

Classifying Material Implications over Minimal Logic

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Abstract

The so-called paradoxes of material implication have motivated the development of many non-classical logics over the years [2–5, 11]. In this note, we investigate some of these paradoxes and classify them, over minimal logic. We provide proofs of equivalence and semantic models separating the paradoxes where appropriate. A number of equivalent groups arise, all of which collapse with unrestricted use of double negation elimination. Interestingly, the principle *ex falso quodlibet*, and several weaker principles, turn out to be distinguishable, giving perhaps supporting motivation for adopting minimal logic as the ambient logic for reasoning in the possible presence of inconsistency.

Keywords: reverse mathematics; minimal logic; ex falso quodlibet; implication; paraconsistent logic; Peirce’s principle.

1 Introduction

The project of constructive reverse mathematics [6] has given rise to a wide literature where various theorems of mathematics and principles of logic have been classified over intuitionistic logic. What is less well-known is that the subtle difference that arises when the principle of explosion, *ex falso quodlibet*, is dropped from intuitionistic logic (thus giving (Johansson’s) *minimal logic*) enables the distinction of many more principles. The focus of the present paper are a range of principles known collectively (but not exhaustively) as the *paradoxes of material implication*; paradoxes because they illustrate that the usual interpretation of formal statements of the form “ $\dots \rightarrow \dots$ ” as informal statements of the form “if... then...” produces counter-intuitive results.

Some of these principles were hinted at in [9]. Here we present a carefully worked-out chart, classifying a number of such principles over minimal logic. These principles hold classically, and intuitionistically either hold or are equivalent to one of three well-known principles (see Section 6). As it turns out, over minimal logic these principles divide cleanly into a small number of distinct categories. We hasten to add that the principles we classify here are considered as *formula schemas*, and not individual instances. For example, when we write the formula $\neg\neg\varphi \rightarrow \varphi$ for double negation elimination (DNE), we mean that this should apply to all well-formed formulae φ . The work presented here is thus not a narrowly-focused investigation of what-instance-implies-what-instance, but rather a broad-stroke painting that classifies formula schemas as a whole.

This paper may be received in two ways: straightforwardly, as a contribution to reverse mathematics over non-classical logics; or more subtly as providing some insight into the kinds of distinctions that a good paraconsistent logic might contribute. Highlights of the paper include many refinements over [14, chapter 6].

In what follows, we take: \rightarrow to be minimal implication; \perp a logical constant not further defined (with the usual identification of $\neg\alpha$ with $\alpha \rightarrow \perp$ for any well-formed formula α); \top a logical constant interchangeable with $\alpha \rightarrow \alpha$ for some (arbitrary) well-formed formula α (that is, \top is always satisfied).

We will be interested in propositional axiom schemas, and instances of such schemas. A (*propositional*) *axiom schema* is any well-formed formula, where the propositional variables are interpreted as

ranging over well-formed formulas. An *instance* of a formula schema is the schema with well-formed formulae consistently substituting for propositional variables in the schema in the intuitively obvious way. For example, the well-formed formula

$$\neg(\alpha \wedge \beta) \rightarrow (\alpha \wedge \beta \rightarrow \neg\gamma)$$

is an instance of the formula schema

$$\neg\varphi \rightarrow (\varphi \rightarrow \psi)$$

(5 in what follows), where φ is replaced with $\alpha \wedge \beta$, and ψ is replaced with $\neg\gamma$. We also say that formula schema Φ *implies* formula schema Ψ if, given any instance ψ of Ψ , there are finitely many instances $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of Φ such that

$$\varphi_1, \varphi_2, \dots, \varphi_n \implies \psi,$$

where \implies is understood as the (meta-theoretic) minimal logic consequence relation. We also say that Φ and Ψ are equivalent if both Φ implies Ψ and vice versa. Often it is theoretically useful to distinguish instance-wise implication from schematic implication [7], which is why we restrict ourselves to instance-wise proofs.

2 Paradoxes of material implication

The paradoxes we classify are the following schemas. Where $\varphi, \psi, \beta, \vartheta$ are any well-formed formulas,

1. $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \vartheta)$ (linearity, strong form)
2. $(\varphi \wedge \neg\varphi) \rightarrow \psi$ (*ex contradictione quodlibet*; the paradox of entailment)
3. $\varphi \rightarrow (\psi \vee \neg\psi)$
4. $(\varphi \rightarrow \psi) \vee \neg\psi$
5. $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$
6. $(\neg\varphi \rightarrow \varphi) \rightarrow \psi$ (*consequentia mirabilis*; Clavius's law)¹
7. $((\varphi \wedge \psi) \rightarrow \vartheta) \rightarrow ((\varphi \rightarrow \vartheta) \vee (\psi \rightarrow \vartheta))$
8. $((\varphi \rightarrow \vartheta) \wedge (\psi \rightarrow \beta)) \rightarrow ((\varphi \rightarrow \beta) \vee (\psi \rightarrow \vartheta))$
9. $(\neg(\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \neg\psi)$ (the counterexample principle)
10. $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$ (Peirce's Principle (PP))
11. $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ (Dirk Gently's Principle² (DGP))
12. $(\neg\neg\varphi \vee \varphi) \rightarrow \varphi$ ³
13. $\psi \vee (\psi \rightarrow \vartheta)$ (Tarski's formula)
14. $(\neg\varphi \rightarrow \neg\psi) \vee (\neg\psi \rightarrow \neg\varphi)$ (weak Dirk Gently's Principle)
15. $(\varphi \rightarrow \psi \vee \vartheta) \rightarrow ((\varphi \rightarrow \psi) \vee (\varphi \rightarrow \vartheta))$
16. $\neg(\varphi \rightarrow \neg\varphi) \rightarrow \varphi$ (a form of Aristotle's law⁴)

¹The version $(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$ is used in [12]. However, since *ex falso quodlibet falsum* ($\perp \rightarrow \neg\phi$) holds in minimal logic, that version is provable over minimal logic. This suggests that while the present work illustrates many distinctions, there are still more distinctions that are not apparent here.

²Our name for this is based on the guiding principle of the protagonist of Douglas Adam's novel *Dirk Gently's Holistic Detective Agency* [1] who believes in the "fundamental interconnectedness of all things." It also appears as an axiom in Gödel–Dummett logic, and is more commonly known as (weak) linearity.

³The Wikipedia page at https://en.wikipedia.org/wiki/Consequentia_mirabilis actually lists this as equivalent to 6, however that is not quite correct as we can see below. If we add the assumption that $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$, then they do turn out to be equivalent. However *that* statement—interpreting \rightarrow as material implication—is minimally at least as strong as LEM.

⁴Connexive logics are closely related. There, instead, the antecedent is taken as axiom schema; thus connexive logics are entirely non-classical, since $\neg(\varphi \rightarrow \neg\varphi)$ is not classical valid.

Strictly speaking, these are of course axiom schemes rather than single axioms. These sentences are all classical theorems, where the arrow \rightarrow is interpreted as material implication.⁵ Initially, we are interested how these principles relate to the following basic logical principles, the universal applicability of which is well-known to be rejected by intuitionistic (or, in the third case, minimal) logic:

- DNE** $\neg\neg\varphi \rightarrow \varphi$ (double negation elimination)
LEM $\varphi \vee \neg\varphi$ (law of excluded middle)
EFQ $\perp \rightarrow \varphi$ (ex falso quodlibet)⁶
WLEM $\neg\varphi \vee \neg\neg\varphi$ (weak law of excluded middle)⁷

It will turn out that there are two further important distinguishable classes; those related to Peirce's Principle, and those related to Dirk Gently's Principle.

We will also consider the following versions of De Morgan's laws

- DM1** $\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$
DM2 $\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$
DM1' $\neg(\neg\varphi \wedge \neg\psi) \leftrightarrow \varphi \vee \psi$
DM2' $\neg(\neg\varphi \vee \neg\psi) \leftrightarrow \varphi \wedge \psi$

Notice that DM2 can be proven in minimal logic.

3 Equivalents

Proposition 1. *The following are equivalent:*

- DNE, 9, 12, 16, DM1', and DM2';*
- LEM, 3, 4, and 6;*
- EFQ, 2, and 5;*
- WLEM, 14, and DM1*

Proof. a) Clearly $\text{DNE} \Leftrightarrow 12$ and $\text{DNE} \Leftrightarrow 16$. To see that $\text{DNE} \Rightarrow 9$, assume DNE and suppose that

$$\neg(\varphi \rightarrow \psi). \quad (1)$$

Suppose also, for contradiction, that

$$\neg(\varphi \wedge \neg\psi). \quad (2)$$

Suppose further that φ and $\neg\psi$. Adjunction then gives $\varphi \wedge \neg\psi$, contradicting (2); and so $\neg\neg\psi$ (discharging the assumption $\neg\psi$). DNE yields ψ , whence $\varphi \rightarrow \psi$ (discharging φ). This contradicts (1), and so $\neg\neg(\varphi \wedge \neg\psi)$. Another application of DNE yields 9.

For the converse, suppose $\neg\neg\varphi$; that is, $\neg(\varphi \rightarrow \perp)$. Then by 9, $\varphi \wedge \neg\perp$. Hence, φ .

Clearly DNE implies both versions of De Morgan's laws. Conversely we see that for $\psi = \varphi$ they both reduce to DNE.

- $\text{LEM} \Leftrightarrow 3$, $\text{LEM} \Leftrightarrow 4$ and $\text{LEM} \Rightarrow 6$ are straightforward.⁸ To see that $6 \Rightarrow \text{LEM}$, it is enough to show that $\neg(\varphi \vee \neg\varphi) \rightarrow \varphi \vee \neg\varphi$. So assume $\neg(\varphi \vee \neg\varphi)$. Then φ leads to a contradiction, whence $\neg\varphi$ holds. This can be weakened to $\varphi \vee \neg\varphi$ and we are done.
- Clearly, $\text{EFQ} \Rightarrow 2 \Rightarrow 5$. To see that $5 \Rightarrow \text{EFQ}$, note that from \perp we may deduce $\top \rightarrow \perp$ by weakening; that is, $\neg\top$. Applying 5, $\top \rightarrow \varphi$, whence φ . The deduction theorem then yields EFQ.

⁵Since all of these axioms are not provable in minimal logic, deducing them in classical logic is not mechanic and they are therefore good exercises for students.

⁶Of course this is intuitionistically provable/definitional.

⁷An axiom in Jankov's logic, and De Morgan logic.

⁸With judicious substitutions—in particular, using $\varphi \equiv \top$ in the right-to-left implications.

d) First we show $\text{WLEM} \Leftrightarrow 14$. We have that either $\neg\psi$ or $\neg\neg\psi$ holds. In the first case we also have $\neg\varphi \rightarrow \neg\psi$. In the second case assume that $\neg\psi$ holds. So we have \perp , which gives $\neg\varphi$ by weakening.

Conversely we either have $\neg\varphi \rightarrow \neg\neg\varphi$ or $\neg\neg\varphi \rightarrow \neg\varphi$. In the first case the assumption that $\neg\varphi$ holds leads to \perp , and hence $\neg\neg\varphi$. In the second case, assuming φ , we get $\neg\neg\varphi$ and thus $\neg\varphi$, which gives \perp by detachment; therefore, $\neg\varphi$.

Next we show $\text{WLEM} \Leftrightarrow \text{DM1}$. Assume DM1 and let φ be arbitrary. Since $\neg(\neg\varphi \wedge \varphi)$ is provable in intuitionistic logic we have $\neg\neg\varphi \vee \neg\varphi$; that is WLEM holds.

Conversely assume that $\neg(\varphi \wedge \psi)$. By WLEM either $\neg\varphi$ or $\neg\neg\varphi$. It is easy to see that in the second case the assumption that ψ holds leads to a contradiction. Hence $\neg\psi$ and we are done. \square

Next, we single out Peirce's Principle (10, PP), and Dirk Gently's Principle (11, DGP). As will be shown in the next section, these principles are strictly distinguishable from the others.

Proposition 2. *The following are equivalent:*

(a) PP, 1, and 13;

(b) DGP, 7, 8, and 15.

Proof. (a) We show $1 \Rightarrow \text{PP} \Rightarrow 13 \Rightarrow 1$.

First, consider φ , and ψ such that $(\varphi \rightarrow \psi) \rightarrow \varphi$. By 1 either $\top \rightarrow \varphi$ or $\varphi \rightarrow \psi$. In the first case φ holds. In the second case we can use the assumption to show that, also, φ holds. Together

$$((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$$

so that PP holds.

Next, assume PP, and let ψ and ϑ be arbitrary well-formed formulas. By PP,

$$(((\psi \vee (\psi \rightarrow \vartheta)) \rightarrow \vartheta) \rightarrow \psi \vee (\psi \rightarrow \vartheta)) \rightarrow \psi \vee (\psi \rightarrow \vartheta).$$

We show that the antecedent (and hence, by modus ponens, the consequent) of this holds. So assume $(\psi \vee (\psi \rightarrow \vartheta)) \rightarrow \vartheta$. Furthermore assume ψ . Then ϑ , so (discharging the assumption ψ) we have $\psi \rightarrow \vartheta$. This weakens to $\psi \vee (\psi \rightarrow \vartheta)$, so 13 follows.

Last, since \rightarrow weakens, clearly 13 implies 1.

(b) First, we show $\text{DGP} \Leftrightarrow 7$. Assume that $(\varphi \wedge \psi) \rightarrow \vartheta$. Now if DGP holds then either $\varphi \rightarrow \psi$ or $\psi \rightarrow \varphi$. In the first case, if φ holds, then also $\varphi \wedge \psi$, and hence ϑ holds. Together that means that in the first case we have $\varphi \rightarrow \vartheta$. Similarly, in the second case $\psi \rightarrow \vartheta$. Conversely, apply 7 to $\vartheta \equiv \varphi \wedge \psi$. Then the antecedent is \top , and so $(\varphi \rightarrow (\varphi \wedge \psi)) \vee (\psi \rightarrow (\varphi \wedge \psi))$. Hence the desired $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ holds.

Next, we show $\text{DGP} \Leftrightarrow 8$. Assume that $(\varphi \rightarrow \vartheta) \wedge (\psi \rightarrow \beta)$. By DGP either $\varphi \rightarrow \beta$ and we are done, or $\beta \rightarrow \varphi$. But in that second case if we assume ψ also β holds, which in turn implies φ , which in turn implies ϑ . Together, in the second case, $\psi \rightarrow \vartheta$. Conversely, apply 8 to $\vartheta \equiv \varphi$ and $\beta \equiv \psi$, which yields

$$((\varphi \rightarrow \varphi) \wedge (\psi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)) .$$

Since the antecedent is $(\top \wedge \top) \equiv \top$, we get the desired $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$.

Last, $\text{DGP} \Leftrightarrow 15$. For the forward direction, suppose that $\varphi \rightarrow \psi \vee \vartheta$. DGP gives $(\psi \rightarrow \vartheta) \vee (\vartheta \rightarrow \psi)$. In the first case, assuming φ , we get $\psi \vee \vartheta$, which (by modus ponens on $\psi \rightarrow \vartheta$) is ϑ in this case. So $\varphi \rightarrow \vartheta$, which weakens to $(\varphi \rightarrow \psi) \vee (\varphi \rightarrow \vartheta)$. In the second case, a similar argument also shows $(\varphi \rightarrow \psi) \vee (\varphi \rightarrow \vartheta)$. Either way the consequent of 15 holds; whence, by the deduction theorem, 15. Conversely, apply 15 to $\varphi \vee \psi$, φ , and ψ to get:

$$(\varphi \vee \psi \rightarrow \varphi \vee \psi) \rightarrow ((\varphi \vee \psi \rightarrow \varphi) \vee (\varphi \vee \psi \rightarrow \psi)) .$$

Now clearly the antecedent is always satisfied. So we have that

$$(\varphi \vee \psi \rightarrow \varphi) \vee (\varphi \vee \psi \rightarrow \psi),$$

which is equivalent to the desired $(\psi \rightarrow \varphi) \vee (\varphi \rightarrow \psi)$. □

We are now in a position to lay out how these principles fit together.

Proposition 3. *The implications in Figure 1 hold:*

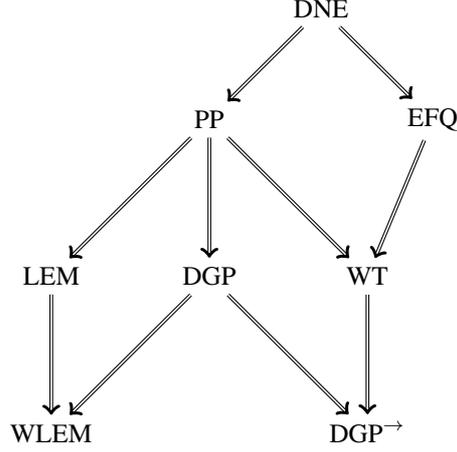


Figure 1: Some principles distinguishable over minimal logic. As is shown in Section 5, none of the arrows can be reversed and no arrow can be added.

That $DNE \Rightarrow EFQ$ and $LEM \Rightarrow WLEM$ is clear; we prove the remaining implications.

Proof. • $DNE \Rightarrow PP$: Assume $(\varphi \rightarrow \psi) \rightarrow \varphi$, and $\neg\varphi$. Modus tollens gives $\neg(\varphi \rightarrow \psi)$ whence, by counterexample (9) (see Proposition 1), $\varphi \wedge \neg\psi$; so φ . With $\neg\varphi$, this gives \perp ; and hence (discharging the assumption $\neg\varphi$) $\neg\neg\varphi$. Applying DNE gives φ .

• $PP \Rightarrow LEM$: Assume $\neg(\varphi \vee \neg\varphi)$; that is, $(\varphi \vee \neg\varphi) \rightarrow \perp$. Then φ leads to a contradiction; so $\neg\varphi$. But then $\varphi \vee \neg\varphi$. So we have $((\varphi \vee \neg\varphi) \rightarrow \perp) \rightarrow (\varphi \vee \neg\varphi)$. Applying PP, $\varphi \vee \neg\varphi$.

• $PP \Rightarrow DGP$: By PP,

$$(((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)) \rightarrow \varphi) \rightarrow ((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)).$$

We show that the antecedent holds. Assume

$$((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)) \rightarrow \varphi, \tag{3}$$

and suppose ψ . Then $\varphi \rightarrow \psi$, so by (3), φ . Thus (discharging the assumption of ψ), $\psi \rightarrow \varphi$. But then again by (3), φ , which weakens to $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ and we are done.

• $DGP \Rightarrow WLEM$: By DGP, we have $(\varphi \rightarrow \neg\varphi) \vee (\neg\varphi \rightarrow \varphi)$. In the former case, assuming φ gives \perp , whence $\neg\varphi$. In the latter case, assuming $\neg\varphi$ gives \perp , whence $\neg\neg\varphi$. □

4 The implicational fragment

For various technical reasons it is often interesting to work only with formulas built up from propositional symbols including \perp with \rightarrow . Of course, we still use \neg as an abbreviation for $\rightarrow \perp$. Assuming classical logic (DNE) this is no restriction, since there \vee and \wedge are definable from \rightarrow and \neg . More precisely we can define $\varphi \vee \psi$ as $\neg\varphi \rightarrow \psi$, but over minimal logic this validates EFQ.⁹ A more faithful (but slightly weaker) translation is:

$$\varphi \vee \psi := (\varphi \rightarrow \perp) \rightarrow (\psi \rightarrow \perp) \rightarrow \perp \quad [\equiv \neg\varphi \rightarrow \neg\psi]. \quad (4)$$

Notice that we might also translate $\varphi \vee \psi$ as $\neg\psi \rightarrow \neg\neg\varphi$, whose equivalence to (4) is minimally provable; we will use whichever of the two translations is more expedient. Moreover, conjunction can be removed entirely also: $\varphi \wedge \psi \rightarrow \vartheta$ translates to $\varphi \rightarrow \psi \rightarrow \vartheta$,¹⁰ and $\vartheta \rightarrow \varphi \wedge \psi$ translates to the two separate cases $\vartheta \rightarrow \varphi$ and $\vartheta \rightarrow \psi$.¹¹ Translates of formulas are then defined in the obvious way. We denote the translation of a formula φ into the implication-only fragment by φ^{\rightarrow} .

The following are of special note:

- **13**, a strong form of linearity, which (by Proposition 2 is equivalent to DNE. Its translation into implicative form is:

$$\neg\psi \rightarrow \neg\neg(\psi \rightarrow \vartheta), \quad (\text{Weak Tarski's Formula, WT})$$

which is an abbreviation for $(\psi \rightarrow \perp) \rightarrow ((\psi \rightarrow \vartheta) \rightarrow \perp) \rightarrow \perp$.

- **11**, Dirk Gently's Principle, which translates to:

$$\neg(\varphi \rightarrow \psi) \rightarrow \neg\neg(\psi \rightarrow \varphi). \quad (\text{Implicative Dirk Gently's Principle, DGP}^{\rightarrow})$$

The above principles are closely related, but distinct. A separation result can be found in Section 5.

Proposition 4. *The following implications hold:*

- (a) $PP \Rightarrow WT$
- (b) $DGP \Rightarrow DGP^{\rightarrow}$
- (c) $EFQ \Rightarrow WT$
- (d) $WT \Rightarrow DGP^{\rightarrow}$

Proof. (a) We use PP in the form $((\perp \rightarrow \vartheta) \rightarrow \perp) \rightarrow \perp$. For modus ponens we need to establish that $(\perp \rightarrow \vartheta) \rightarrow \perp$. For the purpose of applying the deduction theorem to show this (and also WT), assume:

- (i) $\psi \rightarrow \perp$;
- (ii) $(\psi \rightarrow \vartheta) \rightarrow \perp$; and
- (iii) $\perp \rightarrow \vartheta$.

Then by transitivity on (i) and (iii), $\psi \rightarrow \vartheta$. Using (ii), \perp . The deduction theorem (discharging (iii)) yields $(\perp \rightarrow \vartheta) \rightarrow \perp$. Applying PP gives \perp , whence (discharging (ii) in another application of the deduction theorem) $((\psi \rightarrow \vartheta) \rightarrow \perp) \rightarrow \perp$. The conclusion follows by yet another application of the deduction theorem.

- (b) Assume that $\neg(\varphi \rightarrow \psi)$. By DGP either $\varphi \rightarrow \psi$ or $\psi \rightarrow \varphi$. In the first case we get \perp by modus ponens, and therefore also $\neg\neg(\psi \rightarrow \varphi)$. In the second case, since minimally $\alpha \rightarrow \neg\neg\alpha$, also $\neg\neg(\psi \rightarrow \varphi)$. Thus in both cases the conclusion holds.

⁹Simply use \vee -introduction and the proposed translation; EFQ follows. To translate $\varphi \vee \psi$ as $\neg\varphi \rightarrow \psi$ would therefore be disingenuous.

¹⁰Note that negated conjunction is a special case.

¹¹It is not clear, however, that this makes no difference in *proofs*; more on this later.

- (c) The proof is similar to (a), but simpler, since (iii) is now no longer an assumption but an instance of EFQ (and so we do not need to explicitly apply PP).
- (d) Assume that $\neg(\varphi \rightarrow \psi)$. Then, assuming ψ and weakening leads to a contradiction; so $\neg\psi$. By WT, $\neg\neg(\psi \rightarrow \varphi)$. The conclusion follows from the deduction theorem. \square

In Section 5 we show that these implications are strict. For the paradoxes of material implication, using the translation (4) (and the comments following it), we have:

- LEM^{\rightarrow} is $\neg\varphi \rightarrow \neg\neg\neg\varphi$, which is provable in minimal logic. Similarly, $\text{WLEM}^{\rightarrow}$ is provable.
- 1^{\rightarrow} is $\neg(\varphi \rightarrow \psi) \rightarrow \neg\neg(\psi \rightarrow \vartheta)$.
- 2^{\rightarrow} is $\varphi \rightarrow \neg\varphi \rightarrow \psi$.
- 3^{\rightarrow} is $\varphi \rightarrow \neg\psi \rightarrow \neg\neg\neg\psi$, a weakening of double negation introduction and hence provable.
- 4^{\rightarrow} is $\neg(\varphi \rightarrow \psi) \rightarrow \neg\neg\neg\psi$, and is provable.
- 7^{\rightarrow} is $(\varphi \rightarrow \psi \rightarrow \vartheta) \rightarrow \neg(\varphi \rightarrow \vartheta) \rightarrow \neg\neg(\psi \rightarrow \vartheta)$.
- 8^{\rightarrow} is $(\varphi \rightarrow \vartheta) \rightarrow (\psi \rightarrow \beta) \rightarrow \neg(\varphi \rightarrow \beta) \rightarrow \neg\neg(\psi \rightarrow \vartheta)$.
- 9^{\rightarrow} is (a) $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ and (b) $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$. The latter is provable.
- 11^{\rightarrow} is DGP^{\rightarrow} .
- 12^{\rightarrow} is $(\neg\neg\neg\varphi \rightarrow \neg\neg\varphi) \rightarrow \varphi$.
- 13^{\rightarrow} is WT.
- 14^{\rightarrow} is $\neg(\neg\varphi \rightarrow \neg\psi) \rightarrow \neg\neg(\neg\psi \rightarrow \neg\varphi)$. This is provable: assume $\neg(\neg\varphi \rightarrow \neg\psi)$ and $\neg\psi$. Weakening gives $\neg\varphi \rightarrow \neg\psi$, a contradiction; whence $\neg\varphi$. So $\neg\psi \rightarrow \neg\varphi$, which by double negation introduction gives the conclusion.
- 15^{\rightarrow} is $(\varphi \rightarrow \neg\psi \rightarrow \neg\neg\vartheta) \rightarrow \neg(\varphi \rightarrow \psi) \rightarrow \neg\neg(\varphi \rightarrow \vartheta)$.
- DM1^{\rightarrow} is $(\varphi \rightarrow \neg\psi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\neg\psi)$, which is provable.
- DM2^{\rightarrow} is $\neg(\neg\varphi \rightarrow \neg\neg\psi) \rightarrow \neg\varphi$, $\neg(\neg\varphi \rightarrow \neg\neg\psi) \rightarrow \neg\psi$, and $\neg\varphi \rightarrow \neg\psi \rightarrow \neg(\neg\varphi \rightarrow \neg\neg\psi)$, each of which is provable.
- $\text{DM1}'^{\rightarrow}$ is $(\neg\varphi \rightarrow \neg\neg\psi) \leftrightarrow (\neg\varphi \rightarrow \neg\neg\neg\psi)$, which is \top (always satisfied).
- $\text{DM2}'^{\rightarrow}$ is (a) $\neg(\neg\neg\varphi \rightarrow \neg\neg\neg\psi) \rightarrow \varphi$, (b) $\neg(\neg\neg\varphi \rightarrow \neg\neg\neg\psi) \rightarrow \psi$, and (c) $\varphi \rightarrow \psi \rightarrow \neg(\neg\neg\varphi \rightarrow \neg\neg\neg\psi)$. The latter, (c), is provable.

It is often technically useful to know when an operator may be pulled back through an implication, so two further sentences of interest are:

17. $(\varphi \rightarrow \neg\neg\psi) \rightarrow \neg\neg(\varphi \rightarrow \psi)$.
18. $\neg\neg(\varphi \rightarrow \psi) \rightarrow (\neg\neg\varphi \rightarrow \psi)$.

Note:

- The converse of 17 is provable. For, assume $\neg\neg(\varphi \rightarrow \psi)$. Further assume
 - (i) φ ;
 - (ii) $\neg\psi$; and
 - (iii) $\varphi \rightarrow \psi$.
 (i) and (iii) lead to \perp ; whence (discharging (iii)) $\neg(\varphi \rightarrow \psi)$. But then \perp again, so (discharging (ii)) $\neg\neg\psi$. Applying the deduction theorem twice gives the converse of 17.
- Likewise, the converse of 18 is provable. Assume $\neg\neg\varphi \rightarrow \psi$ and φ . Then $\neg\neg\varphi$, so ψ and hence, in fact, $\varphi \rightarrow \psi$, which is stronger than the converse of 18.

We now turn to classifying the foregoing sentences, whenever they are not provable in minimal logic alone.

Proposition 5. *The following are equivalent:*

- (a) DNE, 9^{\rightarrow} (a), 12^{\rightarrow} , $DM2'^{\rightarrow}$ (a), $DM2'^{\rightarrow}$ (b), 18.
- (b) EFQ, 2^{\rightarrow} .
- (c) WT, 1^{\rightarrow} , 15^{\rightarrow} , 17.

Proof. (a) Since full classical logic is obtained by adding DNE to minimal logic, it suffices to prove that each numbered principle implies DNE.

Observe that DNE is a special case of 9^{\rightarrow} (a)—namely, the case where $\psi = \perp$.

To see that $12^{\rightarrow} \Rightarrow$ DNE, assume $\neg\neg\varphi$. By weakening, $\neg\neg\varphi \rightarrow \neg\neg\varphi$, and so by 12^{\rightarrow} , φ .

To see that $DM2'^{\rightarrow}$ (a), $DM2'^{\rightarrow}$ (b) each imply DNE, substitute φ for ψ ; then, assuming $\neg\neg\varphi$, in each case the antecedent is satisfied, and in each case the consequent is φ .

To see that 18 implies DNE, substitute $\varphi := \psi$; then the antecedent of 18 is always satisfied, and the consequent is DNE.

- (b) It is clear that $EFQ \Rightarrow 2^{\rightarrow}$. For the converse, suppose we wish to show that $\perp \rightarrow \vartheta$. Assume \perp . Then (weakening) $\neg\top$. In particular, substituting \top for φ and ϑ for ψ in 2^{\rightarrow} gives

$$\top \rightarrow \neg\top \rightarrow \vartheta.$$

Two applications of modus ponens followed by the deduction theorem gives the desired conclusion.

- (c) Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$ is provable, transitivity with WT shows that $WT \Rightarrow 1^{\rightarrow}$. Conversely, suppose that $\neg\psi$, and $\neg(\psi \rightarrow \vartheta)$. In view of the deduction theorem, we aim to derive \perp . Consider the following form of 1^{\rightarrow} :

$$\neg(\neg\psi \rightarrow \psi) \rightarrow \neg(\psi \rightarrow \vartheta).$$

Using the assumption $\neg(\psi \rightarrow \vartheta)$, by the provable version of contraposition we conclude $\neg\neg(\neg\psi \rightarrow \psi)$. Since the converse of 17 is provable, $\neg\psi \rightarrow \neg\neg\psi$. But then, using the assumption $\neg\psi$, $\neg\neg\psi$, which gives \perp and we are done.

To see that $WT^{\rightarrow} \Rightarrow 15^{\rightarrow}$, assume:

- (i) $\varphi \rightarrow (\neg\psi \rightarrow \neg\vartheta)$;
- (ii) $\neg(\varphi \rightarrow \psi)$; and
- (iii) $\neg(\varphi \rightarrow \vartheta)$.T

In light of the deduction theorem, it is enough to prove \perp . Suppose φ . Using (i) now yields $\neg\psi \rightarrow \neg\vartheta$. From (ii) we see $\neg\psi$; whence $\neg\vartheta$. But from (iii), $\neg\vartheta$, a contradiction. Therefore, $\neg\varphi$. Applying WT, $\neg\neg(\varphi \rightarrow \psi)$, which contradicts (ii). Hence \perp , and we are done.

Conversely, suppose that $\neg\psi$ and (for the purpose of deriving a contradiction) $\neg(\psi \rightarrow \vartheta)$. Suppose that ψ . Then \perp , so that $\neg\vartheta$, which weakens to $\neg\vartheta \rightarrow \neg\neg\vartheta$. Discharging the assumption, $\psi \rightarrow \neg\vartheta \rightarrow \neg\neg\vartheta$. Then invoke 15^{\rightarrow} in the form

$$(\psi \rightarrow \neg\vartheta \rightarrow \neg\neg\vartheta) \rightarrow (\neg(\psi \rightarrow \vartheta) \rightarrow \neg(\psi \rightarrow \vartheta))$$

and detach twice (using the assumptions still in play) to get $\neg\neg(\psi \rightarrow \vartheta)$. This contradicts the assumption, so \perp . Hence WT .¹²

To see that $WT^{\rightarrow} \Rightarrow 17$, suppose that $\varphi \rightarrow \neg\neg\psi$, and (for contradiction) that $\neg(\varphi \rightarrow \psi)$. From the latter, $\neg\psi$; whence, using the former, $\neg\varphi$. Applying WT, $\neg\neg(\varphi \rightarrow \psi)$, which contradicts the assumption; so \perp .

Conversely, suppose that $\neg\psi$. Assuming ψ , we get \perp , so we may conclude $\neg\neg\vartheta$ and therefore $\psi \rightarrow \neg\neg\vartheta$. Applying 17, $\neg\neg(\psi \rightarrow \vartheta)$ and we are done. □

¹²It may be interesting to note that at least one instance of *contraction* is used in this proof.

Proposition 6. *The following implications hold:*

- (a) $WT \Rightarrow 7^{\rightarrow}$;
- (b) $DGP^{\rightarrow} \Leftrightarrow 8^{\rightarrow}$;
- (c) $7^{\rightarrow} \Rightarrow DGP^{\rightarrow}$ (using \wedge).

Proof. (a) Assume that $\varphi \rightarrow \psi \rightarrow \vartheta$. By interderivability of $\neg\alpha \rightarrow \neg\neg\beta$ and $\neg\beta \rightarrow \neg\neg\alpha$, and in view of the deduction theorem, we may show that

$$\neg(\psi \rightarrow \vartheta) \rightarrow \neg\neg(\varphi \rightarrow \vartheta).^{13}$$

Assume the antecedent. Transitivity over the assumptions gives $\neg\varphi$. Applying WT, $\neg\neg(\varphi \rightarrow \vartheta)$ and we are done.

- (b) First consider φ, ψ, ϑ , and β such that $\varphi \rightarrow \vartheta, \psi \rightarrow \beta$, and $\neg\varphi \rightarrow \beta$. We want to show that $\neg\neg(\psi \rightarrow \vartheta)$. By DGP^{\rightarrow} it is enough to show $\neg(\vartheta \rightarrow \psi)$. So assume also $\vartheta \rightarrow \psi$, which together with the above assumption implies $\varphi \rightarrow \beta$, but that contradicts the third of the initial assumptions. Thus we can derive \perp and are done.

Conversely, a special case of 8^{\rightarrow} is

$$(\varphi \rightarrow \varphi) \rightarrow (\psi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \psi) \rightarrow \neg\neg(\psi \rightarrow \varphi).$$

Since the first two assumptions are tautologies we have

$$\neg(\varphi \rightarrow \psi) \rightarrow \neg\neg(\psi \rightarrow \varphi),$$

which is DGP^{\rightarrow} .

- (c) Let φ and ψ be such that $\neg(\varphi \rightarrow \psi)$ and $\neg(\psi \rightarrow \varphi)$. Notice that then also $\neg(\varphi \rightarrow \varphi \wedge \psi)$ and $\neg(\psi \rightarrow \varphi \wedge \psi)$. We want to derive a contradiction.

Applying 7^{\rightarrow} to φ, ψ , and $\varphi \wedge \psi$ gives us

$$(\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))) \rightarrow \neg(\varphi \rightarrow (\varphi \wedge \psi)) \rightarrow \neg\neg(\psi \rightarrow (\varphi \wedge \psi)).$$

Notice that the antecedent is provable minimally, so we have

$$\neg(\varphi \rightarrow (\varphi \wedge \psi)) \rightarrow \neg(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow \perp,$$

and therefore, applying this to our assumptions we get the desired \perp . □

We will comment on the strange status of statement 7^{\rightarrow} in the last section.

5 Separation results (semantics)

To show the strictness of the implications summed up in Figure 1 we will use models rather than proof theoretic methods, which is the route taken in [14]. We base our semantics for minimal logic on the one described in [9]. More precisely, we consider $(\mathcal{W}, \sqsubseteq, Q)$ where $(\mathcal{W}, \sqsubseteq)$ is a partial order and $Q \subseteq \mathcal{W}$ is a *cone*—that is, an upwards closed set. A *valuation* v is a monotone mapping from \mathcal{W} to \mathcal{P} (PROP). We will call the elements of \mathcal{W} *worlds*.

A *model* is a pair (\mathcal{W}, v) and the forcing relation between a model and a formula is defined in almost the same way as for Kripke semantics. That is we set

$$u \Vdash P \iff P \in v(u)$$

¹³Substitute $\varphi \rightarrow \vartheta$ for α , and $\psi \rightarrow \vartheta$ for β

for propositional formulas and then inductively

$$\begin{aligned} u \Vdash \varphi \wedge \psi &\iff u \Vdash \varphi \text{ and } u \Vdash \psi \\ u \Vdash \varphi \vee \psi &\iff u \Vdash \varphi \text{ or } u \Vdash \psi \\ u \Vdash \varphi \rightarrow \psi &\iff \forall y \in \mathcal{W} : (u \sqsubseteq y \wedge y \Vdash \varphi \implies y \Vdash \psi) . \end{aligned}$$

A point of difference with the usual Kripke semantics is that we do not assume that \perp is never forced, but that we have

$$u \Vdash \perp \iff u \in Q .$$

The intuition behind Kripke style semantics is that we have a multitude of possible worlds ordered by \sqsubseteq with the requirement that if a formula is true in some world it is also true in worlds “above” it relative to the order. Each world by itself behaves like a classical model—of course, apart from our treatment of \perp . We will call worlds u such that $u \in Q$ *abnormal*, and the otherwise *normal*. Since we have defined $\neg\varphi$ as $\varphi \rightarrow \perp$,¹⁴ we get

$$u \Vdash \neg\varphi \iff \forall y \in \mathcal{W} : (u \sqsubseteq y \wedge y \Vdash \varphi \implies y \in Q) ,$$

that is $\neg\varphi$ holds in some world if the only worlds above in which φ holds are abnormal ones.

We write $(\mathcal{W}, v) \Vdash \varphi$ or simply $\mathcal{W} \Vdash \varphi$, if φ is forced in all worlds of \mathcal{W} . As usual, the valuation v should be clear from the context and will not be mentioned. There is another reason why v for us is actually irrelevant: in the present paper, we only consider *full models*. That is, we assume that for every upward closed set $U \subseteq \mathcal{W}$ there exists a propositional symbol P_U such $u \Vdash P_U \iff u \in U$. This excludes many pathological cases of considering complicated structures with trivial valuations. It also means that for any well formed formula α there exists a propositional symbol P_α such that

$$\mathcal{W} \Vdash \alpha \iff \mathcal{W} \Vdash P_\alpha ,$$

which has the advantage, that we only need to consider our axiom schemes to range over propositional symbols and not arbitrary formulas.

Having an arbitrary partial order as our underlying structure is different from [7, 16] which uses tree like-orders—or to be precise finitely branching trees. There is a subtle difference between the two notions as we will explain. On first glance the differences seem minuscule:

Proposition 7. *Assume that (\mathcal{W}, v) is a model that is tame in the sense that if L is a maximal chain with maximal element u , and $u \sqsubseteq v$, then there exists some maximal chain L' having v as a maximal element and $L \subset L'$.¹⁵*

Then there exists a tree-like model (\mathcal{W}^t, v^t) such that

$$\mathcal{W} \Vdash \alpha \iff \mathcal{W}^t \Vdash \alpha \tag{5}$$

for any formula α . Furthermore, if (\mathcal{W}, v) is finite (i.e. \mathcal{W} is finite as a set) then \mathcal{W}^t is a finite and finitely branching tree.

Proof. Define \mathcal{W}^t to be

$$\{ (u, L) \in \mathcal{W} \times \mathcal{P}(\mathcal{W}) \mid L \text{ is a maximal chain with } u \text{ as a maximal element} \} ,$$

and set

$$(u, L) \sqsubseteq (v, L') \iff L \subset L' .$$

Notice that if $(u, L) \sqsubseteq (v, L')$ then $u \in L'$ and that, in particular, $u \sqsubseteq v$. As our new valuation set $v^t(u, L) = v(u)$.

¹⁴This is a point of departure for many other non-classical logics, such as relevant logics, logics of formal inconsistency, and the like. As mentioned, this is a preliminary investigation into the realm of non-classical reverse mathematics more generally; so we stick fairly close to the usual, classical, interpretation of negation.

¹⁵This holds if, for example, \mathcal{W} is finite, or one assumes Zorn’s Lemma.

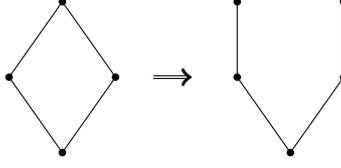


Figure 2: The construction of this proposition removes any joins.

We will prove that for any formula α for all $(u, L) \in \mathcal{W}^t$

$$(u, L) \Vdash \alpha \iff u \Vdash \alpha \quad (6)$$

by induction on the formula α . If α is a propositional symbol (including \perp) then we have 6 by our definition of v^t . The connectives \wedge and \vee are straightforward to deal with. So let $\alpha \equiv \beta \rightarrow \gamma$.

For the direction “ \Leftarrow ” fix $(u, L) \in \mathcal{W}^t$ and assume that $u \Vdash \beta \rightarrow \gamma$. Now consider (v, L') such that $(u, L) \sqsubseteq (v, L')$ and that $(v, L') \Vdash \beta$. By our induction hypothesis that means $v \Vdash \beta$, which implies that $v \Vdash \gamma$. Using the induction hypothesis again we get the desired $(v, L') \Vdash \gamma$.

For the direction “ \Rightarrow ” assume that $(u, L) \Vdash \beta \rightarrow \gamma$, and that v is such that $u \sqsubseteq v$ and $v \Vdash \beta$. The only really non-trivial step in this entire proof is to use the tame-ness assumption to find L' which has v as a maximal element and is such that $L \subseteq L'$. Then $(u, L) \sqsubseteq (v, L')$. By our induction hypothesis we have that $(v, L') \Vdash \beta$, which means that $(v, L') \Vdash \gamma$. Using our induction hypothesis yet again this means that $v \Vdash \gamma$ and we are done. \square

The differences between tree-like and non-tree-like models stem from our definition of a full model. Notice that even if \mathcal{W} is full \mathcal{W}^t might not be. In the example sketched in Figure 2, in \mathcal{W}^t there is no proposition that is forced at one of the top nodes, but not the other; all propositions are either forced at both top-nodes at the same time or not forced at both nodes.

If one does prefer to work with tree-like models one cannot restrict to fullness. For example, as one can easily see, any intuitionistic (i.e. $Q = \emptyset$) tree-like model containing a branching, i.e. that is not v -free, does not satisfy WLEM, which means that *intuitionistic* tree-like models cannot distinguish between WLEM and DGP.¹⁶ So for a structural analysis of what principles hold depending on the underlying partial order it makes more sense to consider arbitrary partial orders rather than tree-like ones. This also excludes Veldman’s explosive nodes [13], which do not add further distinctions here.

It is straightforward to see that we have soundness [9, Proposition 2.3.2], which means we can use these models to show the underivability of formulas in minimal logic. Models for intuitionistic logic, that is minimal logic together with EFQ, are exactly the ones where \perp is never forced (in this case we recover the usual Kripke semantics):

Proposition 8. $\mathcal{W} \Vdash EFQ$ if and only if $Q = \emptyset$.

Proof. Clearly, if $Q = \emptyset$ then $\mathcal{W} \Vdash EFQ$. Conversely, if there is an abnormal world $u \in Q$, then by fullness we can consider the propositional symbol P_\emptyset for which $u \Vdash \perp$, but $u \not\Vdash P_\emptyset$; i.e. $u \not\Vdash \perp \rightarrow P_\emptyset$. \square

Given a structure $(\mathcal{W}, \sqsubseteq, Q)$ we define its *loBOTomy* \mathcal{W}^\perp to be $(\mathcal{W}, \sqsubseteq, \mathcal{W})$, that is we are making all worlds are abnormal. Even this quite trivial construction has very useful consequences for us.

Proposition 9. For any (full) model we have

1. $\mathcal{W}^\perp \Vdash LEM$
2. $\mathcal{W}^\perp \not\Vdash DNE$
3. $\mathcal{W}^\perp \not\Vdash EFQ$

¹⁶Nor can they distinguish between DNE, LEM, and PP (see Proposition 12), but these are all known to be intuitionistically equivalent anyway; see Sec. 6.

Proof. The proofs are easy after one notices that $\mathcal{W}^\perp \Vdash \neg\alpha$ for any formula α . It is also worth pointing out that we need the fullness assumption for the second and third part to ensure that there is a propositional symbol P such that $\mathcal{W} \not\models P$ and therefore $\mathcal{W}^\perp \not\models P$. \square

Proposition 10. *Let \mathcal{W}, v be any full model, and let α be a formula not containing \perp . Then*

$$\mathcal{W} \Vdash \alpha \iff \mathcal{W}^\perp \Vdash \alpha$$

Proof. Induction on the complexity of formulas. \square

We say that a partial order is *v-free* if it doesn't contain a, b, c such $a \sqsubseteq b$, $a \sqsubseteq c$, and b and c incomparable.

Proposition 11. *Let $(\mathcal{W}, \sqsubseteq, Q)$ be a structure.*

1. $\mathcal{W} \Vdash \text{DGP}$ if and only if \mathcal{W} is v-free.
2. $\mathcal{W} \Vdash \text{WLEM}$ if $\mathcal{W} \setminus Q$ is v-free.¹⁷

Proof. 1. Assume \mathcal{W} is v-free and let P and Q be arbitrary propositional symbols. Consider an arbitrary world $u \in \mathcal{W}$. If $P \in v(u)$ we have that $u \Vdash Q \rightarrow P$. Similarly if $Q \in v(u)$ we have that $u \Vdash P \rightarrow Q$. In both cases $u \Vdash P \rightarrow Q \vee Q \rightarrow P$. So assume that neither $P, Q \in v(u)$. Now consider the sets $A = \{y \in \mathcal{W} \mid u \sqsubseteq y \wedge P \in v(y)\}$ and $B = \{y \in \mathcal{W} \mid u \sqsubseteq y \wedge Q \in v(y)\}$. We must have that either $A \subset B$ or $B \subset A$: for assume there were z, z' such that $z \in A$, $z \notin B$, $z' \in B$, and $z' \notin A$. We must have that $z \not\sqsubseteq z'$, since $z \sqsubseteq z'$ implies that $z' \in B$. Similarly $z' \not\sqsubseteq z$, but then u, z, z' contradict the assumption that \mathcal{W} is v-free. If $A \subset B$ we have that for every $u \Vdash Q \rightarrow P$, and if $B \subset A$ we get $u \Vdash P \rightarrow Q$. In both cases $u \Vdash P \rightarrow Q \vee Q \rightarrow P$. Hence we have shown $\mathcal{W} \Vdash \text{DGP}$.

Conversely we will show that if \mathcal{W} is not v-free then $\mathcal{W} \not\models \text{DGP}$. So assume there is $a, b, c \in \mathcal{W}$ such that $a \sqsubseteq b$, $a \sqsubseteq c$, but neither $b \sqsubseteq c$ nor $c \sqsubseteq b$. Let $P_{b\uparrow}$ and $P_{c\uparrow}$ be the propositional symbols corresponding to the upwards closed sets $\{x \in \mathcal{W} : b \sqsubseteq x\}$ and $\{x \in \mathcal{W} : c \sqsubseteq x\}$, respectively. Notice that $b \Vdash P_{b\uparrow}$, $b \not\models P_{c\uparrow}$, $c \Vdash P_{c\uparrow}$, and $c \not\models P_{b\uparrow}$. Assume $a \Vdash (P_{b\uparrow} \rightarrow P_{c\uparrow}) \vee (P_{c\uparrow} \rightarrow P_{b\uparrow})$. Then either $a \Vdash (P_{b\uparrow} \rightarrow P_{c\uparrow})$ or $a \Vdash (P_{c\uparrow} \rightarrow P_{b\uparrow})$; w.l.o.g. the first case. Then by monotonicity $b \Vdash (P_{b\uparrow} \rightarrow P_{c\uparrow})$. Since also $b \Vdash P_{b\uparrow}$ we have $b \Vdash P_{c\uparrow}$; a contradiction.

2. Assume $\mathcal{W} \setminus Q$ is v-free and let P be an arbitrary propositional symbol. Consider an arbitrary world $u \in \mathcal{W}$. If $u \in Q$ we have $u \Vdash \neg\alpha$ for any α , so we may assume that $u \in \mathcal{W} \setminus Q$. If there is no $u \sqsubseteq y$ such that $P \in v(y)$ and $y \notin Q$ then $u \Vdash \neg P$. So assume that there is $u \sqsubseteq y$ such that $y \notin Q$ and $P \in v(y)$. We want to show that $u \Vdash \neg\neg P$. To do this we will show that for any $u \sqsubseteq z$ if $z \Vdash \neg P$ then $z \in Q$. So assume that $u \sqsubseteq z$, and $z \notin Q$. Because $\mathcal{W} \setminus Q$ was assumed to be v-free we either have $z \sqsubseteq y$ or $y \sqsubseteq z$. In the second case, by monotonicity, we get $z \Vdash P$ and therefore $z \Vdash \perp$; a contradiction. In the first case, similarly, by monotonicity $v \Vdash \neg P$ and therefore $y \Vdash \perp$; again a contradiction. So $z \in Q$. Hence we are done, since either $u \Vdash \neg P$ or $u \Vdash \neg\neg P$, so together $u \Vdash \neg P \vee \neg\neg P$. \square

Proposition 12. *Let $(\mathcal{W}, \sqsubseteq, Q)$ be a structure.*

1. $\mathcal{W} \Vdash \text{PP}$ if and only if \mathcal{W} consists of pair-wise incomparable points.
2. $\mathcal{W} \Vdash \text{LEM}$ if and only if $\mathcal{W} \setminus Q$ consists of pair-wise incomparable points.
3. $\mathcal{W} \Vdash \text{DNE}$ if and only if \mathcal{W} consists of pair-wise incomparable points and $Q = \emptyset$.

Proof. 1. Assume that \mathcal{W} consists of pair-wise incomparable points, let P, S be arbitrary propositional symbols, and consider an arbitrary world $u \in \mathcal{W}$. If $S \in v(u)$ we have that $u \Vdash ((S \rightarrow P) \rightarrow S) \rightarrow S$. If $S \notin v(u)$ we have that $u \Vdash (S \rightarrow P)$, and therefore $u \not\models ((S \rightarrow P) \rightarrow S)$. Hence also in that case $u \Vdash ((S \rightarrow P) \rightarrow S) \rightarrow S$.

¹⁷The converse does not hold. See the model \mathcal{W}_4 in Section 6.

Conversely assume that there are distinct $u, y \in \mathcal{W}$ with $u \sqsubseteq y$. Now let $U = \{u \sqsubseteq y \mid u \neq y\}$, and consider P_U as above, and let $S = P_\emptyset$. Then for every $u \sqsubseteq y$ with $u \neq y$ we have $y \Vdash (P_U \rightarrow S) \rightarrow P_U$. For u we have that $u \not\Vdash P_U \rightarrow S$ (since $y \not\Vdash P_U \rightarrow S$) and hence $u \Vdash (P_U \rightarrow S) \rightarrow P_U$. Now, if $u \Vdash \text{PP}$ we would also have that $u \Vdash P_U$; a contradiction. So we have shown that if \mathcal{W} contains two comparable worlds then PP does not hold, or equivalently that if PP holds then \mathcal{W} does not contain comparable worlds.

2. Assume that $\mathcal{W} \setminus Q$ consists of pair-wise incomparable points, let P be an arbitrary propositional symbol, and consider an arbitrary world $u \in \mathcal{W}$. Now either $P \in v(u)$ and therefore $u \Vdash P$ or $P \notin v(u)$. Since the only worlds above $u \sqsubseteq y$, $u \neq y$ are such that $y \in Q$ and therefore $y \Vdash \perp$ we have that for all $u \sqsubseteq y$ with $y \Vdash P$ also $y \Vdash \perp$. Hence in that case $u \Vdash \neg P$.

Conversely assume that there are distinct $u, y \in \mathcal{W} \setminus Q$ with $u \sqsubseteq y$. Consider $P_{y \uparrow}$ as above. Then $u \not\Vdash P_{y \uparrow}$. But also $u \not\Vdash \neg P_{y \uparrow}$, since if $u \Vdash \neg P_{y \uparrow}$ also $y \Vdash \neg P_{y \uparrow}$ which would imply $y \Vdash \perp$; a contradiction to $y \notin Q$. Hence $u \not\Vdash \text{LEM}$. Equivalently we have shown that if $\mathcal{W} \Vdash \text{LEM}$ then $\mathcal{W} \setminus Q$ cannot contain two comparable worlds.

3. This follows from Proposition 8 together with either of the previous two items and Proposition 13 further down. \square

To start with one of the most trivial models imaginable, consider $\mathcal{W}_1 = \{0\}$ and $v_1(0) = \emptyset$. That is there is only one world. By Proposition 9 this means that

$$\text{LEM} \not\Rightarrow \text{DNE}$$

and

$$\text{LEM} \not\Rightarrow \text{EFQ}.$$

The second structure we are considering is $\mathcal{W}_2 = (\{1, 2\}, \leq, \emptyset)$. Since $Q = \emptyset$ this is a model of EFQ. It does not, however, force DNE, since, if P is such that $\llbracket P \rrbracket = \{2\}$ then no node forces $\neg P$, whence $\mathcal{W}_2 \Vdash \neg\neg P$. However, of course, $1 \not\Vdash P$. The same P also ensures that $\mathcal{W}_2 \not\Vdash \text{LEM}$. A useful variant of \mathcal{W}_2 is $\mathcal{W}'_2 = (\{1, 2\}, \leq, \{2\})$; in that case, with $\llbracket P \rrbracket = \{2\}$ again, DGP^\rightarrow is forced since it is v -free; however, WT is not forced. To see this, note that since \perp is forced whenever P is forced, $1 \Vdash \neg P$. However, since $P \rightarrow S$ is nowhere forced, $1 \Vdash \neg(P \rightarrow S)$. Since $1 \not\Vdash \perp$, $1 \not\Vdash \neg\neg(P \rightarrow S)$. Hence $\mathcal{W}'_2 \not\Vdash \text{WT}$. Note that LEM, DGP, and WLEM are also forced in this model, separating WT and DGP^\rightarrow from these principles.

The third structure $(\mathcal{W}_3, \emptyset)$ is a simple v shape, so, for example, $(\{\emptyset, \{1\}, \{2\}\}, \subset, \emptyset)$. \mathcal{W}_3 does not satisfy WLEM, since for $P = P_{\{1\}}$ we have that $\emptyset \not\Vdash \neg P \vee \neg\neg P$. Like \mathcal{W}_2 , a useful variant is $\mathcal{W}'_3 = (\{\emptyset, \{1\}, \{2\}\}, \subset, \{\{1\}, \{2\}\})$. This model does not force DGP^\rightarrow . To see this, assign $P = P_{\{1\}}, S = S_{\{2\}}$. Then since $\emptyset \not\Vdash P \rightarrow S$ and $\emptyset \not\Vdash S \rightarrow P$, and since both $\{1\} \Vdash \perp$ and $\{2\} \Vdash \perp$, $\emptyset \Vdash \neg(S \rightarrow P)$ and $\emptyset \Vdash \neg(P \rightarrow S)$. But since $\emptyset \not\Vdash \perp$, we have $\emptyset \not\Vdash \neg\neg(P \rightarrow S)$. Hence $\mathcal{W}'_3 \not\Vdash \text{DGP}^\rightarrow$.

Figure 3 includes the Hasse diagrams of the underlying orders of these models.

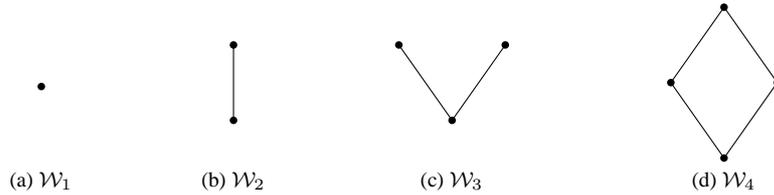


Figure 3: The Hasse diagrams of the underlying orders of our models

These models, together with their \perp versions and useful variants (and \mathcal{W}_4 ; see the next section), show that all the implications in Figure 1 are strict. These models also show that—apart from the transitive closure—no arrows can be added to Figure 1. For simplicity we include a table. Further,

	DNE	EFQ	LEM	DGP	PP	WLEM	WT	DGP [→]
\mathcal{W}_1	✓	✓	✓	✓	✓	✓	✓	✓
\mathcal{W}_1^\perp	✗	✗	✓	✓	✓	✓	✓	✓
\mathcal{W}_2	✗	✓	✗	✓	✗	✓	✓	✓
\mathcal{W}_2^\perp	✗	✗	✓	✓	✗	✓	✓	✓
\mathcal{W}'_2	✗	✗	✓	✓	✗	✓	✗	✓
\mathcal{W}_3	✗	✓	✗	✗	✗	✗	✓	✓
\mathcal{W}_3^\perp	✗	✗	✓	✗	✗	✓	✓	✓
\mathcal{W}'_3	✗	✗	✓	✗	✗	✓	✗	✗
\mathcal{W}_4	✗	✓	✗	✗	✗	✓	✓	✓
\mathcal{W}_4^\perp	✗	✗	✓	✗	✗	✓	✓	✓

Figure 4: An overview, over all models over minimal logic. See next section for \mathcal{W}_4 .

note that since every principle fails to be enforced in at least one of the models, none of the principles outlined are provable in minimal logic alone.

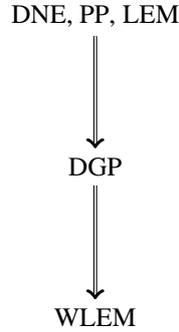
6 The intuitionistic case

It is well known that:

Proposition 13. *LEM and EFQ imply DNE*

Proof. Let φ be such that $\neg\neg\varphi$ holds. Thus if we have $\neg\varphi$ then \perp , which by EFQ implies φ . Together with LEM that means that $\varphi \vee \neg\varphi$ implies φ . \square

Thus under the assumption of EFQ our hierarchy collapses to the following simple one:



The model \mathcal{W}_2 shows that the first of these implications is strict, however none of the models considered so far satisfy EFQ and WLEM, but not DGP. To do so we consider the diamond-shaped structure $(\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \subset, \emptyset)$. As $Q = \emptyset$ this is a model of EFQ. As \mathcal{W}_4 is not \vee -free it does not satisfy DGP (Proposition 11).

Finally we need to check that WLEM holds. So let P be an arbitrary propositional symbol. If $P \notin v(\{1, 2\})$ then $\mathcal{W}_4 \Vdash P \rightarrow \perp$, since P is never forced. If $P \in v(\{1, 2\})$ then $P \rightarrow \perp$ is never forced in any world, so $\mathcal{W}_4 \Vdash \neg\neg P$ vacuously. In both cases $\mathcal{W}_4 \Vdash \neg P \vee \neg\neg P$.

7 Open Questions and Concluding Remarks

7.1 Statement 7^{\rightarrow}

As shown in Section 4 we have

$$\text{WT} \implies 7^{\rightarrow} \implies \text{DGP}^{\rightarrow}.$$

One can check that 7^{\rightarrow} holds in exactly the models discussed which validate DGP^{\rightarrow} , so it is provably weaker than WT and one would naturally conjecture that it is equivalent to that statement. However we could not find a proof of $\text{DGP}^{\rightarrow} \implies 7^{\rightarrow}$. It is also interesting that we could not find a proof of the converse that does not rely on \wedge .

7.2 Kreisel Putnam and Scott logic

In the fabulously titled (even for German speakers) paper [8] it is shown that the formula

$$(\neg\varphi \rightarrow \psi \vee \vartheta) \rightarrow (\neg\varphi \rightarrow \psi) \vee (\neg\varphi \rightarrow \vartheta) \quad (\text{KP})$$

is not derivable in intuitionistic logic. One can check that it holds in exactly the same of the models discussed as DGP^{\rightarrow} . It is completely unclear, though, whether either one implies the other. In the same paper [8] also the following formula, which is not derivable intuitionistically, and which is due to Dana Scott is mentioned

$$((\neg\neg\varphi \rightarrow \varphi) \rightarrow (\varphi \vee \neg\varphi)) \rightarrow (\neg\neg\varphi \vee \neg\varphi).$$

This formula is clearly implied by WLEM. It is, however, weaker, since it actually holds in all models we have considered in this paper. It would also be interesting to have intuitionistic models (i.e. ones validating EFQ) rejecting these principles.

7.3 SmL

It is also worth mentioning another important formula characterising an superintuitionistic logic, the so called Smetanich's logic [15]. It is obtained by adding the following axiom scheme to intuitionistic logic.

$$(\neg\psi \rightarrow \varphi) \rightarrow (((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi). \quad (\text{SmL})$$

It is easily seen that this is implied by LEM. It also implies WLEM which we can see if we apply it to $\neg\varphi \vee \neg\neg\varphi$ and $\neg\varphi$. However we can check that it holds in \mathcal{W}_2 and does not hold in \mathcal{W}'_3 which means that it neither implies LEM nor is it implied by or implies DGP (and therefore neither by WLEM). That means we have the following extension to the left bottom corner of Figure 1.

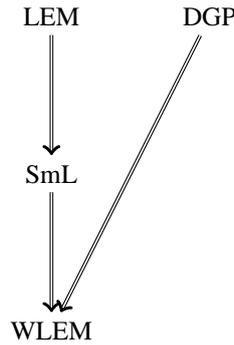


Figure 5: A small extension in the lower left corner of Figure 1.

It is not clear, however, how SmL relates to DGP intuitionistically. One can see that DGP does not imply SmL even under the assumption of EFQ, since the former holds in the model $(\mathcal{W}_5, \emptyset, \leq)$, where $\mathcal{W}_5 = (\{1, 2, 3\})$ and \leq is the usual order.

7.4 Concluding Remarks

The distinctions and equivalences in this paper represent, of course, only the tip of the proverbial iceberg. Beyond what can be distinguished in other non-classical logics, of special mention are substructural logics [10]. The proofs presented here assume all the usual structural rules. An analysis of which proofs still go through in the various substructural logics will clearly shed further light on computational aspects.

References

- [1] D. Adams, *Dirk Gently's Holistic Detective Agency*, UK: William Heinemann Ltd., 1987.
- [2] Alan Ross Anderson and Nuel D. Belnap, *Entailment: The Logic of Relevance and Necessity*, Vol. 1, Princeton University Press, Princeton, 1975.
- [3] A. I. Arruda, *Aspects of the Historical Development of Paraconsistent Logic*, 99–130. In Priest, Routley & Norman, *Paraconsistent Logic: Essays on the Inconsistent*.
- [4] Nuel D. Belnap, *A Useful Four-Valued Logic* (1977), 8–37. In Dunn & Epstein (eds.), *Modern Uses of Multiple-Valued Logics*.
- [5] Susan Haack, *Deviant Logic, Fuzzy Logic: Beyond the Formalism*, Cambridge University Press, 1996.
- [6] Hajime Ishihara, *Reverse mathematics in Bishop's constructive mathematics*, *Philosophia Scientiæ Cahier spécial* **6** (2006), 43–59.
- [7] Hajime Ishihara and Helmut Schwichtenberg, *Embedding classical in minimal implicational logic*, *Mathematical Logic Quarterly* (2016), n/a–n/a.
- [8] G. Kreisel and H. Putnam, *Eine Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül*, *Archiv für mathematische Logik und Grundlagenforschung* **3** (1957), no. 3, 74–78.
- [9] S. Odintsov, *Constructive negations and paraconsistency*, *Trends in Logic*, Springer Netherlands, 2008.
- [10] F. Paoli, *Substructural logics: A primer*, *Trends in Logic*, Springer Netherlands, 2013.
- [11] Graham Priest and Richard Routley and Jean Norman (ed.), *Paraconsistent Logic: Essays on the Inconsistent*, Philosophia Verlag, Munich, 1989.
- [12] R.M. Sainsbury, *Paradoxes*, Cambridge University Press, 1995.
- [13] Wim Veldman, *An intuitionistic completeness theorem for intuitionistic predicate logic*, *The Journal of Symbolic Logic* **41** (1976), no. 1, 159–166.
- [14] Pedro F. Valencia Vizcaíno, *Some uses of cut elimination*, Ph.D. Thesis, 2013.
- [15] Frank Wolter and Michael Zakharyashev, *Modal decision problems*, *Handbook of modal logic*, 2007, pp. 427–489.
- [16] Klaus Weich, *Improving proof search in intuitionistic propositional logic*, Ph.D. Thesis, 2001.