Tanaka's Theorem Revisited

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Abstract

Tanaka (1997) proved a powerful generalization of Friedman's self-embedding theorem that states that given a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of the subsystem WKL_0 of second order arithmetic, and any element m of $\mathcal{M}$, there is a self-embedding j of $(\mathcal{M}, \mathcal{A})$ onto a proper initial segment of itself such that j fixes every predecessor of m.

Here we extend Tanaka's work by establishing the following results for a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0 and a proper cut I of $\mathcal{M}$:

Theorem A. *The following conditions are equivalent*:
(a) I *is closed under exponentiation.*
(b) *There is a self-embedding j of $(\mathcal{M}, \mathcal{A})$ onto a proper initial segment of itself such that I is the longest initial segment of fixed points of j.*

Theorem B. *The following conditions are equivalent*:
(a) I *is a strong cut of* $\mathcal{M}$ *and* $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$.
(b) *There is a self-embedding j of $(\mathcal{M}, \mathcal{A})$ onto a proper initial segment of itself such that* I *is the set of all fixed points of j.*

1 Introduction

One of the fundamental results concerning nonstandard models of Peano arithmetic (PA) is Friedman's theorem [4, Theorem 4.4] that states every countable nonstandard model of PA is isomorphic to a proper initial segment of itself. A notable generalization of Friedman's theorem was established by Tanaka [13] for models of the well-known subsystem WKL_0 of second order arithmetic, who established:

Theorem 1.1. (Tanaka) *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard model of* WKL_0.
(a) *There is a proper initial segment* I *of* $\mathcal{M}$ *and an isomorphism* j *between* $(\mathcal{M}, \mathcal{A})$ *and* $(\mathrm{I}, \mathcal{A}_\mathrm{I})$*, where* $\mathcal{A}_\mathrm{I} := \{\mathrm{A} \cap \mathrm{I} : \ \mathrm{A} \in \mathcal{A}\}$.
(b) *Given any prescribed* m *in* $\mathcal{M}$*, there is an* I *and* j *as in (a) such that* $j(x) = x$ *for all* $x \leq m$.

Tanaka's principal motivation in establishing Theorem 1.1 was the development of non-standard methods within WKL_0 in the context of the reverse mathematics research program; for example Tanaka and Yamazaki [14] used Theorem 1.1 to show that the Haar measure over compact groups can be implemented in WKL_0 via a detour through nonstandard models. This is in contrast to the previously known constructions of the Haar measure whose implementation required the stronger subsystem ACA_0. Other notable applications of the methodology of nonstandard models can be found in the work of Sakamato and Yokoyama [11], who showed that over the subsystem RCA_0 the Jordan curve theorem and the Schönflies theorem are equivalent to WKL_0; and in the work of Yokoyama and Horihata [8], who established the equivalence of ACA_0 and Riemann's mapping theorem for Jordan regions over WKL_0.

Here we continue our work [1] on the study of fixed point sets of self-embeddings of countable nonstandard models of $\mathrm{I}\Sigma_1$ by focusing on the behavior of fixed point sets in Tanaka's theorem. Our methodology can be generally described as an amalgamation of Enayat's strategy for proving Tanaka's theorem [3] with some ideas and results from [1].[1] Before stating our results, recall that j is said to be a proper initial self-embedding of $(\mathcal{M}, \mathcal{A})$ if there is a proper initial segment I of $\mathcal{M}$ and an isomorphism j between $(\mathcal{M}, \mathcal{A})$ and $(\mathrm{I}, \mathcal{A}_\mathrm{I})$, where $\mathcal{A}_\mathrm{I} := \{\mathrm{A} \cap \mathrm{I} : \ \mathrm{A} \in \mathcal{A}\}$; $\mathrm{I}_{\mathrm{fix}}(j)$ is the longest initial segment of fixed points of j; and $\mathrm{Fix}(j)$ is the fixed point set of j, in other words:

$$\mathrm{I}_{\mathrm{fix}}(j) := \{m \in M : \forall x \leq m \ j(x) = x\}, \text{ and}$$
$$\mathrm{Fix}(j) := \{m \in M : j(m) = m\}.$$

Our main results are Theorems A and B below. Note that Theorem A is a strengthening of Tanaka's Theorem (see Section 3 for more detail).

Theorem A. *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard model of* WKL_0. *The following conditions are equivalent for a proper cut* I *of* $\mathcal{M}$:

[1] Enayat's paper [3] provides a complete proof of part (a) of Theorem 1.1, and an outline of the proof of part (b) of Theorem 1.1.

(1) *There is a self-embedding j of $(\mathcal{M}, \mathcal{A})$ such that* $\mathrm{I}_{\mathrm{fix}}(j) = \mathrm{I}$.

(2) I *is closed under exponentiation.*

(3) *There is a proper initial self-embedding j of $(\mathcal{M}, \mathcal{A})$ such that* $\mathrm{I}_{\mathrm{fix}}(j) = \mathrm{I}$.

Theorem B. *Suppose $(\mathcal{M}, \mathcal{A})$ is a countable nonstandard model of* WKL_0. *The following conditions are equivalent for a proper cut* I *of* $\mathcal{M}$:

(1) *There is a self-embedding j of $(\mathcal{M}, \mathcal{A})$ such that* $\mathrm{Fix}(j) = \mathrm{I}$.

(2) I *is a strong cut of* $\mathcal{M}$ *and* $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$.

(3) *There is a proper initial self-embedding j of $(\mathcal{M}, \mathcal{A})$ such that* $\mathrm{Fix}(j) = \mathrm{I}$.

Theorem A is established in Section 3, and Section 4 is devoted to proving Theorem B.

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2 Preliminaries

In this section we review some definitions and basic results that are relevant to the statements and proofs of our main results.

- WKL_0 is the second order theory whose models are of the form of $(\mathcal{M}, \mathcal{A})$, where $(\mathcal{M}, \mathcal{A})$ satisfies (1) Induction for Σ^0_1 formulas; (2) Comprehension for Δ^0_1-formulas; and (3) Weak König's Lemma (which asserts that every infinite subtree of the full binary tree has an infinite branch).

- $\mathrm{Exp} := \forall x \exists y\, \mathrm{Exp}(x, y)$, where $\mathrm{Exp}(x, y)$ is a Δ_0-formula that expresses $2^x = y$ within $\mathrm{I}\Delta_0$.

- The binary Δ_0-formula xEy, known as *Ackermann's membership relation*, expresses "the x-th bit of the binary expansion of y is 1" within $\mathrm{I}\Delta_0$.

- A subset X of M is *coded* in $\mathcal{M}$ iff there is some $a \in \mathrm{M}$ such that $\mathrm{X} = (a_E)^{\mathcal{M}} := \{x \in \mathrm{M} : \mathcal{M} \models xEa\}$.
  Given $a \in \mathrm{M}$, by $\underline{a}$ we mean the set $\{x \in \mathrm{M} : x < a\}$. Note that $\underline{a}$ is coded in $\mathcal{M}$, where $\mathcal{M}$ is a model of $\mathrm{I}\Delta_0$, provided 2^a exists in $\mathcal{M}$.
  It is well-known that for any $n > 0$ and $\mathcal{M} \models \mathrm{I}\Sigma_n$, if $\varphi(x, a)$ is an

unary Σ_n-formula, where a is a parameter from $\mathcal{M}$, then the set $\varphi^{\mathcal{M}}(a) := \{b \in \mathrm{M}\colon \mathcal{M} \models \varphi(b,a)\}$ is *piece-wise coded* in $\mathcal{M}$; i.e. for every $c \in \mathrm{M}$, $\varphi^{\mathcal{M}}(a) \cap \underline{c}$ is coded in $\mathcal{M}$. More specifically, there is some element less than 2^c which codes $\varphi^{\mathcal{M}}(a) \cap \underline{c}$.
Moreover, the above statement holds for $n = 0$ if $\mathcal{M} \models \mathrm{I}\Delta_0 + \mathrm{Exp}$.

- For every cut I of $\mathcal{M}$, the I-*standard system* of $\mathcal{M}$, presented by $\mathrm{SSy}_I(\mathcal{M})$, is the family consisting of sets of the form $a_E \cap \mathrm{I}$, where $a \in \mathrm{M}$; in other words:

$$\mathrm{SSy}_{\mathrm{I}}(\mathcal{M}) = \{(a_E)^{\mathcal{M}} \cap \mathrm{I} : \ a \in \mathrm{M}\}.$$

  When I is the standard cut, i.e. $\mathrm{I} = \mathbb{N}$, we simply write $\mathrm{SSy}(\mathcal{M})$ instead of $\mathrm{SSy}_{\mathbb{N}}(\mathcal{M})$.

- Given a proper cut I of $\mathcal{M}$, I is called a *strong cut*, if for every coded function f in $\mathcal{M}$ whose domain contains I, there exists some $s \in \mathrm{M}$ such that for every $i \in \mathrm{I}$ it holds that $f(i) \notin \mathrm{I}$ iff $s < f(i)$.

- Sat_{Σ_n} is the arithmetical formula defining the satisfaction predicate for Σ_n-formulas within $\mathrm{I}\Delta_0 + \mathrm{Exp}$. It is well-known that for each positive $n \in \omega$, Sat_{Σ_n} can be expressed by a Σ_n-formula in $\mathrm{I}\Sigma_n$; furthermore, within a model of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ (with the help of a nonstandard parameter if the model is nonstandard), Sat_{Σ_0} (which is also written as Sat_{Δ_0}) is expressible both as a Σ_1 and Π_1-formula.

- The strong Σ_n-*Collection scheme* consists of formulas of the following form where φ is a Σ_n-formula:

  $\forall w\ \forall v\ \exists z \forall x < v(\exists y\ \varphi(x,y,w) \to \exists y < z\ \varphi(x,y,w))$.

  It is well-known that the strong Σ_n-Collection scheme is provable in $\mathrm{I}\Sigma_n$ for every $n > 0$.

- Every model $\mathcal{M}$ of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ satisfies the *Coded Pigeonhole Principle*, i.e. if $b \in \mathrm{M}$, and $f : \underline{b+1} \to \underline{b}$ is a coded function in $\mathcal{M}$, then f is not injective.

- By an embedding j from second order model $(\mathcal{M}, \mathcal{A})$ into $(\mathcal{N}, \mathcal{B})$, we mean that j is an embedding from $\mathcal{M}$ into $\mathcal{N}$ such that for every $X \subseteq M$, $X \in \mathcal{A}$ iff $j(X) = Y \cap j(M)$ for some $Y \in \mathcal{B}$.

- When $\mathcal{M}$ and $\mathcal{N}$ are models of arithmetic and b is in M, we write $\mathcal{M} \subseteq_{\mathrm{end},\Pi_{1,\leq b}} \mathcal{N}$, when $\mathcal{N}$ is an end extension of $\mathcal{M}$ and all Π_1-formulas whose parameters are in $\underline{b+1}$ are absolute in the passage between $\mathcal{M}$ and $\mathcal{N}$, i.e., $\mathrm{Th}_{\Pi_1}(\mathcal{M}, m)_{m\leq b} = \mathrm{Th}_{\Pi_1}(\mathcal{N}, m)_{m\leq b}$.

Theorem 2.1. ([3, Theorem 3.2]) *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* WKL_0 *and let* $b \in \mathrm{M}$. *Then* $\mathcal{M}$ *has a countable recursively saturated proper end extension* $\mathcal{N}$ *satisfying* $\mathrm{I}\Delta_0 + \mathrm{Exp} + \mathrm{B}\Sigma_1$ *such that* $\mathrm{SSy}_{\mathrm{M}}(\mathcal{N}) = \mathcal{A}$, *and* $\mathcal{M} \subseteq_{\mathrm{end},\Pi_1,\leq \mathrm{b}} \mathcal{N}$.

Remark 2.2. The proofs of our main results take advantage of the following additional features of the model $\mathcal{N}$ constructed in Enayat's *proof* of Theorem 2.1, namely: given b in $\mathcal{M}$, there is an elementary chain of models $(\mathcal{N}_n : n \in \omega)$ satisfying the following three properties:

(*i*) $\mathcal{N} = \cup_{n\in\omega}\mathcal{N}_n$;

(*ii*) For every $n \in \omega$, $\mathcal{M} \subseteq_{\mathrm{end},\Pi_1,\leq \mathrm{b}} \mathcal{N}_n \prec \mathcal{N}$;

(*iii*) For every $n \in \omega$ the elementary diagram of $(\mathcal{N}_n, a)_{a\in \mathrm{N}_n}$ is available in $(\mathcal{M}, \mathcal{A})$ via some $\mathrm{ED}_n \in \mathcal{A}$. Note that $\mathrm{Th}\left((\mathcal{N}_n, a)_{a\in \mathrm{N}_n}\right)$ is a proper subset of ED_n since ED_n includes sentences of nonstandard length.

Remark 2.3. Enayat [3] noted that if $(\mathcal{M}, \mathcal{A})$ is a model of WKL_0 and b is in M, and there is some end extension $\mathcal{N}$ of $\mathcal{M}$ such that (1) $\mathcal{N} \models I\Delta_0$+Exp, (2) $\mathrm{SSy}_{\mathrm{M}}(\mathcal{N}) = \mathcal{A}$, and (3) there is an initial self-embedding j_1 of $\mathcal{N}$ onto an initial segment that is bounded above by b, then the restriction j of j_1 to $\mathcal{M}$ is an embedding of $\mathcal{M}$ onto an an initial segment J of $\mathcal{M}$ that is below b which has the important feature that j is an isomorphism between $(\mathcal{M}, \mathcal{A})$ and $(\mathrm{J}, \mathcal{A}_{\mathrm{J}})$. Note that if I is a proper cut of $\mathcal{M}$, then $\mathrm{I}_{\mathrm{fix}}(j_1) = \mathrm{I}$ implies that $\mathrm{I}_{\mathrm{fix}}(j) = \mathrm{I}$; and $\mathrm{Fix}(j_1) = \mathrm{I}$ implies that $\mathrm{Fix}(j) = \mathrm{I}$.

- The following theorem summarizes some of the results about $\mathrm{I}_{\mathrm{fix}}(j)$ and $\mathrm{Fix}(j)$ from [1] which will be employed in this paper:

Theorem 2.4. Suppose $\mathcal{M} \models \mathrm{I}\Delta_0$+Exp *and* j *is a nontrivial self-embedding of* $\mathcal{M}$. *Then:*
(a) $\mathrm{I}_{\mathrm{fix}}(j) \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathrm{Exp}$.
(b) *If* $\mathcal{M} \models \mathrm{I}\Sigma_1$ *then* $\mathrm{Fix}(j)$ *is a* Σ_1*-elementary submodel of* $\mathcal{M}$. *Moreover, if* $\mathrm{Fix}(j)$ *is a proper initial segment of* $\mathcal{M}$, *then it is a strong cut* of $\mathcal{M}$.

- Given two countable nonstandard model $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{I}\Sigma_1$ which share a common proper cut I, the following theorem from [1, Cor. 3.3.1] provides a useful sufficient condition for existence of a proper initial embedding between $\mathcal{M}$ and $\mathcal{N}$ which fixes each element of I:

Theorem 2.5. *Let $\mathcal{M}$, $\mathcal{N}$ and I be as above such that I is closed under exponentiation. The following are equivalent*:

(1) *There is a proper initial embedding f of $\mathcal{M}$ into $\mathcal{N}$ such that $f(i) = i$ for all $i \in I$.*

(2) $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, i)_{i\in I} \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N}, i)_{i\in I}$ *and* $\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathcal{N})$.

- Another prominent subsystem of second order arithmetic is ACA_0, in which the comprehension scheme is restricted to formulas with no second order quantifier. The following results of Paris and Kirby [9] and Gaifman [6, Thm. 4.9-4.11] concerning ACA_0 are employed in the proof of Theorem B.[2]

Theorem 2.6. (Paris and Kirby) Suppose $\mathcal{M} \models \mathrm{I}\Delta_0$. *The following are equivalent for a proper cut* I *of* $\mathcal{M}$:

(a) I *is a strong cut of* $\mathcal{M}$.

(b) $(\mathrm{I}, \mathrm{SSy}_\mathrm{I}(\mathcal{M})) \models \mathrm{ACA}_0$.

Theorem 2.7. (Gaifman) *Given a countable model $(\mathcal{M}, \mathcal{A})$ of* ACA_0 *and a linear order $\mathbb{L}$, there exists an end extension $\mathcal{M}_\mathbb{L}$ of $\mathcal{M}$ such that there is an isomorphic copy $\mathbb{L}' = \{c_l : l \in \mathbb{L}\}$ of $\mathbb{L}$ in* $\mathrm{M}_\mathbb{L}\backslash\mathrm{M}$, *and there is a composition preserving embedding $j \mapsto \widehat{j}$ from the semi-group of initial self-embeddings of $\mathbb{L}$ into the semi-group of initial self-embeddings of $\mathcal{M}_\mathbb{L}$ that satisfy the following properties*:

(a) $\mathrm{SSy}_\mathrm{M}(\mathcal{M}_\mathbb{L}) = \mathcal{A}$ *and* $\mathrm{Fix}(\widehat{j}) = M$ *for each initial self-embedding j of $\mathbb{L}$ that is fixed point free.*

(b) *For each initial self-embedding j of $\mathbb{L}$, $\widehat{j}$ is an* ***elementary*** *initial self-embedding of $\mathcal{N}_\mathbb{L}$, i.e. $\widehat{j}(\mathcal{M}_\mathbb{L}) \preceq_{\mathrm{end}} \mathcal{N}_\mathbb{L}$.*

(c) $\mathbb{L}'$ *is downward cofinal in $M_\mathbb{L}\backslash M$ if $\mathbb{L}$ has no first element.*

(d) *For any $l_0 \in \mathbb{L}$, l_0 is a strict upper bound for $j(\mathbb{L})$ iff c_{l_0} is a strict upper bound for $\widehat{j}(M_\mathbb{L})$.*

[2]Gaifman couched his results in terms of arbitrary models of $\mathrm{PA}(\mathcal{L})$ for countable $\mathcal{L}$. Note that if $(\mathcal{M}, \mathcal{A}) \models \mathrm{ACA}_0$, then the expansion $(\mathcal{M}, A)_{A\in\mathcal{A}}$ of $\mathcal{M}$ is a model of $\mathrm{PA}(\mathcal{L})$, where $\mathcal{L}$ is the extension of $\mathcal{L}_A$ by predicate symbols for each $A \in \mathcal{A}$. Moreover, the collection of subsets of M that are parametrically definable in $(\mathcal{M}, A)_{A\in\mathcal{A}}$ coincides with $\mathcal{A}$.

3 The longest cut fixed by self-embeddings

This section is devoted to the proof of Theorem A. Recall that if $(\mathcal{M}, \mathcal{A})$ is a model of WKL_0 there are arbitrary large as well as arbitrary small nonstandard cuts in $\mathcal{M}$ that are closed under exponentiation. More specifically, for every nonstandard a in M, there are nonstandard cuts I_1 and I_2 (as defined below) such that $\mathrm{I}_1 < a \in \mathrm{I}_2$, and both are closed under exponentiation:

$$\mathrm{I}_1 := \{x \in M : 2^x_n < a \text{ for all } n \in \omega\}, \text{ where } 2^x_0 := x, \text{ and for every } n \in \omega, \; 2^x_{n+1} := 2^{2^x_n};$$

$$\mathrm{I}_2 := \{x \in M : x < 2^a_n \text{ for some } n \in \omega\}.$$

So Theorem A implies that $\mathrm{I}_{\mathrm{fix}}(j)$ can be arranged to be as high or as low in the nonstandard part of $\mathcal{M}$ as desired. In particular, Theorem A is a strengthening of Tanaka's Theorem.

Proof of Theorem A. (1) $\Rightarrow$ (2) is an immediate consequence of Theorem 2.4.(a), and (3) $\Rightarrow$ (1) is trivial so we concentrate on establishing (2) $\Rightarrow$ (3).

Assume that I is closed under exponentiation and fix some $a \in \mathrm{M} \setminus \mathrm{I}$. We leave it as an exercise for the reader to use strong Σ_1-Collection along with the fact that Sat_{Δ_0} has a Σ_1-description in $\mathcal{M}$ to show that there is some $b \in \mathrm{M}$ such that:

($\sharp$) $\quad \mathcal{M} \models \forall w < a\, (\exists z\ \delta(z, w) \to \exists z < b\ \delta(z, w))$, for all Δ_0-formulas δ.

Next we invoke Theorem 2.1 to get hold of a countable recursively saturated proper end extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathrm{Exp}$, $\mathrm{SSy}_{\mathrm{M}}(\mathcal{N}) = \mathcal{A}$, and $\mathcal{M} \subseteq_{\mathrm{end}, \Pi_1, \leq \mathrm{b}} \mathcal{N}$. Moreover, we will safely assume that the model $\mathcal{N}$ additionally satisfies the three properties listed in Remark 2.2. In light of Remark 2.3, in order to establish (3) it suffices to construct a proper initial self-embedding j of $\mathcal{N}$ such that $j(\mathrm{N}) < b$ and $\mathrm{I}_{\mathrm{fix}}(j) = \mathrm{I}$. The construction of the desired j is the novel element of the proof of Theorem A, which we now turn to.

To construct j we will employ a modification of the strategy employed in the proof of (2) $\Rightarrow$ (3) of [1, Theorem 4.1], using a 3-level back-and-forth method. A modification is needed since we need to overcome the fact that $\mathrm{I}\Sigma_1$ need not hold in $\mathcal{N}$; instead we will rely on recursive saturation of $\mathcal{N}$ and the properties of $\mathcal{N}$ listed in Remark 2.2. First, note that ($\sharp$) together with the fact that $\mathrm{Th}_{\Pi_1}(\mathcal{M}, x)_{x \leq b} = \mathrm{Th}_{\Pi_1}(\mathcal{N}, x)_{x \leq b}$, implies:

$(*) \quad \mathcal{N} \models \forall w < a\,(\exists z\ \delta(z,w) \to \exists z < b\ \delta(z,w))$, for all Δ_0-formula δ.

Since I is closed under exponentiation, we can choose $\{c_n : n \in \omega\}$ that is downward cofinal in M $\setminus$ I such that $c_0 = a$ and $2^{c_{n+1}} < c_n$ for all $n \in \omega$.

The proof will be complete once we recursively construct finite sequences $\bar{u} := (u_0, ..., u_{m-1})$ and $\bar{v} := (v_0, ..., v_{m-1})$ of elements of $\mathcal{N}$ for all $n \in \omega$ such that:

(i) $u_0 = 0 = v_0$.

(ii) For every c in N there is some $n \in \omega$ such that $c = u_n$.

(iii) For every $n \in \omega$, $v_n < b$, and if for some c in N it holds that $c < v_n$, then there is some $m \in \omega$ such that $c = v_m$.

(iv) For every $m \in \omega$ the following condition holds:
$(*_m): \quad \mathcal{N} \models \forall w < c_m\ (\exists z\ \delta(z, w, \bar{u}) \to \exists z < b\ \delta(z, w, \bar{v}))$,
for every Δ_0-formula δ.

(v) For every $m \in \omega$, there is some $n \in \omega$ such that $u_n < c_m$ and $u_n \neq v_n$.

Note that $(*_0)$ holds thanks to $(*)$ since $c_0 = a$. Let $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ respectively be enumerations of element of N and $\underline{b}$. By statement (i) and $(*_0)$ the first step of induction holds. Suppose for $m \in \omega$, $\bar{u}$ and $\bar{v}$ are constructed such that $(*_m)$ holds. In order to find suitable u_{m+1} and v_{m+1}, by considering congruence modulo 3 we have three cases for $m+1$: Case 0 takes care of (ii) and (iv), Case 1 takes care of (iii) and (iv), and Case 2 takes care of (v) and (iv).

CASE 0 ($\mathbf{m+1=3k}$, for some $\mathbf{k} \in \omega$): In this case if a_k is one of the elements of $\bar{u}$, put $u_{m+1} = u_m$ and $v_{m+1} = v_m$. Otherwise, put $u_{m+1} = a_k$ and define:

$$p(y) := \{y < b\} \cup$$
$$\{\forall w < c_{m+1}(\exists z\ \delta(z, w, \bar{u}, a_k) \to \exists z < b\ \delta(z, w, \bar{v}, y)) : \delta \text{ is a } \Delta_0\text{−formula}\}.$$

Note that $p(y)$ is a recursive type. Since $\mathcal{N}$ is recursively saturated, it suffices to prove that $p(y)$ is finitely satisfiable and let v_{m+1} be one of the realizations of $p(y)$ in $\mathcal{N}$. Since $p(y)$ is closed under conjunctions we only need to show that each formula in $p(y)$ is satisfiable. For this purpose, suppose δ is a Δ_0-formula, and let

$$\mathrm{D} := \left\{ w \in \underline{c_{m+1}} : \mathcal{N} \models \exists z \ \delta(z, w, \bar{u}, a_k) \right\}.$$

We claim that there is some $d < 2^{c_{m+1}}$ which codes D in $\mathcal{N}$. To see this, we note that in the above definition, $\mathcal{N}$ can be safely replaced by some $\mathcal{N}_n$, where n is large enough to contain the parameters $\bar{u}$ and a_k (thanks to properties (i) and (ii) in Remark 2.2). On the other hand, by property (iii) in Remark 2.2, there is some $\mathrm{ED}_n \in \mathcal{A}$ such that:

$$\mathrm{D} = \left\{ w \in \underline{c_{m+1}} : \ulcorner \exists z \ \delta(z, w, \bar{u}, a_k) \urcorner \in \mathrm{ED}_n \right\}.$$

Since $(\mathcal{M}, \mathcal{A})$ satisfies $\mathrm{I}\Sigma^0_1$, the above characterization of D shows that D is coded in $\mathcal{M}$ (and therefore in $\mathcal{N}$) by some $d < 2^{c_{m+1}}$ (recall that the code of each subset of $\underline{m}$ is below 2^m). Therefore we have:

(1) $\mathcal{N} \models \forall w < c_{m+1} \, (wEd \to \exists z \ \delta(z, w, \bar{u}, a_k))$;

By putting (1) together with $\mathrm{B}\Sigma_1$ in $\mathcal{N}$, and existentially quantifying a_k we obtain:

(2) $\mathcal{N} \models \exists t, x \ \forall w < c_{m+1} \, (wEd \to \exists z < t \ \delta(z, w, \bar{u}, x))$.

On the other hand, coupling (2) with $(*_m)$ yields:

(3) $\mathcal{N} \models \exists t, x < b \ \forall w < c_{m+1}((wEd \to \exists z < t \ \delta(z, w, \bar{v}, x)))$,

which makes it clear that each formula in $p(y)$ is satisfiable in $\mathcal{N}$.

CASE 1 ($\mathbf{m+1 = 3k+1}$, for some $\mathbf{k} \in \omega$): In this case if $b_k \geq \mathrm{Max}\{\bar{v}\}$ or if it is one of the elements of $\bar{v}$, put $u_{m+1} = u_m$ and $v_{m+1} = v_m$. Otherwise, put $v_{m+1} = b_k$ and define:

$$q(x) := \{\forall w < c_{m+1}(\forall z < b \ \neg\delta(z, w, \bar{v}, b_k) \to \forall z \ \neg\delta(z, w, \bar{u}, x)) : \delta \text{ is a } \Delta_0\text{−formula}\}.$$

$q(x)$ is clearly a recursive type and closed under conjunctions, so by recursive saturation of $\mathcal{N}$ it suffices to verify that each formula in $q(x)$ is satisfiable in $\mathcal{N}$, and let u_{m+1} be one of the realizations of $q(x)$ in $\mathcal{N}$. Suppose some formula in q is not realizable in $\mathcal{N}$, then for some Δ_0-formula δ we have:

$$\mathcal{N} \models \forall x \, (\exists w < c_{m+1} \, (\forall z < b \ \neg\delta(z, w, \bar{v}, b_k) \wedge \exists z \ \delta(z, w, \bar{u}, x))).$$

Let:

$$\mathrm{R} := \left\{ w \in \underline{c_{m+1}} : \mathcal{N} \models \forall z < b \ \neg\delta(z, w, \bar{v}, b_k) \right\}.$$

Since R is Δ_0-definable in $\mathcal{N}$ there exists some $r < 2^{c_{m+1}}$ which codes R in $\mathcal{N}$. Therefore,

(4) $\mathcal{N} \models \forall x < \mathrm{Max}\{\bar{u}\}\,(\exists w < c_{m+1}\,(wEr \wedge \exists z\ \delta(z,w,\bar{u},x)))\,,$

which by Σ_1-Collection in $\mathcal{N}$ implies:

(5) $\mathcal{N} \models \exists t\ \forall x < \mathrm{Max}\{\bar{u}\}\,(\exists w < c_{m+1}\,(wEr \wedge \exists z < t\ \delta(z,w,\bar{u},x)))\,.$

Putting (5) together with $(*_m)$ yields:

(6) $\mathcal{N} \models \exists t < b\ \forall x < \mathrm{Max}\{\bar{v}\}\,(\exists w < c_{m+1}\,(wEr \wedge \exists z < t\ \delta(z,w,\bar{v},x)))\,.$

By substituting b_k for x in (6) we obtain:

(7) $\mathcal{N} \models \exists t < b\ \forall x < \mathrm{Max}\{\bar{v}\}\,(\exists w < c_{m+1}\,(wEr \wedge \exists z < t\ \delta(z,w,\bar{v},b_k)))\,.$

But (7) contradicts the assumption that r codes R. So $q(x)$ is finitely satisfiable.

CASE 2 ($\mathbf{m+1=3k+2}$, for some $\mathbf{k}\in\omega$): Consider the type $l(x,y) := \{x \neq y, x \leq c_k\} \cup l_0(x,y)$, where:

$$l_0(x,y) :=$$
$$\{\forall w < c_{m+1}\,(\exists z\ \delta(z,w,\bar{u},x) \rightarrow \exists z < b\ \delta(z,w,\bar{v},y)) : \delta \text{ is a } \Delta_0\text{-formula}\}\,.$$

Once we demonstrate that $l(x,y)$ is realized in $\mathcal{N}$ we can define (u_{m+1}, v_{m+1}) as any realization in $\mathcal{N}$ of $l(x,y)$. Since $l_0(x,y)$ is closed under conjunctions and $\mathcal{N}$ is recursively saturated, to show that $l(x,y)$ is realized in $\mathcal{N}$ it suffices to demonstrate that the conjunction of $x \neq y$ and $x \leq c_k$, and each formula in $l_0(x,y)$ is satisfiable in $\mathcal{N}$. So suppose δ is a Δ_0-formula and for each $s < c_k$ consider the map F from $\underline{c_k}$ to the power set of $\underline{c_{m+1}}$ by:

$$F(s) := \{w \in \underline{c_{m+1}} :\ \mathcal{N} \models \exists z\ \delta(z,w,\bar{u},s)\}.$$

Thanks to properties (i) through (iii) of $\mathcal{N}$ listed in Remark 2.2, there is some $\mathrm{ED}_n \in \mathcal{A}$ such that:

$$F(s) = \{w \in \underline{c_{m+1}} : \ulcorner \exists z\ \delta(z,w,\bar{u},s) \urcorner \in \mathrm{ED}_n\}.$$

Since $(\mathcal{M},\mathcal{A})$ satisfies $\mathrm{I}\Sigma^0_1$, the above characterization of $F(s)$ together with the veracity of $\mathrm{I}\Sigma^0_1$ in $(\mathcal{M},\mathcal{A})$ makes it clear that F is coded in $\mathcal{M}$ by some f (and therefore in $\mathcal{N}$) that codes a function from $\underline{c_k}$ to $\underline{2^{c_{m+1}}}$ with $f(s) := \sum_{l \in F(s)} 2^l$. On the other hand the definition of $f(s)$ and the assumption that $2^{c_{n+1}} < c_n$ for all $n \in \omega$ makes it clear that:

$$f(s) \leq \sum_{l<c_{m+1}} 2^l = 2^{c_{m+1}} - 1 < 2^{c_{m+1}} < c_m < c_k.$$

So by the coded pigeonhole principle there are **distinct** $s, s' < c_k$ such that $f(s) = f(s')$, in other words:

$$\mathcal{N} \models \forall w < c_{m+1}(\exists z\ \delta(z, w, \bar{u}, s) \leftrightarrow \exists z\ \delta(z, w, \bar{u}, s')).$$

Now by repeating the argument used in Case 0 for $(\bar{u}, s, s')$ we can find some $t, t' < b$ such that:

$$\mathcal{N} \models \forall w < c_{m+1}(\exists z\ \delta(z, w, \bar{u}, s, s') \rightarrow \exists z < b\ \delta(z, w, \bar{v}, t, t')).$$

Since $s \neq s'$, either $s \neq t$ or $s \neq t'$. So the conjunction of $x \neq y$, $x \leq c_k$, and each formula in $l_0(x, y)$ is satisfiable in $\mathcal{N}$, and the proof is complete. $\square$

4 Cuts which are fixed-point sets of self-embeddings

In this section we present the proof of Theorem B. But before going through the proof, let us point out that a model of WKL_0 does not necessarily carry a cut satisfying statement (2) of Theorem B (see [1, Remark 5.1.1] for an explanation). However, if $\mathcal{M} \models \mathrm{PA}$, there are arbitrarily high strong cuts I in $\mathcal{M}$ such that $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$. To see this when $\mathcal{M}$ is a countable model of PA, let $\mathcal{A}$ be the family of definable subsets of $\mathcal{M}$. Since $(\mathcal{M}, \mathcal{A}) \models \mathrm{ACA}_0$ and WKL_0 is a subsystem of ACA_0, by Theorem 1.1 (Tanaka's theorem), for every $a \in \mathrm{M}$ there is a cut I containing a such that $(\mathcal{M}, \mathcal{A}) \cong (\mathrm{I}, \mathcal{A}_{\mathrm{I}})$ and $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$. Furthermore, I is strong cut of $\mathcal{M}$ by Theorem 2.6 since $\mathcal{A}_{\mathrm{I}} = \mathrm{SSy}_{\mathrm{I}}(\mathcal{M})$.

Proof of Theorem B. (1) $\Rightarrow$ (2) is an immediate consequence of Theorem 2.4.(b), and (3) $\Rightarrow$ (1) is trivial; so we concentrate on the proof of (2) $\Rightarrow$ (3).

Suppose I is a strong cut in $\mathcal{M}$ and $\mathrm{I} \prec_{\mathrm{end},\ \Sigma_1} \mathcal{M}$. The proof of (3) is inspired by the proof of [1, Theorem 5.1] and consists of the following four stages:

Stage 1: Fix some $b_0 \in \mathrm{M} \setminus \mathrm{I}$. Using Theorem 2.1, let $\mathcal{N}$ be a model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathrm{Exp}$ such that $\mathrm{SSy}_{\mathrm{M}}(\mathcal{N}) = \mathcal{A}$, $\mathcal{M} \subseteq_{\mathrm{end},\Pi_1,\leq \mathrm{b}_0} \mathcal{N}$, and the three conditions specified in Remark 2.2 hold for $\mathcal{N}$.

Stage 2: Let $\mathbb{Q}$ be the set of rational numbers with its natural ordering. Since I is a strong cut in $\mathcal{M}$, by Theorem 2.6, and the case $\mathbb{L} = \mathbb{Q}$ of Theorem 2.7, we can find an elementary end extension $\mathrm{I}_{\mathbb{Q}}$ of I such that

$\mathrm{SSy}_{\mathrm{I}}(\mathrm{I}_{\mathbb{Q}}) = \mathcal{A}$ and $\mathrm{I}_{\mathbb{Q}} \backslash \mathrm{I}$ contains a copy of $\mathbb{Q}' := \{c_q : q \in \mathbb{Q}\}$ of $\mathbb{Q}$, and there is a composition preserving embedding $j \mapsto \widehat{j}$ from the semi-group of initial self-embeddings of $\mathbb{Q}$ into the semi-group of initial self-embeddings of $\mathrm{I}_{\mathbb{Q}}$ that satisfies conditions (a) through (d) of Theorem 2.7. In particular $\mathbb{Q}'$ is downward cofinal in $\mathrm{I}_{\mathbb{Q}} \backslash \mathrm{I}$.

Stage 3: An initial embedding $k : \mathcal{N} \to \mathrm{I}_{\mathbb{Q}}$ is constructed such that k fixes each element of I. Note that Theorem 2.5 cannot be invoked for this purpose since $\mathrm{I}\Sigma_1$ need not hold in $\mathcal{N}$; instead, we will take advantage of recursive saturation of $\mathcal{N}$, and the properties of $\mathcal{N}$ listed in Remark 2.2. We will go through construction of k after describing stage 4 of the proof.

Stage 4: The desired self-embedding j satisfying (3) of Theorem B can be readily constructed as follows: Fix some $c_{q_1} < k(b_0)$ in $\mathbb{Q}'$ and let j_1 be a fixed-point free initial embedding of $\mathbb{Q}$ such that $j_1(\mathbb{Q}) < q_1$. Then define $h := k^{-1}\widehat{j_1}k$, and let j be the restriction of h to $\mathcal{M}$. First, note that by the way j_1 is chosen, h is well-defined and $h(\mathrm{N}) < b_0$. Therefore, j is an isomorphism between $\mathcal{M}$ and a proper cut J of $\mathcal{M}$. Moreover, Since $\mathrm{Fix}(\widehat{j_1}) = \mathrm{I}$ (as arranged in Stage 2) and k fixes each element of I (as arranged in Stage 3), by Remark 2.3 we may conclude that $\mathrm{Fix}(j) = \mathrm{I}$ and j is an isomorphism between $(\mathcal{M}, \mathcal{A})$ and $(\mathrm{J}, \mathcal{A}_{\mathrm{J}})$.

The above description of the four stages of the proof should make it clear that the proof of condition (3) of Theorem B will be complete once we verify that Stage 3 can be carried out, so we focus on the construction of an initial embedding k of $\mathcal{N}$ into $\mathrm{I}_{\mathbb{Q}}$ that fixes each element of I. To do so, we first note that since (i) $\mathrm{I}\Sigma_1$ holds in both $\mathcal{M}$ and $\mathrm{I}_{\mathbb{Q}}$, (ii) $\mathrm{SSy}_{\mathrm{I}}(\mathcal{M}) = \mathrm{SSy}_{\mathrm{I}}(\mathrm{I}_{\mathbb{Q}})$, and (iii) $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, i)_{i\in\mathrm{I}} = \mathrm{Th}_{\Sigma_1}(\mathcal{N}_{\mathbb{Q}}, i)_{i\in\mathrm{I}}$ (because $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$, and $\mathrm{I} \prec \mathrm{I}_{\mathbb{Q}}$) by Theorem 2.5 there is a proper initial embedding $f : \mathcal{M} \to \mathrm{I}_{\mathbb{Q}}$ such that $f(i) = i$ for each $i \in I$. In particular, $f(\mathrm{M}) < e$ for some $e \in \mathrm{I}_{\mathbb{Q}}$. Moreover, since $\mathrm{Th}_{\Pi_1}(\mathcal{M}, x)_{x\leq b_0} = \mathrm{Th}_{\Pi_1}(\mathcal{N}, x)_{x\leq b_0}$ we have:

$$(*_0): \quad \mathcal{N} \models \exists z\ \delta(z, w) \Rightarrow \mathrm{I}_{\mathbb{Q}} \models \exists z < e\ \delta(z, f(w)), \text{ for all } \Delta_0\text{-formulas } \delta \text{ and all } w < b_0.$$

Now choose $\{b_n : n \in \omega\}$ to be a decreasing sequence in $\mathrm{M} \setminus \mathrm{I}$ such that b_0 is the element chosen in Stage 1, and $2^{b_{n+1}} < b_n$ for all $n \in \omega$. In order to construct k, we recursively build finite sequences $\bar{u} := (u_0, ..., u_m)$ of elements of N and $\bar{v} := (v_0, ..., v_m) < e$ for each $m \in \omega$ such that:

(i) $u_0 = 0 = v_0$.

(ii) For every c in N there is some $n \in \omega$ such that $c = u_n$.

(*iii*) For every $n \in \omega$, $v_n < b$, and if for some c in $\mathrm{I}_{\mathbb{Q}}$ it holds that $c < v_n$, then there is some $m \in \omega$ such that $c = v_m$.

(*iv*) For every $m \in \omega$ the following condition holds:

$(*_m)$: $\mathcal{N} \models \exists z\ \delta(z, w, \bar{u}) \Rightarrow \mathrm{I}_{\mathbb{Q}} \models \exists z < e\ \delta(z, f(w), \bar{v})$ for all Δ_0-formulas δ and all $w < b_m$

Let $\{a_n : n \in \omega\}$ and $\{d_n : n \in \omega\}$ respectively be enumerations of element of N and $\underline{e} \subset \mathrm{I}_{\mathbb{Q}}$, and $\langle \delta_r : r \in \mathrm{M}\rangle$ be a canonical enumeration of Δ_0-formulas in $\mathcal{M}$. The first step of induction holds thanks to $(*_0)$ and the choice of u_0 and v_0 in statement (*i*). Next, suppose $\bar{u} := (u_0, ..., u_m) \in \mathrm{N}$ and $\bar{v} := (v_0, ..., v_m) < e$ are constructed, for given $m \in \omega$. In order to build u_{m+1} and v_{m+1} we distinguish two cases, one handling the 'forth' step and the other handling the 'back' step of the back-and-forth construction:

CASE 0 ($\mathbf{m+1 = 2k}$, for some $\mathbf{k} \in \omega$): In this case if a_k is one of elements of $\bar{u}$, put $u_{m+1} = u_m$ and $v_{m+1} = v_m$. Otherwise, put $u_{m+1} = a_k$ and define:

$$\mathrm{A} := \{\langle r, w\rangle < b_{m+1} : \mathcal{N} \models \exists z\ \mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}, a_k))\}.$$

Note that in the above definition, $\mathcal{N}$ can be safely replaced by some $\mathcal{N}_n$, where n is large enough to contain the parameters $\bar{u}$ and a_k (thanks to properties (*i*) and (*ii*) in Remark 2.2). On the other hand, by property (*iii*) in Remark 2.2, there is some $\mathrm{ED}_n \in \mathcal{A}$ such that:

$$\mathrm{A} = \{\langle r, w\rangle < b_{m+1} : \ulcorner \exists z\ \mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}, a_k)\urcorner \in \mathrm{ED}_n\}\,.$$

Since $(\mathcal{M}, \mathcal{A})$ satisfies $\mathrm{I}\Sigma^0_1$, the above characterization of A shows that A is coded in $\mathcal{N}$ by some $a < 2^{b_{m+1}}$. Therefore we have:

(1) $\mathcal{N} \models \forall \langle r, w\rangle < b_{m+1}(\langle r, w\rangle E a \to \exists z\ \mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}, a_k)))$.

Recall that $\mathrm{B}\Sigma_1$ holds in $\mathcal{N}$, and Sat_{Δ_0} has a Σ_1-description in $\mathcal{N}$, so (1) allows us to conclude:

(2) $\mathcal{N} \models \exists t\ \forall \langle r, w\rangle < b_{m+1}(\langle r, w\rangle E a \to \exists z < t\ \mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}, a_k)))$.

By quantifying out a_k in (2) and coupling it with $(*_m)$, we obtain:

(3) $\mathrm{I}_{\mathbb{Q}} \models \exists x, t < e\, \forall \langle r, w\rangle < f(b_{m+1})\, (\langle r, w\rangle E f(a) \to \exists z < t\, \mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{v}, x)))$.

Clearly any element of $\mathrm{I}_{\mathbb{Q}}$ that witnesses x in (3) can serve as a suitable candidate for v_{m+1}.

CASE 1 ($\mathbf{m+1 = 2k+1}$, for some $\mathbf{k} \in \omega$): In this case if $d_k \geq \mathrm{Max}\{\bar{v}\}$ or if it is one of the elements of $\bar{v}$, put $u_{m+1} = u_m$ and $v_{m+1} = v_m$. Otherwise, put $v_{m+1} = b_k$ and define:

$$\mathrm{B} := \{\langle r, w\rangle < f(b_{m+1}) : \mathrm{I}_{\mathbb{Q}} \models \forall z(\mathrm{Sat}_{\Delta_0}(\delta_r(z, w, \bar{v}, d_k)) \rightarrow b < z)\}.$$

Note that B is Σ_1-definable in $\mathrm{I}_{\mathbb{Q}}$, so there is some $b < 2^{f(b_{m+1})} = f(2^{b_{m+1}})$ which codes B in $\mathrm{I}_{\mathbb{Q}}$. Therefore $b = f(c)$ for some $c < 2^{b_{m+1}}$. Define:

$$p(x) := \{\forall w < b_{m+1}(\langle \ulcorner \delta \urcorner, w\rangle Ec \rightarrow \forall z\ \neg\delta(z, w, \bar{u}, x)) : \delta \text{ is a } \Delta_0\text{−formula}\}.$$

Since $\mathcal{N}$ is recursively saturated and $p(x)$ is recursive, in order to find a suitable element in N which serves as u_{m+1} , it suffices to prove that $p(x)$ is finitely satisfiable. So suppose $p(x)$ is not finitely satisfiable. It can be readily checked that $p(x)$ is closed under conjunction, so we can safely assume there is a Δ_0-formula δ such that:

(4) $\mathcal{N} \models \forall x(\exists w < b_{m+1}(\langle \ulcorner \delta \urcorner, w\rangle Ec \wedge \exists z\ \delta(z, w, \bar{u}, x)))$.

Clearly (4) implies:

(5) $\mathcal{N} \models \forall x < \mathrm{Max}\{\bar{u}\}\ (\exists w < b_{m+1}(\langle \ulcorner \delta \urcorner, w\rangle Ec \wedge \exists z\ \delta(z, w, \bar{u}, x)))$.

We can bound variable z in (5) by using $\mathrm{B}\Sigma_1$ in $\mathcal{N}$, and next employ $(*_m)$ to deduce:

(6) $\mathrm{I}_{\mathbb{Q}} \models \exists t < e \forall x < \mathrm{Max}\{\bar{v}\}(\exists w < f(b_{m+1})(\langle \ulcorner \delta \urcorner, w\rangle Ef(c) \wedge \exists z < t\ \delta(z, w, \bar{v}, x)))$.

By replacing x in (6) with d_k, we obtain:

(7) $\mathrm{I}_{\mathbb{Q}} \models \exists t < e\ (\exists w < f(b_{m+1})(\langle \ulcorner \delta \urcorner, w\rangle Eb \wedge \exists z < t\ \delta(z, w, \bar{v}, d_k)))$.

But (7) contradicts the assumption that b codes B in $\mathrm{I}_{\mathbb{Q}}$. So $p(x)$ is finitely satisfiable. $\square$

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