# REVERSIBILITY OF EXTREME RELATIONAL STRUCTURES 

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#### Abstract

A relational structure $\mathbb{X}$ is called reversible iff each bijective homomorphism from $\mathbb{X}$ onto $\mathbb{X}$ is an isomorphism, and linear orders are prototypical examples of such structures. One way to detect new reversible structures of a given relational language $L$ is to notice that the maximal or minimal elements of isomorphism-invariant sets of interpretations of the language $L$ on a fixed domain $X$ determine reversible structures. We isolate certain syntactical conditions providing that a consistent $L_{\infty \omega}$-theory defines a class of interpretations having extreme elements on a fixed domain and detect several classes of reversible structures. In particular, we characterize the reversible countable ultrahomogeneous graphs.


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## 1 Introduction

Generally speaking, a structure is reversible iff each bijective endomorphism of that structure is an automorphism. Several prominent structures have this property; for example, each compact Hausdorff space $\mathcal{X}$ is reversible (because each continuous bijection $f: \mathcal{X} \rightarrow \mathcal{X}$ is a closed mapping and, hence, a homeomorphism) and, similarly, each linear order $\mathbb{X}$ is a reversible relational structure (since an increasing bijection $f: \mathbb{X} \rightarrow \mathbb{X}$ must be an isomorphism).

The reversible structures mentioned above are extreme: compact Hausdorff topologies are, on one hand, maximal compact and, on the other hand, minimal Hausdorff topologies, and linear orders are maximal partial orders. In this paper, searching for reversible structures, we investigate this phenomenon in the class of relational structures. So throughout the paper we assume that $L=\left\langle R_{i}: i \in I\right\rangle$ is a relational language, where $\operatorname{ar}\left(R_{i}\right)=n_{i} \in \mathbb{N}$, for $i \in I$, that $X$ is a non-empty set and $\operatorname{Int}_{L}(X)=\prod_{i \in I} P\left(X^{n_{i}}\right)$ the set of all interpretations of the language $L$, over the domain $X$. An interpretation $\rho=\left\langle\rho_{i}: i \in I\right\rangle \in \operatorname{Int}_{L}(X)$ will be called reversible iff $\langle X, \rho\rangle$ is a reversible structure.

First in Section 3] we easily establish the reversibility of minimal and maximal elements of the poset $\langle\mathcal{C}, \subset\rangle$, where $\mathcal{C} \subset \operatorname{Int}_{L}(X)$ is an isomorphism-invariant set, and, in particular, if $\mathcal{C}$ is of the form $\operatorname{Int}_{L}^{\mathcal{T}}(X)=\left\{\rho \in \operatorname{Int}_{L}(X):\langle X, \rho\rangle \models \mathcal{T}\right\}$, for some set $\mathcal{T}$ of sentences of the infinitary language $L_{\infty \omega}$. Of course, there are sets of the form $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ having neither minimal nor maximal elements, and, hence, in Section 4 we isolate a class of formulas $\mathcal{F}$ such that the set of maximal elements of the poset $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ is co-dense, whenever $\mathcal{T} \subset \mathcal{F}$, and prove a dual statement concerning minimal elements. We note that it is not our goal to find a syntactical characterization of the largest class $\mathcal{F}$ with the property mentioned above, because, for example, for a countable language $L$, each isomorphism-invariant set $\mathcal{C} \subset \operatorname{Int}_{L}(\omega)$ is of the form $\operatorname{Int}_{L}^{\{\varphi\}}(\omega)$, where $\varphi$ is the disjunction of the Scott sentences of the structures belonging to $\mathcal{C}$ and, trivially, the $\operatorname{set}^{\operatorname{Int}}{ }_{L}^{\left\{\varphi \vee \varphi_{m}\right\}}(\omega)$, where $\varphi_{m}:=\bigwedge_{i \in I} \forall \bar{v} R_{i}(\bar{v})$ has a largest element, $\left\langle X^{n_{i}}: i \in I\right\rangle$. Our goal is to find a reasonable class of sentences providing relevant examples of reversible structures.

Sections 5 and 6 contain some applications of the results mentioned above. In particular it is shown that the concept of "forbidden finite substructures" provides a large class of extreme (and, hence, reversible) structures. Clearly, one thing is to prove that extreme interpretations exist and the other is to find (or characterize) them. Some results on this topic are given in examples.

## 2 Preliminaries

The algebra of interpretations Abusing notation, for $\rho, \sigma \in \operatorname{Int}_{L}(X)$ we will write $\rho \subset \sigma$ iff $\rho_{i} \subset \sigma_{i}$, for all $i \in I$. Clearly $\left\langle\operatorname{Int}_{L}(X), \subset\right\rangle$ is a Boolean lattice and, abusing notation again, the operations in the corresponding Boolean algebra will be denoted in the following way: if $\rho^{j} \in \operatorname{Int}_{L}(X)$, for $j \in J$, then $\bigcap_{j \in J} \rho^{j}:=$ $\left\langle\bigcap \bigcap_{j \in J} \rho_{i}^{j}: i \in I\right\rangle, \bigcup_{j \in J} \rho^{j}:=\left\langle\bigcup_{j \in J} \rho_{i}^{j}: i \in I\right\rangle, \rho^{c}:=\left\langle X^{n_{i}} \backslash \rho_{i}: i \in I\right\rangle$, $0:=\langle\emptyset: i \in I\rangle$ and $1:=\left\langle X^{n_{i}}: i \in I\right\rangle$.

Direct and inverse images of interpretations If $X$ and $Y$ are non-empty sets and $n \geq 2$, the $n$-th power of a mapping $f: X \rightarrow Y$ is the mapping $f^{n}: X^{n} \rightarrow Y^{n}$ defined by $f^{n}\left(\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)=\left\langle f\left(x_{1}\right), \ldots, f\left(x_{n}\right)\right\rangle$, for each $\left\langle x_{1}, \ldots, x_{n}\right\rangle \in X^{n}$. Clearly, $f$ is an injection (surjection) iff $f^{n}$ is an injection (surjection).

For $L$-interpretations $\rho=\left\langle\rho_{i}: i \in I\right\rangle \in \operatorname{Int}_{L}(X)$ and $\sigma=\left\langle\sigma_{i}: i \in I\right\rangle \in$ $\operatorname{Int}_{L}(Y)$ the interpretations $f[\rho] \in \operatorname{Int}_{L}(Y)$ and $f^{-1}[\sigma] \in \operatorname{Int}_{L}(X)$ are defined by

$$
\begin{equation*}
f[\rho]:=\left\langle f^{n_{i}}\left[\rho_{i}\right]: i \in I\right\rangle \text { and } f^{-1}[\sigma]:=\left\langle\left(f^{n_{i}}\right)^{-1}\left[\sigma_{i}\right]: i \in I\right\rangle, \tag{1}
\end{equation*}
$$

and these operators have all properties of direct and inverse images: $\rho \subset f^{-1}[f[\rho]]$, $f\left[f^{-1}[\sigma]\right]=\sigma \cap f[1], f\left[\bigcap_{j \in J} \rho^{j}\right] \subset \bigcap_{j \in J} f\left[\rho^{j}\right], f^{-1}\left[\rho^{c}\right]=\left(f^{-1}[\rho]\right)^{c}$, etc.

Morphisms Bijective homomorphisms will be called condensations. Working with elements of $\operatorname{Int}_{L}(X)$, instead of $\operatorname{Hom}(\langle X, \rho\rangle,\langle X, \sigma\rangle)$ we will write $\operatorname{Hom}(\rho, \sigma)$. Also, instead of $\langle X, \rho\rangle \cong\langle X, \sigma\rangle$ we will shortly write $\rho \cong \sigma$ and regard $\cong$ as an equivalence relation on the set $\operatorname{Int}_{L}(X)$. Let $[\rho] \cong:=\left\{\sigma \in \operatorname{Int}_{L}(X): \sigma \cong \rho\right\}$.

Fact 2.1 For each $\rho=\left\langle\rho_{i}: i \in I\right\rangle, \sigma=\left\langle\sigma_{i}: i \in I\right\rangle \in \operatorname{Int}_{L}(X)$ we have:
(a) $\operatorname{Hom}(\rho, \sigma)=\left\{f \in{ }^{X} X: f[\rho] \subset \sigma\right\}$;
(b) $[\rho] \cong=\{f[\rho]: f \in \operatorname{Sym}(X)\}$.

The condensation order and reversibility if $\mathbb{P}=\langle P, \leq\rangle$ is a partial order, a subset $C$ of $P$ is called convex iff $p \leq q \leq r$ and $p, r \in C$ implies $q \in C$. A set $A \subset P$ is called an antichain iff different elements of $A$ are incomparable. Clearly, each antichain is convex and $\operatorname{Conv}_{\mathbb{P}}(A)=\left\{p \in P: \exists a^{\prime}, a^{\prime \prime} \in A \quad a^{\prime} \leq p \leq a^{\prime \prime}\right\}$ is the minimal convex set containing the set $A \subset P$ (the convex closure of $A$ ).

Here we recall some facts from [8, 9, 10]. Let $\preccurlyeq_{c}$ be the pre-order on the set $\operatorname{Int}_{L}(X)$ defined by: $\rho \preccurlyeq_{c} \sigma$ iff there is a condensation $f:\langle X, \rho\rangle \rightarrow\langle X, \sigma\rangle$. The corresponding antisymmetric quotient, the poset $\left\langle\operatorname{Int}_{L}(X) / \sim_{c}, \leq_{c}\right\rangle$, where $\rho \sim_{c} \sigma \Leftrightarrow \rho \preccurlyeq_{c} \sigma \wedge \sigma \preccurlyeq_{c} \rho$ and $[\rho]_{\sim_{c}} \leq_{c}[\sigma]_{\sim_{c}} \Leftrightarrow \rho \preccurlyeq_{c} \sigma$, for $\rho, \sigma \in \operatorname{Int}_{L}(X)$, is called the condensation order. Defining $[\rho]_{\sim_{c}}:=\left\{\sigma \in \operatorname{Int}_{L}(X): \sigma \sim_{c} \rho\right\}$, for each $\rho \in \operatorname{Int}_{L}(X)$ we have

$$
\begin{equation*}
[\rho]_{\cong} \subset[\rho]_{\sim_{c}}=\operatorname{Conv}_{\left\langle\operatorname{Int}_{L}(X), \subset\right\rangle}\left([\rho]_{\cong}\right) . \tag{2}
\end{equation*}
$$

Fact 2.2 For each interpretation $\rho \in \operatorname{Int}_{L}(X)$ the following is equivalent:
(a) $\rho$ is reversible, that is $\operatorname{Cond}(\rho)=\operatorname{Aut}(\rho)$,
(b) $[\rho] \cong$ is an antichain in the Boolean lattice $\left\langle\operatorname{Int}_{L}(X), \subset\right\rangle$,
(c) there is no $\sigma \in[\rho]_{\cong}$ such that $\rho \nsubseteq \sigma$,
(d) there is no $\sigma \in[\rho] \cong$ such that $\sigma \nsubseteq \rho$,
(e) $\rho^{c}$ is reversible.

An interpretation $\rho \in \operatorname{Int}_{L}(X)$ will be called strongly reversible iff $[\rho] \cong=\{\rho\}$; weakly reversible iff $[\rho]_{\cong}$ is a convex set in the Boolean lattice $\left\langle\operatorname{Int}_{L}(X), \subset\right\rangle$. Clearly we have $\operatorname{sRev}_{L}(X) \subset \operatorname{Rev}_{L}(X) \subset \operatorname{wRev}_{L}(X)$, where $\operatorname{Rev}_{L}(X)$ (resp. $\left.\operatorname{sRev}_{L}(X), \operatorname{wRev}_{L}(X)\right)$ denotes the set of all reversible (resp. strongly reversible, weakly reversible) interpretations $\rho \in \operatorname{Int}_{L}(X)$.

It is easy to see that both reversibility and its two variations are $\sim_{c}$-invariants and, hence, $\cong$-invariants. (A property $\mathcal{P}$ is called $\sim$-invariant iff for each $\rho, \sigma \in$ $\operatorname{Int}_{L}(X)$ we have: if $\rho$ has $\mathcal{P}$ and $\sigma \sim \rho$, then $\sigma$ has $\mathcal{P}$ ). In addition, weakly
reversible interpretations have the Cantor-Schröder-Bernstein property for condensations (if $\rho$ is weakly reversible and there are condensations $f:\langle X, \sigma\rangle \rightarrow\langle X, \rho\rangle$ and $g:\langle X, \rho\rangle \rightarrow\langle X, \sigma\rangle$, then $\sigma \cong \rho)$.

Concerning strong reversibility we have: an interpretation $\rho \in \operatorname{Int}_{L}(X)$ is strongly reversible iff for each $i \in I$, the relation $\rho_{i}$ is a subset of $X^{n_{i}}$ which is definable by a first-order formula of the empty language without parameters.

Example 2.3 If $L_{b}=\langle R\rangle$ is the binary language (i.e. $\operatorname{ar}(R)=2$ ) and $X \neq \emptyset$, then the only strongly reversible elements of $\operatorname{Int}_{L_{b}}(X)$ are: $\emptyset$ (the empty relation), $\Delta_{X}$ (the diagonal), $X^{2} \backslash \Delta_{X}$ (the complete graph) and $X^{2}$ (the full relation).

Partial orders If $\mathbb{P}=\langle P, \leq\rangle$ is a partial order, by $\operatorname{Min} \mathbb{P}($ resp. Max $\mathbb{P})$ we denote the set of minimal (resp. maximal) elements of $\mathbb{P}$. A set $D \subset P$ is called dense (resp. co-dense) in $\mathbb{P}$ iff for each $p \in P$ there is $q \in D$ such that $q \leq p$ (resp. $p \leq q$ ). A set of $L$-interpretations $\mathcal{C} \subset \operatorname{Int}_{L}(X)$ will be called union-complete (resp. intersection-complete) iff $\cup \mathcal{L} \in \mathcal{C}$ (resp. $\cap \mathcal{L} \in \mathcal{C}$ ) for each chain $\mathcal{L} \subset \mathcal{C}$. The partial order $\langle\mathcal{C}, \subset\rangle$ will be shortly denoted by $\mathcal{C}$, when it is convenient.

Fact 2.4 If $\mathcal{C} \subset \operatorname{Int}_{L}(X)$ is a union-complete (resp. intersection-complete) set, then $\operatorname{Max} \mathcal{C}($ resp. $\operatorname{Min} \mathcal{C})$ is a co-dense (resp. dense) subset of $\mathcal{C}$.

Proof. If $\mathcal{C}$ is union-complete and $\rho \in \mathcal{C}$, then, by the Hausdorff maximal principle, there is a maximal chain $\mathcal{L}$ in $\mathcal{C}$ such that $\rho \in \mathcal{L}$. By the union-completeness of $\mathcal{C}$, we have $\bigcup \mathcal{L} \in \mathcal{C}$, by the maximality of $\mathcal{L}$ we have $\bigcup \mathcal{L} \in \operatorname{Max}\langle\mathcal{C}, \subset\rangle$ and, since $\rho \in \mathcal{L}$, we have $\rho \subset \bigcup \mathcal{L}$. The proof for intersection-complete sets is dual.

Infinitary languages Let $L=\left\langle R_{i}: i \in I\right\rangle$ be a relational language, $\kappa$ an infinite cardinal and $\operatorname{Var}=\left\{v_{\alpha}: \alpha \in \kappa\right\}$ a set of variables. By $\mathrm{At}_{L}$ we denote the corresponding set of atomic formulas, that is,

$$
\operatorname{At}_{L}=\left\{v_{\alpha}=v_{\beta}: \alpha, \beta \in \kappa\right\} \cup\left\{R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\} .
$$

The class of $L_{\infty \omega}$-formulas is the class $\operatorname{Form}_{L_{\infty \omega}}=\bigcup_{\xi \in \text { Ord }}$ Form $_{\xi}$, where

$$
\begin{aligned}
\operatorname{Form}_{0} & =\operatorname{At}_{L}, \\
\operatorname{Form}_{\xi+1} & =\operatorname{Form}_{\xi} \cup\left\{\neg \varphi: \varphi \in \operatorname{Form}_{\xi}\right\} \\
& \cup\left\{\forall v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \operatorname{Form}_{\xi}\right\} \cup\left\{\exists v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \operatorname{Form}_{\xi}\right\} \\
& \cup\left\{\bigwedge \Phi: \Phi \subset \text { Form }_{\xi}\right\} \cup\left\{\bigvee \Phi: \Phi \subset \operatorname{Form}_{\xi}\right\}, \\
\text { Form }_{\gamma} & =\bigcup_{\xi<\gamma} \text { Form }_{\xi}, \text { for a limit ordinal } \gamma .
\end{aligned}
$$

Let $\mathbb{X}=\left\langle X,\left\langle R_{i}^{\mathbb{X}}: i \in I\right\rangle\right.$ be an $L$-structure and $\vec{x}=\left\langle x_{\alpha}: \alpha \in \kappa\right\rangle \in{ }^{\kappa} X$ a valuation. If $\beta \in \kappa$ and $x^{\prime} \in X$, by $\vec{x}_{\left\langle\beta, x^{\prime}\right\rangle}$ we denote the valuation $\vec{y} \in{ }^{\kappa} X$ defined by: $y_{\alpha}=x_{\alpha}$, for all $\alpha \neq \beta$; and $y_{\beta}=x^{\prime}$. The satisfiability relation for $L_{\infty \omega^{-}}$ formulas is defined in a standard way, for example, $\mathbb{X} \models\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)[\vec{x}]$ iff $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \in R_{i}^{\mathbb{X}} ; \mathbb{X} \models(\neg \varphi)[\vec{x}]$ iff $\mathbb{X} \models \varphi[\vec{x}]$ is not true; $\mathbb{X} \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$ iff $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, for each $x \in X ; \mathbb{X} \models(\bigvee \Phi)[\vec{x}]$ iff $\mathbb{X} \models \varphi[\vec{x}]$, for some $\varphi \in \Phi$.
$L_{\infty}$-formulas $\varphi$ and $\psi$ are called logically equivalent, in notation $\varphi \leftrightarrow \psi$ iff they are equivalent in all $L$-structures, that is, iff for each $L$-structure $\mathbb{X}$ we have:

$$
\begin{equation*}
\forall \vec{x} \in{ }^{\kappa} X \quad(\mathbb{X} \models \varphi[\vec{x}] \Leftrightarrow \mathbb{X} \models \psi[\vec{x}]) \tag{3}
\end{equation*}
$$

If $\mathbb{X}$ and $\mathbb{Y}$ are $L$-structures, a mapping $f: X \rightarrow Y$ preserves an $L_{\infty \omega}$-formula $\varphi$ iff

$$
\begin{equation*}
\forall \vec{x} \in{ }^{\kappa} X \quad(\mathbb{X} \models \varphi[\vec{x}] \Rightarrow \mathbb{Y} \models \varphi[f \vec{x}]) \tag{4}
\end{equation*}
$$

where $f \vec{x}=\left\langle f\left(x_{\alpha}\right): \alpha \in \kappa\right\rangle$. We say that the formula $\varphi$ is absolute under $f$ iff in (4) we have " $\Leftrightarrow$ " instead of " $\Rightarrow$ ".

## 3 Reversibility of maximal and minimal structures

A set $\mathcal{C} \subset \operatorname{Int}_{L}(X)$ is said to be isomorphism-invariant or, shortly, $\cong$-invariant iff

$$
\begin{equation*}
\forall \rho \in \mathcal{C} \quad[\rho] \cong \subset \mathcal{C} . \tag{5}
\end{equation*}
$$

By $\mathcal{C}^{c}$ we will denote the set $\left\{\rho^{c}: \rho \in \mathcal{C}\right\} .\left(\right.$ Clearly, $\mathcal{C}^{c} \neq \operatorname{Int}_{L}(X) \backslash \mathcal{C}$.)
Theorem 3.1 If $\mathcal{C} \subset \operatorname{Int}_{L}(X)$ is an $\cong$-invariant set and $\tau \in \operatorname{Max} \mathcal{C}$ (respectively, $\tau \in \operatorname{Min} \mathcal{C}$ ), then
(a) $\tau$ is a reversible interpretation;
(b) $[\tau] \cong=[\tau]_{\sim_{c}}$ is an antichain in $\mathcal{C}$ and $[\tau] \cong \subset \operatorname{Max\mathcal {C}}$ (resp. $[\tau] \cong \subset \operatorname{Min\mathcal {C}}$ );
(c) The set $\mathcal{C}^{c}$ is $\cong$-invariant and $\tau \in \operatorname{Max} \mathcal{C}$ iff $\tau^{c} \in \operatorname{Min}\left(\mathcal{C}^{c}\right)$, for any $\tau$.

Proof. (a) Suppose that $\tau$ is not reversible. Then, by Fact 2.2, there is $\sigma \in[\tau] \cong$ such that $\tau \nsubseteq \sigma$ and, by (5), $\sigma \in \mathcal{C}$, which is impossible, by the maximality of $\tau$.
(b) By (a) and Fact $2.2,[\tau] \cong$ is an antichain and, by (2),$[\tau] \cong=[\tau]_{\sim_{c}}$. Suppose that there are $\tau_{1} \in[\tau] \cong$ and $\rho \in \mathcal{C}$ such that $\tau_{1} \varsubsetneqq \rho$. Then, by Fact[2.1) (b) there is $f \in \operatorname{Sym}(X)$ such that $f\left[\tau_{1}\right]=\tau$, which, together with (5) implies $\tau \nsubseteq f[\rho] \in \mathcal{C}$. But, by the maximality of $\tau$, this is impossible. For $\tau \in \operatorname{Min} \mathcal{C}$ the proof is dual.
(c) For $\rho \in \mathcal{C}$ we show that $\left[\rho^{c}\right]_{\cong} \subset \mathcal{C}^{c}$. So, if $\sigma \in\left[\rho^{c}\right]_{\cong}$, then, by Fact 2.1(b), there is a bijection $f: X \rightarrow X$ such that $\sigma=f\left[\rho^{c}\right]=f[\rho]^{c}$ and, since $f[\rho] \in[\rho] \cong \subset \mathcal{C}$, we have $\sigma \in \mathcal{C}^{c}$. Let $\tau \in \operatorname{Int}_{L}(X)$. Then $\tau^{c} \in \operatorname{Min}\left(\mathcal{C}^{c}\right)$ iff
$\tau^{c}=\rho^{c}$, for some $\rho \in \mathcal{C}$, and for each $\rho \in \mathcal{C}$ we have $\rho^{c} \subset \tau^{c} \Rightarrow \rho^{c}=\tau^{c}$. In other words, $\tau=\rho$, for some $\rho \in \mathcal{C}$ and for each $\rho \in \mathcal{C}$ we have $\tau \subset \rho \Rightarrow \rho=\tau$, which means that $\tau \in \operatorname{Max} \mathcal{C}$.

We note that, in fact, an interpretation $\tau \in \operatorname{Int}_{L}(X)$ is reversible iff $\tau \in \operatorname{Max} \mathcal{C}$ for some $\cong$-invariant set $\mathcal{C} \subset \operatorname{Int}_{L}(X)$. Namely, the remaining implication is trivial: if $\tau$ is reversible, then by Fact 2.2 we have $\tau \in[\tau] \cong=\operatorname{Max}[\tau] \cong$.

Now we consider the sets of interpretations satisfying $L_{\infty}$-sentences. In order to make a correspondence between interpretations and their complements to each $L_{\infty \omega}$-formula $\varphi$ we adjoin the formula $\varphi^{c}$ defined in the following way:

$$
\left(v_{\alpha}=v_{\beta}\right)^{c}:=v_{\alpha}=v_{\beta} \text { and }\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}:=\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) ;
$$

If $\xi \in \operatorname{Ord}$ and $\varphi^{c}$ is defined for a formula $\varphi \in \operatorname{Form}_{\xi}$, then
$(\neg \varphi)^{c}:=\neg \varphi^{c},\left(\forall v_{\alpha} \varphi\right)^{c}:=\forall v_{\alpha} \varphi^{c}$ and $\left(\exists v_{\alpha} \varphi\right)^{c}:=\exists v_{\alpha} \varphi^{c} ;$
If $\Phi \subset \operatorname{Form}_{\xi}$ and $\varphi^{c}$ is defined for each formula $\varphi \in \Phi$, then
$(\bigwedge \Phi)^{c}:=\bigwedge \Phi^{c}$ and $(\bigvee \Phi)^{c}:=\bigvee \Phi^{c}$,
where, for a set $\Phi$ of $L_{\infty}$-formulas $\Phi^{c}$ denotes the set $\left\{\varphi^{c}: \varphi \in \Phi\right\}$.
Theorem 3.2 If $X$ is a non-empty set and $\mathcal{T}$ a set of $L_{\infty}$-sentences, then
(a) The set $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ is $\cong$-invariant;
(b) $\left(\operatorname{Int}_{L}^{\mathcal{T}}(X)\right)^{c}=\operatorname{Int}_{L}^{\mathcal{T}^{c}}(X)$ and this set is $\cong$-invariant;
(c) Maximal and minimal elements of $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ are reversible interpretations; (d) $\tau \in \operatorname{Max}_{\operatorname{Int}}^{\mathcal{T}}(X)$ iff $\tau^{c} \in \operatorname{Min} \operatorname{Int}_{L}^{\mathcal{T}^{c}}(X)$, for $\tau \in \operatorname{Int}_{L}(X)$.

Proof. (a) If $\rho \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$ and $\sigma \in[\rho]_{\cong}$, then there exists an isomorphism $f$ : $\langle X, \rho\rangle \rightarrow\langle X, \sigma\rangle$. Since for each sentence $\varphi \in \mathcal{T}$ we have $\langle X, \rho\rangle \models \varphi$, by a standard fact that each $L_{\infty}$-formula is absolute under each isomorphism we have $\langle X, \sigma\rangle \models \varphi$ as well. Thus $\sigma \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$ and (5) is true.
(b) First, using induction we prove the following auxiliary statement.

Claim 3.3 For each L-structure $\langle X, \rho\rangle$ and each formula $\varphi \in \operatorname{Form}_{L_{\infty} \omega}$ we have

$$
\begin{equation*}
\forall \vec{x} \in{ }^{\kappa} X\left(\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}[\vec{x}] \Leftrightarrow\langle X, \rho\rangle \models \varphi[\vec{x}]\right) . \tag{6}
\end{equation*}
$$

Proof. Let $\vec{x} \in{ }^{\kappa} X$. Then $\left\langle X, \rho^{c}\right\rangle \models\left(v_{\alpha}=v_{\beta}\right)^{c}[\vec{x}]$ iff $\left\langle X, \rho^{c}\right\rangle \models\left(v_{\alpha}=v_{\beta}\right)[\vec{x}]$ iff $x_{\alpha}=x_{\beta}$ iff $\langle X, \rho\rangle \models\left(v_{\alpha}=v_{\beta}\right)[\vec{x}]$. Also, $\left\langle X, \rho^{c}\right\rangle \models\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}[\vec{x}]$ iff $\left\langle X, \rho^{c}\right\rangle \models \neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)[\vec{x}]$ iff $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \notin \rho^{c}$ iff $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \in \rho$ iff $\langle X, \rho\rangle \models R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)[\vec{x}]$.

Suppose that (6) holds for a formula $\varphi$ and let $\vec{x} \in{ }^{\kappa} X$. Then $\left\langle X, \rho^{c}\right\rangle \models$ $(\neg \varphi)^{c}[\vec{x}]$ iff not $\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}[\vec{x}]$, iff not $\langle X, \rho\rangle \models \varphi[\vec{x}]$, iff $\langle X, \rho\rangle \models(\neg \varphi)[\vec{x}]$. $\left\langle X, \rho^{c}\right\rangle \vDash\left(\forall v_{\alpha} \varphi\right)^{c}[\vec{x}]$ iff $\left\langle X, \rho^{c}\right\rangle \models \forall v_{\alpha} \varphi^{c}[\vec{x}]$ iff for each $x \in X$ we have $\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, that is, by (6), $\langle X, \rho\rangle \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, iff $\langle X, \rho\rangle \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$.
$\left\langle X, \rho^{c}\right\rangle \models\left(\exists v_{\alpha} \varphi\right)^{c}[\vec{x}]$ iff $\left\langle X, \rho^{c}\right\rangle \models \exists v_{\alpha} \varphi^{c}[\vec{x}]$ iff there is $x \in X$ such that $\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, that is, by (6), $\langle X, \rho\rangle \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, iff $\langle X, \rho\rangle \models\left(\exists v_{\alpha} \varphi\right)[\vec{x}]$.

Let $\Phi \subset$ Form $_{L_{\infty} \omega}$ and suppose that (6) holds for each formula $\varphi \in \Phi$. $\left\langle X, \rho^{c}\right\rangle \models(\bigwedge \Phi)^{c}[\vec{x}]$ iff $\left\langle X, \rho^{c}\right\rangle \models \bigwedge\left\{\varphi^{c}: \varphi \in \Phi\right\}[\vec{x}]$, iff for each $\varphi \in \Phi$ we have $\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}[\vec{x}]$, that is, by (6), $\langle X, \rho\rangle \models \varphi[\vec{x}]$, iff $\langle X, \rho\rangle \models(\bigwedge \Phi)[\vec{x}]$. $\left\langle X, \rho^{c}\right\rangle \models(\bigvee \Phi)^{c}[\vec{x}]$ iff $\langle X, \rho\rangle \models \bigvee\left\{\varphi^{c}: \varphi \in \Phi\right\}[\vec{x}]$ iff for some $\varphi \in \Phi$ we have $\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}[\vec{x}]$, that is, by (6), $\langle X, \rho\rangle \models \varphi[\vec{x}]$, iff $\langle X, \rho\rangle \models(\bigvee \Phi)[\vec{x}]$.

By Claim 3.3, the sets $\left(\operatorname{Int}_{L}^{\mathcal{T}}(X)\right)^{c}=\left\{\rho^{c}: \forall \varphi \in \mathcal{T}\langle X, \rho\rangle \models \varphi\right\}$ and $\operatorname{Int}_{L}^{\mathcal{T}^{c}}(X)=\left\{\rho^{c}: \forall \varphi \in \mathcal{T}\left\langle X, \rho^{c}\right\rangle \models \varphi^{c}\right\}$ are equal.

Statements (c) and (d) follow from (a), (b) and Theorem 3.1(a) and (c).
Example 3.4 Reversibility, complete theories and elementary equivalence.
If $\rho \in \operatorname{Int}_{L}(X)$ and $\operatorname{Th}(\langle X, \rho\rangle)$ is the corresponding first-order theory, then $[\rho]_{\equiv}:=\operatorname{Int}_{L}^{\operatorname{Th}(\langle X, \rho\rangle)}(X)$ is the set of interpretations $\sigma \in \operatorname{Int}_{L}(X)$ such that the structures $\langle X, \rho\rangle$ and $\langle X, \sigma\rangle$ are elementarily equivalent. We show that, regarding the relationship between the sets $[\rho]_{\equiv}$ and $\operatorname{Rev}_{L}(X)$, everything is possible.

1. If $\mathbb{Q}=\langle Q, \rho\rangle$ is the rational line, then $\operatorname{Th}(\mathbb{Q})$ is the theory of dense linear orders without end points which is $\omega$-categorical and, hence, $[\rho]_{\equiv=[\rho] \cong \subset} \subset$ $\operatorname{Rev}_{L}(Q)$. By Fact $2.2[\rho] \cong$ is an antichain so each element of the set $\operatorname{Int}_{L}{ }^{\operatorname{Th}(\mathbb{Q})}(Q)$ is both a maximal and a minimal element of the set $\operatorname{Int}_{L}^{\operatorname{Th}(\mathbb{Q})}(Q)$.
2. If $\mathbb{G}=\langle G, \rho\rangle$ is the countable universal homogeneous graph (also called the Rado graph, the Erdős-Rényi graph [3]), then the theory $\operatorname{Th}(\mathbb{G})$ is $\omega$-categorical and the structure $\mathbb{G}$ is not reversible (since deleting of one of its edges produces an isomorphic copy of $\mathbb{G}$, see [ $[1]$ ). Thus $\operatorname{Int}_{L}^{\operatorname{Thh}(G)}(G) \cap \operatorname{Rev}_{L}(G)=\emptyset$ and the set $\operatorname{Int}_{L}^{\operatorname{Th}(\mathbb{G})}(G)$ has neither minimal nor maximal elements.
3. It is well known that the theory $\mathcal{T}$ of one equivalence relation having exactly one equivalence class of size $n$, for each $n \in \mathbb{N}$, is complete. For a cardinal $\kappa \leq \omega$ let $\mathbb{E}_{\kappa}=\left\langle\omega, \rho_{\kappa}\right\rangle$ be a countable model of $\mathcal{T}$ having exactly $\kappa$-many infinite equivalence classes. It is known that $\operatorname{Int}_{L}^{\mathcal{T}}(\omega)=\bigcup_{\kappa \leq \omega}\left[\rho_{\kappa}\right] \cong\left(=\left[\rho_{0}\right]_{\equiv}\right)$ and, hence, $\mathcal{T}$ is not an $\omega$-categorical theory. By [10], an equivalence relation is reversible iff the number of equivalence classes of the same size is finite or all equivalence classes are finite and their sizes form a reversible sequence. Thus the structures $\mathbb{E}_{n}, n<\omega$, are reversible, while $\mathbb{E}_{\omega}$ is not (even weakly) reversible. So we have $\operatorname{Int}_{L}^{\mathcal{T}}(\omega) \cap \operatorname{Rev}_{L}(\omega)=\bigcup_{n \in \omega}\left[\rho_{n}\right] \cong \operatorname{and}_{\operatorname{Int}_{L}^{\mathcal{T}}}^{\mathcal{T}}(\omega) \backslash \operatorname{Rev}_{L}(\omega)=\left[\rho_{\omega}\right] \cong$.

We show that $\operatorname{Max}\left(\operatorname{Int}_{L}^{\mathcal{T}}(\omega)\right)=\left[\rho_{0}\right] \cong \cup\left[\rho_{1}\right] \cong$. Suppose that $n \in\{0,1\}$ and $\rho_{n} \nsubseteq \sigma \in \operatorname{Int}_{L}^{\mathcal{T}}(\omega)$. For $k \in \mathbb{N}$, let $C_{k}$ be the unique equivalence class of size $k$ determined by $\rho_{n}$. Since $\rho_{n} \nsubseteq \sigma$, the equivalence classes corresponding to $\sigma$ are unions of those corresponding to $\rho$ and, in addition, there is the minimal $k_{0}$ such that, in $\langle\omega, \sigma\rangle$, the class $C_{k_{0}}$ is joined with some another $\rho$-class. But then
there is no $\sigma$-class of size $k_{0}$, which contradicts our assumption that $\sigma \in \operatorname{Int}_{L}^{\mathcal{T}}(\omega)$. If $\kappa \geq 2$, then $\rho_{\kappa} \subset \sigma$, for some $\sigma \cong \rho_{1}$ (we join $\kappa$ infinite classes into one). Similarly we show that $\operatorname{Min}\left(\operatorname{Int}_{L}^{\mathcal{T}}(\omega)\right)=\left[\rho_{0}\right] \cong$ and that for $\kappa \geq 1$ there are no minimal elements of $\operatorname{Int}_{L}^{\mathcal{T}}(\omega)$ below $\rho_{\kappa}$ (split infinite $\rho_{\kappa}$-classes into infinite parts).

Since $\rho_{m} \preccurlyeq_{c} \rho_{n}$, for $1 \leq n \leq m \leq \omega$, the suborder $\left\{[\rho]_{\sim_{c}}: \rho \in \operatorname{Int}_{L}^{\mathcal{T}}(\omega)\right\}=$ $\left\{\left[\rho_{n}\right]_{\cong}: n \in \omega\right\} \cup\left\{\left[\rho_{\omega}\right]_{\sim_{c}}\right\}$ of the condensation order $\left\langle\operatorname{Int}_{L}(\omega) / \sim_{c}, \leq_{c}\right\rangle$ is isomorphic to the disjoint union of the one element poset (corresponding to $\left[\rho_{0}\right] \cong$ ) and the chain of the type $1+\omega^{*}$, with the maximum $\left[\rho_{1}\right]_{\cong}$ and minimum $\left[\rho_{\omega}\right]_{\sim_{c}}$.

## 4 Theories having extreme interpretations

Example 3.4 shows that some sets of the form $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ have neither minimal nor maximal elements. In this section we give some syntactical conditions providing extreme interpretations in that sense. First, in order to provide maximal interpretations we define the class of $R$-positive $L_{\infty \omega}$-formulas by $\mathcal{P}:=\bigcup_{\xi \in \mathrm{Ord}} \mathcal{P}_{\xi}$, where

$$
\begin{aligned}
\mathcal{P}_{0} & =\mathrm{At}_{L} \cup\left\{\neg v_{\alpha}=v_{\beta}: \alpha, \beta \in \kappa\right\}, \\
\mathcal{P}_{\xi+1} & =\mathcal{P}_{\xi} \cup\left\{\forall v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{P}_{\xi}\right\} \cup\left\{\exists v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{P}_{\xi}\right\} \\
& \cup\left\{\bigwedge \Phi: \Phi \subset \mathcal{P}_{\xi}\right\} \cup\left\{\bigvee \Phi: \Phi \subset \mathcal{P}_{\xi}\right\}, \\
\mathcal{P}_{\gamma} & =\bigcup_{\xi<\gamma} \mathcal{P}_{\xi}, \text { for a limit ordinal } \gamma,
\end{aligned}
$$

and the class of $L_{\infty \omega}$-formulas $\mathcal{F}:=\bigcup_{\xi \in \mathrm{Ord}} \mathcal{F}_{\xi}$, where

$$
\begin{aligned}
\mathcal{F}_{0} & =\mathcal{P} \cup\left\{\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\}, \\
\mathcal{F}_{\xi+1} & =\mathcal{F}_{\xi} \cup\left\{\forall v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{F}_{\xi}\right\} \\
& \cup\left\{\bigwedge \Phi: \Phi \subset \mathcal{F}_{\xi}\right\} \cup\left\{\bigvee \Phi: \Phi \subset \mathcal{F}_{\xi} \wedge|\Phi|<\omega\right\}, \\
\mathcal{F}_{\gamma} & =\bigcup_{\xi<\gamma} \mathcal{F}_{\xi}, \text { for a limit ordinal } \gamma .
\end{aligned}
$$

Concerning minimal interpretations, let us define the class of $R$-negative $L_{\infty} \omega^{-}$ formulas by $\mathcal{N}=\bigcup_{\xi \in \text { Ord }} \mathcal{N}_{\xi}$, where

$$
\begin{aligned}
\mathcal{N}_{0} & =\left\{\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\} \\
& \cup\left\{v_{\alpha}=v_{\beta}: \alpha, \beta \in \kappa\right\} \cup\left\{\neg v_{\alpha}=v_{\beta}: \alpha, \beta \in \kappa\right\}, \\
\mathcal{N}_{\xi+1} & =\mathcal{N}_{\xi} \cup\left\{\forall v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{N}_{\xi}\right\} \cup\left\{\exists v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{N}_{\xi}\right\} \\
& \cup\left\{\bigwedge \Phi: \Phi \subset \mathcal{N}_{\xi}\right\} \cup\left\{\bigvee \Phi: \Phi \subset \mathcal{N}_{\xi}\right\}, \\
\mathcal{N}_{\gamma} & =\bigcup_{\xi<\gamma} \mathcal{N}_{\xi}, \text { for a limit ordinal } \gamma,
\end{aligned}
$$

and let $\mathcal{G}$ be the class of $L_{\infty \omega}$-formulas $\bigcup_{\xi \in \text { Ord }} \mathcal{G}_{\xi}$, where

$$
\mathcal{G}_{0}=\mathcal{N} \cup\left\{R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\},
$$

$$
\begin{aligned}
\mathcal{G}_{\xi+1} & =\mathcal{G}_{\xi} \cup\left\{\forall v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \mathcal{G}_{\xi}\right\} \\
& \cup\left\{\bigwedge \Phi: \Phi \subset \mathcal{G}_{\xi}\right\} \cup\left\{\bigvee \Phi: \Phi \subset \mathcal{G}_{\xi} \wedge|\Phi|<\omega\right\} \\
\mathcal{G}_{\gamma} & =\bigcup_{\xi<\gamma} \mathcal{G}_{\xi}, \text { for a limit ordinal } \gamma
\end{aligned}
$$

Theorem 4.1 Let $L$ be a relational language, $X$ a non-empty set and $\mathcal{T}$ a set of $L_{\infty}$-sentences such that $\operatorname{Int}_{L}^{\mathcal{T}}(X) \neq \emptyset$. Then
(a) If $\mathcal{T} \subset \mathcal{F}$, then the set $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ is union-complete and $\operatorname{Max}\left(\operatorname{Int}_{L}^{\mathcal{T}}(X)\right)$ is its co-dense subset consisting of reversible interpretations;
(b) If $\mathcal{T} \subset \mathcal{G}$, then $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ is intersection-complete and $\operatorname{Min}\left(\operatorname{Int}_{L}^{\mathcal{T}}(X)\right)$ is its dense subset consisting of reversible interpretations.

A proof is given in the sequel. First by induction we prove the following claim.
Claim 4.2 (a) The formulas from the class $\mathcal{P}$ are preserved under condensations.
(b) For each formula $\varphi \in \mathcal{F}$, each chain $\mathcal{L} \subset \operatorname{Int}_{L}(X)$ and valuation $\vec{x} \in{ }^{\kappa} X$ we have:

$$
\begin{equation*}
(\forall \rho \in \mathcal{L}\langle X, \rho\rangle \models \varphi[\vec{x}]) \Rightarrow\langle X, \bigcup \mathcal{L}\rangle \models \varphi[\vec{x}] . \tag{7}
\end{equation*}
$$

Proof. (a) Let $\mathbb{X}$ and $\mathbb{Y}$ be $L$-structures and $f: X \rightarrow Y$ a condensation. By induction we show that (4) holds for each formula $\varphi \in \mathcal{P}$. First, clearly, homomorphisms preserve all atomic formulas. If $\vec{x} \in{ }^{\kappa} X$ and $\mathbb{X} \models\left(\neg v_{\alpha}=v_{\beta}\right)[\vec{x}]$, that is $x_{\alpha} \neq x_{\beta}$, then, since $f$ is an injection, $f\left(x_{\alpha}\right) \neq f\left(x_{\beta}\right)$, that is $\mathbb{Y} \models\left(v_{\alpha}=v_{\beta}\right)[f \vec{x}]$.

Suppose that (4) holds for a formula $\varphi \in \mathcal{P}$; let $\vec{x} \in{ }^{\kappa} X$. If $\mathbb{X} \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$, then for each $x \in X$ we have $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ and, by (4), $\mathbb{Y} \models \varphi\left[(f \vec{x})_{\langle\alpha, f(x)\rangle}\right]$. Since $f$ is a surjection, for each $y \in Y$ there is $x \in X$ such that $y=f(x)$ and, hence, $\mathbb{Y} \models \varphi\left[(f \vec{x})_{\langle\alpha, y\rangle}\right]$. Thus $\mathbb{Y} \models\left(\forall v_{\alpha} \varphi\right)[f \vec{x}]$.

If $\mathbb{X} \models\left(\exists v_{\alpha} \varphi\right)[\vec{x}]$, then for some $x \in X$ we have $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ which by (4) implies $\mathbb{Y} \models \varphi\left[(f \vec{x})_{\langle\alpha, f(x)\rangle}\right]$. Thus, for $y=f(x) \in Y$ we have $\mathbb{Y} \models \varphi\left[(f \vec{x})_{\langle\alpha, y\rangle}\right]$ and, hence, $\mathbb{Y} \models\left(\exists v_{\alpha} \varphi\right)[f \vec{x}]$.

Let $\Phi \subset \mathcal{P}$, suppose that (4) holds for each formula $\varphi \in \Phi$ and let $\vec{x} \in{ }^{\kappa} X$. If $\mathbb{X} \models(\bigwedge \Phi)[\vec{x}]$, then for each $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi[\vec{x}]$ and, by (4), $\mathbb{Y} \models \varphi[f \vec{x}]$, which means that $\mathbb{Y} \models(\bigwedge \Phi)[f \vec{x}]$. If $\mathbb{X} \models(\bigvee \Phi)[\vec{x}]$, then for some $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi[\vec{x}]$, that is $\mathbb{Y} \models \varphi[f \vec{x}]$, which implies that $\mathbb{Y} \models(\bigvee \Phi)[f \vec{x}]$.
(b) Let $\mathcal{L}$ be a non-empty chain $\operatorname{in~}_{\operatorname{Int}_{L}(X) \text { and } \vec{x} \in{ }^{\kappa} X \text {. (We note that then }}$ $\bigcup \mathcal{L}=\left\langle\bigcup_{\rho \in \mathcal{L}} \rho_{i}: i \in I\right\rangle=\left\langle\tau_{i}: i \in I\right\rangle=: \tau \in \operatorname{Int}_{L}(X)$ and, for an $i \in I$, the set $\mathcal{L}_{i}:=\left\{\rho_{i}: \rho \in \mathcal{L}\right\}$ is a chain in the algebra $\left.\left\langle\operatorname{Int}_{\left\langle R_{i}\right\rangle}(X), \subset\right\rangle.\right)$

Let $\varphi \in \mathcal{P}$ and suppose that $\langle X, \rho\rangle \models \varphi[\vec{x}]$, for each $\rho \in \mathcal{L}$. If $\rho \in \mathcal{L}$, then by Fact [2.1 a) the identity mapping $\operatorname{id}_{X}:\langle X, \rho\rangle \rightarrow\langle X, \cup \mathcal{L}\rangle$ is a condensation and, by (a), preserves $\varphi$. Thus, since $\langle X, \rho\rangle \models \varphi[\vec{x}]$ we obtain $\langle X, \bigcup \mathcal{L}\rangle \models \varphi[\vec{x}]$.

If $\varphi:=\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)$, then for any $\rho \in \operatorname{Int}_{L}(X)$ we have $\langle X, \rho\rangle \models \varphi[\vec{x}]$ iff $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \notin \rho_{i}$. Now, if $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \notin \rho_{i}$, for each $\rho \in \mathcal{L}$, then, $\left\langle x_{\alpha_{1}}, \ldots, x_{\alpha_{n_{i}}}\right\rangle \notin \tau_{i}$, that is $\langle X, \cup \mathcal{L}\rangle \models \varphi[\vec{x}]$.

Suppose that the statement is true for a formula $\varphi \in \mathcal{F}_{\xi}$. Let $\mathcal{L}$ be a chain in $\operatorname{Int}_{L}(X)$ and $\vec{x} \in{ }^{\kappa} X$. If for each $\rho \in \mathcal{L}$ we have $\langle X, \rho\rangle \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$, that is $\langle X, \rho\rangle \models \varphi\left[\vec{x}_{\langle\alpha, y\rangle}\right]$, for all $y \in X$, then for each $y \in X$ and each $\rho \in \mathcal{L}$ we have $\langle X, \rho\rangle \models \varphi\left[\vec{x}_{\langle\alpha, y\rangle}\right]$ and by the inductive hypothesis and (7) it follows that $\langle X, \bigcup \mathcal{L}\rangle \models \varphi\left[\vec{x}_{\langle\alpha, y\rangle}\right]$. This holds for all $y \in X$ so, $\langle X, \bigcup \mathcal{L}\rangle \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$.

Let $\Phi \subset \mathcal{F}_{\xi}$ and suppose that the statement is true for each formula $\varphi \in \Phi$. Let $\mathcal{L}$ be a chain in $\operatorname{Int}_{L}(X)$ and $\vec{x} \in{ }^{\kappa} X$.

If $\Phi=\left\{\psi_{k}: k \leq n\right\}$ and for each $\rho \in \mathcal{L}$ we have $\langle X, \rho\rangle \models\left(\bigvee_{k=1}^{n} \psi_{k}\right)[\vec{x}]$, then there are $k_{0} \leq n$ and a cofinal subset $\mathcal{L}_{0}$ of $\mathcal{L}$ such that $\langle X, \rho\rangle \models \psi_{k_{0}}[\vec{x}]$, for every $\rho \in \mathcal{L}_{0}$ and, by the induction hypothesis, $\left\langle X, \bigcup \mathcal{L}_{0}\right\rangle \models \psi_{k_{0}}[\vec{x}]$. By the cofinality of $\mathcal{L}_{0}$ we have $\cup \mathcal{L}_{0}=\bigcup \mathcal{L}$ and, hence, $\langle X, \bigcup \mathcal{L}\rangle \models\left(\bigvee_{k=1}^{n} \psi_{k}\right)[\vec{x}]$.

If for each $\rho \in \mathcal{L}$ we have $\langle X, \rho\rangle \models(\bigwedge \Phi)[\vec{x}]$, that is $\langle X, \rho\rangle \models \varphi[\vec{x}]$, for all $\varphi \in \Phi$, then, for each $\varphi \in \Phi$ and $\rho \in \mathcal{L}$ we have $\langle X, \rho\rangle \models \varphi[\vec{x}]$, so, by the induction hypothesis, $\langle X, \bigcup \mathcal{L}\rangle \models \varphi[\vec{x}]$. Thus $\langle X, \bigcup \mathcal{L}\rangle \models(\bigwedge \Phi)[\vec{x}]$.
Proof of Theorem 4.1(a) Let $\mathcal{L} \subset \operatorname{Int}_{L}^{\mathcal{T}}(X)$ be a chain. If $\varphi \in \mathcal{T}$, then for each $\rho \in \mathcal{L}$ we have $\rho \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$ and, hence, $\langle X, \rho\rangle \models \varphi$, which, by (7), implies that $\langle X, \bigcup \mathcal{L}\rangle \models \varphi$. So $\bigcup \mathcal{L} \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$ and, thus, the set $\operatorname{Int}_{L}^{\mathcal{T}}(X)$ is union-complete. The second statement follows from Fact 2.4 and Theorem 3.2(c).

Claim 4.3 (a) $\mathcal{N}=\left\{\varphi^{c}: \varphi \in \mathcal{P}\right\}$;
(b) $\mathcal{G}=\left\{\varphi^{c}: \varphi \in \mathcal{F}\right\}$, up to logical equivalence.

Proof. (a) ( $\supset$ ) We show that for each $\xi \in \operatorname{Ord}$ and each $\varphi \in \mathcal{P}_{\xi}$ we have $\varphi^{c} \in \mathcal{N}_{\xi}$. For $\xi=0$ we have: $\left(v_{\alpha}=v_{\beta}\right)^{c}:=v_{\alpha}=v_{\beta} \in \mathcal{N}_{0},\left(\neg v_{\alpha}=v_{\beta}\right)^{c}:=\neg\left(v_{\alpha}=\right.$ $\left.v_{\beta}\right)^{c}:=\neg v_{\alpha}=v_{\beta} \in \mathcal{N}_{0}$, and $\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}:=\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) \in \mathcal{N}_{0}$.

Suppose that the statement is true for all $\xi<\zeta$. If $\zeta$ is a limit ordinal, then, clearly, the statement is true for $\zeta$. Let $\zeta=\xi+1$. If $\varphi \in \mathcal{P}_{\xi}$, then $\varphi^{c} \in \mathcal{N}_{\xi}$ and, hence, $\left(\forall v_{\alpha} \varphi\right)^{c}:=\forall v_{\alpha} \varphi^{c} \in \mathcal{N}_{\xi+1}$ and $\left(\exists v_{\alpha} \varphi\right)^{c}:=\exists v_{\alpha} \varphi^{c} \in \mathcal{N}_{\xi+1}$.

If $\Phi \subset \mathcal{P}_{\xi}$, then $\varphi^{c} \in \mathcal{N}_{\xi}$, for all $\varphi \in \Phi$, and, hence, we have $(\bigwedge \Phi)^{c}:=$ $\bigwedge\left\{\varphi^{c}: \varphi \in \Phi\right\} \in \mathcal{N}_{\xi+1}$, and $(\bigvee \Phi)^{c}:=\bigvee\left\{\varphi^{c}: \varphi \in \Phi\right\} \in \mathcal{N}_{\xi+1}$.
( $\subset)$ We show that for each $\xi \in$ Ord and each $\psi \in \mathcal{N}_{\xi}$ there is $\varphi \in \mathcal{P}_{\xi}$ such that $\psi=\varphi^{c}$. So $v_{\alpha}=v_{\beta}$ is the formula $\left(v_{\alpha}=v_{\beta}\right)^{c}, \neg v_{\alpha}=v_{\beta}$ is the formula $\left(\neg v_{\alpha}=v_{\beta}\right)^{c}$, and $\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)$ is the formula $\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}$.

Suppose that the statement is true for all $\xi<\zeta$. If $\zeta$ is a limit ordinal, then, clearly, the statement is true for $\zeta$. Let $\zeta=\xi+1$. If $\psi \in \mathcal{N}_{\xi}$, then there is $\varphi \in \mathcal{P}_{\xi}$ such that $\psi=\varphi^{c}$. Now $\forall v_{\alpha} \psi=\forall v_{\alpha} \varphi^{c}=\left(\forall v_{\alpha} \varphi\right)^{c}$, and $\forall v_{\alpha} \varphi \in \mathcal{P}_{\xi+1}$. Also $\exists v_{\alpha} \psi=\exists v_{\alpha} \varphi^{c}=\left(\exists v_{\alpha} \varphi\right)^{c}$, and $\exists v_{\alpha} \varphi \in \mathcal{P}_{\xi+1}$.

If $\Phi \subset \mathcal{N}_{\xi}$, then for each $\psi \in \Phi$ there is $\varphi_{\psi} \in \mathcal{P}_{\xi}$ such that $\psi=\varphi_{\psi}^{c}$. So $\bigwedge\left\{\varphi_{\psi}: \psi \in \Phi\right\} \in \mathcal{P}_{\xi+1}$ and $\left(\bigwedge\left\{\varphi_{\psi}: \psi \in \Phi\right\}\right)^{c}=\bigwedge\left\{\varphi_{\psi}^{c}: \psi \in \Phi\right\}=\bigwedge \Phi$. Also $\bigvee\left\{\varphi_{\psi}: \psi \in \Phi\right\} \in \mathcal{P}_{\xi+1}$ and $\left(\bigvee\left\{\varphi_{\psi}: \psi \in \Phi\right\}\right)^{c}=\bigvee\left\{\varphi_{\psi}^{c}: \psi \in \Phi\right\}=\bigvee \Phi$.
(b) ( $\supset$ ) We show that for each $\xi \in$ Ord and each $\varphi \in \mathcal{F}_{\xi}$ we have $\varphi^{c} \in \mathcal{G}_{\xi}$. For $\xi=0$, if $\varphi \in \mathcal{P}$, then, by (a), $\varphi^{c} \in \mathcal{N} \subset \mathcal{G}_{0}$ and $\left(\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}$ is the formula $\neg \neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)$, which is equivalent to $R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) \in \mathcal{G}_{0}$.

Suppose that the statement is true for all $\xi<\zeta$. If $\zeta$ is a limit ordinal, then, clearly, the statement is true for $\zeta$. Let $\zeta=\xi+1$. If $\varphi \in \mathcal{F}_{\xi}$, then $\varphi^{c} \in \mathcal{G}_{\xi}$ and, hence, $\left(\forall v_{\alpha} \varphi\right)^{c}:=\forall v_{\alpha} \varphi^{c} \in \mathcal{G}_{\xi+1}$.

If $\Phi \subset \mathcal{F}_{\xi}$, then $\varphi^{c} \in \mathcal{G}_{\xi}$, for all $\varphi \in \Phi$, and, hence, we have $(\bigwedge \Phi)^{c}:=$ $\bigwedge\left\{\varphi^{c}: \varphi \in \Phi\right\} \in \mathcal{G}_{\xi+1}$, and $(\bigvee \Phi)^{c}:=\bigvee\left\{\varphi^{c}: \varphi \in \Phi\right\} \in \mathcal{G}_{\xi+1}$, if $|\Phi|<\omega$.
(C) We show that for each $\xi \in$ Ord and each $\psi \in \mathcal{G}_{\xi}$ there is $\varphi \in \mathcal{F}_{\xi}$ such that $\psi=\varphi^{c}$. For $\xi=0$, if $\psi \in \mathcal{N}$, then we apply (a). Also, $R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)$ is equivalent to the formula $\left(\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{c}$.

Suppose that the statement is true for all $\xi<\zeta$. If $\zeta$ is a limit ordinal, then, clearly, the statement is true for $\zeta$. Let $\zeta=\xi+1$. If $\psi \in \mathcal{G}_{\xi}$, then there is $\varphi \in \mathcal{F}_{\xi}$ such that $\psi=\varphi^{c}$. Now $\forall v_{\alpha} \psi=\forall v_{\alpha} \varphi^{c}=\left(\forall v_{\alpha} \varphi\right)^{c}$, and $\forall v_{\alpha} \varphi \in \mathcal{F}_{\xi+1}$.

If $\Phi \subset \mathcal{G}_{\xi}$, then for each $\psi \in \Phi$ there is $\varphi_{\psi} \in \mathcal{F}_{\xi}$ such that $\psi=\varphi_{\psi}^{c}$. So $\bigwedge\left\{\varphi_{\psi}: \psi \in \Phi\right\} \in \mathcal{F}_{\xi+1}$ and $\left(\bigwedge\left\{\varphi_{\psi}: \psi \in \Phi\right\}\right)^{c}=\bigwedge\left\{\varphi_{\psi}^{c}: \psi \in \Phi\right\}=\bigwedge \Phi$. If $|\Phi|<\omega$, then $\bigvee\left\{\varphi_{\psi}: \psi \in \Phi\right\} \in \mathcal{F}_{\xi+1}$ and $\left(\bigvee\left\{\varphi_{\psi}: \psi \in \Phi\right\}\right)^{c}=\bigvee \Phi$.

Proof of Theorem 4.1(b) If $\mathcal{L}$ is a chain in the poset $\left\langle\operatorname{Int}_{L}^{\mathcal{T}}(X), \subset\right\rangle$, then, by Theorem 3.2(b) we have $\mathcal{L}^{c}=\left\{\rho^{c}: \rho \in \mathcal{L}\right\} \subset\left\{\rho^{c}: \rho \in \operatorname{Int}_{L}^{\mathcal{T}}(X)\right\}=$ : $\left(\operatorname{Int}_{L}^{\mathcal{T}}(X)\right)^{c}=\operatorname{Int}_{L}^{\mathcal{T}^{c}}(X)$. Since $\mathcal{T} \subset \mathcal{G}$, by Claim 4.3(b) w.l.o.g. we assume that $\mathcal{T} \subset\left\{\varphi^{c}: \varphi \in \mathcal{F}\right\}$ and, hence, $\mathcal{T}^{c} \subset\left\{\left(\varphi^{c}\right)^{c}: \varphi \in \mathcal{F}\right\}$. By Claim 3.3 for each interpretation $\rho \in \operatorname{Int}_{L}(X)$ and each $L_{\infty} \omega$-sentence $\varphi$ we have: $\langle X, \rho\rangle \models \varphi$ iff $\langle X, \rho\rangle \vDash\left(\varphi^{c}\right)^{c}$ so, w.l.o.g. again, we suppose that $\mathcal{T}^{c} \subset \mathcal{F}$. Now, clearly, $\mathcal{L}^{c}$ is a chain in the poset $\left\langle\operatorname{Int}_{L}^{\mathcal{T}^{c}}(X), \subset\right\rangle$ and, by Theorem4.1(a), $\cup \mathcal{L}^{c}=\bigcup_{\rho \in \mathcal{L}} \rho^{c}=$ $\left(\bigcap_{\rho \in \mathcal{L}} \rho\right)^{c} \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$ and, by Theorem 3.2, $\bigcap_{\rho \in \mathcal{L}} \rho=\bigcap \mathcal{L} \in \operatorname{Int}_{L}^{\mathcal{T}}(X)$. The second statement follows from Fact 2.4 and Theorem 3.2(c).

Example 4.4 Extreme partial orders. Clearly, for the set of axioms of the theory of strict partial orders $\mathcal{T}_{\text {poset }}=\left\{\varphi_{\text {irr }}, \varphi_{\text {tr }}\right\} \subset \operatorname{Sent}_{L_{b}}$, where $\varphi_{\text {irr }}:=\forall v_{0} \neg R\left(v_{0}, v_{0}\right)$ and $\varphi_{t r}:=\forall v_{0}, v_{1}, v_{2}\left(\neg R\left(v_{0}, v_{1}\right) \vee \neg R\left(v_{1}, v_{2}\right) \vee R\left(v_{0}, v_{2}\right)\right)$ we have $\mathcal{T}_{\text {poset }} \subset$ $\mathcal{F} \cap \mathcal{G}$ and, hence, the poset $\mathbb{P}:=\left\langle\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {poset }}}(X), \subset\right\rangle$ of all strict partial orders on $X$ has all the properties from (a) and (b) of Theorem 4.1. It is evident that $\operatorname{Min} \mathbb{P}=\{\emptyset\}$ and, by Example 2.3, this antichain order is the unique strongly reversible strict partial order on $X$.

The maximal elements of the poset $\mathbb{P}$ are exactly the strict linear orders. Namely, it is clear that if $\langle X, \rho\rangle$ is a strict linear order and $\rho \nsubseteq \rho^{\prime}$, then $\rho^{\prime}$ is not a strict par-
tial order. On the other hand, by the Order extension principle (i.e. the Szpilrajn extension theorem [13], following from Zorn's lemma), if $\rho$ is a strict partial order on $X$, then there is a strict linear order $\rho^{\prime}$ on $X$ such that $\rho^{\prime} \supset \rho$.

We note that, by a well-known theorem of Dushnik and Miller [2], the poset $\mathbb{P}$ has the following property: each interpretation $\rho \in \operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {poset }}}(X)$ is the intersection of a family of maximal elements of $\mathbb{P}$ and the minimal size of such a family is called the Dushnik-Miller dimension of the poset $\langle X, \rho\rangle$. In [2] a poset is called reversible iff it is of dimension $\leq 2$, but it is easy to check that the poset $\mathbb{X}=\langle Z,<\rangle$, where $Z$ is the set of integers and $<:=\{\langle 2 n-1,2 n\rangle: n \in \mathbb{N}\}$ is of dimension 2 , but not reversible in our sense. In [7] Kukiela has shown that Boolean lattices are reversible posets (in our sense), but, clearly, lot of them have dimension $>2$.

Example 4.5 The poset of interpretations of countable connected graphs is unioncomplete but not intersection-complete, although the minimal elements are dense in it. For the set of axioms of graph theory $\mathcal{T}_{\text {graph }}=\left\{\varphi_{\text {irr }}, \varphi_{\text {sym }}\right\}$, where $\varphi_{\text {irr }}:=$ $\forall v_{0} \neg R\left(v_{0}, v_{0}\right)$ and $\varphi_{\text {sym }}:=\forall v_{0}, v_{1}\left(\neg R\left(v_{0}, v_{1}\right) \vee R\left(v_{1}, v_{0}\right)\right)$ we have $\mathcal{T}_{\text {graph }} \subset \mathcal{F}$ and the $L_{\infty}$-sentence $\varphi_{\text {conn }}$ given by

$$
\forall u, v\left(u=v \vee \bigvee_{n \geq 2} \exists v_{1}, \ldots, v_{n}\left(u=v_{1} \wedge v=v_{n} \wedge \bigwedge_{k=1}^{n-1} R\left(v_{k}, v_{k+1}\right)\right)\right.
$$

and expressing that a graph is connected belongs to $\mathcal{P}$. So $\mathcal{T}_{\text {graph }} \cup\left\{\varphi_{\text {conn }}\right\} \subset \mathcal{F}$ and, by Theorem 4.1(a), the poset $\left\langle\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\varphi_{\text {conn }}\right\}}(\omega), \subset\right\rangle$ is union-complete. Since a graph is a tree iff it is a minimal connected graph, the minimal elements of our poset are exactly the tree graph relations on $X$. Since every connected graph admits a spanning tree (it is an easy application of Zorn's lemma; see [12]), our poset has dense set of minimal elements. For $k \in \omega$, let $\mathbb{G}_{k}=\left\langle\omega \cup\{\omega\}, \rho_{k}\right\rangle$, where $\rho_{k}=\{\{n, n+1\}: n \in \omega\} \cup\{\{n, \omega\}: n \geq k\}$. It is evident that the graphs $\mathbb{G}_{k}$ are connected and $\rho_{0} \supsetneq \rho_{1} \supsetneq \rho_{2} \supsetneq \ldots$, but the graph $\mathbb{G}_{\omega}=\left\langle\omega \cup\{\omega\}, \bigcap_{k \in \omega} \rho_{k}\right\rangle$ is disconnected and, hence, the poset is not intersection-complete.

## 5 Omitting finite substructures

A class $\mathcal{K} \subset \operatorname{Mod}_{L}$ is called a universal class iff it is axiomatizable by a finite set of universal $\left(\Pi_{1}^{0}\right)$ sentences iff there exists a finite set of finite $L$-structures $\left\{\mathbb{F}_{k}: k \leq n\right\} \subset \operatorname{Mod}_{L}$ such that $\mathbb{X} \in \mathcal{K}$ iff $\mathbb{F}_{k} \nrightarrow \mathbb{X}$, for all $k \leq n$ (see [14, 15, 4, 5]). Here, using that concept, we show that forbidding finite structures provides a large zoo of reversible structures.
Fact 5.1 For each finite L-structure $\mathbb{F}$ there is an $L_{\infty \omega}$-sentence $\psi_{\mathbb{F} \hookrightarrow}$ such that for each L-structure $\mathbb{Y}$ we have: $\mathbb{F} \hookrightarrow \mathbb{Y}$ iff $\mathbb{Y} \models \psi_{\mathbb{F} \hookrightarrow . I f, ~ i n ~ a d d i t i o n, ~ t h e ~ l a n g u a g e ~}$ $L$ is finite, then the sentence $\neg \psi_{\mathbb{F} \hookrightarrow}$ is logically equivalent to a $\Pi_{1}^{0}$ sentence $\eta_{\mathbb{F} \hookrightarrow}$.

Proof. Let $L=\left\langle R_{i}: i \in I\right\rangle$, where $\operatorname{ar}\left(R_{i}\right)=n_{i}$, for $i \in I$, and w.l.o.g. suppose that $\mathbb{F}=\left\langle m,\left\langle R_{i}^{\mathbb{F}}: i \in I\right\rangle\right\rangle \in \operatorname{Mod}_{L}$, where $m=\{0, \ldots, m-1\} \in \mathbb{N}$. Let $\chi_{R_{i}^{\mathbb{E}}}: m^{n_{i}} \rightarrow 2, i \in I$, be the characteristic functions of the sets $R_{i}^{\mathbb{F}} \subset m^{n_{i}}$ and let $\varphi_{\mathbb{F}}\left(v_{0}, \ldots, v_{m-1}\right)$ be the $L_{\infty}$-formula defined by

$$
\begin{equation*}
\varphi_{\mathbb{F}}(\bar{v}):=\bigwedge_{0 \leq j<k<m} v_{j} \neq v_{k} \wedge \bigwedge_{i \in I} \bigwedge_{\bar{x} \in m^{n_{i}}} R_{i}\left(v_{x_{0}}, \ldots, v_{x_{n_{i}-1}}\right)^{\chi_{R_{\bar{F}}^{F}}(\bar{x})}, \tag{8}
\end{equation*}
$$

where, by definition, $\eta^{1}:=\eta$ and $\eta^{0}:=\neg \eta$. We show first that

$$
\begin{equation*}
\mathbb{F} \models \varphi_{\mathbb{F}}[0,1, \ldots, m-1] . \tag{9}
\end{equation*}
$$

For $j<m$, under the valuation $\langle 0,1, \ldots, m-1\rangle$ the variable $v_{j}$ obtains the value $j$ and, hence, $\mathbb{F} \models\left(\bigwedge_{0 \leq j<k<m} v_{j} \neq v_{k}\right)[0,1, \ldots, m-1]$ is true. Let $i \in I$ and $\bar{x}=$ $\left\langle x_{0}, \ldots, x_{n_{i}-1}\right\rangle \in m^{n_{i}}$. Then $\mathbb{F} \models R_{i}\left(v_{x_{0}}, \ldots, v_{x_{n_{i}-1}}\right)^{\chi_{R_{i}^{\mathbb{F}}}(\bar{x})}[0,1, \ldots, m-1]$ iff $\mathbb{F} \models R_{i}\left[x_{0}, \ldots, x_{n_{i}-1}\right]_{{R_{i}^{\mathbb{F}}}_{i}(\bar{x})}^{\text {iff }}\left(\chi_{R_{i}^{\mathbb{F}}}(\bar{x})=1 \wedge \bar{x} \in R_{i}^{\mathbb{F}}\right) \vee\left(\chi_{R_{i}^{\mathbb{F}}}(\bar{x})=0 \wedge \bar{x} \notin R_{i}^{\mathbb{F}}\right)$ which is true. Thus (9) is proved.

Let $\psi_{\mathbb{F} \hookrightarrow}:=\exists \bar{v} \varphi_{\mathbb{F}}(\bar{v})$. If $\mathbb{Y} \in \operatorname{Mod}_{L}$ and $f: \mathbb{F} \hookrightarrow \mathbb{Y}$, then by (9) we have $\mathbb{Y} \models \varphi_{\mathbb{F}}[f(0), \ldots, f(m-1)]$ and, since $\bar{y}:=\langle f(0), \ldots, f(m-1)\rangle \in Y^{m}$ we have $\mathbb{Y} \models \exists \bar{v} \varphi_{\mathbb{F}}(\bar{v})$, that is, $\mathbb{Y} \models \psi_{\mathbb{F} \hookrightarrow}$. Conversely, let $\bar{y}=\left\langle y_{0}, \ldots y_{m-1}\right\rangle \in Y^{m}$ and $\mathbb{Y} \models \varphi_{\mathbb{F}}[\bar{y}]$. Since under the valuation $\bar{y}$ the variable $v_{j}$ obtains the value $y_{j}$, by (8) $y_{0}, \ldots y_{m-1}$ are different elements of $Y$ and, hence, the mapping $f: m \rightarrow Y$ defined by $f(j)=y_{j}$, for $j<m$, is an injection. For a proof that $f: \mathbb{F} \rightarrow \mathbb{Y}$ is a strong homomorphism we take $i \in I$ and $\bar{x}:=\left\langle j_{0}, \ldots, j_{n_{i}-1}\right\rangle \in m^{n_{i}}$ and show that

$$
\left\langle j_{0}, \ldots, j_{n_{i}-1}\right\rangle \in R_{i}^{\mathbb{F}} \Leftrightarrow\left\langle y_{j_{0}}, \ldots, y_{j_{n_{i}-1}}\right\rangle \in R_{i}^{\mathbb{Y}} .
$$

Since $\mathbb{Y} \models \varphi_{\mathbb{F}}[\bar{y}]$, by (8) for $\bar{x}$ we have $\mathbb{Y} \models R_{i}\left(v_{j_{0}}, \ldots, v_{j_{n_{i}-1}}\right)^{\chi_{R_{i}^{\mathbb{P}}}(\bar{x})}[\bar{y}]$, that is $\mathbb{Y} \models R_{i}\left[y_{j_{0}}, \ldots, y_{j_{n_{i}-1}}\right]^{\chi_{R_{i}^{\mathbb{F}}}\left(\left\langle j_{0}, \ldots, j_{n_{i}-1}\right\rangle\right)}$, thus $\left\langle y_{j_{0}}, \ldots, y_{j_{n_{i}-1}}\right\rangle \in R_{i}^{\mathbb{Y}}$ if and only if $\chi_{R_{i}^{\mathbb{F}}}\left(\left\langle j_{0}, \ldots, j_{n_{i}-1}\right\rangle\right)=1$ iff $\left\langle j_{0}, \ldots, j_{n_{i}-1}\right\rangle \in R_{i}^{\mathbb{F}}$ and that's it.

If $|L|<\omega$, then the sentence $\neg \psi_{\mathbb{F} \hookrightarrow}$ is equivalent to the $\Pi_{1}^{0}$ sentence $\eta_{\mathbb{F} \hookrightarrow \rightarrow}:=$ $\forall \bar{v}\left(\bigvee_{0 \leq j<k<m} v_{j}=v_{k} \vee \bigvee_{i \in I} \bigvee_{\bar{x} \in m^{n_{i}}} R_{i}\left(v_{x_{0}}, \ldots, v_{x_{n_{i}-1}}\right)^{1-\chi_{R_{\bar{~}}^{\mathbb{F}}}(\bar{x})}\right)$.
Theorem 5.2 Let $L$ be a finite language, $\mathcal{T}$ an $L_{\infty \omega}$-theory and $\mathbb{F}_{j}, j \in J$, finite $L$-structures such that the poset $\mathbb{P}:=\left\langle\operatorname{Int}_{L}^{\mathcal{T} \cup\left\{\eta_{\left.\mathbb{F}_{j} \nrightarrow: j \in J\right\}}\right.}(X), \subset\right\rangle$ is non-empty.
(a) If $\mathcal{T} \subset \mathcal{F}$, then the poset $\mathbb{P}$ is union-complete and $\operatorname{Max} \mathbb{P}$ is a co-dense set in $\mathbb{P}$ consisting of reversible interpretations;
(b) If $\mathcal{T} \subset \mathcal{G}$, then the poset $\mathbb{P}$ is intersection-complete and $\operatorname{Min} \mathbb{P}$ is a dense set in $\mathbb{P}$ consisting of reversible interpretations;

$$
\text { (c) } \tau \in \operatorname{Max}\left(\operatorname{Int}_{L}^{\mathcal{T} \cup\left\{\eta_{\left.\mathbb{F}_{j} \ngtr: j \in J\right\}}\right.}(X)\right) \text { iff } \tau^{c} \in \operatorname{Min}\left(\operatorname{Int}_{L}^{\mathcal{T}^{c} \cup\left\{\eta_{\mathbb{F}_{j} \nless}: j \in J\right\}}(X)\right) \text {. }
$$

Proof. Since $\Pi_{1}^{0} \subset \mathcal{F} \cap \mathcal{G}$, (a) and (b) follow from Fact 5.1 and Theorem4.1.
(c) It is easy to check that for each $p \in\{0,1\}$ we have $\left(R_{i}(\bar{v})^{1-p}\right)^{c} \leftrightarrow R_{i}(\bar{v})^{p}$ and, also, that $\chi_{R_{i}^{\mathrm{F}}}(\bar{x})=1-\chi_{R_{i}^{\mathrm{Ec}}}(\bar{x})$, which implies that $\left(\eta_{\mathbb{F} \nrightarrow}\right)^{c} \leftrightarrow \eta_{\mathbb{F}^{c} \nsim}$. So the statement follows from Theorem 3.2 (d).

Maximal $\mathbb{K}_{n}$-free graphs In the sequel, for convenience, for a graph $\mathbb{X}=\langle X, \rho\rangle$ the relation $\rho$ will be identified with the corresponding set of two-element subsets of $X,\left\{\{x, y\} \in[X]^{2}:\langle x, y\rangle \in \rho\right\}$ and $\mathbb{X}^{g c}$ will denote the graph-complement, $\left\langle X,[X]^{2} \backslash \rho\right\rangle$, of the graph $\mathbb{X}$. For a set $Y \subset X$, the subgraph $\langle Y, \rho \upharpoonright Y\rangle$ of $\mathbb{X}$ will be sometimes denoted by $Y$. For a cardinal $\nu, \mathbb{K}_{\nu}$ will denote the complete graph of size $\nu$, and $\mathbb{E}_{\nu}$ the graph with $\nu$ vertices and no edges. Clearly, $\mathbb{E}_{\nu}=\mathbb{K}_{\nu}^{g c}$.

If $\mathbb{F}$ is a finite graph which is not complete, then, trivially, $X^{2} \backslash \Delta_{X}$ is the unique maximal element of the poset $\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\eta_{\mathbb{H} \nrightarrow}\right\}}(X)$ and here we consider what forbidding $\mathbb{K}_{n}$ 's produce. By Theorem 5.2, the poset $\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\eta_{\left.\mathbb{K}_{n} \nrightarrow\right\}}\right\}}(X)$ has maximal elements, they are reversible and, clearly, different from $X^{2} \backslash \Delta_{X}$. We recall that a graph is called $\mathbb{K}_{n}$-free iff it has no subgraphs isomorphic to $\mathbb{K}_{n}$; trivially, the graphs $\mathbb{K}_{m}, m<n$, are maximal $\mathbb{K}_{n}$-free graphs.

Claim 5.3 Let $n \geq 3$ and let $\mathbb{X}=\langle X, \rho\rangle$ be a $\mathbb{K}_{n}$-free graph. Then
(a) $\mathbb{X}$ is a maximal $\mathbb{K}_{n}$-free graph iff

$$
\begin{equation*}
\forall\{x, y\} \in[X]^{2} \backslash \rho \exists K \in[X]^{n}[K]^{2} \backslash \rho=\{\{x, y\}\} . \tag{10}
\end{equation*}
$$

(b) If $\mathbb{X}$ is a maximal $\mathbb{K}_{n}$-free graph and $|X| \geq n-1$, then

$$
\begin{equation*}
\forall x \in X \quad \exists K \in[X \backslash\{x\}]^{n-2} \quad\{x\} \cup K \cong \mathbb{K}_{n-1} . \tag{11}
\end{equation*}
$$

(c) If $\mathbb{X}$ is a maximal $\mathbb{K}_{n}$-free graph, $|X| \geq n-1,\left\{Y_{x}: x \in X\right\}$ is a family of non-empty sets, $Y:=\bigcup_{x \in X}\{x\} \times Y_{x}$ and

$$
\begin{equation*}
\sigma=\left\{\left\{\langle x, y\rangle,\left\langle x^{\prime}, y^{\prime}\right\rangle\right\} \in[Y]^{2}:\left\{x, x^{\prime}\right\} \in \rho\right\}, \tag{12}
\end{equation*}
$$

then $\mathbb{Y}=\langle Y, \sigma\rangle$ is a maximal $\mathbb{K}_{n}-$ free graph.
(d) $\mathbb{X}$ is a maximal $\mathbb{K}_{n}$-free graph iff $\mathbb{X}^{c}$ is a minimal $\left\langle n, \Delta_{n}\right\rangle$-free reflexive graph iff $\mathbb{X}^{g c}$ is a minimal $\mathbb{E}_{n}$-free graph.

Proof. (a) If $|X|<n$, then (10) holds iff $\rho=[X]^{2}$ iff $\langle X, \rho\rangle \cong \mathbb{K}_{|X|}$. Let $|X| \geq n$.
If $\mathbb{X}$ is maximal and $\{x, y\} \in[X]^{2} \backslash \rho$, then the graph $\langle X, \rho \cup\{\{x, y\}\}\rangle$ is not $\mathbb{K}_{n}$-free, which means that there is a set $K \in[X]^{n}$ such that $x, y \in K$ and $\langle K,(\rho \cup\{\{x, y\}\}) \upharpoonright K\rangle \cong \mathbb{K}_{n}$, which implies that $[K]^{2} \backslash \rho=\{\{x, y\}\}$.

Conversely, if (10) holds, then for any $\{x, y\} \in[X]^{2} \backslash \rho$ there is $K \in[X]^{n}$ such that $\langle K,(\rho \cup\{\{x, y\}\}) \upharpoonright K\rangle \cong \mathbb{K}_{n}$, thus $\mathbb{X}$ is a maximal $\mathbb{K}_{n}$-free graph.
(b) If $|X|=n-1$, then $\mathbb{X} \cong \mathbb{K}_{n-1}$ and (11) is evident. Let $|X| \geq n$ and $x \in X$. If $\{x, y\} \notin \rho$, for some $y \in X \backslash\{x\}$, then by (10) there is a set $K^{\prime}=\left\{x, y, x_{1}, \ldots, x_{n-2}\right\} \in[X]^{n}$ such that $\left[K^{\prime}\right]^{2} \backslash \rho=\{\{x, y\}\}$ and for $K:=\left\{x_{1}, \ldots, x_{n-2}\right\} \in[X \backslash\{x\}]^{n-2}$ we have $\{x\} \cup K \cong \mathbb{K}_{n-1}$.

If $\{x, y\} \in \rho$, for all $y \in X \backslash\{x\}$, then, since $|X| \geq n$, there is a pair $\{u, v\} \in[X \backslash\{x\}]^{2} \backslash \rho$ and, by (10), there is a set $K=\left\{u, v, x_{1}, \ldots, x_{n-2}\right\} \in$ $[X]^{n}$ such that $[K]^{2} \backslash \rho=\{\{u, v\}\}$. Now, if $x \notin\left\{x_{1}, \ldots, x_{n-2}\right\}$, then $\{x\} \cup$ $\left\{x_{1}, \ldots, x_{n-2}\right\} \cong \mathbb{K}_{n-1}$ and (11) is true. If $x=x_{j}$, for some $j \leq n-2$, then $\{x\} \cup\left\{u, x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-2}\right\} \cong \mathbb{K}_{n-1}$ and (11) is true again.
(c) Suppose that $\left\{\left\langle x_{i}, y_{i}\right\rangle: 1 \leq i \leq n\right\}$ is a copy of $\mathbb{K}_{n}$ in $\mathbb{Y}$; then, by (12), $\left\{x_{i}: 1 \leq i \leq n\right\}$ would be a copy of $\mathbb{K}_{n}$ in $\mathbb{X}$, which contradicts our assumption. Thus $\mathbb{Y}$ is a $\mathbb{K}_{n}$-free graph.

Suppose that $\langle Y, \tau\rangle$ is a $\mathbb{K}_{n}$-free graph, where $\sigma \varsubsetneqq \tau$. Let $\left\{\langle x, y\rangle,\left\langle x^{\prime}, y^{\prime}\right\rangle\right\} \in$ $\tau \backslash \sigma$. If $x=x^{\prime}$, then by (b) there is a set $K=\left\{x_{1}, \ldots, x_{n-2}\right\} \in[X \backslash\{x\}]^{n-2}$ such that $\{x\} \cup K \cong \mathbb{K}_{n-1}$. For $j \leq n-2$ we choose $y_{i} \in Y_{x_{i}}$ and, by (12), $\left\{\langle x, y\rangle,\left\langle x, y^{\prime}\right\rangle\right\} \cup\left\{\left\langle x_{j}, y_{j}\right\rangle: j \leq n-2\right\}$ is a copy of $\mathbb{K}_{n}$ in $\langle Y, \tau\rangle$, contrary to our assumption.

If $x \neq x^{\prime}$, then $\left\{x, x^{\prime}\right\} \in[X]^{2} \backslash \rho$ and by (a) there is $K=\left\{x, x^{\prime}, x_{1}, \ldots, x_{n-2}\right\}$ $\in[X]^{n}$ such that $[K]^{2} \backslash \rho=\left\{\left\{x, x^{\prime}\right\}\right\}$. Again, for $j \leq n-2$ we choose $y_{i} \in Y_{x_{i}}$ and, by (12), $\left\{\langle x, y\rangle,\left\langle x^{\prime}, y^{\prime}\right\rangle\right\} \cup\left\{\left\langle x_{j}, y_{j}\right\rangle: j \leq n-2\right\}$ is a copy of $\mathbb{K}_{n}$ in $\langle Y, \tau\rangle$, contrary to our assumption. Thus $\mathbb{Y}$ is a maximal $\mathbb{K}_{n}$-free graph.
(d) Clearly, up to logical equivalence we have $\mathcal{T}_{\text {graph }}^{c}=\left\{\varphi_{\text {refl }}, \varphi_{\text {sym }}\right\}$ and $\mathbb{K}_{n}^{c} \cong\left\langle n, \Delta_{n}\right\rangle$. Now the first claim follows from Theorem 5.2(c) and the second claim follows from the first one.

Example 5.4 Claim 5.3 provides a large jungle of extreme and, hence, reversible structures. So if $n \geq 3, \mathbb{X} \cong \mathbb{K}_{n-1}$ and $\left\{Y_{x}: x \in X\right\}$ is family of non-empty sets, then the graph $\mathbb{Y}$ defined in Claim [5.3(c) is a maximal $\mathbb{K}_{n}$-free graph. The reader will notice that $\mathbb{Y}$ is in fact the complete $(n-1)$-partite graph and that $\mathbb{Y}^{g c}$ is a disjoint union of $n-1$ complete graphs, which is a minimal $\mathbb{E}_{n}$-free graph. If $\left|Y_{x}\right|=\omega$, for all $x \in X$, then $\mathbb{Y}^{g c}$ is a reversible countable ultrahomogeneous graph from the list of Lachlan and Woodrow (see Remark 5.6).

For $n=3$, the complete bi-partite graphs $\mathbb{K}_{\nu, \omega}, \nu \leq \omega$, are maximal countable triangle-free graphs. In particular, the star graph $\mathbb{S}_{\omega}:=\mathbb{K}_{1, \omega}$, is a maximal triangle-free graph. Furthermore, some maximal triangle-free graphs are not bipartite, for example the cycle graph $\mathbb{C}_{5}$. Also, by taking $\mathbb{X} \cong \mathbb{C}_{5}$ in Claim 5.3(c) we obtain infinite maximal $\mathbb{K}_{3}$-free graphs which are not bi-partite.

Of course, there are reversible $\mathbb{K}_{3}$-free graphs which are not maximal $\mathbb{K}_{3}$-free. For example, the linear graph $\mathbb{G}_{\omega}=\langle\omega, \tau\rangle$, where $\tau=\{\{n, n+1\}: n \in \omega\}$, is reversible since deleting an edge produces a disconnected graph.

Maximal $\mathbb{K}_{n}$-free graphs with all vertices of infinite degree In the context of graph theory, the sentence

$$
\varphi_{\infty}:=\forall v \quad \bigwedge_{n \in \mathbb{N}} \exists v_{1}, \ldots, v_{n}\left(\bigwedge_{1 \leq i<j \leq n} v_{i} \neq v_{j} \wedge \bigwedge_{1 \leq i \leq n} R\left(v, v_{i}\right)\right)
$$

says that each vertex of a graph has infinitely many neighbors. Since $\varphi_{\infty} \in \mathcal{P}$, by Theorem 5.2] the poset $\operatorname{Int}_{L b}^{\mathcal{T}_{\text {graph }} \cup\left\{\varphi_{\infty}, \eta_{\left.\mathbb{K}_{n} \nrightarrow\right\}}\right\}}(X)$ has a co-dense set of maximal elements and they are reversible. Some such interpretations are given in Example 5.4.

Example 5.5 The Henson graph $\mathbb{H}_{n}$ is a maximal $\mathbb{K}_{n}$-free graph with all vertices of infinite degree. For $n \geq 3, \mathbb{H}_{n}$ denotes the unique countable homogeneous universal $\mathbb{K}_{n}$-free graph (the Henson graph, see [6]). In order to recall a convenient characterization of $\mathbb{H}_{n}$ we introduce the following notation: if $\mathbb{G}=\langle G, \rho\rangle$ is a graph and $n \geq 3$ let $C_{n}(\mathbb{G}):=\left\{\langle H, K\rangle: K \subset H \in[G]^{<\omega} \wedge K\right.$ is $\mathbb{K}_{n-1}$-free $\}$ and for $\langle H, K\rangle \in C_{n}(\mathbb{G})$ let

$$
G_{K}^{H}:=\{v \in G \backslash H: \forall k \in K\{v, k\} \in \rho \wedge \forall h \in H \backslash K\{v, h\} \notin \rho\} .
$$

Now, by [6] we have: a countable graph $\mathbb{G}=\langle G, \rho\rangle$ is isomorphic to $\mathbb{H}_{n}$ iff $\mathbb{G}$ is $\mathbb{K}_{n}$-free and $G_{K}^{H} \neq \emptyset$, for each $\langle H, K\rangle \in C_{n}(\mathbb{G})$.

We show that the Henson graph $\mathbb{H}_{n}=\langle G, \rho\rangle$ is a maximal $\mathbb{K}_{n}$-free graph. Suppose that $\left\langle G, \rho^{\prime}\right\rangle$ is a $\mathbb{K}_{n}$-free graph, where $\rho \nsubseteq \rho^{\prime}$ and $\left\{a_{1}, a_{2}\right\} \in \rho^{\prime} \backslash \rho$. By recursion we construct different elements $a_{3}, \ldots, a_{n} \in G \backslash\left\{a_{1}, a_{2}\right\}$ such that

$$
\begin{equation*}
\forall k \in\{3, \ldots, n\} \forall i \in\{1,2, \ldots, k-1\} \quad\left\{a_{i}, a_{k}\right\} \in \rho . \tag{13}
\end{equation*}
$$

Let $k \in\{3, \ldots, n\}$ and suppose that the sequence $a_{1}, a_{2}, \ldots, a_{k-1}$ satisfies (13). Then, since $\left\{a_{1}, a_{2}\right\} \notin \rho$, for $H=K:=\left\{a_{1}, a_{2}, \ldots, a_{k-1}\right\}$ we have $\mathbb{K}_{n-1} \nLeftarrow$ $\langle K, \rho \upharpoonright K\rangle$ and, hence, $\langle H, K\rangle \in C_{n}\left(\mathbb{H}_{n}\right)$ so, by the characterization mentioned above, there is $a_{k} \in G \backslash\left\{a_{1}, a_{2}, \ldots, a_{k-1}\right\}$ such that $\left\{a_{i}, a_{k}\right\} \in \rho$, for all $i<k$. So, the sequence $a_{1}, a_{2}, \ldots, a_{k}$ satisfies (13) and the recursion works. But, since $\left\{a_{1}, a_{2}\right\} \in \rho^{\prime}$ the vertices $a_{1}, \ldots, a_{n}$ determine a subgraph of the graph $\left\langle G, \rho^{\prime}\right\rangle$ isomorphic to $\mathbb{K}_{n}$, which contradicts our assumption.

Since the star graph $\mathbb{S}_{\omega}$ (see Example [5.4) is $\mathbb{K}_{n}$-free, by the universality of $\mathbb{H}_{n}$ there is a copy of $\mathbb{S}_{\omega}$ in $\mathbb{H}_{n}$ and, hence, $\mathbb{H}_{n}$ contains a vertex of infinite degree. By the homogeneity of $\mathbb{H}_{n}$ all vertices of $\mathbb{H}_{n}$ are of infinite degree.

Remark 5.6 By a well-known characterization of Lachlan and Woodrow [11], each countable ultrahomogeneous graph is isomorphic to one of the following:

- $\mathbb{G}_{\mu \nu}$, the union of $\mu$ disjoint copies of $\mathbb{K}_{\nu}$, where $\mu \nu=\omega$, - reversible iff $\mu<\omega$ or $\nu<\omega$ [10];
- $\mathbb{G}_{\text {Rado }}$, the Rado graph - non-reversible (see Example 3.4);
- $\mathbb{H}_{n}$, the Henson graph, for $n \geq 3$ - reversible (see Example 5.5);
- the graph-complements of these graphs - a graph is reversible iff its graphcomplement is reversible (it is an easy consequence of Fact 2.2].

Omitting extreme finite structures Clearly, the minimal elements of the set $\operatorname{Int}_{L}^{\left\{\eta_{\mathbb{F} \nLeftarrow H}\right\}}(X)$, in the sequel denoted shortly by $\operatorname{Int}_{L}^{\eta_{\mathbb{F} \nrightarrow}}(X)$, will be different from the trivial one, $\langle\emptyset: i \in I\rangle$, iff the forbidden structure $\mathbb{F}$ is minimal, that is isomorphic to $\langle m,\langle\emptyset: i \in I\rangle\rangle$, for some $m \in \mathbb{N}$. Dually, $\operatorname{Max}\left(\operatorname{Int}_{L}^{\eta_{\mathrm{F} \nrightarrow}}(X)\right) \neq$ $\left\{\left\langle X,\left\langle X^{n_{i}}: i \in I\right\rangle\right\rangle\right\}$ iff $\mathbb{F} \cong\left\langle m,\left\langle m^{n_{i}}: i \in I\right\rangle\right\rangle$. We give some examples of such restrictions.

Claim 5.7 Let $m, n \in \mathbb{N}$, $L_{n}=\langle R\rangle$, where $\operatorname{ar}(R)=n$. Then
(a) If $\rho \in \operatorname{Int}_{L}^{\eta_{\langle m, \emptyset\rangle \nrightarrow}}(X)$, then $\rho \in \operatorname{Min}\left(\operatorname{Int}_{L_{n}}^{\eta_{\langle m, \emptyset\rangle}}(X)\right)$ iff

$$
\begin{equation*}
\forall \bar{x} \in \rho \exists K \in[X]^{m} \quad \rho \cap K^{n}=\{\bar{x}\} ; \tag{14}
\end{equation*}
$$

(b) If $\rho \in \operatorname{Int}_{L_{n}}^{\eta_{\left\langle m, m^{n}\right\rangle \nrightarrow}}(X)$, then $\rho \in \operatorname{Max}\left(\operatorname{Int}_{L_{n}}^{\eta_{\left\langle m, m^{n}\right\rangle \nrightarrow}}(X)\right)$ iff

$$
\begin{equation*}
\forall \bar{x} \in X^{n} \backslash \rho \exists K \in[X]^{m} \quad K^{n} \backslash \rho=\{\bar{x}\} \tag{15}
\end{equation*}
$$

Proof. (a) If there exists $\bar{x} \in \rho$ such that $\rho \cap K^{n} \backslash\{\bar{x}\} \neq \emptyset$, for each $K \in[X]^{m}$ satisfying $\bar{x} \in K^{n}$, then $\rho \backslash\{\bar{x}\} \in \operatorname{Int}_{L}^{\eta_{\langle m, \emptyset\rangle \psi}}(X)$ so, $\rho$ is not minimal.

Suppose that (14) holds and that $\rho \supseteq \sigma \in \operatorname{Int}_{L}^{\eta_{\langle m, 9\rangle \leftrightarrow}}(X)$. Then, by (14), for $\bar{x} \in \rho \backslash \sigma$ there is $K \in[X]^{m}$ such that $\rho \cap K^{n}=\{\bar{x}\}$ and, hence, $\sigma$ cap $K^{n}=\emptyset$, which is impossible because $\langle m, \emptyset\rangle\langle\langle\langle X, \sigma\rangle$.
(b) follows from (a) and Theorem 5.2(c).

Now we show that the minimal binary structures omitting the minimal structure $\langle m, \emptyset\rangle$ can be characterized using maximal $\mathbb{K}_{m}$-free graphs.

Claim 5.8 If $|X| \geq m \geq 2$, then $\rho \in \operatorname{Min}\left(\operatorname{Int}_{L_{b}}^{\eta_{\langle m, q\rangle \psi}}(X)\right)$ iff $\rho$ is of the form

$$
\rho=\sigma_{X \backslash R} \cup \Delta_{R},
$$

where $R \subset X,|X \backslash R| \geq m-1$, and $\sigma_{X \backslash R}$ is an orientation of the graphcomplement of a maximal $\mathbb{K}_{m}$-free graph $\left\langle X \backslash R, \tau_{X \backslash R}\right\rangle$.

Proof. $(\Rightarrow)$ Let $\rho \in \operatorname{Min}\left(\operatorname{Int}_{L_{b}}^{\eta_{\langle m, \emptyset\rangle \nrightarrow}}(X)\right)$ and let $R:=\{x \in X:\langle x, x\rangle \in \rho\}$. $|X \backslash R| \leq m-2$ would imply that for each $K \in[X]^{m}$ we have $|K \cap R| \geq 2$ and, hence, $\left|\rho \cap K^{2}\right| \geq 2$, which is impossible by (14); so, $|X \backslash R| \geq m-1$.

By (14), for $\langle x, y\rangle \in \rho \cap(R \times X)$ there is $K \in[X]^{m}$ such that $\rho \cap K^{2}=$ $\{\langle x, y\rangle\}$ and, since $\langle x, x\rangle \in \rho \cap K^{2}$, we have $x=y$. Thus $\rho \cap(R \times X)=\Delta_{R}$ and, similarly, $\rho \cap(X \times R)=\Delta_{R}$, which means that $\rho=\Delta_{R} \cup \sigma_{X \backslash R}$, where $\sigma_{X \backslash R}:=\rho \cap(X \backslash R)^{2}$.

By (14), for $\langle x, y\rangle \in \sigma_{X \backslash R}$ there is $K \in[X]^{m}$ such that $\rho \cap K^{2}=\{\langle x, y\rangle\}$ and, since $x \neq y$, we have $\langle y, x\rangle \notin \sigma_{X \backslash R}$; thus $\sigma_{X \backslash R} \cap \sigma_{X \backslash R}^{-1}=\emptyset$. Moreover, since $x \neq y$, we have $K \cap R=\emptyset$, that is, $K \in[X \backslash R]^{m}$; so, by Claim 5.7(a) $\sigma_{X \backslash R} \in \operatorname{Min}\left(\operatorname{Int}_{L_{n}}^{\eta_{\langle m, Q\rangle \nrightarrow}}(X \backslash R)\right)$. Thus $\left\langle X \backslash R, \sigma_{X \backslash R}\right\rangle$ is a minimal digraph omitting $\langle m, \emptyset\rangle$ and, hence, its symmetrization $\left\langle X \backslash R, \sigma_{X \backslash R} \cup \sigma_{X \backslash R}^{-1}\right\rangle$ is a minimal $\mathbb{E}_{m}$-free graph. By Claim[5.3(d), the graph-complement $\tau_{X \backslash R}$ of $\sigma_{X \backslash R} \cup \sigma_{X \backslash R}^{-1}$ is a maximal $\mathbb{K}_{m}$-free graph and $\sigma_{X \backslash R}$ is an orientation of its graph-complement.
$(\Leftarrow)$ Let $K \in[X]^{m}$. If $K \cap R \neq \emptyset$, then $\rho \cap K^{2} \neq \emptyset$, and if $K \cap R=$ $\emptyset$, then $\rho \cap K^{2} \neq \emptyset$ because otherwise the graph $\tau_{X \backslash R}$ would not be $\mathbb{K}_{m}$-free. Hence $\rho \in \operatorname{Int}_{L_{b}}^{\eta_{\langle m,( \rangle\rangle \nrightarrow}}(X)$. Let $\rho^{\prime} \varsubsetneqq \rho$ and $\langle x, y\rangle \in \rho \backslash \rho^{\prime}$. If $x=y$, take $Z \in[X \backslash R]^{m-1}$, such that $\rho \cap Z^{2}=\emptyset$ (such $Z$ exists because $|X \backslash R| \geq m-1$ and the graph $\tau_{X \backslash R}$ is maximal $\mathbb{K}_{m}$-free, and thus not $\mathbb{K}_{m-1}$-free by Claim 5.3 (b)). Then $\rho^{\prime} \cap(Z \cup\{x\})^{2}=\emptyset$, that is $\rho^{\prime} \notin \operatorname{Int}_{L_{b}}^{\eta_{\langle m, \phi\rangle \nrightarrow}}(X)$. If $x \neq y$, then $x, y \in X \backslash R$, $\{x, y\} \notin \tau_{X \backslash R}$, and since the graph $\tau_{X \backslash R}$ is maximal $\mathbb{K}_{m}$-free, there is $Z \subset X \backslash R$ such that $x, y \in Z$ and $\left\langle Z,\left(\tau_{X \backslash R} \cup\{x, y\}\right) \cap Z^{2}\right\rangle \cong \mathbb{K}_{m}$. Now $\rho^{\prime} \cap Z^{2}=\emptyset$ that is $\rho^{\prime} \notin \operatorname{Int}_{L_{b}}^{\eta_{\langle m, \emptyset\rangle \nrightarrow}}(X)$. Therefore $\rho \in \operatorname{Min}\left(\operatorname{Int}_{L_{b}}^{\eta_{\langle m, \emptyset\rangle \nrightarrow}}(X)\right)$.

In particular, for $m=2$, we have that $\rho \in \operatorname{Min}\left(\operatorname{Int}_{L_{b}}^{\eta_{\langle 2, \emptyset\rangle \nrightarrow}}(X)\right)$ if and only if there is a set $R \nsubseteq X$ and a tournament relation $\sigma_{X \backslash R}$ on the set $X \backslash R$ such that

$$
\begin{equation*}
\rho=\sigma_{X \backslash R} \cup \Delta_{R} \tag{16}
\end{equation*}
$$

( $\langle X, \rho\rangle$ is a disjoint union of a nonempty tournament and isolated reflexive points).
Dually we have: $\rho \in \operatorname{Max}\left(\operatorname{Int}_{L_{b}}^{\eta_{\left\langle\left(2,2^{2}\right\rangle \nrightarrow\right.}}(X)\right)$ iff there is a set $R \varsubsetneqq X$ and a tournament relation $\sigma_{X \backslash R}$ on the set $X \backslash R$ such that

$$
\begin{equation*}
\rho=X^{2} \backslash\left(\sigma_{X \backslash R} \cup \Delta_{R}\right) \tag{17}
\end{equation*}
$$

For example, from (16) and (17) we obtain the reversibility of tournaments and reflexivized tournaments (for $R=\emptyset$ ). In particular, strict linear orders and reflexive linear orders are reversible. If we take $R=X \backslash\{x\}$, for some $x \in X$, then we obtain the diagonal without one point and complete graph with one reflexivized point. We note that complete graphs with $n$ reflexivized points are also reversible, but for $n \geq 2$ they contain a copy of $\left\langle 2,2^{2}\right\rangle$.

Maximal graphs without cycle subgraphs For $n \geq 4$, and concerning infinite graphs, the full graph is the only maximal graph which do not contain a copy of $\mathbb{C}_{n}$, while for $n=3$ we obtain the triangle-free graphs considered above. If $3 \in A \subset \omega \backslash 3$, then we obtain non-trivial maximal interpretations which do not contain copies of $\mathbb{C}_{n}$, for $n \in A$, and in the extreme situation, when we take $A=\omega \backslash 3$, we obtain graphs without cycles. The maximal such graphs are the trees (connected graphs without cycles). The bipartite graphs are obtained if we take $A=\{3,5,7, \ldots\}$ and the maximal ones are the complete bipartite graphs.

Local cardinal bounds Let $L=\left\langle R_{i}: i \in I\right\rangle$ be a finite language, where $\operatorname{ar}\left(R_{i}\right)=n_{i}$, for $i \in I$, let $M \subset \mathbb{N}$ and let $k=\left\langle k_{m}^{i}: m \in M \wedge i \in I\right\rangle$ and $l=\left\langle l_{m}^{i}: m \in M \wedge i \in I\right\rangle$ be sequences in $\omega$ such that for each $m \in M$ and $i \in I$ we have $0 \leq k_{m}^{i} \leq l_{m}^{i} \leq m^{n_{i}}$. Then the set of $L$-sentences

$$
\mathcal{T}_{M}^{k, l}:=\bigcup_{m \in M}\left\{\eta_{\langle m, \sigma\rangle \nrightarrow}: \sigma \in \operatorname{Int}_{L}(m) \wedge \exists i \in I\left(\left|\sigma_{i}\right|<k_{m}^{i} \vee\left|\sigma_{i}\right|>l_{m}^{i}\right)\right\}
$$

is a $\Pi_{1}^{0}$ theory and for a non-empty set $X$ and $\rho \in \operatorname{Int}_{L}(X)$ we have

$$
\rho \in \operatorname{Int}_{L}^{\mathcal{T}_{M}^{k, l}}(X) \Leftrightarrow \forall m \in M \quad \forall K \in[X]^{m} \forall i \in I \quad k_{m}^{i} \leq\left|\rho_{i} \cap K^{n_{i}}\right| \leq l_{m}^{i}
$$

(the size of the components of $\rho$ restricted to $m$-element subsets of $X$ is bounded). By Theorem 5.2, if $\mathcal{T}$ is an $L_{\infty \omega}$-theory and the poset $\operatorname{Int}_{L}^{\mathcal{T} \cup \mathcal{T}_{M}^{k, l}}(X)$ is non-empty, then it has a dense set of minimal and co-dense set of maximal elements.

Example 5.9 Graph theory does not admit two non-trivial bounds. If $\langle m, \sigma\rangle$ is a graph, then, by irreflexivity, $0 \leq|\sigma| \leq m^{2}-m$. Let $L=L_{b}, M=\{3\}$ and $0<k \leq l<6$. If $\mathcal{T}_{\{3\}}^{k, l}=\left\{\eta_{\langle 3, \sigma\rangle \nrightarrow}: \sigma \in\left[3^{2}\right]^{<k} \cup\left[3^{2}\right]^{>l}\right\}$, then $\mathcal{T}:=\mathcal{T}_{\text {graph }} \cup \mathcal{T}_{\{3\}}^{k, l}$ is a $\Pi_{1}^{0}$ theory and $\rho \in \operatorname{Int}_{L_{b}}^{\mathcal{T}}(\omega)$ iff the structure $\mathbb{X}=\langle\omega, \rho\rangle$ is a graph such that $k \leq\left|\rho \cap K^{2}\right| \leq l$, for each $K \in[\omega]^{3}$, which means that (by symmetry) each 3-element substructure of $\mathbb{X}$ has one or two edges. But this is impossible, because, by the Ramsey theorem, $\mathbb{X}$ must contain an infinite empty or complete subgraph. On the other hand, if we take $k=0$, then for $l \in\{4,5\}$ the condition $\left|\rho \cap K^{2}\right| \leq l$ means that the graph is triangle-free and some maximal interpretations with that property are described in Examples ?? and 5.5. For $l \in\{2,3\}$ maximal interpretations satisfying $\left|\rho \cap K^{2}\right| \leq l$ are $\bigcup_{\omega} \mathbb{K}_{2}$ and $\mathbb{K}_{1} \cup \bigcup_{\omega} \mathbb{K}_{2}$.

## 6 Uniform upper bounds for definable sets

If $\varphi\left(v_{1}, \ldots, v_{p}, w_{1}, \ldots, w_{q}\right)$ is an $L_{\infty}$-formula and $n \in \mathbb{N}$, then, denoting $p$ tuples by ${ }^{-}$and $q$-tuples by ${ }^{\sim}$, by $\exists \leq n \widetilde{w} \varphi(\bar{v}, \widetilde{w})$ we denote the formula

$$
\begin{aligned}
& \forall w_{1}^{1}, \ldots, w_{q}^{1}, \ldots, w_{1}^{n+1}, \ldots, w_{q}^{n+1} \\
& \left(\bigvee_{k \leq n+1} \neg \varphi\left(v_{1}, \ldots, v_{p}, w_{1}^{k}, \ldots, w_{q}^{k}\right) \vee \bigvee_{1 \leq k<l \leq n+1} \bigwedge_{1 \leq j \leq q} w_{j}^{k}=w_{j}^{l}\right),
\end{aligned}
$$

where $\left\{w_{1}^{1}, \ldots, w_{q}^{1}, \ldots, w_{1}^{n+1}, \ldots, w_{q}^{n+1}\right\}$ is a set of $q(n+1)$ different variables and $\varphi\left(v_{1}, \ldots, v_{p}, w_{1}^{k}, \ldots, w_{q}^{k}\right)$ is the formula obtained from $\varphi$ by replacement of each free occurrence of $w_{j}$ by $w_{j}^{k}$. Or, shortly,
$\exists_{\leq n} \widetilde{w} \varphi(\bar{v}, \widetilde{w}):=\forall \widetilde{w}^{1}, \ldots, \widetilde{w}^{n+1}\left(\bigvee_{k \leq n+1} \neg \varphi\left(\bar{v}, \widetilde{w}^{k}\right) \vee \bigvee_{1 \leq k<l \leq n+1} \widetilde{w}^{k}=\widetilde{w}^{l}\right)$.
Clearly, $\psi:=\forall \bar{v} \exists_{\leq_{n}} \widetilde{w} \varphi(\bar{v}, \widetilde{w})$ is an $L_{\infty}$-sentence and, if $\mathbb{X}$ is an $L$-structure, then $\mathbb{X} \models \psi$ iff for each $\bar{x} \in X^{p}$ the set $D_{\mathbb{X}, \varphi, \bar{x}, q}:=\left\{\widetilde{y} \in X^{q}: \mathbb{X} \models \varphi[\bar{x}, \widetilde{y}]\right\}$ is of size $\leq n$. $\left(D_{\mathbb{X}, \varphi, \bar{x}, q}\right.$ is the $q$-ary relation on the set $X$ definable in the structure $\mathbb{X}$ by the formula $\varphi$ with the parameters $x_{1}, \ldots, x_{p}$.)

Let $\neg \mathcal{F}$ be the class of $L_{\infty \omega}$-formulas $\bigcup_{\xi \in \text { Ord }} \neg \mathcal{F}_{\xi}$, where

$$
\begin{aligned}
\neg \mathcal{F}_{0} & =\mathcal{N} \cup\left\{R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\}, \\
\neg \mathcal{F}_{\xi+1} & =\neg \mathcal{F}_{\xi} \cup\left\{\exists v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \neg \mathcal{F}_{\xi}\right\} \\
& \cup\left\{\bigvee \Phi: \Phi \subset \neg \mathcal{F}_{\xi}\right\} \cup\left\{\bigwedge \Phi: \Phi \subset \neg \mathcal{F}_{\xi} \wedge|\Phi|<\omega\right\}, \\
\neg \mathcal{F}_{\gamma} & =\bigcup_{\xi<\gamma} \neg \mathcal{F}_{\xi}, \text { for a limit ordinal } \gamma .
\end{aligned}
$$

Also we define the class of $L_{\infty}$-formulas $\neg \mathcal{G}=\bigcup_{\xi \in \text { Ord }} \neg \mathcal{G}_{\xi}$, where

$$
\begin{aligned}
\neg \mathcal{G}_{0} & =\mathcal{P} \cup\left\{\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right): i \in I \wedge\left\langle\alpha_{1}, \ldots, \alpha_{n_{i}}\right\rangle \in \kappa^{n_{i}}\right\}, \\
\neg \mathcal{G}_{\xi+1} & =\neg \mathcal{G}_{\xi} \cup\left\{\exists v_{\alpha} \varphi: \alpha \in \kappa \wedge \varphi \in \neg \mathcal{G}_{\xi}\right\} \\
& \cup\left\{\bigvee \Phi: \Phi \subset \neg \mathcal{G}_{\xi}\right\} \cup\left\{\bigwedge \Phi: \Phi \subset \neg \mathcal{G}_{\xi} \wedge|\Phi|<\omega\right\}, \\
\neg \mathcal{G}_{\gamma} & =\bigcup_{\xi<\gamma} \neg \mathcal{G}_{\xi}, \text { for a limit ordinal } \gamma .
\end{aligned}
$$

Theorem 6.1 Let $\varphi(\bar{v}, \widetilde{w})$ be an $L_{\infty \omega}$-formula and $\mathcal{T}$ an $L_{\infty}$-theory such that the poset $\left.\mathbb{P}:=\left\langle\operatorname{Int}_{L}^{\mathcal{T} \cup\{\forall \bar{v}} \exists_{\leq n} \widetilde{w} \varphi(\bar{v}, \widetilde{w})\right\}(X), \subset\right\rangle$ is non-empty. Then
(a) If $\varphi(\bar{v}, \widetilde{w}) \in \neg \mathcal{F}$ and $\mathcal{T} \subset \mathcal{F}$, then $\mathbb{P}$ is a union-complete poset and $\operatorname{Max} \mathbb{P}$ is a co-dense set in $\mathbb{P}$ consisting of reversible interpretations.
(b) If $\varphi(\bar{v}, \widetilde{w}) \in \neg \mathcal{G}$ and $\mathcal{T} \subset \mathcal{G}$, then $\mathbb{P}$ is an intersection-complete poset and $\operatorname{Min} \mathbb{P}$ is a dense set in $\mathbb{P}$ consisting of reversible interpretations.

A proof of the theorem is given after some preliminary work. First, to each $L_{\infty \omega^{-}}$ formula $\varphi$ we adjoin a formula $\varphi\urcorner$ as follows:

$$
\left(v_{\alpha}=v_{\beta}\right)^{\neg}:=\neg v_{\alpha}=v_{\beta} \text { and }\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{\neg}:=\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) ;
$$

If $\xi \in \operatorname{Ord}$ and $\varphi\urcorner$ is defined for a formula $\varphi \in \operatorname{Form}_{\xi}$, then
$\left.(\neg \varphi)\urcorner:=\varphi,\left(\forall v_{\alpha} \varphi\right)^{\urcorner}:=\exists v_{\alpha} \varphi\right\urcorner$ and $\left.\left.\left(\exists v_{\alpha} \varphi\right)\right\urcorner:=\forall v_{\alpha} \varphi\right\urcorner ;$
If $\Phi \subset \operatorname{Form}_{\xi}$ and $\left.\varphi\right\urcorner$ is defined for each formula $\varphi \in \Phi$, then
$(\bigwedge \Phi)\urcorner:=\bigvee \Phi\urcorner$ and $(\bigvee \Phi)\urcorner:=\bigwedge \Phi\urcorner$,
where, for a set $\Phi$ of $L_{\infty}$-formulas by $\left.\Phi\right\urcorner$ we denote the set $\left.\{\varphi\urcorner: \varphi \in \Phi\right\}$.
Fact 6.2 Let $\varphi$ be an $L_{\infty \omega}$-formula. Then
(a) $\varphi^{\urcorner} \leftrightarrow \neg \varphi$;
(b) If $\varphi \in \mathcal{N}$, then $\varphi\urcorner \in \mathcal{P}$;
(c) If $\varphi \in \neg \mathcal{F}$, then $\varphi\urcorner \in \mathcal{F}$.

Proof. (a) Let $\mathbb{X} \in \operatorname{Mod}_{L}$. By induction we show that for each $\varphi \in \operatorname{Form}_{L_{\infty} \omega}$ we have

$$
\begin{equation*}
\forall \vec{x} \in{ }^{\kappa} X\left(\mathbb{X} \models \varphi^{\urcorner}[\vec{x}] \Leftrightarrow \mathbb{X} \models(\neg \varphi)[\vec{x}]\right) \tag{18}
\end{equation*}
$$

For $\varphi \in \mathrm{At}_{L}$ by definition we have $\left.\varphi\right\urcorner=\neg \varphi$, so (18) is true.
Let $\varphi \in \operatorname{Form}_{\xi}$ and suppose that (18) is true. For $\vec{x} \in{ }^{\kappa} X$ we have $\mathbb{X} \models$ $\left.\left(\forall v_{\alpha} \varphi\right)\right\urcorner[\vec{x}]$ iff $\left.\mathbb{X} \models\left(\exists v_{\alpha} \varphi\right\urcorner\right)[\vec{x}]$ iff for some $x \in X$ we have $\left.\left.\mathbb{X} \models \varphi^{\urcorner}\right\urcorner \vec{x}_{\langle\alpha, x\rangle}\right]$, that is, by (18), $\mathbb{X} \models(\neg \varphi)\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, that is, $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ is not true, iff it is not true that for all $x \in X$ we have $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ iff $\mathbb{X} \models\left(\forall v_{\alpha} \varphi\right)[\vec{x}]$ is not true iff $\mathbb{X} \models\left(\neg \forall v_{\alpha} \varphi\right)[\vec{x}]$. Also $\left.\mathbb{X} \vDash\left(\exists v_{\alpha} \varphi\right)\right\urcorner[\vec{x}]$ iff $\mathbb{X} \vDash\left(\forall v_{\alpha} \varphi^{\urcorner}\right)[\vec{x}]$ iff for each $x \in X$ we have $\mathbb{X} \models \varphi\urcorner\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, that is, by (18), $\mathbb{X} \models(\neg \varphi)\left[\vec{x}_{\langle\alpha, x\rangle}\right]$, that is, $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ is not true, iff it is not true that for some $x \in X$ we have $\mathbb{X} \models \varphi\left[\vec{x}_{\langle\alpha, x\rangle}\right]$ iff $\mathbb{X} \models\left(\exists v_{\alpha} \varphi\right)[\vec{x}]$ is not true iff $\mathbb{X} \models\left(\neg \exists v_{\alpha} \varphi\right)[\vec{x}]$. Finally, $\mathbb{X} \models(\neg \varphi) \neg[\vec{x}]$ iff $\mathbb{X} \models \varphi[\vec{x}]$ iff $\mathbb{X} \models \neg(\neg \varphi)[\vec{x}]$.

Let $\Phi \subset \operatorname{Form}_{\xi}$ and suppose that $\varphi$ satisfies (18), for all $\varphi \in \Phi$. For $\vec{x} \in{ }^{\kappa} X$ we have $\mathbb{X} \models(\bigwedge \Phi)\urcorner[\vec{x}]$ iff $\mathbb{X} \models \bigvee\{\varphi\urcorner: \varphi \in \Phi\}[\vec{x}]$, iff for some $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi\urcorner[\vec{x}]$, that is, by (18), $\mathbb{X} \models(\neg \varphi)[\vec{x}]$, iff it is not true that for all $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi[\vec{x}]$, iff it is not true that $\mathbb{X} \models(\bigwedge \Phi)[\vec{x}]$ iff $\mathbb{X} \models(\neg \bigwedge \Phi)[\vec{x}]$. Also, $\mathbb{X} \models(\bigvee \Phi)\urcorner[\vec{x}]$ iff $\mathbb{X} \models \bigwedge\left\{\varphi^{\urcorner}: \varphi \in \Phi\right\}[\vec{x}]$, iff for each $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi^{\urcorner}[\vec{x}]$, that is, by $(\overline{18)}, \mathbb{X} \models(\neg \varphi)[\vec{x}]$, iff it is not true that for some $\varphi \in \Phi$ we have $\mathbb{X} \models \varphi[\vec{x}]$, iff it is not true that $\mathbb{X} \models(\bigvee \Phi)[\vec{x}]$ iff $\mathbb{X} \models(\neg \bigvee \Phi)[\vec{x}]$.
(b) First we have $\left.\left(v_{\alpha}=v_{\beta}\right)\right\urcorner:=\neg v_{\alpha}=v_{\beta} \in \mathcal{P}$ and $\left.\left(\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)\right\urcorner:=$ $R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) \in \mathcal{P}$ and, also, $\left.\left(\neg v_{\alpha}=v_{\beta}\right)\right\urcorner:=v_{\alpha}=v_{\beta} \in \mathcal{P}$. If $\varphi \in \mathcal{N}_{\xi}$ and $\varphi\urcorner \in \mathcal{P}$, then the formulas $\left.\left.\left(\forall v_{\alpha} \varphi\right)\right\urcorner:=\exists v_{\alpha} \varphi\right\urcorner$ and $\left.\left.\left(\exists v_{\alpha} \varphi\right)\right\urcorner:=\forall v_{\alpha} \varphi\right\urcorner$ belong to $\mathcal{P}$. If $\Phi \subset \mathcal{N}_{\xi}$ and $\left.\varphi\right\urcorner \in \mathcal{P}$, for all $\varphi \in \Phi$, then $\left.\Phi\right\urcorner \subset \mathcal{P}$ and, hence, $(\bigwedge \Phi)\urcorner:=\bigvee \Phi\urcorner \in \mathcal{P}$ and $(\bigvee \Phi)\urcorner:=\bigwedge \Phi\urcorner \in \mathcal{P}$.
(c) By (b) the claim is true for all $\varphi \in \mathcal{N}$. Also we have $\left(R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right)\right)^{\neg}:=$ $\neg R_{i}\left(v_{\alpha_{1}}, \ldots, v_{\alpha_{n_{i}}}\right) \in \mathcal{F}$. If $\varphi \in \neg \mathcal{F}_{\xi}$ and $\left.\varphi\right\urcorner \in \mathcal{F}$, then $\left.\left.\left(\exists v_{\alpha} \varphi\right)\right\urcorner:=\forall v_{\alpha} \varphi\right\urcorner \in$ $\mathcal{F}$. If $\Phi \subset \neg \mathcal{F}_{\xi}$ and $\left.\varphi\right\urcorner \in \mathcal{F}$, for all $\varphi \in \Phi$, then $\left.\Phi\right\urcorner \subset \mathcal{F}$ and, hence, $(\bigvee \Phi)\urcorner:=\bigwedge \Phi\urcorner \in \mathcal{F}$. If $\Phi$ is a finite set, then, again, $(\bigwedge \Phi)\urcorner:=\bigvee \Phi\urcorner \in \mathcal{F}$.

Proof of Theorem 6.1, (a) It is a standard fact that, if $\psi$ is an $L_{\infty \omega}$-formula, $\varphi$ its subformula, $\varphi \leftrightarrow \varphi^{\prime}$ and $\psi^{\prime}$ the formula obtained from $\psi$ by replacement of $\varphi$ by $\varphi^{\prime}$, then $\psi \leftrightarrow \psi^{\prime}$. So, by Fact 6.2(a), the sentence $\psi:=\forall \bar{v} \exists_{\leq_{n}} \widetilde{w} \varphi(\bar{v}, \widetilde{w})$ is logically equivalent to the sentence

$$
\begin{equation*}
\left.\psi^{\prime}:=\forall \bar{v} \forall \widetilde{w}^{1}, \ldots \widetilde{w}^{n+1}\left(\bigvee_{k \leq n+1} \varphi\right\urcorner\left(\bar{v}, \widetilde{w}^{k}\right) \vee \bigvee_{1 \leq k<l \leq n+1} \widetilde{w}^{k}=\widetilde{w}^{l}\right) \tag{19}
\end{equation*}
$$

By Fact 6.2(c) we have $\varphi^{\urcorner} \in \mathcal{F}$ so the sentence $\psi^{\prime}$ belongs to $\mathcal{F}$ and the statement follows from Theorem 4.1 (a).
(b) It is easy to check that for an $L_{\infty \omega}$-formula $\varphi$, up to logical equivalence, we have: $\varphi \in \mathcal{P}$ iff $\varphi^{c} \in \mathcal{N}$ and, also, $\varphi \in \neg \mathcal{G}$ iff $\varphi^{c} \in \neg \mathcal{F}$. So, if $\varphi(\bar{v}, \widetilde{w}) \in \neg \mathcal{G}$, then $\varphi^{c}(\bar{v}, \widetilde{w}) \in \neg \mathcal{F}$ and, by (a), $\left.\mathbb{P}^{\prime}:=\left\langle\operatorname{Int}_{L}^{\mathcal{T}^{c} \cup\{\forall \bar{v}} \exists_{\leq n} \widetilde{w} \varphi^{c}(\bar{v}, \widetilde{w})\right\}(X), \subset\right\rangle$ has the properties from (a). Since $\forall \bar{v} \exists_{\leq_{n}} \widetilde{w} \varphi^{c}(\bar{v}, \widetilde{w})$ is the formula $\left(\forall \bar{v} \exists_{\leq_{n}} \widetilde{w} \varphi^{c}(\bar{v}, \widetilde{w})\right)^{c}$, by Theorem 3.2(b) we have $\mathbb{P}^{\prime}=\left\{\rho^{c}: \rho \in \operatorname{Int}_{L}^{\mathcal{T} \cup\left\{\forall \bar{v} \exists_{\leq n} \tilde{w} \varphi(\bar{v}, \tilde{w})\right\}}(X)\right\}$, which means that $\mathbb{P}$ is the reverse of $\mathbb{P}^{\prime}$ and, hence, has the mentioned properties.

Maximal graphs of finite degree If $n \in \omega$, a graph $\mathbb{G}=\langle X, \rho\rangle$ is of degree $\leq n$ iff $\operatorname{deg}(x):=|\{y \in X:\{x, y\} \in \rho\}| \leq n$, for all $x \in X$. Since the atomic $L_{b}$-formula $R(v, w)$ belongs to the class $\neg \mathcal{F}$, by Theorem 6.1 the poset $\mathbb{P}=\left\langle\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }}} \cup\left\{\psi_{\operatorname{deg} \leq n}\right\}(X), \subset\right\rangle$, where $\psi_{\operatorname{deg} \leq n}:=\forall v \exists_{\leq n} w R(v, w)$ is the $L_{b^{-}}$ sentence saying that a graph is of degree $\leq n$, is union-complete and $\operatorname{Max} \mathbb{P} \subset$ $\operatorname{Rev}_{L_{b}}(X)$ is a co-dense set in $\mathbb{P}$.

Example 6.3 Maximal graphs of degree $\leq 2$. We recall that, for $n \in \omega$, a graph $\mathbb{G}=\langle X, \rho\rangle$ is called $n$-regular iff $\operatorname{deg}(x)=n$, for all $x \in X$. The linear graphs $\langle\omega,\{\{n, n+1\}: n \in \omega\}\rangle$ and $\langle Z,\{\{n, n+1\}: n \in Z\}\rangle$, where $Z$ is the set of integers, will be denoted by $\mathbb{G}_{\omega}$ and $\mathbb{G}_{\mathbb{Z}}$, respectively.

Claim 6.4 A graph $\mathbb{X}$ is a maximal graph of degree $\leq 2$ iff $\mathbb{X} \cong \mathbb{Y} \cup \mathbb{Z}$, where

- $\mathbb{Y}$ is $\emptyset$, or a 2-regular graph,
- $\mathbb{Z}$ is $\emptyset$, or $\mathbb{K}_{1}$, or $\mathbb{K}_{2}$, or $\mathbb{G}_{\omega}$.

Proof. The implication $\Leftarrow$ is evident.
$(\Rightarrow)$ Let $\mathbb{X}=\langle X, \sim\rangle$ be a maximal graph of degree $\leq 2$. Suppose that there are three different vertices $x, y, z \in X$ of degree $<2$. Then the substructure of $\mathbb{X}$ determined by $\{x, y, z\}$ is not a complete graph, say $\{x, y\} \notin \sim$ and the graph
$\langle X, \sim \cup\{\{x, y\}\}\rangle$ is of degree $\leq 2$, which contradicts the maximality of $\mathbb{X}$. If all the vertices of $\mathbb{X}$ are of degree 2 , then $\mathbb{X}=\mathbb{Y} \dot{\cup} \emptyset$, where $\mathbb{Y}$ is a 2 -regular graph.

If two vertices of $\mathbb{X}$, say $x$ and $y$, are of degree $<2$, then by the maximality of $\mathbb{X},\{x, y\} \in \sim$ and, hence, $\mathbb{X}=\mathbb{Y} \cup \mathbb{K}_{2}$, where $\mathbb{Y}$ is a 2-regular graph or $\emptyset$.

If exactly one vertex of $\mathbb{X}$, say $x$, is of degree $<2$ and $\operatorname{deg}(x)=0$, then $\mathbb{X}=\mathbb{Y} \cup \mathbb{K}_{1}$, where $\mathbb{Y}$ is a 2-regular graph or $\emptyset$. Otherwise we have $\operatorname{deg}(x)=1$ and, hence, there is $y \in X \backslash\{x\}$ such that $\{x, y\} \in \sim$. If $C_{x}$ is the connectivity component of $\mathbb{X}$ containing $x$, then in $C_{x}$ we have $\operatorname{deg}(x)=1$ and $\operatorname{deg}(z)=2$, for all $z \in C_{x} \backslash\{x\}$. Now defining $x_{0}=x, x_{1}=y$ and $x_{n+1}$ as the unique neighbor of $x_{n}$ different from $x_{n-1}$ we have $\left\{x_{n}: n \in \omega\right\} \subset C_{x}$ and, by the connectedness of $C_{x}$ we have the equality. Thus $C_{x} \cong \mathbb{G}_{\omega}$. Now, if the graph $\mathbb{X}$ is connected, then $\mathbb{X} \cong \emptyset \dot{\cup} \mathbb{G}_{\omega}$. Otherwise, the graph induced on the set $X \backslash C_{x}$ is 2-regular and we have $\mathbb{X} \cong \mathbb{Y} \dot{\cup} \mathbb{G}_{\omega}$, where $\mathbb{Y}$ is a 2 -regular graph.

It is known that 2-regular graphs are characterized as disjoint unions of copies of $\mathbb{G}_{\mathbb{Z}}$ and $\mathbb{C}_{n}$, for $n \geq 3$. Thus there are $\mathfrak{c}$-many non-isomorphic maximal countable graphs of degree $\leq 2$; so, the poset $\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\psi_{\text {deg } \leq 2}\right\}}(\omega)$ has $\mathfrak{c}$-many nonisomorphic maximal elements; they are reversible and characterized in Claim6.4,

Example 6.5 Maximal connected graphs of degree $\leq n$. Since $\varphi_{\text {conn }} \in \mathcal{P}$, by Theorem 6.1 maximal elements of the poset $\left\langle\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\psi_{\operatorname{deg} \leq n}, \varphi_{c o n n}\right\}}(X), \subset\right\rangle$ form a co-dense set in it consisting of reversible interpretations. By the analysis from Example 6.3, $\operatorname{Max}\left(\operatorname{Int}_{L_{b}}^{\mathcal{T}_{\text {graph }} \cup\left\{\psi_{\operatorname{deg} \leq 2}, \varphi_{\text {conn }}\right\}}(\omega)\right)=\left[\mathbb{G}_{\omega}\right] \cong \cup\left[\mathbb{G}_{\mathbb{Z}}\right]_{\cong}$.

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