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Factorizing the Top–Loc adjunction through positive topologies

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Abstract

We characterize the category of Sambin's positive topologies as the result of the Grothendieck construction applied to a doctrine over the category **Loc** of locales. We then construct an adjunction between the category of positive topologies and that of topological spaces **Top**, and show that the well-known adjunction between **Top** and **Loc** factors through the constructed adjunction.

Keywords Grothendieck constructions · Suplattices · Locales · Formal topologies

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1 Introduction

Positive topologies are introduced by Sambin [22] (see also [7]) as a natural structure for developing constructive pointfree topology. The category **PTop** of positive topologies can be regarded as a natural extension of the category **Loc** of locales; actually **Loc** is a reflective subcategory of **PTop** (see e.g. [7]). In a predicative setting, the role of a locale is played by a formal cover $(S, \lhd)$, sometimes called a formal topology,

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which can be read as a presentation of a frame by generators and relations, see e.g. [5]. A positive topology is then a formal cover endowed with a positivity relation, that is a relation $\ltimes$ between S and $\mathcal{P}(S)$ such that for every $a \in S$ and $U, V \subseteq S$

1. $a \ltimes U \implies a \in U$;
2. $a \ltimes U \wedge (\forall b \in S)(b \ltimes U \to b \in V) \implies a \ltimes V$;
3. $a \lhd U \wedge a \ltimes V \implies (\exists b \in U)(b \ltimes V)$.

The motivating example of a positive topology is built from a topological space in such a way as to keep the information about its closed subsets (classically, all such information is already encoded by the opens); see Sect. 5.2.

In [8] the first author and Vickers characterize positive topologies as locales endowed with a suitable family of suplattice homomorphisms. Here we show that this characterization can be organized into a fibration arising from a doctrine[1] over **Loc** via the so-called Grothendieck construction (see, e.g. [11]).

We will then use this representation of **PTop** to give an adjunction between the category **Top** of topological spaces and **PTop**; in particular, the notion of sobriety provided by this adjunction coincides with the one introduced in [22], which is known [1] to be constructively weaker than the notion of sobriety provided by the usual **Top**–**Loc** adjunction [13]. Moreover, the **Top**–**Loc** adjunction can be factorized as the composition of the **Top**–**PTop** adjunction above and the reflection **PTop**–**Loc**.

As a by-product, we get the completeness and cocompleteness of the category **PTop** and of the wider category **BTop** of basic topologies, which can be similarly characterized as a Grothendieck construction over the category of suplattices. This completes the picture in [10], where the pointwise counterparts of **BTop** and **PTop** were shown to be complete and cocomplete.

Our foundational framework is intuitionistic and impredicative, like that provided by the internal language of a topos. We use the term "constructive" in this sense.

2 Basic topologies and positive topologies

A **suplattice** (or *complete join semilattice*) is a poset $(L, \leq)$ with all joins, that is, $\bigvee X$ exists for all subsets $X \subseteq L$.[2] A map $f : L \to M$ between two suplattices *preserves joins* if

$$f\Big(\bigvee_{i\in I} x_i\Big) = \bigvee_{i\in I} f(x_i)$$

[1] In this paper we will always use the term "doctrine" to mean an indexed preorder, that is a contravariant functor towards the category **PreOrd** of preorders and monotone maps. The "logical" intuition behind a doctrine $\mathbf{Q} : \mathbb{C}^{op} \to \mathbf{PreOrd}$ is that an object A of $\mathbb{C}$ can be seen as a type, an element $\varphi \in \mathbf{Q}(A)$ can be seen as a proposition in context $\varphi(x)\,[x : A]$, an arrow $f : B \to A$ of $\mathbb{C}$ represents a term in context $f(y) : A\,[y : B]$, and the map $\mathbf{Q}(f)$ represents the substitution operation $\varphi(x) \mapsto \varphi(f(y))$.
Such a categorical approach to logic, which dates back to Lawvere [16], is still a topic of interest as witnessed by many recent works such as [17,18] and [9].

[2] In particular, L has least element 0, namely the empty join. Moreover, L has also all meets and, in particular, the top element 1, namely the empty meet.

for every family $(x_i)_{i\in I}$ in L. Suplattices and join-preserving maps form a category **SL**. We hence refer to join-preserving maps between suplattices as suplattice homomorphisms.

If X is a set and L is (the carrier of) a suplattice, then the collection of maps $\mathbf{Set}(X, L)$ has a natural suplattice structure where joins are computed pointwise, that is,

$$\Big(\bigvee_{i\in I}\varphi_i\Big)(x) := \bigvee_{i\in I}\big(\varphi_i(x)\big).$$

If X has a suplattice structure, then $\mathbf{SL}(X, L)$ is a sub-suplattice of $\mathbf{Set}(X, L)$.

A *base* for a suplattice L is a subset $S \subseteq L$ such that $p = \bigvee\{a \in S \mid a \leq p\}$ for all $p \in L$. For instance, the powerset $\mathcal{P}(S)$ of a set S is a suplattice (with respect to union) and a base for $\mathcal{P}(S)$ is given by all singletons.[3] Given a base S, let $\triangleleft \subseteq S \times \mathcal{P}(S)$ be the relation defined as $a \triangleleft U$ iff $a \leq \bigvee U$. It is easy to check that $\triangleleft$ satisfies the following properties:

1. $a \in U \implies a \triangleleft U$;
2. $a \triangleleft U \;\wedge\; (\forall u \in U)(u \triangleleft V) \implies a \triangleleft V$;

for every $a \in S$ and $U, V \subseteq S$. A pair $(S, \triangleleft)$ satisfying 1 and 2 above is called a **basic cover**. A basic cover has to be understood as a presentation of a suplattice by generators and relations. Indeed, any basic cover induces an equivalence relation $=_\triangleleft$ on $\mathcal{P}(S)$ where $U =_\triangleleft V$ is

$$(\forall u \in U)(u \triangleleft V) \wedge (\forall v \in V)(v \triangleleft U).$$

The quotient $\mathcal{P}(S)/{=_\triangleleft}$ is a suplattice (with a base indexed by S) where joins $\bigvee_i [U_i]$ can be computed as $[\bigcup_i U_i]$. To complete the picture, one should note that the basic cover induced by a suplattice L (with any base S) presents a suplattice which is isomorphic to L itself.

Two basic covers $\mathcal{S}_1 = (S_1, \triangleleft_1)$ and $\mathcal{S}_2 = (S_2, \triangleleft_2)$ are isomorphic if they induce isomorphic suplattices. More generally we say that a morphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ is a suplattice homomorphism from $\mathcal{P}(S_2)/=_{\triangleleft_2}$ to $\mathcal{P}(S_1)/=_{\triangleleft_1}$.[4] This corresponds to having a relation $s \subseteq S_1 \times S_2$ which **respects the covers** in the following sense:

$$\text{if } a\, s\, b \text{ and } b \triangleleft_2 V, \text{ then } a \triangleleft_1 s^- V$$

where $s^- V := \{x \in S_1 \mid (\exists v \in V)(x\, s\, v)\}$. Actually, the same homomorphism corresponds to several relations which we want to consider equivalent; explicitly, two relations s and s' are equivalent if $s^- V =_{\triangleleft_1} s'^- V$ for all $V \subseteq S_2$.

Basic covers and their morphisms form a category which is dual to the category **SL** of suplattices, that is, a category equivalent to $\mathbf{SL}^{op}$. We refer the reader to [2] for further details.

[3] Incidentally, note that $\mathcal{P}(S)$ is the free suplattice over the set S.

[4] Contravariance is chosen to match the direction of locales.

2.1 Basic topologies

A **basic topology** [22] is a triple $(S, \lhd, \ltimes)$ where $(S, \lhd)$ is a basic cover and $\ltimes$ is a relation between S and $\mathcal{P}(S)$ such that

1. $a \ltimes U \Longrightarrow a \in U$;
2. $a \ltimes U \wedge (\forall b \in S)(b \ltimes U \to b \in V) \Longrightarrow a \ltimes V$;
3. $a \lhd U \wedge a \ltimes V \Longrightarrow (\exists b \in U)(b \ltimes V)$.

The relation $\ltimes$ is called a **positivity relation** on $(S, \lhd)$. Thus, a basic topology can be regarded as a suplattice together with the extra structure specified by a positivity relation.

The powerset $\Omega := \mathcal{P}(1)$ of a singleton can be identified with the algebra of propositions up to logical equivalence.[5] Condition 3. in the definition above says that the map[6]

$$\begin{array}{rcl} \varphi_Z : \mathcal{P}(S)/=_{\lhd} & \longrightarrow & \Omega \\ {[U]} & \longmapsto & U \between Z \end{array}$$

is well-defined if Z is of the form $\{a \in S \mid a \ltimes V\}$, in which case φ_Z is a suplattice homomorphism. Given any positivity relation $\ltimes$ on $(S, \lhd)$, the collection of all such φ_Z forms a sub-suplattice of $\mathbf{SL}(\mathcal{P}(S)/=_{\lhd}, \Omega)$. The first author and Vickers [8, Theorem 2.3] have shown that there is a bijective correspondence between positivity relations on $(S, \lhd)$ and sub-suplattices of $\mathbf{SL}(\mathcal{P}(S)/=_{\lhd}, \Omega)$. Thus, a basic topology can be identified with a pair (L, Φ) where L is a suplattice and Φ is a sub-suplattice of the collection $\mathbf{SL}(L, \Omega)$ of suplattice homomorphisms from L to Ω.[7]

Let $\mathcal{S}_1 = (S_1, \lhd_1, \ltimes_1)$ and $\mathcal{S}_2 = (S_2, \lhd_2, \ltimes_2)$ be basic topologies, and (L_1, Φ_1) and (L_2, Φ_2) be the corresponding suplattices together with sub-suplattices of suplattice homomorphisms to Ω. According to [22], a morphism between basic topologies $\mathcal{S}_1$ and $\mathcal{S}_2$ is a morphism s between $(S_1, \lhd_1)$ and $(S_2, \lhd_2)$ satisfying the following additional condition

$$\text{if } a\, s\, b \text{ and } a \ltimes_1 U, \text{ then } b \ltimes_2 s\, U$$

for all $a \in S_1$, $b \in S_2$ and $U \subseteq S_1$, where $s\, U := \{y \in S_2 \mid (\exists u \in U)(u\, s\, y)\}$. This corresponds to having a suplattice homomorphism $f : L_2 \to L_1$ such that $\Phi_1 \circ f \subseteq \Phi_2$, where $\Phi_1 \circ f := \{\varphi \circ f \mid \varphi \in \Phi_1\}$; in other words

$$\text{if } L_1 \xrightarrow{\varphi} \Omega \text{ belongs to } \Phi_1, \text{ then } L_2 \xrightarrow{f} L_1 \xrightarrow{\varphi} \Omega \text{ belongs to } \Phi_2$$

[5] Two relevant facts about Ω will be essential later: (i) for every $a, b \in \Omega$, $a \leq b$ if and only if $a = 1$ implies $b = 1$; (ii) for every set-indexed family $(a_i)_{i \in I}$ of elements of Ω, $\bigvee_{i \in I} a_i = 1$ if and only if there exists $i \in I$ such that $a_i = 1$.

[6] For $U, V \subseteq S$, we use Sambin's "overlap" symbol $U \between V$ to mean that $U \cap V$ is inhabited. Clearly $U \between V$ implies $U \cap V \neq \emptyset$. The converse is equivalent to the law of excluded middle, as it is clear by considering the case of Ω. In that case, $p \between q$ means $p = q = 1$, and so $p \between p$ is just $p = 1$. On the contrary, $p \cap p \neq 0$ is $p \neq 0$, that is $(\neg\neg p) = 1$.

[7] When L is a frame, suplattice homomorphisms from L to Ω are known to correspond to the overt weakly closed sublocales of L [4] (classically, these are just the closed sublocales).

(see [8, Proposition 2.9]).

Let **BTop** be the category whose objects are pairs (L, Φ) of a suplattice L and a sub-suplattice Φ of $\mathbf{SL}(L, \Omega)$, and whose arrows $f : (L_1, \Phi_1) \to (L_2, \Phi_2)$ are suplattice homomorphisms $f : L_2 \to L_1$ such that $\Phi_1 \circ f \subseteq \Phi_2$. Apart from the impredicativity involved, **BTop** is equivalent to the category of basic topologies in [22].

2.2 Positive topologies

A **positive topology** [22] is a basic topology $(S, \lhd, \ltimes)$ such that the underlying basic cover $(S, \lhd)$ is a formal cover [5] (sometimes called a formal topology). This means that the suplattice presented by $(S, \lhd)$ is a **frame**, that is, binary meets distribute over arbitrary joins.

By an observation similar to the one we made for a basic topology in Sect. 2.1, a positive topology can be identified with a pair (L, Φ) where L is a frame and Φ is a sub-suplattice of $\mathbf{SL}(L, \Omega)$. A morphism between such pairs (L, Φ) and (M, Ψ) is a frame homomorphism $f : M \to L$ such that $\Phi \circ f \subseteq \Psi$, which corresponds to a formal map between positive topologies as described in [22].

Let **PTop** be the subcategory of **BTop** consisting of those objects whose underlying suplattice is a frame and arrows which are frame homomorphisms between the underlying frames. The category **PTop** is thus equivalent to that of positive topologies in [22].

3 A categorical characterization of BTop and PTop

In this section, we are going to give a categorical characterization of **BTop** and **PTop** in terms of Grothendieck constructions over two doctrines on the opposite of the category of suplattices and on the category of locales, respectively.

3.1 A doctrine on $\mathbf{SL}^{op}$

For L a suplattice, the (contravariant) hom-functor $\mathbf{SL}(\_, L) : \mathbf{SL}^{op} \to \mathbf{Set}$ can be also regarded as a functor

$$\mathbf{SL}(\_, L) : \mathbf{SL} \to \mathbf{SL}^{op}$$

where, for $f \in \mathbf{SL}(X, Y)$ and $\varphi \in \mathbf{SL}(Y, L)$, we have $\mathbf{SL}(f, L)(\varphi) = \varphi \circ f$.

Another well-known contravariant functor is the subobject functor

$$\mathbf{Sub} : \mathbf{SL}^{op} \to \mathbf{PreOrd}$$

which sends each suplattice L to the preorder (actually a poset) $\mathbf{Sub}(L)$ of subobjects of L in **SL**. Recall that a suboject of L can be represented as a subset $I \subseteq L$ closed under joins in L, that is a sub-suplattice of L. Given $f : M \to L$ in **SL** and $I \in \mathbf{Sub}(L)$,

$\mathbf{Sub}(f)$ sends I to the pullback $\{x \in M \mid f(x) \in I\}$ of I along f.

$$\begin{tikzcd} \mathbf{Sub}(f)(I) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, tail] \\ M \arrow[r, "f"'] & L \end{tikzcd}$$

The composition $\mathbf{Sub} \circ \mathbf{SL}(\_, \Omega)$ is a functor

$$\mathbf{P} : \mathbf{SL} \to \mathbf{PreOrd}$$

which, of course, can also be read as a contravariant functor on $\mathbf{SL}^{op}$

$$\mathbf{P} : (\mathbf{SL}^{op})^{op} \to \mathbf{PreOrd},$$

that is, a doctrine on $\mathbf{SL}^{op}$.

As the result of the so-called Grothendieck construction [11, Definition 1.10.1],[8] we get a category $\int \mathbf{P}$ whose objects are pairs (L, Φ) with L a suplattice and Φ a subobject of $\mathbf{SL}(L, \Omega)$ in $\mathbf{SL}$. An arrow $(L, \Phi) \to (M, \Psi)$ in $\int \mathbf{P}$ is a suplattice homomorphism $f : M \to L$ such that

$$\Phi \subseteq \mathbf{P}(f)(\Psi).$$

Since $\mathbf{P}(f)(\Psi) = \{\varphi \in \mathbf{SL}(L, \Omega) \mid \varphi \circ f \in \Psi\}$ by definition, such a condition is equivalent to the following

$$\Phi \circ f \subseteq \Psi$$

Therefore, $\int \mathbf{P}$ is exactly the category **BTop** of basic topologies which we introduced in Sect. 2.1 above.

This construction yields a forgetful functor $\mathbf{U} : \int \mathbf{P} \to \mathbf{SL}^{op}$, which is in fact a fibration (see [11]). This functor has a right adjoint, the *constant object functor*

$$\mathbf{\Delta} : \mathbf{SL}^{op} \to \int \mathbf{P},$$

[8] The Grothendieck construction (which is formulated in [11] for the general case of an indexed category) applied to a doctrine $\mathbf{Q} : \mathbb{C}^{op} \to \mathbf{PreOrd}$ provides a category $\int \mathbf{Q}$ where objects are pairs (A, φ) with A an object of $\mathbb{C}$ and $\varphi \in \mathbf{Q}(A)$, and arrows from (A, φ) to (B, ψ) are arrows $f : A \to B$ of $\mathbb{C}$ such that $\varphi \leq \mathbf{Q}(f)(\psi)$ in $\mathbf{Q}(A)$ (with composition of arrows inherited from $\mathbb{C}$).
Recalling what we said in footnote 1, one can understand an object (A, φ) of $\int \mathbf{Q}$ as an object of $\mathbb{C}$ together with a distinguished "subset" $\{x \in A \mid \varphi\}$ obtained by separation by means of the proposition φ, and an arrow $f : (A, \varphi) \to (B, \psi)$ as an operation from A to B such that the image of $\{x \in A \mid \varphi\}$ is included in $\{x \in B \mid \psi\}$.

which sends each suplattice L to the object $(L, \mathbf{SL}(L, \Omega))$ and each $f : L \to M$ in $\mathbf{SL}^{op}$ to itself as an arrow from $\mathbf{\Delta}(L)$ to $\mathbf{\Delta}(M)$ in $\int \mathbf{P}$. So $\mathbf{\Delta}$ is full.

$$\begin{tikzcd} \mathbf{BTop} = \int \mathbf{P} \arrow[r, bend left, "\mathbf{U}"] \arrow[r, phantom, "\perp"] & \mathbf{SL}^{op} \arrow[l, bend left, "\mathbf{\Delta}"] \end{tikzcd}$$

Moreover $\mathbf{U} \circ \mathbf{\Delta}$ is just the identity functor on $\mathbf{SL}^{op}$. Thus, $\mathbf{\Delta}$ is full, faithful and injective on objects, and so $\mathbf{SL}^{op}$ can be regarded as a reflective subcategory of $\int \mathbf{P}$. In this way, we recover the result in [6].

Note that the monad T induced by the adjunction $\mathbf{U} \dashv \mathbf{\Delta}$ is an idempotent monad. By the results in Sect. 4.2 of [3], we have that $\mathbf{SL}^{op}$ is equivalent both to the category of free algebras (the Kleisli category) and to the category of algebras (the Eilenberg–Moore category) on T. Hence the adjunction $\mathbf{U} \dashv \mathbf{\Delta}$ is monadic.

Remark Since in a suplattice arbitrary meets always exist, if $(L, \leq)$ is a suplattice, then $(L, \leq)^{op} := (L, \geq)$ is a suplattice as well. Moreover, every suplattice homomorphism $f : X \to Y$ has a right adjoint (as a monotone function) $f^{op} : Y \to X$ which preserves all meets. This determines a contravariant functor $(\_)^{op}$, which is in fact an isomorphism between $\mathbf{SL}$ and $\mathbf{SL}^{op}$. In particular, $\mathbf{SL}(X, Y) \cong \mathbf{SL}(Y^{op}, X^{op})$ for all X and Y.

Classically, $\mathbf{SL}(\_, \Omega)$ is naturally isomorphic to the functor $(\_)^{op}$ because $\Omega^{op} \cong \Omega$ so that $\mathbf{SL}(L, \Omega) \cong \mathbf{SL}(\Omega, L^{op}) \cong L^{op}$.[9] Therefore, for every L, $\mathbf{P}(L) = \mathbf{Sub}(\mathbf{SL}(L, \Omega)) \cong \mathbf{Sub}(L^{op})$ which is isomorphic to the lattice of suplattice quotients of L. In other words, an object (L, Φ) corresponds to an epimorphism $e : L \to \Phi^{op}$, and an arrow $(L, \Phi) \to (M, \Psi)$ is a suplattice homomorphism $f : M \to L$ such that $e \circ f : M \to \Phi^{op}$ preserves the congruence relation on M corresponding to Ψ.

3.2 The case of frames (and locales)

The category $\mathbf{Frm}$ of frames is the subcategory of $\mathbf{SL}$ whose objects are frames and whose arrows preserve finite meets (in addition to arbitrary joins). The category $\mathbf{Loc}$ of locales is defined as $\mathbf{Frm}^{op}$. By restricting the functor $\mathbf{P}$ to $\mathbf{Frm}$, we get a doctrine

$$\widetilde{\mathbf{P}} : \mathbf{Loc}^{op} = \mathbf{Frm} \longrightarrow \mathbf{PreOrd}$$

on $\mathbf{Loc}$, which gives rise to a fibration $\mathbf{U} : \int \widetilde{\mathbf{P}} \to \mathbf{Loc}$ fitting in a pullback square of categories as follows.

[9] This cannot hold constructively, for if φ were an order-isomorphism between $(\Omega, \leq)$ and $(\Omega, \geq)$, then we could prove $\neg\neg p \leq p$ for every $p \in \Omega$ as follows. If $\varphi(p) = 1 = \varphi(0)$, then $p = 0$ and so $\varphi(\neg\neg p) = \varphi(0) = 1$. This shows that $\varphi(p) = 1$ implies $\varphi(\neg\neg p) = 1$, that is, $\varphi(p) \leq \varphi(\neg\neg p)$. Since φ is an isomorphism, this would entail $\neg\neg p \leq p$.

$$\begin{tikzcd} \int \widetilde{\mathbf{P}} \arrow[r, tail] \arrow[d, "\mathbf{U}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathbf{P} \arrow[d, "\mathbf{U}"] \\ \mathbf{Loc} \arrow[r, tail] & \mathbf{SL}^{op} \end{tikzcd}$$

Here $\int \widetilde{\mathbf{P}}$ is exactly the category **PTop** as introduced in Sect. 2.2.

As we have shown before in the case of $\mathbf{SL}^{op}$ and $\int \mathbf{P}$, there is an adjunction $\mathbf{U} \dashv \mathbf{\Delta}$ between $\int \widetilde{\mathbf{P}}$ and **Loc** with $\mathbf{\Delta}$ full, faithful and injective on objects. Thus, the category **Loc** can be regarded as a reflective subcategory of **PTop**, as already shown in [7].

4 Weakly sober spaces

4.1 Irreducible closed subsets

The open sets of a topological space (X, τ) form a frame with respect to set-theoretic unions and intersections. A subset $C \subseteq X$ is **closed** if

$$(\forall I \in \tau)(x \in I \;\Rightarrow\; C \between I) \Longrightarrow x \in C$$

for all $x \in X$. The collection $\mathsf{Closed}(X, \tau)$ of closed subsets of (X, τ) is a complete lattice (where meets are given by intersections, and joins are given by closure of unions), but it need not be a co-frame constructively.[10]

As usual, it makes sense to define the closure $\mathsf{cl}D$ of a subset $D \subseteq X$ as the intersection of all closed subsets containing D.

Every closed subset C of X determines a map

$$\begin{aligned} \varphi_C : \tau &\longrightarrow \Omega \\ I &\longmapsto C \between I \end{aligned}$$

which preserves joins, that is, $\varphi_C \in \mathbf{SL}(\tau, \Omega)$. Note that φ_D makes sense also when D is an arbitrary subset; however $\varphi_D = \varphi_{\mathsf{cl}D}$ because $I \between D$ if and only if $I \between \mathsf{cl}D$ for every $I \in \tau$. So the mapping

$$\begin{aligned} \mathsf{Closed}(X, \tau) &\longrightarrow \mathbf{SL}(\tau, \Omega) \\ C &\longmapsto \varphi_C \end{aligned}$$

is injective and preserves joins. Thus $\mathsf{Closed}(X, \tau)$ is a sub-suplattice of $\mathbf{SL}(\tau, \Omega)$.[11]

A closed subset $C \subseteq X$ is **irreducible** if any of the following equivalent conditions holds:

[10] For a Brouwerian counterexample consider the discrete space and recall that the so-called "constant domain axiom" $\forall x(\varphi \vee \psi) \to \varphi \vee \forall x\, \psi$, with x not free in φ, is not provable constructively.

[11] Classically, every $\varphi \in \mathbf{SL}(\tau, \Omega)$ is of the form φ_C: take C to be the closed subset $X \setminus \bigcup\{I \in \tau \mid \varphi(I) = 0\}$. Hence $\mathsf{Closed}(X, \tau) \cong \mathbf{SL}(\tau, \Omega)$. This cannot be the case constructively, as we will see in Sect. 4.2.

1. φ_C preserves finite meets;
2. C is inhabited and for every $I, J \in \tau$, if $I \between C$ and $J \between C$, then $(I \cap J) \between C$;
3. $\{I \in \tau \mid I \between C\}$ is a completely-prime filter of opens.

In other words, a closed subset C is irreducible if and only if φ_C is a frame homomorphism, that is, a *point* in the sense of locale theory. However we cannot show constructively that all frame homomorphisms $\tau \to \Omega$ arise in this way; see Sect. 4.2.

Classically, C is irreducible if and only if it is non-empty and cannot be written as a disjoint union of two non-empty closed subsets [13]; moreover $\{C \subseteq X \mid C \text{ is irreducible closed}\}$ can be identified with $\mathbf{Frm}(\tau, \Omega)$.

4.2 Weak sobriety

Recall that a space is $T0$ or Kolmogorov if $x = y$ follows from the assumption that $\mathsf{cl}\{x\} = \mathsf{cl}\{y\}$. Since $\mathsf{cl}\{x\}$ is always irreducible, we have the following embeddings for a $T0$ space (X, τ):

$$X \hookrightarrow \{C \subseteq X \mid C \text{ is irreducible closed}\} \hookrightarrow \mathbf{Frm}(\tau, \Omega).$$

A $T0$ space is **weakly sober** if every irreducible closed subset is the closure of a singleton, that is, if the embedding $X \hookrightarrow \{C \subseteq X \mid C \text{ is irreducible closed}\}$ is a bijection. It is **sober** if the embedding $X \hookrightarrow \mathbf{Frm}(\tau, \Omega)$ is a bijection. Note that every weakly sober space is sober classically.

Constructively, every $T2$ space is weakly sober [1, Proposition 11.27], provided that the $T2$ separation property for (X, τ) is understood as the following statement: $(\forall I \in \tau)(\forall J \in \tau)(x \in I \wedge y \in J \longrightarrow I \between J) \longrightarrow x = y$, for all $x, y \in X$.

However, if every weakly sober space were sober, then the non-constructive principle LPO (the Limited Principle of Omniscience) would be derivable [1, Proposition 11.25]. Thus, we cannot prove that all $\varphi \in \mathbf{SL}(\tau, \Omega)$ are of the form φ_C for some closed subset C; otherwise $\mathbf{Frm}(\tau, \Omega)$ could be identified with the irreducible closed subsets, which would make sobriety and weak sobriety coincide.

5 Factorizing the Top–Loc adjunction

The usual $\mathbf{\Omega} \dashv \mathbf{Pt}$ adjunction between the category **Top** of topological spaces and the category **Loc** of locales does not compose with the adjunction $\mathbf{U} \dashv \mathbf{\Delta}$ between **Loc** and **PTop** ($= \int \widetilde{\mathbf{P}}$) to give an adjunction between **Top** and **PTop**.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\mathbf{\Omega}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, bend left, "\mathbf{Pt}"] \arrow[r, bend right, "\mathbf{\Delta}"'] \arrow[r, phantom, "\perp"] & \mathbf{PTop} \arrow[l, bend right, "\mathbf{U}"'] \end{tikzcd}$$

Nevertheless, a meaningful adjunction between **Top** and **PTop** can be given, as explained in the following, through which the usual **Top**–**Loc** adjunction factors.

5.1 Points of a positive topology

The suplattice Ω is an initial frame, that is, a terminal locale. Hence $\mathbf{\Delta}(\Omega)$ is a terminal object in **PTop**. We define a **point** of a positive topology (L, Φ) as a global point $\mathbf{\Delta}(\Omega) \to (L, \Phi)$ in **PTop**, and we write $\mathbf{Pt}^+(L, \Phi)$ instead of $\mathbf{PTop}(\mathbf{\Delta}(\Omega), (L, \Phi))$. Thus, a point of (L, Φ) is a frame homomorphism $f : L \to \Omega$ such that $\mathbf{SL}(\Omega, \Omega) \circ f \subseteq \Phi$. Since $\mathbf{SL}(\Omega, \Omega)$ contains the identity map, we have $f \in \Phi$. Conversely, if $f \in \Phi$ and $\varphi \in \mathbf{SL}(\Omega, \Omega)$, then we have $\varphi \circ f = \bigvee\{g \in \{f\} | \varphi = \mathsf{id}_\Omega\} \in \Phi$. In other words, the points of (L, Φ) are exactly those points of the locale L that are in Φ. Hence, $\mathbf{Pt}^+(L, \Phi)$ can be regarded as a subspace of the topological space $\mathbf{Pt}(L)$.

The construction $\mathbf{Pt}^+$ can be extended to a functor from **PTop** to **Top** as follows. Given an arrow $(L, \Phi) \to (M, \Psi)$ with underlying frame homomorphism $f : M \to L$, the continuous map $\mathbf{Pt}(f) : \mathbf{Pt}(L) \to \mathbf{Pt}(M)$, which sends a point $p : L \to \Omega$ to the point $p \circ f : M \to \Omega$, can be restricted to a continuous map $\mathbf{Pt}^+(L, \Phi) \to \mathbf{Pt}^+(M, \Psi)$ because $\Phi \circ f \subseteq \Psi$.

5.2 The canonical positive topology associated to a space

As shown in Sect. 4.1, the closed subsets $\mathsf{Closed}(X, \tau)$ of a topological space (X, τ) can be seen as a sub-suplattice of $\mathbf{SL}(\tau, \Omega)$ via the mapping $C \mapsto \varphi_C$. Thus, we can define a functor $\mathbf{\Lambda} : \mathbf{Top} \to \mathbf{PTop}$ whose object part is

$$\mathbf{\Lambda}(X, \tau) \; = \; \big(\tau, \{\varphi_C \mid C \text{ is closed}\}\big).$$

For a continuous map $f : (X, \tau_X) \to (Y, \tau_Y)$, the **PTop**-morphism $\mathbf{\Lambda}(f)$ is just the locale morphism corresponding to the frame homomorphism $f^{-1} : \tau_Y \to \tau_X$. This makes sense because for any closed subset $C \subseteq X$, the suplattice homomorphism $\varphi_C \circ f^{-1} : \tau_Y \to \Omega$ is precisely φ_D, where $D = \mathsf{cl}\, f(C)$.

5.3 The adjunction between $\mathbf{Pt}^+$ and $\mathbf{\Lambda}$

Theorem *The following hold:*

1. $\mathbf{Pt} = \mathbf{Pt}^+ \circ \mathbf{\Delta}$;
2. $\mathbf{\Omega} = \mathbf{U} \circ \mathbf{\Lambda}$;
3. $\mathbf{\Lambda} \dashv \mathbf{Pt}^+$.

As a consequence, the usual adjunction between **Top** *and* **Loc** *factors through an adjunction between* **PTop** *and* **Loc**.

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, bend left=15, "\mathbf{\Lambda}"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=40, "\mathbf{\Omega}"] & \mathbf{PTop} \arrow[l, bend left=15, "\mathbf{Pt}^+"] \arrow[r, bend left=15, "\mathbf{U}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, bend left=15, "\mathbf{\Delta}"] \arrow[ll, bend left=40, "\mathbf{Pt}"]
\end{tikzcd}$$

Proof For every locale L, $\mathbf{Pt}(L) = \mathbf{Pt}(L) \cap \mathbf{SL}(L, \Omega) = \mathbf{Pt}^+(\boldsymbol{\Delta}(L))$, and for every topological space (X, τ), $\mathbf{U}(\boldsymbol{\Lambda}(X, \tau)) = \tau = \boldsymbol{\Omega}(X, \tau)$. Hence 1 and 2 hold.

For 3, if $f : \boldsymbol{\Lambda}(X, \tau) \to (L, \Phi)$ in **PTop**, then one can define a continuous map $\widetilde{f}$ from (X, τ) to $\mathbf{Pt}^+(L, \Phi)$ as follows:

$$\widetilde{f}(x) := \varphi_{\mathsf{cl}\{x\}} \circ f,$$

that is, for every $y \in L$, $\widetilde{f}(x)(y) := \mathsf{cl}\{x\} \between f(y) \in \Omega$.

Conversely, if g is a continuous map from (X, τ) to $\mathbf{Pt}^+(L, \Phi)$, then an arrow $\widehat{g}$ from $\boldsymbol{\Lambda}(X, \tau)$ to (L, Φ) in **PTop** is defined as follows:

$$\widehat{g}(y) := g^{-1}(\{\varphi \in \mathbf{Pt}(L) \cap \Phi \mid \varphi(y) = 1\}) \in \tau$$

for every $y \in L$. This is an arrow in **PTop** because it preserves arbitrary joins and finite meets, and for every closed subset $C \subseteq X$ we have $\varphi_C \circ \widehat{g} = \bigvee\{\varphi \in \mathbf{Pt}^+(L, \Phi) \mid \varphi \in g(C)\} \in \Phi$.

One can show that the maps $\widetilde{(\_)}$ and $\widehat{(\_)}$ define a natural isomorphism between the functors $\mathbf{PTop}(\boldsymbol{\Lambda}(\_), \_)$ and $\mathbf{Top}(\_, \mathbf{Pt}^+(\_))$. $\square$

Since $\mathbf{Pt}^+(\boldsymbol{\Lambda}(X, \tau))$ is the space of irreducible closed subsets of X and $\mathbf{Pt}(\boldsymbol{\Omega}(X, \tau))$ is the space of frame homomorphisms from τ to Ω, a topological space (X, τ) is weakly sober when the unit of the adjunction $\boldsymbol{\Lambda} \dashv \mathbf{Pt}^+$ gives a homeomorphism between (X, τ) and $\mathbf{Pt}^+(\boldsymbol{\Lambda}(X, \tau))$, while it is sober when the unit of the adjunction $\boldsymbol{\Omega} \dashv \mathbf{Pt}$ gives a homeomorphism between (X, τ) and $\mathbf{Pt}(\boldsymbol{\Omega}(X, \tau))$.

Classically, $\mathbf{SL}(\tau, \Omega) = \{\varphi_C \mid C \text{ is closed}\}$ holds (see footnote 11). Hence $\boldsymbol{\Lambda} = \boldsymbol{\Delta} \circ \boldsymbol{\Omega}$, and thus $\mathbf{Pt}^+ \circ \boldsymbol{\Lambda} = \mathbf{Pt}^+ \circ \boldsymbol{\Delta} \circ \boldsymbol{\Omega} = \mathbf{Pt} \circ \boldsymbol{\Omega}$. Therefore, as already noted, sobriety and weak sobriety coincide classically.

Remark A positivity relation on a formal cover is also called a *binary positivity* [21,22], which is often explained as generalization of a (unary) positivity predicate. Impredicatively, formal covers with a unary positivity predicate (often called just formal topologies [20]) correspond to *open* locales [12,14,15], which are also called *overt* locales [23].

Overt locales form a coreflective subcategory **oLoc** of **Loc** [19]. On the other hand, our result above presents **Loc** as a reflective subcategory of **PTop**. Thus, the relation between **oLoc** and **Loc** and that between **PTop** and **Loc** seem to be of different kinds. In particular, the two adjunctions **oLoc**–**Loc** and **PTop**–**Loc** do not compose to give any adjunction between **PTop** and **oLoc**, apart from the fact that **oLoc** embeds into **PTop** (via the embedding $\boldsymbol{\Delta}$). Moreover, classically, every locale is overt so that **oLoc** and **Loc** coincide, but this is clearly not the case for **PTop** .

The relation between **PTop** and **oLoc** has much to be clarified. However, the above observation suggests that the result of this section seems to be independent from what is already known about **oLoc**.

6 Limits and colimits in BTop and PTop

Let $\mathbf{Q} : \mathbb{C}^{op} \to \mathbf{PreOrd}$ be a doctrine which factors through the embedding of the category of inflattices (that is, the category whose objects are posets having all meets and whose arrows are functions preserving them) in **PreOrd**.

Under this assumption, if $\mathbb{C}$ is complete, then the Grothendieck construction $\int \mathbf{Q}$ gives a complete category. Indeed, it is easy to check that if $(A_i, \varphi_i)_{i \in I}$ is a set-indexed family of objects in $\int \mathbf{Q}$, its product is given by the object

$$\prod_{i \in I}(A_i, \varphi_i) \; := \; \left(\prod_{i \in I} A_i, \bigwedge_{i \in I} \mathbf{Q}(\pi_i)(\varphi_i) \right)$$

together with the projections π_i inherited from $\mathbb{C}$; and the equalizer of two parallel arrows $f, g : (A, \varphi) \to (B, \psi)$ in $\int \mathbf{Q}$ is $e : \big(E, \mathbf{Q}(e)(\varphi)\big) \to (A, \varphi)$, where $e : E \to A$ is the equalizer of f and g in $\mathbb{C}$.

On the other hand, if $\mathbb{C}$ is cocomplete, then $\int \mathbf{Q}$ is cocomplete as well. Indeed, if we denote with $\exists_f$ the left adjoint to $\mathbf{Q}(f)$ for every arrow f of $\mathbb{C}$, the coproduct of a family of objects $(A_i, \varphi_i)_{i \in I}$ in $\int \mathbf{Q}$ is given by the object

$$\sum_{i \in I}(A_i, \varphi_i) \; := \; \left(\sum_{i \in I} A_i, \bigvee_{i \in I} \exists_{j_i}(\varphi_i) \right)$$

together with the injections j_i inherited from $\mathbb{C}$; and the coequalizer of two arrows $f, g : (A, \varphi) \to (B, \psi)$ is $q : (B, \psi) \to (Q, \exists_q(\psi))$, where $q : B \to Q$ is the coequalizer of f and g in $\mathbb{C}$.

The doctrines $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ introduced in Sects. 3.1 and 3.2, respectively, satisfy the above requirements. Indeed, every $\mathbf{P}(L)$ and every $\widetilde{\mathbf{P}}(L)$ is an inflattice because an arbitrary intersection of sub-suplattices is a sub-suplattice. Moreover, every $\mathbf{P}(f)$ has a left adjoint, namely $\exists_f(\Phi) := \Phi \circ f$, essentially by the very definition of $\mathbf{P}$; hence every $\mathbf{P}(f)$ preserves meets. Finally, it is well known that both $\mathbf{SL}^{op}$ and **Loc** are complete and cocomplete [13]. Thus, the categories **PTop** and **BTop** are complete and cocomplete.

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