrightarrow \beta$ , each  $f_i, 0 \leq i \leq n$  is such that for each  $\beta^{+i} \leq \gamma < \beta^{+i+1}, \{(\zeta, \eta) : (\gamma, \zeta, \eta) \in f_i\}$  defines a bijection  $\beta^{+i} \leftrightarrow \gamma$ . It is easily seen that  $\mathcal{M}^+$  is a  $(\beta^{+n+1}, \beta)$ -model for  $T^+$ .

Conversely assume that  $\mathcal{M}^+$  is a  $(\beta^{+n+1},\beta)$ -model for  $T^+$ . Consider the model  $\mathcal{M}$  which is obtained from  $\mathcal{M}^+ \upharpoonright \mathcal{L}$ , by replacing its universe with  $W_n^{\mathcal{M}^+}$ . It follows from (1) that  $\mathcal{M}$  is a model of T. We show that it is a  $(\beta^{+n},\beta)$ -model. We have  $U^{\mathcal{M}} = U^{\mathcal{M}^+}$ , which has size  $\beta$ . On the other hand, axioms (4)-(6) can be used to show that  $|W_0^{\mathcal{M}^+}| = \beta$ ,  $|W_{i+1}^{\mathcal{M}^+}| \leq |W_i^{\mathcal{M}^+}|^+$  and  $|W_m^{\mathcal{M}^+}| \geq \beta^{+n}$ , so by induction on  $i \leq n$ , we have  $|W_i^{\mathcal{M}^+}| = \beta^{+i}$ . In particular  $|W_n^{\mathcal{M}^+}| = \beta^{+n}$ , and the result follows.

**Corollary 3.2.** For each  $n \geq 1$ , the gap-(n + 1)-cardinal transfer principle implies the gap-n-cardinal transfer principle.

Remark 3.3. In personal communication, Ali Enayat informed us that Corollary 3.2 is an immediate consequence of the downward Löwenheim-Skolem theorem, i.e., the fact that if  $\mathcal{M} = (M, \ldots)$  is an infinite structure in a countable language and X is any subset of M, then there is an elementary substructure  $\mathcal{M}_0 = (M_0, \ldots)$  of  $\mathcal{M}$  that includes X and whose cardinality is  $\max\{\aleph_0, |X|\}$ . Using this theorem, it is easy to see that every model  $\mathcal{M}$  that exhibits a gap-m model, say  $(\kappa^{+m}, \kappa)$ , for some m > 0 has an elementary sub-model  $\mathcal{M}_0$  that exhibits a gap-m model  $(\kappa^{+n}, \kappa)$  for all n < m.

3.2. **Proof of Theorem 1.4.** In this subsection we complete the proof of Theorem 1.4. Let L[G\*H] be the model obtained in Subsection 2.2. So in L[G\*H] both transfer principles  $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$  and  $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$  fail. So, by induction, and using Lemma 3.1, for

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each  $n \geq 2$ , the transfer principles

$$(\aleph_n, \aleph_0) \to (\aleph_{n+1}, \aleph_1)$$

and

$$(\aleph_{n+1}, \aleph_1) \to (\aleph_n, \aleph_0)$$

fail in L[G\*H].

## 4. The case of gap-1 and some problems

In general, we can not hope to prove a result as above for gap-1-cardinal transfer principles. This is because of Vaught's theorem [7] that the transfer principle  $(\beta^+, \beta) \to (\aleph_1, \aleph_0)$  is a theorem of ZFC. However we do not know the answer to the following question:

**Question 4.1.** Is it consistent that both transfer principles  $(\aleph_2, \aleph_1) \to (\aleph_3, \aleph_2)$  and  $(\aleph_3, \aleph_2) \to (\aleph_2, \aleph_1)$  fail simultaneously.

As we showed in Corollary 3.2, the gap-(n + 1)-cardinal transfer principle implies the gap-n-cardinal transfer principle.

On the other hand if L[G] is a generic extension of L by the Levy collapse of an inaccessible cardinal  $\kappa$  to  $\aleph_2$ , then it follows from results of Vaught [7], Chang [1] and Jensen [2] that the gap-1-cardinal transfer principle holds in L[G], while by Silver's result stated in the introduction, the gap-2-cardinal transfer principle fails in L[G]. We do not know the answer for higher gaps.

**Question 4.2.** Assume n > 1. Is it consistent that the gap-n-cardinal transfer principle holds while the gap-(n + 1)-cardinal transfer principle fails?

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School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran.

E-mail address: golshani.m@gmail.com

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran.

E-mail address: sh.mohsenipour@gmail.com