**Fields with automorphism** 

 and valuation

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### 1 Introduction

The question has been of interest for decades: If a modelcomplete theory T is augmented with axioms for an automorphism  $\sigma$  of its models, does the resulting theory  $T_{\sigma}$  have a model-companion?

It does have, when T is the theory ACF of algebraically closed fields. The model companion of  $T_{\sigma}$  in this case is ACFA, studied by Macintyre [14] and Chatzidakis and Hrushovski [6] and others. However,  $T_{\sigma}$  is not companionable when T is ACFA itself [13].

A more general result, established by Kikyo [12], is that if  $T_{\sigma}$  is companionable, and T is dependent, then T must also be stable. In particular then,  $T_{\sigma}$  cannot be companionable when T is the theory ACVF of algebraically closed valued fields in the signature of fields with a predicate for a valuation ring. Note that an automorphism  $\sigma$  of a valued field induces an automorphism  $\sigma_v$  of the the value group  $\Gamma$  and an automorphism  $\bar{\sigma}$  of the residue field. Then  $T_{\sigma}$  is companionable when T is the model companion of the theory of any of the following classes of valued fields:

- 1) valued *D*-fields [21],
- 2) isometric valued difference fields, where  $\sigma_v(\gamma) = \gamma$  for all  $\gamma$  in  $\Gamma$  [3, 2],
- 3) contractive valued difference fields, where  $\sigma_v(\gamma) > n\gamma$  for all positive  $\gamma$  in  $\Gamma$  and n in  $\omega$  [1],
- 4) multiplicative valued fields, where  $\sigma_v(\gamma) = \rho \gamma$  for all  $\gamma$  in  $\Gamma$ , for a certain constant  $\rho$  [15].

Moreover,

5)  $T_{\sigma} \cup T_{v}$  is companionable when T is ACVF and  $T_{v}$  is a companionable theory of ordered abelian groups equipped with an automorphism [7] (in this case

$$(\Gamma, \sigma_v) \models T_v).$$

The corresponding model companion in each of the five cases satisfies an analogue of Hensel's lemma for  $\sigma$ -polynomials (see [1, Definition 4.2]).

In this paper we consider the theory FAV of a valued field equipped with an automorphism of the field alone. There is no required interaction of valuation and the automorphism: the automorphism need not fix the valuation ring (setwise). The similar case of a differential field with an automorphism of the field alone was treated in [16]. Our main theorem, Theorem 11, is that FAV has a model companion, FAV<sup>\*</sup>.

There is an obvious candidate for  $FAV^*$ , since FAV is included in the union of two model-complete theories, namely ACFA and ACVF. However, we show, as Theorem 12, that ACFA  $\cup$  ACVF is *not* model complete.

Our paper is organized as follows. In §2 we give axioms of FAV. In §3 preliminaries about companionable theories are explained. Then in §4, Theorem 8 establishes a geometric axiomatization of ACFA. Using this, in §5 we prove Theorems 11 and 12.

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# 2 Fields with an automorphism and a valuation

A signature sufficient for a first-order axiomatization of fields with an automorphism and a valuation is the signature  $\{+, -, \times, 0, 1\}$  of fields, augmented with

- 1) a singulary operator  $\sigma$  for the automorphism and
- 2) a singulary predicate  $\in \mathfrak{O}$  for membership in the valuation ring.

We shall write the last two symbols *after* their arguments. The fields with an automorphism and a valuation are then axiomatized by the field axioms, along with axioms

$$(x+y)^{\sigma} = x^{\sigma} + y^{\sigma}, \quad (x \cdot y)^{\sigma} = x^{\sigma} \cdot y^{\sigma}, \quad \exists y \ y^{\sigma} = x$$

for a surjective endomorphism (which for a field is an automorphism), and axioms

$$0 \in \mathfrak{O},$$
  
$$x \in \mathfrak{O} \land y \in \mathfrak{O} \Rightarrow -x \in \mathfrak{O} \land x + y \in \mathfrak{O} \land x \cdot y \in \mathfrak{O},$$
  
$$\exists y \ (x \notin \mathfrak{O} \Rightarrow x \cdot y = 1 \land y \in \mathfrak{O})$$

for a valuation ring. It will be our habit, as here, to suppress outer universal quantifiers. For convenience, we introduce a singulary predicate  $\in \mathfrak{M}$  for membership in the unique maximal ideal of the valuation ring. This means requiring

$$x \in \mathfrak{M} \Leftrightarrow \exists y \ \big( x = 0 \lor (x \cdot y = 1 \land y \notin \mathfrak{O}) \big),$$

or equivalently

$$x \notin \mathfrak{M} \Leftrightarrow \exists y \ (x \cdot y = 1 \land y \in \mathfrak{O}).$$
(1)

Because both the new predicate and its negation can thus be given existential definitions, use of the predicate does not affect the existence of a *model-companion* of the theory being axiomatized [17, Lem. 1.1, p. 427]. Let us denote this theory by FAV; officially, its signature is

$$\{+,-,\times,0,1,\sigma,\in\mathfrak{O},\in\mathfrak{M}\}.$$

For later use, we note that the identity

$$x \notin \mathfrak{O} \Leftrightarrow \exists y \ (x \cdot y = 1 \land y \in \mathfrak{M}) \tag{2}$$

holds in FAV.

For a valuation as such, we can introduce a new sort having signature  $\{+, 0, \infty, >\}$ , so that the valuation is a *surjective* function val from the original sort to the new sort that satisfies also

$$\operatorname{val}(x) + \operatorname{val}(y) = \operatorname{val}(x \cdot y), \quad (3)$$
$$0 = \operatorname{val}(1),$$
$$\infty = \operatorname{val}(0),$$
$$\operatorname{val}(x) > \operatorname{val}(y) \Leftrightarrow \exists z \ (y \cdot z = 1 \land x \cdot z \in \mathfrak{M}).$$

These rules ensure that the new sort is an ordered additive abelian group—the *value group*—with an additional element  $\infty$  that is greater than all others, and

$$\infty + x = \infty = x + \infty.$$

Also, val restricts to a homomorphism from the multiplicative group of units of the field onto the value group, and the kernel of this homomorphism is  $\mathfrak{O} \smallsetminus \mathfrak{M}$ , which is the group of units

of the valuation ring  $\mathfrak{O}$ . Moreover,

$$\begin{aligned} \operatorname{val}(x) &\ge 0 \Leftrightarrow x \in \mathfrak{O}, \\ \operatorname{val}(x) &> 0 \Leftrightarrow x \in \mathfrak{M}, \\ \operatorname{val}(x) &< 0 \Leftrightarrow x \notin \mathfrak{O}. \end{aligned}$$

As with the maximal ideal  $\mathfrak{M}$ , so with the value group, its official status does not matter for our purposes. Officially we shall not use the value group, and so we may write a typical model of FAV as  $(K, \sigma, \mathfrak{O})$ . However, the value group may be useful for thinking things through.

# 3 Model-companions and -completions

The (Robinson) diagram of a structure  $\mathfrak{A}$  in a signature  $\mathscr{S}$  is the theory diag( $\mathfrak{A}$ ), in the signature  $\mathscr{S}(A)$  (where A is the domain of  $\mathfrak{A}$ ), of structures in which  $\mathfrak{A}$  embeds. This means diag( $\mathfrak{A}$ ) is axiomatized by all of the quantifier-free sentences of  $\mathscr{S}$  with parameters from (the underlying set A of)  $\mathfrak{A}$  that are true in  $\mathfrak{A}$ . Thus diag( $\mathfrak{A}$ ) is also axiomatized simply by the atomic and negated atomic sentences of  $\mathscr{S}(A)$  that are true in  $\mathfrak{A}$ .

When it exists, a **model-companion** of a theory  $T_0$  is a theory  $T_1$  in the same signature such that

- for each i, every model of T<sub>i</sub> embeds in a model of T<sub>1-i</sub>, that is, T<sub>1-i</sub>∪diag(𝔅) is consistent whenever 𝔅 is a model of T<sub>i</sub>; and
- 2)  $T_1$  is **model-complete**, that is,  $T_1 \cup \text{diag}(\mathfrak{A})$  is complete whenever  $\mathfrak{A} \models T_1$ .

The model-companion of a theory is unique when it exists. It was "introduced by Barwise, Eklof, Robinson, and Sabbagh in 1969" [5, p. 609], these four logicians being known collectively as Eli Bers [10, p. 410]. The model-companion generalizes an earlier notion of Robinson [20, §4.3, p. 109]: the theory  $T_1$  is a **model-completion** of  $T_0$  in case  $T_0 \subseteq T_1$  and  $T_1 \cup$ diag( $\mathfrak{A}$ ) is consistent and complete whenever  $\mathfrak{A} \models T_0$ . If  $T_0$ has the model-companion  $T_1$ , then  $T_1$  is a model-completion of  $T_0$  just in case  $T_0$  has the **amalgamation property**, that is, two models having a common submodel have a common supermodel (this is an exercise in Hodges [10, §8.4, exer. 9, p. 390] attributed to Eli Bers [8, Lem. 2.1, p. 254]).

We say that a theory is **inductive** if every union of a chain of models is a model. Robinson's name for such a theory was  $\sigma$ -persistent; but since we are already using the symbol  $\sigma$ for a field automorphism, we prefer the simpler term for the kind of theory in question. By the Chang–Łoś–Suszko Theorem [10, 6.5.9, p. 297], A theory T is inductive if and only if it is precisely the theory  $T_{\forall\exists}$  axiomatized by the  $\forall\exists$  (or  $\forall_2$ ) consequences of T.

By a **system** we shall mean a (finite) conjunction of atomic and negated atomic formulas. For theories, having a modelcompanion or -completion means having an appropriate condition for when systems over a given model have solutions in a larger model. We recall first Robinson's equivalent formulation of when inductive theories have model-completions; we review also the proof, for the sake of the variations that we shall state and use.

**Theorem 1** (Robinson [20, §5.5, p. 128]). For an inductive theory T to have a model-completion, a sufficient and necessary condition is that, for every system  $\varphi(\mathbf{x}, \mathbf{z})$  in the signature of T, there is a quantifier-free formula  $\vartheta(\boldsymbol{x}, \boldsymbol{y})$  in that signature such that, for all models  $\mathfrak{M}$  of T, for all tuples  $\boldsymbol{a}$  of parameters from M having the same length as  $\boldsymbol{x}$ , the following conditions are equivalent:

(i)  $\varphi(\boldsymbol{a}, \boldsymbol{z})$  is soluble in some model of  $T \cup \operatorname{diag}(\mathfrak{M})$ ,

(ii)  $\vartheta(\boldsymbol{a}, \boldsymbol{y})$  is soluble in  $\mathfrak{M}$  itself.

When such  $\vartheta$  do exist, then the model-completion of T is the theory  $T^*$  axiomatized by the sentences

$$\exists \boldsymbol{y} \ \vartheta(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \exists \boldsymbol{z} \ \varphi(\boldsymbol{x}, \boldsymbol{z}), \tag{4}$$

along with axioms for T itself.

The sentence (4) is equivalent to the  $\forall \exists$  sentence  $\exists \boldsymbol{z} (\vartheta(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \varphi(\boldsymbol{x}, \boldsymbol{z}))$ , outer universal quantifiers being suppressed.

Proof of Robinson's theorem. For the necessity of the given condition, suppose T has the model-completion  $T^*$ . For every system  $\varphi(\boldsymbol{x}, \boldsymbol{z})$  in the signature of T, for every model  $\mathfrak{M}$ of T, for every tuple  $\boldsymbol{a}$  of parameters from M such that (i) holds, since every model of T embeds in a model of  $T^*$ , we can conclude from the completeness of  $T^* \cup \text{diag}(\mathfrak{M})$  that

$$T^* \cup \operatorname{diag}(\mathfrak{M}) \vdash \exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z}).$$

By Compactness and the Lemma on Constants [10, 2.3.2, p. 43], there is a quantifier-free formula  $\vartheta_{(\mathfrak{M},\boldsymbol{a})}(\boldsymbol{x},\boldsymbol{y})$  of the signature of T such that

$$T^* \vdash \exists \boldsymbol{y} \ \vartheta_{(\mathfrak{M},\boldsymbol{a})}(\boldsymbol{x},\boldsymbol{y}) \Rightarrow \exists \boldsymbol{z} \ \varphi(\boldsymbol{x},\boldsymbol{z}).$$
(5)

By Compactness again, for the given system  $\varphi$ , there is a disjunction  $\vartheta$  of finitely many of the formulas  $\vartheta_{(\mathfrak{M}, \boldsymbol{a})}$  such that for

every model  $\mathfrak{M}$  of T, for every tuple  $\boldsymbol{a}$  of parameters from M having the length of  $\boldsymbol{x}$ , if (i), then (ii). If conversely (ii), then  $\varphi(\boldsymbol{a}, \boldsymbol{z})$  is soluble in every model of  $T^* \cup \text{diag}(\mathfrak{M})$ , by (5); but such a model is a model of  $T \cup \text{diag}(\mathfrak{M})$ , and so (i) holds.

For the sufficiency of Robinson's condition, we first show that every model  $\mathfrak{M}$  of T embeds in a model of the theory  $T^*$ having the axioms (4) in addition to those of T. Here we shall use (ii) implies (i), but not the converse. For every system  $\varphi(\boldsymbol{x}, \boldsymbol{z})$  in the signature of T, for all  $\boldsymbol{a}$  and  $\boldsymbol{b}$  from M such that  $\mathfrak{M} \models \vartheta(\boldsymbol{a}, \boldsymbol{b})$ , for some model  $\mathfrak{N}$  of  $T \cup \text{diag}(\mathfrak{M})$ , the sentence

$$\exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z})$$

is true in  $\mathfrak{N}$ . This sentence being existential and thus preserved in larger models, by Zorn's Lemma and inductivity of T, we can move the last of the three bold quantifiers to the front: in some model  $\mathfrak{M}'$  of  $T \cup \operatorname{diag}(\mathfrak{M})$ , for all systems  $\varphi(\boldsymbol{x}, \boldsymbol{z})$ , for all  $\boldsymbol{a}$  and  $\boldsymbol{b}$  from M, the sentence

$$\exists \boldsymbol{z} \ \big( \vartheta(\boldsymbol{a}, \boldsymbol{b}) \Rightarrow \varphi(\boldsymbol{a}, \boldsymbol{z}) \big).$$

is true in  $\mathfrak N.$  Now we can form the chain

$$\mathfrak{M} \subseteq \mathfrak{M}' \subseteq \mathfrak{M}'' \subseteq \cdots,$$

whose limit is a model of  $T^*$ . Thus  $T^* \cup \text{diag}(\mathfrak{M})$  is consistent.

We now show  $T^* \cup \operatorname{diag}(\mathfrak{M})$  is complete by induction on the complexity of sentences. The theory is complete with respect to existential sentences, namely  $\exists_1$  sentences, since (i) implies (ii). Indeed, suppose the sentence  $\exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z})$  is true in some model of  $T^* \cup \operatorname{diag}(\mathfrak{M})$ , where  $\varphi$  is quantifier-free in the signature of T, and  $\boldsymbol{a}$  is from M. Since  $\varphi$  is a disjunction of systems, it is enough to assume  $\varphi$  itself is a system. Since  $T \subseteq T^*$ , we have (i) and therefore (ii). Since the formula  $\vartheta$  here is quantifier-free, the sentence  $\vartheta(\boldsymbol{a}, \boldsymbol{b})$  belongs to diag $(\mathfrak{M})$  for some  $\boldsymbol{b}$  from M. Since the sentence (4) is an axiom of  $T^*$ , we conclude

$$T^* \cup \operatorname{diag}(\mathfrak{M}) \vdash \exists \boldsymbol{y} \ \varphi(\boldsymbol{a}, \boldsymbol{y}).$$

Thus  $T^* \cup \operatorname{diag}(\mathfrak{M})$  is complete with respect to  $\exists_1$  sentences.

Suppose now that for some positive integer n, for all models  $\mathfrak{M}$  of T, the theory  $T^* \cup \operatorname{diag}(\mathfrak{M})$  is complete with respect to  $\exists_n$  sentences. For an arbitrary model  $\mathfrak{M}$  of T, let  $\varphi(\boldsymbol{x}, \boldsymbol{z})$  be an  $\forall_n$  formula, and let  $\boldsymbol{a}$  be a tuple of parameters from M such that the  $\exists_{n+1}$  sentence  $\exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z})$  is true in some model  $\mathfrak{N}$  of  $T^* \cup \operatorname{diag}(\mathfrak{M})$ . Then for some  $\boldsymbol{c}$  from N, the sentence  $\varphi(\boldsymbol{a}, \boldsymbol{c})$  is true in  $\mathfrak{N}$ . Since  $\mathfrak{N} \models T$ , by inductive hypothesis we have

$$T^* \cup \operatorname{diag}(\mathfrak{N}) \vdash \exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z}).$$

By Compactness, there is a quantifier-free formula  $\psi(\boldsymbol{x}, \boldsymbol{y})$  such that

$$\mathfrak{N}\models \exists \boldsymbol{y}\; \psi(\boldsymbol{a},\boldsymbol{y}), \qquad T^*\vdash \exists \boldsymbol{y}\; \psi(\boldsymbol{x},\boldsymbol{y}) \Rightarrow \exists \boldsymbol{z}\; \varphi(\boldsymbol{x},\boldsymbol{z}).$$

Since again  $\mathfrak{N}$  is a model of  $T^* \cup \operatorname{diag}(\mathfrak{M})$ , which is complete with respect to existential sentences, we can conclude

$$T^* \cup \operatorname{diag}(\mathfrak{M}) \vdash \exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z}).$$

Thus  $T^* \cup \operatorname{diag}(\mathfrak{M})$  is complete with respect to  $\exists_{n+1}$  sentences.

By induction,  $T^* \cup \text{diag}(\mathfrak{M})$  is complete.

A model  $\mathfrak{M}$  of a theory T is **existentially closed** if  $T \cup \text{diag}(\mathfrak{M})$  is complete with respect to existential formulas. For Theorem 1 then, the proof of completeness of  $T^* \cup \text{diag}(\mathfrak{M})$  is a generalization of the proof of the following.

**Porism 2** (Robinson's Test [20, 4.2.1, p. 92]). A theory T is model-complete, provided that all of its models are existentially closed.

For an inductive theory T with a model  $\mathfrak{M}$ , for every system  $\varphi(\boldsymbol{x}, \boldsymbol{z})$  in the signature of T, for all  $\boldsymbol{a}$  from M, for some model  $\mathfrak{N}$  of  $T \cup \operatorname{diag}(\mathfrak{M})$ , if  $T \cup \operatorname{diag}(\mathfrak{M}) \cup \{\exists \boldsymbol{z} \ \varphi(\boldsymbol{a}, \boldsymbol{z})\}$  is consistent, then  $\varphi(\boldsymbol{a}, \boldsymbol{z})$  is solved in  $\mathfrak{N}$ . By the method of the proof that every model of T embeds in a model of  $T^*$ , we can again put the last bold quantifier in front and go on to obtain the following.

**Porism 3.** Every model of an inductive theory embeds in an existentially closed model.

The two porisms lead to the following result, now standard [5, 3.5.15, p. 198].

**Theorem 4** (Eklof and Sabbagh [8, 7.10–3, pp. 286–8]). An inductive theory T has a model companion  $T^*$  if and only if the models of  $T^*$  are precisely the existentially closed models of T.

Robinson used Theorem 1 to prove that the theory  $\mathsf{DF}_0$ of fields of characteristic 0 with a derivation had a modelcompletion,  $\mathsf{DCF}_0$ . But there are simpler practical approaches to obtaining model-completions; simpler still, if all we want are model-companions. First of all, in the proof of Theorem 1, we did not really need to extract the finite disjunction  $\vartheta$  from all of the formulas  $\vartheta_{(\mathfrak{M}, \mathbf{a})}$ . Moreover, the proof that models of T embed in models of  $T^*$  did not require the formulas  $\vartheta$  to be quantifier-free. Neither is this required for the observation that the models of  $T^*$  are the existentially closed models of T. Thus we have the following. **Porism 5.** For an inductive theory T to have a modelcompletion, a sufficient and necessary condition is that, for every system  $\varphi(\boldsymbol{x}, \boldsymbol{z})$  in the signature of T, there is a set  $\Theta$ of quantifier-free formulas  $\vartheta(\boldsymbol{x}, \boldsymbol{y})$  in that signature such that, for all models  $\mathfrak{M}$  of T, for all tuples  $\boldsymbol{a}$  of parameters from M having the same length as  $\boldsymbol{x}$ , the following conditions are equivalent:

(i)  $\varphi(\boldsymbol{a}, \boldsymbol{z})$  is soluble in some model of  $T \cup \operatorname{diag}(\mathfrak{M})$ ,

(ii)  $\vartheta(\boldsymbol{a}, \boldsymbol{y})$  is soluble in  $\mathfrak{M}$  itself for some  $\vartheta$  in  $\Theta$ .

When such  $\Theta$  do exist, then the model-completion of T is the theory  $T^*$  axiomatized by the sentences (4), where  $\vartheta$  ranges over  $\Theta$ , along with axioms for T itself. If the formulas in the sets  $\Theta$  are not necessarily quantifier-free, the theory  $T^*$  is still the model-companion of T.

Simpler axiomatizations than Robinson's for  $\mathsf{DCF}_0$  were found by showing that the axioms need not explicitly concern all systems [4, 19]. The general observation can be formulated as follows.

**Porism 6.** Theorem 1 and its Porism 5 still hold, even if  $\varphi$  is constrained to range over a collection of systems containing,

1) for each system  $\psi(\boldsymbol{x}, \boldsymbol{u})$  in the signature of T,

2) for each model  $\mathfrak{M}$  of T,

3) for each tuple  $\boldsymbol{a}$  of parameters from M,

a system  $\varphi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v})$  such that, if  $\exists \boldsymbol{u} \ \psi(\boldsymbol{a}, \boldsymbol{u})$  is consistent with  $T \cup \operatorname{diag}(\mathfrak{M})$ , then so is  $\exists \boldsymbol{u} \ \exists \boldsymbol{v} \ \varphi(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v})$ , and

$$T \cup \operatorname{diag}(\mathfrak{M}) \vdash \varphi(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}) \Rightarrow \psi(\boldsymbol{a}, \boldsymbol{u}).$$
(6)

We may refer to  $\varphi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v})$  as a **refinement** of the system  $\psi(\boldsymbol{x}, \boldsymbol{u})$ . We shall apply Porism 6 when T is FAV or, as in the next section, the theory of *difference fields*.

## 4 Difference fields

A difference field is a field equipped with an automorphism. As was observed in the introduction, the theory of difference fields in the signature  $\{+, -, \times, 0, 1, \sigma\}$  has a model-companion, called ACFA. Perhaps any axiomatization of ACFA can be made to serve present purposes; we shall derive the one that we shall use from Theorem 8. which is in the style of [16, Thm 3.1, p. 1337].

The following lemma will be the reason for condition (7) in Theorem 8. For notational economy, our set  $\omega$  of natural numbers is the set of von Neumann natural numbers, where

$$n = \{0, \dots, n-1\} = \{i \colon i < n\}.$$

**Fact 7.** Suppose  $(K, \sigma)$  is a difference field, and I is a prime ideal of the polynomial ring  $K[X_j: j < n]$ , and  $\tau$  is an embedding of m in n. Write  $X_j + I$  as  $a_j$  whenever j < n. For  $\sigma$  to extend to an automorphism of a field that includes  $K[\mathbf{a}]$  so that  $a_i^{\sigma} = a_{\tau(i)}$  whenever i < m, it is necessary and sufficient that

$$f(a_i : i < m) = 0 \iff f(a_{\tau(i)} : i < m) = 0$$

for all f in  $K[X_i: i < m]$ .

**Theorem 8.** A difference-field  $(K, \sigma)$  is existentially closed among all difference fields if and only if,

- 1) for all m and n in  $\boldsymbol{\omega}$  such that  $m \leq n$ ,
- 2) for every injective function  $\tau$  from m into n,
- 3) for every finite subset  $I_0$  of  $K[X_j: j < n]$ ,

if  $I_0$  generates a prime ideal  $(I_0)$  of  $K[X_j: j < n]$ , and

$$\{ f(X_{\tau(i)} \colon i < m) \colon f \in (I_0) \cap K[X_i \colon i < m] \}$$
  
=  $(I_0) \cap K[X_{\tau(i)} \colon i < m], (7)$ 

then the system

$$\bigwedge_{f \in I_0} f = 0 \land \bigwedge_{i < m} X_i^{\sigma} = X_{\tau(i)}$$
(8)

has a solution in K (the case m = 0 ensures that K is algebraically closed).

*Proof.* We refine an arbitrary system of difference equations and inequations as follows. Over a difference field  $(K, \sigma)$ , suppose a system has a solution  $(a_i: i < k)$  from some larger model. Whenever i < j < k, we may assume that  $a_i \neq a_j$ and that the system has the inequation  $X_i \neq X_j$  as one of its conjuncts. We obtain a refinement having the form (8) as follows.

1. In non-constant terms, repeatly make the replacements

of 
$$(t+u)^{\sigma}$$
,  $(-t)^{\sigma}$ ,  $(t \cdot u)^{\sigma}$   
with  $t^{\sigma} + u^{\sigma}$ ,  $-t^{\sigma}$ ,  $t^{\sigma} \cdot u^{\sigma}$ 

respectively, until  $\sigma$  is applied only to variables and constant terms.

- 2. For every atomic or negated atomic formula  $\varphi$  of the system that is not of the form  $X^{\sigma} = Y$ , but in which  $X^{\sigma}$  appears as an argument, replace that argument with a new variable Y, and introduce the new equation  $X^{\sigma} = Y$ .
- 3. If for some *i* less than *k*, there is not already an equation of the form  $X_i^{\sigma} = Y$ , then introduce such an equation, *Y* being a new variable.
- 4. Replace any polynomial inequation  $f \neq g$  with

$$(f-g)\cdot X=1,$$

where X is a new variable.

After indexing the new variables appropriately, we have that

- 1) for some m and n in  $\boldsymbol{\omega}$  such that  $m \leq n$ ,
- 2) for some function  $\tau$  from m into n,
- 3) for some finite subset  $I_0$  of  $K[X_j: j < n]$ ,

our system has the form of (8). (It may be that some of the hidden parameters are part of compound terms that involve  $\sigma$ ; but such terms can just be understood as standing for the appropriate elements of K.) If  $\tau$  is not injective, then the system must have equations  $X_i^{\sigma} = X_{\ell}$  and  $X_i^{\sigma} = X_{\ell}$ , where  $k \leq i < j < m$ ; but these equations imply  $X_i = X_j$ , and so we can replace  $X_i$  throughout with  $X_i$ . Thus we may assume  $\tau$  is injective. In case the ideal generated by  $I_0$  is not prime, still, for some  $(a_i: k \leq i < n)$  in the larger difference field, the new system has the solution  $(a_i: i < n)$ , and we can then add enough equations  $f(X_i; j < n) = 0$  that are satisfied by  $(a_i: i < n)$  so that  $I_0$  becomes a set of generators of a prime ideal  $\mathfrak{P}$ , and  $(a_i: i < n)$  is a generic point over K of the zero-set of  $\mathfrak{P}$ . In this case (7) is satisfied. For every solution  $(b_i: i < n)$ of the latest system,  $(b_i: i < k)$  solves the original system. Thus if  $(K, \sigma)$  meets the given conditions, it is existentially closed as a difference field.

Conversely, under the given conditions, every system of the form (8) is indeed consistent with  $(K, \sigma)$ , by Fact 7: if  $\boldsymbol{a}$  is a generic zero of  $I_0$ , we can extend  $\sigma$  to an isomorphism from  $K(a_i: i < m)$  to  $K(a_{\tau(i)}: i < m)$ , and then to an automorphism of a field including  $K(\boldsymbol{a})$ . In this way,  $\boldsymbol{a}$  solves (8), so this system must have a solution in K, if  $(K, \sigma)$  is existentially closed as a difference field.

If we did not already know that ACFA existed, the foregoing theorem would prove it by Porism 6, since the conditions that (8) must satisfy are first-order. This is so, because of the existence of appropriate bounds on degrees of polynomials, as established in [22]. In particular, for all n and r in  $\omega$ , there are bounds s and t in  $\omega$  such that, for all fields K, for all min  $\omega$ , for every ideal I of  $K[X_j: j < n]$  generated by a set  $\{f_i: i < m\}$ , each  $f_i$  having degree r or less,

1) the primeness of the ideal can be established by showing

$$gh \in I \& g \notin I \implies h \in I$$

for all polynomials g and h in  $K[X_j: j < n]$  having degree s or less, and

2) membership in I by polynomials like gh having degree 2s or less is established by polynomials of degree t or less, in the sense that, if indeed  $gh \in I$ , then  $gh = \sum_{i < m} g_i \cdot f_i$  for some  $g_i$  having degree t or less.

Because ACF admits full elimination of quantifiers, ACFA is the model-*completion* of the theory of difference fields that are algebraically closed as fields (compare to the last sentence in Porism 5).

# 5 A model-completion

We now consider the class of models  $(K, \sigma, \mathfrak{O})$  of FAV such that

$$\exists x \; x \notin \mathfrak{O}$$

and,

- 1) for all m and n in  $\omega$  such that  $m \leq n$ ,
- 2) for every injective function  $\tau$  from m into n,
- 3) for every finite subset  $I_0$  of  $\mathfrak{O}[X_j: j < n]$ ,
- 4) for all subsets  $\lambda$  of n and  $\kappa$  of  $\lambda$ ,

if

- a)  $I_0$  generates a prime ideal  $(I_0)$  of  $K[X_j: j < n]$  such that the condition (7) in Theorem 8 holds, and
- b) when S is the ring  $\mathfrak{O}[I_0 \cup \{X_\ell : \ell \in \lambda\}]$ , the ideal of S generated by the set  $\mathfrak{M} \cup I_0 \cup \{X_k : k \in \kappa\}$  is proper, that is,

$$\left(\mathfrak{M} \cup I_0 \cup \{X_k \colon k \in \kappa\}\right) S \subsetneq S,\tag{9}$$

then K contains a common solution to the system (8) in Theorem 8 and the system

$$\bigwedge_{\ell \in \lambda} X_{\ell} \in \mathfrak{O} \land \bigwedge_{k \in \kappa} X_k \in \mathfrak{M}.$$
 (10)

The case  $m = 0 = \lambda$  ensures that K is algebraically closed.

As the existentially closed difference-fields, characterized by Theorem 8, are just the models of a certain theory ACFA, so the models of FAV just described are the models of a certain theory, which we shall call FAV<sup>\*</sup>. In particular, the condition (9) is first-order. Indeed, this condition means there are no  $g_f$ and  $h_k$  in S such that

$$\sum_{f \in I_0} g_f \cdot f + \sum_{k \in \kappa} h_k \cdot X_k \equiv 1 \pmod{\mathfrak{M}}.$$
 (11)

The ring S is, for some subset  $I_1$  of  $I_0$ , isomorphic to the quotient of the polynomial ring  $\mathfrak{O}[\{Y_f: f \in I_1\} \cup \{X_\ell: \ell \in \lambda\}]$  by an element of bounded degree. We can also work over the residue field  $\mathfrak{O}/\mathfrak{M}$ , instead of  $\mathfrak{O}$ . Thus, by [22], for we can bound the degrees of the  $g_f$  and  $h_k$  that would make (11) true.

**Lemma 9.** Every model of FAV embeds in a model of  $FAV^*$ .

*Proof.* Let  $(K, \sigma, \mathfrak{O})$  be a model of FAV such that,

- 1) for some m and n in  $\omega$ , where  $m \leq n$ ,
- 2) for some injective function  $\tau$  from m into n,
- 3) for some finite subset  $I_0$  of  $\mathfrak{O}[X_j: j < n]$ ,
- 4) for some sets  $\lambda$  and  $\kappa$ , where  $\kappa \subseteq \lambda \subseteq n$ , we have that
  - a)  $I_0$  generates a prime ideal  $(I_0)$  of  $K[X_j: j < n]$  such that the condition (7) in Theorem 8 holds, and

b) when S is the ring  $\mathfrak{O}[I_0 \cup \{X_\ell : \ell \in \lambda\}]$ , then (9) holds. We already know, as in the proof of Theorem 8, that the system (8) has a solution  $\boldsymbol{a}$  in a difference field  $(L, \tilde{\sigma})$  of which  $(K, \sigma)$  is a substructure; and we may require  $\boldsymbol{a}$  to be a generic solution of the field-theoretic part

$$\bigwedge_{f \in I_0} f = 0$$

of the system. We now show that L has a valuation ring  $\tilde{\mathfrak{O}}$  such that

$$K \cap \mathfrak{O} = \mathfrak{O}$$

and  $\boldsymbol{a}$  solves (10), that is,

$$\bigwedge_{\ell \in \lambda} a_{\ell} \in \widetilde{\mathfrak{O}} \land \bigwedge_{k \in \kappa} a_k \in \widetilde{\mathfrak{M}}.$$
(12)

We can do this by refining the proof of Chevalley's theorem on extending valuations (for which see [9, Thm 3.1.1, p. 57]), or simply by using a refinement [23, Thm 5, p. 12] of the theorem itself. By this refinement, the sub-ring  $S = \mathfrak{O}[I_0 \cup \{X_\ell : \ell \in \lambda\}]$ of  $K(X_j : j < n)$  has a prime ideal  $\mathfrak{P}$  that includes the proper ideal generated by  $\mathfrak{M} \cup I_0 \cup \{X_k : k \in \kappa\}$ ; therefore some valuation ring  $\mathfrak{O}^*$  of  $K(X_j : j < n)$  with maximal ideal  $\mathfrak{M}^*$ satisfies

$$S \subseteq \mathfrak{O}^*, \qquad \mathfrak{P} = \mathfrak{M}^* \cap S.$$

In particular,

$$\{X_{\ell} \colon \ell \in \lambda\} \subseteq \mathfrak{O}^*, \ I_0 \cup \{X_k \colon k \in \kappa\} \subseteq \mathfrak{M}^*, \ \mathfrak{M}^* \cap \mathfrak{O} = \mathfrak{M}.$$

Now we can understand  $\mathfrak{O}^*/(I_0)\mathfrak{O}^*$  as a valuation ring of  $K(\boldsymbol{a})$ . By Chevalley's Theorem, we can extend this valuation ring to a valuation ring  $\widetilde{\mathfrak{O}}$  of L. In this case (12) holds.  $\Box$ 

**Lemma 10.**  $FAV^*$  is model complete.

*Proof.* We proceed as in the proof of Theorem 8. Over a model of FAV, supposing a system of atomic and negated atomic formulas has a solution  $(a_i: i < k)$  from some larger model, we transform the system into the conjunction of a system of the form (8) and a system of the form (10). We proceed as before, but now, since formulas  $f \in \mathfrak{D}$  and  $f \in \mathfrak{M}$  and their negations may appear, we can eliminate negations by applying (1) and (2), and we can replace  $f \in \mathfrak{O}$  and  $f \in \mathfrak{M}$  themselves with  $X \in \mathfrak{O} \land f = X$  and  $X \in \mathfrak{M} \land f = X$  respectively, where X is a new variable.

**Theorem 11.** FAV<sup>\*</sup> is the model companion of FAV and the model completion of the theory of models of FAV whose fields are algebraically closed.

*Proof.* The first part is the content of Lemmas 9 and 10. When the underlying field is required to be algebraically closed, then, by quantifier-elimination in the theory of such fields, the conditions that the systems (8) and (10) are to meet are given by a quantifier-free formula.

**Theorem 12.** The theory of models of ACFA that also have valuations is not the model companion of FAV.

*Proof.* We show that there is a model  $(K, \sigma, \mathfrak{O})$  of FAV that is not a model of FAV<sup>\*</sup>, although the reduct  $(K, \sigma)$  is a model of ACFA.

It is known [11] that every nonprincipal ultraproduct of the algebraic closures of the fields of prime order, each equipped with its Frobenius automorphism, is a model of ACFA. Now let

$$(K, \sigma, \mathfrak{O}) = \prod_{p} \left( \mathbb{F}_{p}(T)^{\mathrm{alg}}, x \mapsto x^{p}, \mathfrak{O}_{T} \right) / \mathscr{U}$$

for some nonprincipal ultrafilter  $\mathscr{U}$  on the set of primes and some valuation ring  $\mathfrak{O}_T$  of each  $\mathbb{F}_p(T)$ . For example,  $\mathfrak{O}_T$  might be the *T*-adic valuation ring, consisting of those elements of  $\mathbb{F}_p(T)$  that, considered as functions of *T*, are well-defined at 0. The structure is as desired since by (3) it satisfies

$$\forall x \ (\operatorname{val}(x) > 0 \Rightarrow \operatorname{val}(x^{\sigma}) > 0),$$

that is,  $\forall x \ (x \in \mathfrak{M} \Rightarrow x^{\sigma} \in \mathfrak{M})$ , while in every model of FAV<sup>\*</sup> the system

$$x \in \mathfrak{M} \wedge x^{\sigma} \notin \mathfrak{M}$$

is soluble.

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