

# An inner model theoretic proof of Becker's theorem

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## Abstract

We re-prove Becker's theorem from [1] by showing that  $AD^{L(\mathbb{R})}$  implies that  $L(\mathbb{R}) \models$  “ $\omega_2$  is  $\delta_1^2$ -supercompact”. Our proof uses inner model theoretic tools instead of Baire category. We also show that  $\omega_2$  is  $< \Theta$ -strongly compact.

This article draws inspiration from the work of Neeman ([5]) who, using inner model theoretic tools, showed that under  $AD^{L(\mathbb{R})}$ ,  $\omega_1$  is  $< \Theta$ -supercompact. We have also been influenced by the work of Becker ([1]), Becker-Jackson ([2]) and Jackson ([4]). In [1], Becker showed that assuming  $AD+V = L(\mathbb{R})$ ,  $\omega_2$  is  $\delta_1^2$ -supercompact. In [2], Becker and Jackson showed that, under  $AD+V = L(\mathbb{R})$ , all projective cardinals are  $\delta_1^2$ -supercompact. Finally, in [4], Jackson showed that under  $AD+V = L(\mathbb{R})$  all Suslin cardinals and their successors are  $\delta_1^2$ -supercompact.

In this short note, we re-prove Becker's theorem using inner model theoretic tools. The paper assumes familiarity with what is commonly called HOD analysis. The reader can find this background exposted in [5] and in [8]. The point of re-proving such results is to find more applications of inner model theory in descriptive set theory. In particular, we strongly believe that connecting iteration sets with Kechris-Woodin generic codes will yield many applications, and thus invite the community to consider Conjecture 3.1.

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## 1 Measures on $\mathcal{P}_{\omega_2}(\lambda)$

We do not want to make the paper artificially long. The paper is aimed at experts of inner model theory, those who are familiar with the terminology of [8].

We assume  $AD + V = L(\mathbb{R})$ . Fix  $\lambda < \Theta$ . Let  $A$  be an  $OD$  set of reals such that  $\gamma_{A,\infty} \geq \lambda$ . Suppose  $\mathcal{R}$  is a suitable premouse that is  $A$ -iterable. It is customary to let  $\delta^{\mathcal{R}}$  be the Woodin cardinal of  $\mathcal{R}$ . Assume that  $\lambda \in \text{rng}(\pi_{(\mathcal{R},A),\infty})$ . We then let  $\lambda^{\mathcal{R}}$  be such that  $\pi_{(\mathcal{R},A),\infty}(\lambda^{\mathcal{R}}) = \lambda$ .

We let  $\text{Code}(A, \lambda) \subseteq \mathbb{R}$  be the set of reals  $x$  such that  $x$  codes a pair  $(\mathcal{R}_x, \alpha_x)$  such that  $\mathcal{R}_x$  is an  $A$ -iterable suitable pre-mouse such that  $\lambda^{\mathcal{R}}$  is defined and  $\alpha_x < \lambda^{\mathcal{R}}$ . Let  $\leq_{A,\lambda}$  be the natural pre-wellordering of  $\text{Code}(A, \lambda)$  given by:  $x \leq_{A,\lambda} y$  if and only if whenever  $S$  is an  $A$ -iterate of  $\mathcal{R}_x$  and an  $A$ -iterate of  $\mathcal{R}_y$ ,  $\pi_{(\mathcal{R}_x,A),(S,A)}(\alpha_x) \leq \pi_{(\mathcal{R}_y,A),(S,A)}(\alpha_y)$ . We have that  $\leq_{A,\lambda}$  has length  $\lambda$ . Given  $x \in \text{Code}(A, \lambda)$  let

$$c(x) = \pi_{(\mathcal{R}_x,A),\infty}(\alpha_x) = |x|_{\leq_{A,\lambda}}.$$

Let  $S$  be a tree of a  $\Sigma_1^2$ -scale on a universal  $\Sigma_1^2$ -set. Given  $x$  and  $y$  we write  $x \sim_S y$  if and only if  $x \in L[S, y]$  and  $y \in L[S, x]$ . We then say that  $d$  is an  $S$ -degree if  $d$  is an  $\sim_S$ -class. We write  $d \leq_S e$  if  $d \in L[S, e]$ . Let now  $C(A, \lambda) = \{d : \text{Code}(A, \lambda) \cap HC^{L[S,d]} \neq \emptyset\}$ . The following are two key points to keep in mind:

1.  $\sim_S$  is an equivalence relation,
2.  $C(A, \lambda)$  contains an  $S$ -cone, i.e., there is an  $S$ -degree  $e$  such that whenever  $e \leq_S d$ ,  $d \in C(A, \lambda)$ .

The following is a corollary to the Harrington-Kechris theorem (see [3], and see [6] and the references there for some uses of it).

**Corollary 1.1** *There is a formula  $\phi$  such that whenever  $d \in C(A, \lambda)$ ,  $g$  is  $< \omega_1^V$ -generic over  $L[S, d]$  and  $\mathcal{R} \in L_{\omega_1^V}[S, d][g]$ ,*

$$\mathcal{R} \text{ is a suitable pre-mouse if and only if } L[S, d][g] \models \phi[\mathcal{R}].$$

Moreover, there is a formula  $\psi$  such that for any  $A$ -iterable suitable  $\mathcal{Q}, \mathcal{R} \in L_{\omega_1^V}[S][g]$  and for any  $\pi$ ,

$$\mathcal{R} \text{ is an } A\text{-iterate of } \mathcal{Q} \text{ and } \pi : H_A^{\mathcal{Q}} \rightarrow H_A^{\mathcal{R}} \text{ is the } A\text{-iteration embedding if and only if } L[S, d][g] \models \psi[\mathcal{Q}, \mathcal{R}, \pi, \tau_A],$$

where  $\tau_A$  is the term relation for  $A$  in  $L[S, d]^{Coll(\omega, < \omega_1^V)}$ .

The formulas  $\phi$  and  $\psi$  essentially repeat the definitions of suitability and  $A$ -iterability. Another important lemma that we need is a consequence of what is usually called *generic comparisons* (see [8]). The proof is a standard generic comparison argument which we leave to the reader.

**Lemma 1.2** *Suppose  $d \in C(A, \lambda)$  and  $g$  is  $< \omega_1^V$ -generic over  $L[S, d]$ . Suppose  $\phi$  is as in Corollary 1.1, and for some  $\mathcal{R} \in L[S, d][g]$ ,  $L[S, d][g] \models \phi[\mathcal{Q}, \mathcal{R}]$ . Then there is an  $\emptyset$ -iterate  $\mathcal{S}$  of  $\mathcal{R}$  such that  $\mathcal{S} \in L_{\omega_1^V}[S, d]$ .*

Given  $d \in C(A, \lambda)$  we let  $B_d$  be the set of  $\beta$  such that there is  $x \in \text{Code}(A, \lambda)$  with the property that  $(\mathcal{R}_x, \alpha_x) \in L[S, d]$  and  $c(\mathcal{R}_x, \alpha_x) = \beta$ . As  $|L_{\omega_1^V}[S, d]| = \omega_1^V$ , we have that  $B_d \in \wp_{\omega_2}(\lambda)$ . Lemma 1.2 has the following easy corollary.

**Corollary 1.3** *Suppose  $d_0 \in C(A, \lambda)$  and  $d$  is a  $S$ -degree such that  $L[S, d]$  is a  $< \omega_1^V$ -generic extension of  $L[S, d_0]$ . Then  $B_{d_0} = B_d$ .*

We now define  $\mu(A, \lambda)$  on  $\wp_{\omega_2}(\lambda)$  by setting  $B \in \mu(A, \lambda)$  if and only if for an  $S$ -cone of  $d$ ,  $B_d \in B$ .

**Lemma 1.4**  *$\mu(A, \lambda)$  is an  $\omega_2$ -complete ultrafilter on  $\wp_{\omega_2}(\lambda)$ .*

*Proof.* Clearly  $\mu(A, \lambda)$  is an ultrafilter. Let  $(B_\xi : \xi < \omega_1)$  be such that  $B_\xi \in \mu(A, \lambda)$  for all  $\xi < \omega_1$ . Let  $WO$  be the set of reals coding a countable ordinal. Using the coding lemma we can find  $y \in \mathbb{R}$  and a  $\Sigma_2^1(y)$ -set  $D \subseteq WO \times \mathbb{R}$  such that

1.  $[y]_S \in C(A, \lambda)$ ,
2. for every  $x \in WO$ ,  $D_x \neq \emptyset$  (here  $D_x = \{z : (x, z) \in D\}$ ),
3. for every  $x \in WO$ ,  $D_x \subseteq \{z : [z]_S \text{ is a base of a cone witnessing that } B_{|x|} \in \mu(A, \lambda)\}^1$ .

Let  $d \in C$  be such that  $y \in L[S, d]$ . We claim that  $B_d \in B_\xi$  for every  $\xi < \omega_1$ . To see this, fix  $\xi < \omega_1$ . Let  $g \subseteq \text{Coll}(\omega, \xi)$  be  $L[S, d]$ -generic and  $u$  be a real such that  $L[S, d][g] = L[S, u]$ . Let  $x \in \mathbb{R}^{L[S, u]}$  be such that  $|x| = \xi$ . Because  $D$  is  $\Sigma_2^1(y)$  we have that there is  $z \in D_x \cap L[S, u]$ . Because  $[z]_S \leq [u]_S$ , we must have that  $B_{[u]_S} \in B_\xi$ . However, it follows from Corollary 1.3 that  $B_d = B_{[u]_S}$ . Hence,  $B_d \in B_\xi$ .

As  $d$  was arbitrary, we have shown that for any  $d$  that is  $S$ -above  $[y]_S$ ,  $B_d \in \cap_{\xi < \omega_1} B_\xi$ . It follows that  $\cap_{\xi < \omega_1} B_\xi \in \mu(A, \lambda)$ .  $\square$

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<sup>1</sup>where  $[z]_S$  is the  $S$ -degree given by  $z$  and  $|x|$  is the ordinal coded by  $x$ .

## 2 $\omega_2$ is $\delta_1^2$ -supercompact and $< \Theta$ -strongly compact

**Proposition 2.1** *For every  $\lambda < \Theta$  and an ordinal definable  $A \subseteq \mathbb{R}$  such that  $\gamma_{A,\infty} \geq \lambda$ ,  $\mu(A, \lambda)$  is superfine, i.e., for every  $B \in \wp_{\omega_2}(\lambda)$ ,*

$$\{D \in \wp_{\omega_2}(\lambda) : B \subseteq D\} \in \mu(A, \lambda).$$

*Proof.* Fix  $B$  and let  $f : \omega_1 \rightarrow B$  be a bijection. Let  $B_\xi = \{x \in \text{Code}(A, \lambda) : c(x) = f(\xi)\}$ . Using the coding lemma find  $y \in \mathbb{R}$  and  $D \subseteq WO \times \mathbb{R}$  such that

1.  $[y]_S \in C(A, \lambda)$ ,
2.  $D \in \Sigma_2^1(y)$ ,
3. for every  $x \in WO$ ,  $D_x \neq \emptyset$ ,
4. for every  $x \in WO$ ,  $D_x \subseteq \{z \in \text{Code}(A, \lambda) : c(z) = f(|x|)\}$ .

We claim that for every  $d$  such that  $[y]_S \leq_S d$ ,  $B \subseteq B_d$ . To see this, fix  $d$  such that  $[y]_S \leq_S d$ . Fix  $\zeta \in B$ . We want to see that  $\zeta \in B_d$ . Let  $\xi = f^{-1}(\zeta)$ , and fix  $u \in \mathbb{R}$  such that  $L[S, u]$  is a generic extension of  $L[S, d]$  and  $\xi$  is countable in  $L[S, u]$ . Fix  $x \in WO \cap L[S, u]$  such that  $|x| = \xi$ . Because  $D \in \Sigma_2^1(y)$ , we have that  $D_x \cap L[S, u] \neq \emptyset$ . Fix then  $z \in \text{Code}(A, \lambda) \cap D_x \in L[S, u]$ . It follows that  $c(z) = f(\xi)$ . Since  $c(z) \in B_{[u]_T} = B_d$ , we have that  $\zeta \in B_d$ .  $\square$

Putting Proposition 1.4 and Proposition 2.1 we get the following corollary.

**Corollary 2.2** *Assume  $AD+V = L(\mathbb{R})$ . Then  $\omega_2$  is a  $< \Theta$ -strongly compact. More precisely, for every  $\lambda < \Theta$  there an  $\omega_2$ -complete superfine ordinal definable ultrafilter on  $\wp_{\omega_2}(\lambda)$ .*

**Theorem 2.3 (Becker, [1])** *Assume  $AD+V = L(\mathbb{R})$ . Then  $\omega_2$  is  $\delta_1^2$ -supercompact.*

*Proof.* Set  $\lambda = \delta_1^2$ . Suppose  $\mathcal{R}$  is an  $\emptyset$ -iterable suitable pre-mouse. Recall that if  $\nu$  is the least cardinal that is  $< \delta^{\mathcal{R}}$ -strong in  $\mathcal{R}$  then  $\pi_{(\mathcal{R}, \emptyset), \infty}(\nu) = \lambda$  (see [7, Chapter 8]). We now want to show that  $\mu =_{\text{def}} \mu(\emptyset, \lambda)$  is an  $\omega_2$ -supercompactness measure. Proposition 1.4 shows that  $\mu$  is  $\omega_2$ -complete and Proposition 2.1 shows that  $\mu$  is fine. It remains to show that  $\mu$  is normal. The following lemma is the first step towards normality. Set  $\text{Code} =_{\text{def}} \text{Code}(\emptyset, \lambda)$  and  $\leq^* = \leq_{\emptyset, \lambda}$ .

**Lemma 2.4** *Suppose  $F : \wp_{\omega_2}(\lambda) \rightarrow \lambda$  is such that for an  $S$ -cone of  $d$ ,  $F(B_d) \in B_d$ . Then for an  $S$ -cone of  $d$  there is  $x \in (\mathbb{R}^{L[S, d]} \cap \text{Code})$  such that  $c(x) = F(B_d)$ .*

*Proof.* Assume not. Fix an  $S$ -degree  $d_0$  such that whenever  $d$  is  $S$ -above  $d_0$ , for every  $x \in (\mathbb{R}^{L[S,d]} \cap \text{Code})$ ,  $c(x) \neq F(B_d)$ . Fix  $(\mathcal{R}, \alpha) \in L[S, d_0]$  such that  $\pi_{(\mathcal{R}, \emptyset), \infty}(\alpha) = F(B_{d_0})$ .

Let  $\nu < \omega_1$  be any cardinal of  $L[S, d_0]$  such that  $(\mathcal{R}, \alpha) \in L_\nu[S, d_0]$  and let  $g \subseteq \text{Coll}(\omega, (\nu^+)^{L[S, d_0]})$  be  $L[S, d_0]$ -generic. Let  $x \in \mathbb{R}$  be such that  $L[S, d_0][g] = L[S, x]$ . We then have that  $B_{d_0} = B_{[x]_S}$  (see Corollary 1.1). This is a contradiction as we can find  $y \in L[S, x] \cap \mathbb{R}$  coding  $(\mathcal{R}, \alpha)$ .  $\square$

**Lemma 2.5**  $\mu$  is normal.

*Proof.* Suppose  $\mu$  is not normal. Let  $F : \wp_{\omega_2}(\lambda) \rightarrow \lambda$  be such that for an  $S$ -cone of  $d$ ,  $F(B_d) \in B_d$  but  $F$  is not constant on a  $\mu$ -measure one set. Let  $e_0 \in C$  be a base for the cone of the previous sentence.

Let  $e \in C$  be  $S$ -above  $e_0$  and such that for every  $d$  such that  $e \in L[S, d]$ , there is  $x \in (\mathbb{R}^{L[S,d]} \cap \text{Code})$  with the property that  $c(x) = F(B_d)$ . We now follow an idea of Becker from [1].

Given an ordinal  $\xi < \lambda$  let  $D_\xi = \{d : F(B_d) \neq \xi\}$ . We have that for each  $\xi$ ,  $D_\xi$  contains an  $S$ -cone. Let then  $C_\xi = \{x \in \mathbb{R} : [x]_S \text{ is a base of a cone contained in } D_\xi\}$ . It follows from the coding lemma that there is a real  $y$  and a set  $D$  such that

1.  $e \leq_S [y]_S$ ,
2.  $H \subseteq \text{Code} \times \mathbb{R}$  is  $\Sigma_1^2(y)$ ,
3. if  $(x, z) \in H$  then  $z \in C_{c(x)}$ ,
4. for every  $x \in \text{Code}$  there is  $z$  such that  $(x, z) \in H$ .

Set  $d = [y]_S$ . Let  $x \in \text{Code} \cap L[S, d]$  be such that  $c(x) = F(B_d)$ . Because  $H$  is  $\Sigma_1^2(y)$  we have that  $H_x \cap L[S, d] \neq \emptyset$ . Let then  $z \in H_x \cap L[S, d]$ . It follows that  $[z]_S \leq_S d$ . Hence,  $d \in H_{c(x)}$ . It follows that  $F(B_d) \neq c(x) = F(B_d)$ , contradiction.  $\square$

$\square$

### 3 A covering conjecture

Again we assume  $AD + V = L(\mathbb{R})$ . Suppose  $\kappa < \lambda < \Theta$  and  $A$  is an ordinal definable set of reals such that  $\gamma_{A,\infty} \geq \lambda$ . Given  $X \in \wp_\kappa(\lambda)$  we say that  $X$  is an  $A$ -iteration set if for every  $\alpha \in X$  there is an  $A$ -iterable  $\mathcal{Q}$  such that  $\alpha \in \text{rng}(\pi_{(\mathcal{Q},A),\infty})$  and  $\pi_{(\mathcal{Q},A),\infty}[\gamma_A^{\mathcal{Q}}] \cap \lambda \subseteq X$ .

**Conjecture 3.1** *Assume  $AD + V = L(\mathbb{R})$ . Suppose  $\kappa$  is either a Suslin cardinal or a successor of a Suslin cardinal. Then for every  $\lambda < \Theta$ , an OD set  $A \subseteq \mathbb{R}$  such that  $\lambda \leq \gamma_{A,\infty}$ , and  $B \in \wp_\kappa(\lambda)$  there is an  $A$ -iteration set  $X \in \wp_\kappa(\lambda)$  such that  $B \subseteq X$ .*

The conjecture is clearly true for  $\kappa = \omega_1$ . Proposition 2.1 shows that the conjecture is true for  $\kappa = \omega_2$ . We expect that the validity of the full conjecture will follow once a link is made between Kechris-Woodin generic codes and iteration sets.

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