

TOWARDS MARTIN'S MINIMUM

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ABSTRACT. We show that it is consistent with $\mathbf{MA} + \neg\mathbf{CH}$ that the Forcing Axiom fails for all forcing notions in the class of ω^ω -bounding forcing notions with norms of [17].

0. INTRODUCTION

A Forcing Axiom $\mathbf{FA}_\kappa(\mathbb{P})$ for a forcing notion \mathbb{P} is a statement guaranteeing existence of directed subsets of \mathbb{P} that meet any member of a pregiven family of size κ of dense subsets of \mathbb{P} . As an axiom, $\mathbf{FA}_\kappa(\mathbb{P})$ is a powerful combinatorial tool that allows one to get some of the properties of forcing extensions. Even more interesting are axioms demanding $\mathbf{FA}_\kappa(\mathbb{P})$ for all forcing notions in a fixed family \mathcal{K} . Here the most popular are Martin's Axiom \mathbf{MA} postulating $\mathbf{FA}_\kappa(\mathbb{P})$ for each ccc partial order \mathbb{P} and any cardinal $\kappa < \mathfrak{c}$ and (capturing more forcing notions) Proper Forcing Axiom \mathbf{PFA} postulating $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ for all proper forcing notions. The quest for giving the largest possible class of forcing notions \mathbb{P} for which the axiom $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ may (simultaneously) hold was successfully accomplished by Foreman, Magidor and Shelah [8], [9], who introduced Martin's Maximum.

In the present paper we want to start investigations in the opposite directions, looking in some sense for the *minimal* version of the standard Martin's Axiom. That is, we would like to have a model in which $\neg\mathbf{CH} + \mathbf{MA}$ holds but $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ fails for as many (necessarily not-ccc) forcing notions as possible. Our attention concentrates on forcing notions built according to the scheme of *norms on possibilities* of Roslanowski and Shelah [17]. (Unfortunately, some familiarity with that paper has to be assumed. In particular, for all definitions related to norms on possibilities we have to refer the reader to [17].)

These lines of investigations have some history already. Steprans proved that $\neg\mathbf{CH} + \mathbf{MA}$ is consistent with the negation of the forcing axiom for the Silver forcing notion (see [6, §2]). Next, Judah, Miller and Shelah [14], and Velickovic [20], showed that $\neg\mathbf{CH} + \mathbf{MA}$ does not imply the forcing axiom for the Sacks forcing notion. It has been a common believe that the arguments of [14] can be repeated for a number of forcing notions in which conditions are finitely branching trees. For example, Brendle wrote in the proof of [5, Proposition 5.1(c)] (about the method of [14]): "it is easy to see that this argument works for any forcing notion with compact trees which doesn't have splitting going on every level". It seems that [17] has provided the right formalism for specifying which forcing notions can be taken care of in this context. However, one would like to cover all of ω^ω -bounding forcing notions from [17] (so avoid the limitations on splittings which in the language of

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[17] would restrict us to t -omittory trees) and get the failure of \mathbf{FA}_{\aleph_1} for all these forcing notions simultaneously. As remarked by Brendle, the obvious modifications of the method of [14] seem to be not applicable here. Therefore, we rather follow the Steprans way slightly generalizing it to be able to deal with a number of forcing notions in the same model.

Notation Our notation is rather standard and compatible with that of classical textbooks on Set Theory (like Bartoszynski and Judah [2]). However in forcing we keep the older convention that *a stronger condition is the larger one*.

- Notation 0.1.*
1. For two sequences η, ν we write $\nu \triangleleft \eta$ whenever ν is a proper initial segment of η , and $\nu \leq \eta$ when either $\nu \triangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\ell g(\eta)$.
 2. The cardinality of the continuum is denoted by \mathfrak{c} .
 3. For a forcing notion \mathbb{P} , $\Gamma_{\mathbb{P}}$ stands for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} . With this one exception, all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a dot above (e.g. \dot{s}, \dot{f}).
 4. Ordinal numbers are denoted by $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \zeta$ (with possible indexes); cardinals will be called κ, μ . The first infinite ordinal is ω . The letters u, v, η, ν, ρ will stand for finite sequences.

Where are the respective definitions in [17]? As we stated before, we have to assume that the reader is familiar with [17]. (Otherwise we would have to give the list of all needed definitions and it could be longer than the rest of the paper.) However, for reader's convenience we list below exact pointers to the descriptions of the cases of norms on possibilities that are used here.

- Weak creatures and weak creating pairs: [17, §1.1], in particular [17, Definitions 1.1.1, 1.1.3].
- Creatures and creating pairs: [17, §1.2], in particular [17, Definitions 1.2.1, 1.2.2, 1.2.4, 1.2.5] (as in [17], we assume here that all creating pairs are nice and smooth).
- Forcing notions $\mathbb{Q}_f^*(K, \Sigma)$, $\mathbb{Q}_{w\infty}^*$: [17, Definitions 1.1.6, 1.1.7, 1.1.10].
- Tree-creatures and tree-creating pairs: [17, §1.3], in particular [17, Definitions 1.3.1, 1.3.3].
- Forcing notions $\mathbb{Q}_0^{\text{tree}}(K, \Sigma)$, $\mathbb{Q}_1^{\text{tree}}(K, \Sigma)$: [17, Def. 1.3.5].
- Creating pairs which capture singletons: [17, Def. 2.1.10, 1.2.5(3)].
- 2-big (tree) creating pairs: [17, Def. 2.2.1, 2.3.2].
- Halving Property: [17, Def. 2.2.7].
- \mathbf{H} -fast function f : [17, Def. 1.1.12].
- Simple creating pairs, gluing creating pairs: [17, Def. 2.1.7].
- t -omittory tree-creating pairs: [17, Def. 2.3.4].
- Strongly finitary creating pairs / tree-creating pairs: [17, Def. 1.1.3 + 3.3.4].

The forcing notions we want to take care of. Here we specify the family of forcing notions for which we want to get the failure of \mathbf{FA}_{\aleph_1} (with keeping \mathbf{MA}).

Definition 0.2. Let (K, Σ) be a weak creating pair for \mathbf{H} . We say that it is *typical* if for each $t \in K$ such that $(\exists u \in \mathbf{basis}(t))(|\text{pos}(u, t)| = 1)$ we have $\mathbf{nor}[t] \leq 1$.

The reason for the above definition is the following.

Proposition 0.3. *Suppose that (K, Σ) is a 2-big creating pair (tree-creating pair, respectively), $t \in K$ is such that $\mathbf{nor}[t] \geq 4$. Let $u \in \mathbf{basis}(t)$. Then there are*

creatures (tree-creatures, resp.) $s_0, s_1 \in \Sigma(t)$ such that $\mathbf{nor}[s_0], \mathbf{nor}[s_1] \geq \mathbf{nor}[t] - 2$ and $\text{pos}(u, s_0) \cap \text{pos}(u, s_1) = \emptyset$.

Proof. Choose a set $A \subseteq \text{pos}(u, t)$ such that

- there is $s^* \in \Sigma(t)$ with $\text{pos}(u, s^*) \subseteq A$ and $\mathbf{nor}[s^*] \geq \mathbf{nor}[t] - 1$, but
- for each $a \in A$ and $s \in \Sigma(t)$, if $\text{pos}(u, s) \subseteq A \setminus \{a\}$ then $\mathbf{nor}[s] < \mathbf{nor}[t] - 1$.

Clearly it is possible; necessarily $|A| \geq 2$ (remember that (K, Σ) is typical). Fix $a \in A$. Applying bigness to s^* we get $s_0 \in \Sigma(s^*) \subseteq \Sigma(t)$ such that $\mathbf{nor}[s_0] \geq \mathbf{nor}[t] - 2$ and $\text{pos}(u, s_0) \subseteq A \setminus \{a\}$. On the other hand, by the choice of the set A (and bigness) we find $s_1 \in \Sigma(t)$ such that $\mathbf{nor}[s_1] \geq \mathbf{nor}[t] - 1$ and $\text{pos}(u, s_1) \subseteq \text{pos}(u, t) \setminus (A \setminus \{a\})$. \square

Definition 0.4. Let \mathcal{K} be the family of all non-trivial forcing notions of one of the following types:

1. $\mathbb{Q}_1^{\text{tree}}(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary 2-big typical tree-creating pair for a function $\mathbf{H} \in \mathcal{H}(\aleph_1)$;
2. $\mathbb{Q}_1^{\text{tree}}(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical t-omittory tree-creating pair for a function $\mathbf{H} \in \mathcal{H}(\aleph_1)$ (note that in this case $\mathbb{Q}_1^{\text{tree}}(K, \Sigma)$ is a dense subforcing of $\mathbb{Q}_0^{\text{tree}}(K, \Sigma)$, see [17, 2.3.5]);
3. $\mathbb{Q}_f^*(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical creating pair for $\mathbf{H} \in \mathcal{H}(\aleph_1)$, $f : \omega \times \omega \rightarrow \omega$ is an \mathbf{H} -fast function, (K, Σ) is $\bar{2}$ -big, has the Halving Property and is either simple or gluing;
4. $\mathbb{Q}_{w\infty}^*(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical creating pair for $\mathbf{H} \in \mathcal{H}(\aleph_1)$, (K, Σ) captures singletons.

Theorem 0.5 (See [17, §2.3]). *All forcing notions in the class \mathcal{K} are proper and ω^ω -bounding.*

Now we may state the main result of this paper.

Theorem 0.6. *Assume $\kappa = \kappa^{<\kappa}$. Then there is a ccc forcing notion \mathbb{P} of size κ such that in $\mathbf{V}^{\mathbb{P}}$:*

- $\mathfrak{c} = \kappa$ and **MA** holds true, but
- **FA** $_{\aleph_1}(\mathbb{Q})$ fails for every forcing notion $\mathbb{Q} \in \mathcal{K}$.

The class \mathcal{K} includes all ω^ω -bounding forcing notions presented in [17] (modulo the demand that they are additionally required to be in $\mathcal{H}(\aleph_1)$). In particular, the following forcing notions are in \mathcal{K} :

1. Sacks-like forcings. Let $\{I_n : n \in \omega\}$ be a sequence of finite sets. Every perfect set $P \subseteq \prod_n I_n$ corresponds to a tree T . Consider the forcing notion \mathbf{P} which consists of trees T such that

$$\forall s \in T \exists t \supseteq s \text{ succ}(t) = I_{|t|},$$

ordered by inclusion. If $I_n = 2$ for all n we get the Sacks forcing \mathbf{S} . If $I_n = k$ ($n \in \omega$) then we get forcing notions \mathbb{D}_k (for $k < \omega$) of Newelski and Roslanowski [16]. Finally, if $I_n = h(n)$ then we get the Shelah forcing \mathbf{Q}_h of [12].

2. Silver-like forcings. For $h \in \omega^\omega$ let $\mathbf{P}_h = \{f : \omega \setminus \text{dom}(f) \in [\omega]^\omega, \forall n (n \in \text{dom}(f) \rightarrow f(n) \leq h(n))\}$. For $f, g \in \mathbf{P}_h$, $f \geq g$ if $g \subseteq f$. If $h(n) = 2$ ($n \in \omega$) then \mathbf{P}_h is Silver forcing and for $h(n) = n$ we get Miller forcing from [15].

3. forcing notion $\mathbb{Q}_{f,g}$ from [3] its siblings from [7] and [13]. Suppose that $f \in \omega^\omega$ and $g \in \omega^{\omega \times \omega}$ are two functions such that
- (a) $f(n) > \prod_{j < n} f(j)$ for $n \in \omega$,
 - (b) $g(n, j+1) > f(n) \cdot g(n, j)$ for $n, j \in \omega$, and
 - (c) $\min \{j \in \omega : g(n, j) > f(n+1)\} \xrightarrow{n \rightarrow \infty} \infty$.
- Define $Seq^f = \{s \in \omega^{<\omega} : \forall j < |s| \ s(j) \leq f(j)\}$ and let $T \in \mathbf{PT}_{f,g}$ if
- (a) T is a perfect subtree of Seq^f , and
 - (b) there exists a function $r \in \omega^\omega$, $\lim_{n \rightarrow \infty} r(n) = \infty$ such that

$$\forall s \in T \ (stem(T) \subseteq s \rightarrow |succ_T(s)| \geq g(|s|, r(|s|))).$$

4. Forcing \mathbf{S}_{g,g^*} from [2]. Let $g, g^* \in \omega^\omega$ be two strictly increasing functions such that $g(0) = 0$, $g^*(0) = 1$ and $g(n) \ll g^*(n) \ll g(n+1)$ for all $n > 0$. For $n \in \omega$ let

$$P_n = \left\{ a \subseteq g(n+1) : |a| = \frac{g(n+1)}{2^n} \right\}.$$

For a set $A \subseteq P_n$ define

$$\mathbf{nor}(A) = \min\{|X| : \forall a \in A \ X \not\subseteq a\}.$$

Let

$$\mathbf{S}_{g,g^*} = \left\{ \langle t_n : n \in \omega \rangle : \forall n \ t_n \in P_n \ \& \ \forall k \ \limsup_{n \rightarrow \infty} \frac{\mathbf{nor}(t_n)}{g^*(n+1)^k} = \infty \right\}.$$

For $p = \langle t_n^0 : n \in \omega \rangle$ and $q = \langle t_n^1 : n \in \omega \rangle$ we define $p \geq q$ if $t_n^0 \subseteq t_n^1$ for all n .

5. More complicated forcing notions from [10] and [4].

1. \mathcal{S} -FAMILIES OF GOOD GRAPHS

In this section we introduce a property of families of graphs that will be one of our main tools later.

Definition 1.1. Let \mathcal{U} be a countable basis of a topology on a set X , $A \subseteq X \times \omega_1$. A triple $\mathcal{G} = (A, \mathcal{U}, E)$ is a *good graph* if the following conditions are satisfied:

- (a) $E \subseteq [A]^2$, and
- (b) if $(x_0, \alpha_0), (x_1, \alpha_1) \in A$ are distinct then $x_0 \neq x_1$, and
- (c) if $(x_0, \alpha_0), (x_1, \alpha_1) \in A$, $x_0 \neq x_1$ and $\{(x_0, \alpha_0), (x_1, \alpha_1)\} \notin E$ then there are disjoint $U_0, U_1 \in \mathcal{U}$ such that $x_0 \in U_0$, $x_1 \in U_1$ and

$$(\forall (x'_0, \alpha'_0), (x'_1, \alpha'_1) \in A)(x'_0 \in U_0 \ \& \ x'_1 \in U_1 \ \Rightarrow \ \{(x'_0, \alpha'_0), (x'_1, \alpha'_1)\} \notin E).$$

Definition 1.2. Suppose that $\mathcal{F} = \{\mathcal{G}_\zeta : \zeta < \xi\}$ is a family of good graphs, $\mathcal{G}_\zeta = (A_\zeta, U_\zeta, E_\zeta)$.

1. Let $0 < m < \omega$. An m -selector for \mathcal{F} is a set $S \subseteq (\bigcup_{\zeta < \xi} A_\zeta)^m$ such that for some (not necessarily distinct) $\zeta_0, \dots, \zeta_{m-1} < \xi$ we have
 - (α) if $\nu \in S$, $\ell < m$ then $\nu(\ell) \in A_{\zeta_\ell}$,
 - (β) if $\nu, \rho \in S$ are distinct, then for some $\ell < m$ we have $\{\nu(\ell), \rho(\ell)\} \in E_{\zeta_\ell}$.
2. \mathcal{F} is called an \mathcal{S} - m -family if there is no uncountable m -selector for \mathcal{F} .
3. \mathcal{F} is an \mathcal{S} -family (of good graphs) if it is an \mathcal{S} - m -family for each $m < \omega$.

Let us show how we are going to use \mathcal{S} -families of good graphs.

Definition 1.3. Let $\mathbb{P} = (\mathbb{P}, \leq)$ be a forcing notion and let $\mathcal{G} = (A, \mathcal{U}, E)$ be a good graph (with $A \subseteq X \times \omega_1$, \mathcal{U} a countable basis of a topology on X). We say that \mathcal{G} is *densely representable by $\perp_{\mathbb{P}}$* if there is a one-to-one mapping $\pi : A \rightarrow \mathbb{P}$ such that

- (a) for each $\alpha < \omega_1$, the set $\{\pi(x, \alpha) : (x, \alpha) \in A\}$ is dense in \mathbb{P} , and
- (b) if $(x_0, \alpha_0), (x_1, \alpha_1) \in A$ are distinct, $\{(x_0, \alpha_0), (x_1, \alpha_1)\} \notin E$ then the conditions $\pi(x_0, \alpha_0), \pi(x_1, \alpha_1)$ are incompatible (in \mathbb{P}).

Proposition 1.4. *Let \mathcal{F} be an \mathcal{S} -family of good graphs. Suppose that \mathbb{P} is a forcing notion such that some $\mathcal{G} \in \mathcal{F}$ is densely representable by $\perp_{\mathbb{P}}$. Then $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ fails.*

Proof. Let $\mathcal{G} = (A, \mathcal{U}, E)$ and let $\pi : A \rightarrow \mathbb{P}$ witness that \mathcal{G} is densely representable by $\perp_{\mathbb{P}}$. Let $H_\xi = \{\pi(x, \xi) : (x, \xi) \in A\}$ (for $\xi < \omega_1$). Then the sets H_ξ are dense in \mathbb{P} by 1.3(a). We claim that they witness the failure of $\mathbf{FA}_{\aleph_1}(\mathbb{P})$. So suppose that $G \subseteq \mathbb{P}$ is a directed set which meets each H_ξ . For every $\xi < \omega_1$ choose $(x_\xi, \xi) \in A$ such that $\pi(x_\xi, \xi) \in G \cap H_\xi$. Look at the set $S = \{\langle (x_\xi, \xi) \rangle : \xi < \omega_1\}$ — it is an uncountable 1-selector from \mathcal{F} , a contradiction. \square

One of problems that we will have to take care of when building the forcing notion needed for 0.6 is preserving “being an \mathcal{S} -family”. A part of this difficulty will be dealt with by “killing the ccc of bad forcing notions”.

Proposition 1.5. *Let \mathcal{F} be an \mathcal{S} -family of good graphs. Suppose that \mathbb{P} is a ccc forcing notion such that*

$$\Vdash_{\mathbb{P}} \text{“}\mathcal{F} \text{ is not an } \mathcal{S}\text{-family”}.$$

Then there is a ccc forcing notion $\mathbb{P}^{\mathcal{F}}$ (of size \aleph_1) such that

$$\Vdash_{\mathbb{P}^{\mathcal{F}}} \text{“}\mathbb{P} \text{ is not ccc and } \mathcal{F} \text{ is an } \mathcal{S}\text{-family”}.$$

Proof. Let $\mathcal{F} = \{\mathcal{G}_\zeta : \zeta < \xi\}$, $\mathcal{G}_\zeta = (A_\zeta, \mathcal{U}_\zeta, E_\zeta)$, \mathcal{U}_ζ a basis of a topology on X_ζ , $A_\zeta \subseteq X_\zeta \times \omega_1$. Assume that some condition in \mathbb{P} forces that \mathcal{F} is not an \mathcal{S} -family. Then we find $m < \omega$, $\zeta_0, \dots, \zeta_{m-1} < \xi$, $p \in \mathbb{P}$ and \mathbb{P} -names $\dot{\nu}_\alpha$ for elements of $(\bigcup_{\ell < m} A_{\zeta_\ell})^m$ (for $\alpha < \omega_1$) such that

$$p \Vdash_{\mathbb{P}} \text{“} (\forall \alpha < \beta < \omega_1)(\dot{\nu}_\alpha \neq \dot{\nu}_\beta) \ \& \ (\forall \alpha < \omega_1)(\forall \ell < m)(\dot{\nu}_\alpha(\ell) \in A_{\zeta_\ell}) \ \& \ (\forall \alpha < \beta < \omega_1)(\exists \ell < m)(\{\dot{\nu}_\alpha(\ell), \dot{\nu}_\beta(\ell)\} \in E_{\zeta_\ell}) \text{”}.$$

For each $\alpha < \omega_1$ choose a sequence $\nu_\alpha \in (\bigcup_{\ell < m} A_{\zeta_\ell})^m$ and a condition $p_\alpha \geq p$ such that $p_\alpha \Vdash \dot{\nu}_\alpha = \nu_\alpha$. (So necessarily $(\forall \alpha < \omega_1)(\forall \ell < m)(\nu_\alpha(\ell) \in A_{\zeta_\ell})$.) Let \mathbb{Q} be the following forcing notion:

a condition in \mathbb{Q} is a finite set $q \subseteq \omega_1$ such that

$$(\forall \{\alpha, \beta\} \in [q]^2)(\forall \ell < m)(\{\nu_\alpha(\ell), \nu_\beta(\ell)\} \notin E_{\zeta_\ell});$$

the order is the inclusion (i.e., $q_1 \leq q_2$ if and only if $q_1 \subseteq q_2$).

Note that if $\alpha, \beta \in q \in \mathbb{Q}$, $\alpha \neq \beta$ then the conditions p_α, p_β are incompatible (we will use it in 1.5.4).

Claim 1.5.1. *Assume $\{q_\varepsilon : \varepsilon < \omega_1\} \subseteq \mathbb{Q}$, $q_\varepsilon = \{\alpha_i^\varepsilon : i < k\}$ (the increasing enumeration; $k < \omega$). Then there is $Y \in [\omega_1]^{\aleph_1}$ such that*

(\otimes) $_Y$ for each $\varepsilon, \delta \in Y$, if q_ε, q_δ are incompatible in \mathbb{Q} then there are $i < k$ and $\ell < m$ such that $\{\nu_{\alpha_i^\varepsilon}(\ell), \nu_{\alpha_i^\delta}(\ell)\} \in E_{\zeta_\ell}$.

Proof of the claim. For each $\varepsilon < \omega_1$, $\ell < m$ and distinct $i, j < k$ choose $U_\ell^{\varepsilon, i, j} \in \mathcal{U}_{\zeta_\ell} \cup \{*\}$ such that if $\nu_{\alpha_i^\varepsilon}(\ell) = \nu_{\alpha_j^\varepsilon}(\ell)$ then $U_\ell^{\varepsilon, i, j} = *$, otherwise $U_\ell^{\varepsilon, i, j}, U_\ell^{\varepsilon, j, i} \in \mathcal{U}_{\zeta_\ell}$ are such that

(\otimes_1) $\nu_{\alpha_i^\varepsilon}(\ell) \in U_\ell^{\varepsilon, i, j} \times \omega_1$, $\nu_{\alpha_j^\varepsilon}(\ell) \in U_\ell^{\varepsilon, j, i} \times \omega_1$, $U_\ell^{\varepsilon, i, j} \cap U_\ell^{\varepsilon, j, i} = \emptyset$, and for all $(x_0, \alpha_0), (x_1, \alpha_1) \in A_{\zeta_\ell}$,

$$x_0 \in U_\ell^{\varepsilon, i, j} \ \& \ x_1 \in U_\ell^{\varepsilon, j, i} \quad \Rightarrow \quad \{(x_0, \alpha_0), (x_1, \alpha_1)\} \notin E_{\zeta_\ell}.$$

(Why possible? Remember the definition of \mathbb{Q} and 1.1(c).) Each \mathcal{U}_{ζ_ℓ} is countable, so there are $U_\ell^{i, j} \in \mathcal{U}_{\zeta_\ell} \cup \{*\}$ and an uncountable set $Y \subseteq \omega_1$ such that

(\otimes_2) $(\forall \varepsilon \in Y)(\forall i, j < k, i \neq j)(\forall \ell < m)(U_\ell^{\varepsilon, i, j} = U_\ell^{i, j})$.

Suppose that $\varepsilon, \delta \in Y$ and the conditions q_ε, q_δ are incompatible. It means that there are $i, j < k$ and $\ell < m$ such that $\{\nu_{\alpha_i^\varepsilon}(\ell), \nu_{\alpha_j^\delta}(\ell)\} \in E_{\zeta_\ell}$. We are going to show that we may demand $i = j$ (what will finish the proof of the claim). So suppose that $i \neq j$. If $\nu_{\alpha_i^\varepsilon}(\ell) \neq \nu_{\alpha_j^\varepsilon}(\ell)$, then (by (\otimes_2)) $\nu_{\alpha_i^\varepsilon}(\ell) \neq \nu_{\alpha_j^\delta}(\ell)$ and (by (\otimes_1) + (\otimes_2)) $\nu_{\alpha_j^\delta}(\ell) \in U_\ell^{\delta, i, j} \times \omega_1 = U_\ell^{\varepsilon, i, j} \times \omega_1$. But applying the last part of (\otimes_1) we may conclude now that $\{\nu_{\alpha_i^\varepsilon}(\ell), \nu_{\alpha_j^\delta}(\ell)\} \notin E_{\zeta_\ell}$, a contradiction. So $\nu_{\alpha_i^\varepsilon}(\ell) = \nu_{\alpha_j^\delta}(\ell)$. But then $\nu_{\alpha_i^\varepsilon}(\ell) = \nu_{\alpha_j^\delta}(\ell)$ and thus $\{\nu_{\alpha_i^\varepsilon}(\ell), \nu_{\alpha_i^\delta}(\ell)\} \in E_{\zeta_\ell}$. \square

Claim 1.5.2. \mathbb{Q} is a ccc forcing notion.

Proof of the claim. Suppose that $\{q_\xi : \xi < \omega_1\} \subseteq \mathbb{Q}$ is an antichain in \mathbb{Q} . We may assume that, for some $k < \omega$, $|q_\xi| = k$ for all $\xi < \omega_1$. Let $q_\xi = \{\alpha_0^\xi, \alpha_1^\xi, \dots, \alpha_{k-1}^\xi\}$ be the increasing enumeration. Using 1.5.1 we may find an uncountable $Y \subseteq \omega_1$ such that for each distinct $\varepsilon, \delta \in Y$ there are $i_{\varepsilon, \delta} < k$ and $\ell_{\varepsilon, \delta} < m$ with $\{\nu_{\alpha_{i_{\varepsilon, \delta}}^\varepsilon}(\ell_{\varepsilon, \delta}), \nu_{\alpha_{i_{\varepsilon, \delta}}^\delta}(\ell_{\varepsilon, \delta})\} \in E_{\zeta_\ell}$. For each $\varepsilon \in Y$ let η_ε be a sequence of length $k \cdot m$ such that

$$\eta_\varepsilon(n) = \nu_{\alpha_i^\varepsilon}(\ell) \quad \text{whenever } n = i \cdot m + \ell, \ i < k, \ \ell < m.$$

Look at the set $\{\eta_\varepsilon : \varepsilon \in Y\} \subseteq (\bigcup_{\zeta < \xi} A_\zeta)^{k \cdot m}$. It should be clear that it is an uncountable $k \cdot m$ -selector for \mathcal{F} (the clause (β) of 1.2(1) for $\varepsilon, \delta \in Y$ is witnessed by $i_{\varepsilon, \delta} \cdot m + \ell_{\varepsilon, \delta}$). A contradiction. \square

Claim 1.5.3. $\Vdash_{\mathbb{Q}}$ “ \mathcal{F} is an \mathcal{S} -family of good graphs”.

Proof of the claim. The only bad thing that could happen after forcing with \mathbb{Q} is that an uncountable m^* -selector was added (for some $m^* < \omega$). If so, then we have $m^* < \omega$, \mathbb{Q} -names $\dot{\eta}_\varepsilon$ (for $\varepsilon < \omega_1$) and a condition $q \in \mathbb{Q}$ such that

$$q \Vdash_{\mathbb{Q}} \text{ “ } \{\dot{\eta}_\varepsilon : \varepsilon < \omega_1\} \text{ is an } m^*\text{-selector for } \mathcal{F}, \ \dot{\eta}_\varepsilon\text{'s are pairwise distinct ”.}$$

Clearly we may require that for some $\zeta_0, \dots, \zeta_{m^*-1}$, for each $\varepsilon < \omega_1$ the condition q forces that $\dot{\eta}_\varepsilon(\ell) \in A_{\zeta_\ell}$ (for all $\ell < m^*$). For each $\varepsilon < \omega_1$ pick a sequence η_ε and a condition $q_\varepsilon \geq q$ such that $q_\varepsilon \Vdash_{\mathbb{Q}} \text{ “ } \dot{\eta}_\varepsilon = \eta_\varepsilon \text{ ”}$. Next, choose an uncountable set $Y \subseteq \omega_1$ and $k < \omega$ such that for $\varepsilon \in Y$, $q_\varepsilon = \{\alpha_i^\varepsilon : i < k\}$ (the increasing

enumeration) and $(\otimes)_Y$ of 1.5.1 holds (possible by 1.5.1). Let ρ_ε (for $\varepsilon \in Y$) be sequences of length $k \cdot m + m^*$ defined by

$$\rho_\varepsilon(n) = \begin{cases} \nu_{\alpha_\varepsilon^i}(\ell) & \text{if } n = i \cdot m + \ell, \ i < k, \ \ell < m, \\ \eta_\varepsilon(\ell) & \text{if } n = k \cdot m + \ell, \ \ell < m^*. \end{cases}$$

Note that for distinct $\varepsilon, \delta \in Y$ we have:

- (\boxtimes_1) if q_ε, q_δ are compatible in \mathbb{Q} , $\varepsilon < \delta < \omega_1$, then for some $\ell < m^*$ we have $\{\rho_\varepsilon(km + \ell), \rho_\delta(km + \ell)\} = \{\eta_\varepsilon(\ell), \eta_\delta(\ell)\} \in E_{\zeta_\ell}$;
- (\boxtimes_2) if q_ε, q_δ are incompatible in \mathbb{Q} then there are $i < k$ and $\ell < m$ so that $\{\rho_\varepsilon(im + \ell), \rho_\delta(im + \ell)\} = \{\nu_{\alpha_\varepsilon^i}(\ell), \nu_{\alpha_\delta^i}(\ell)\} \in E_{\zeta_\ell}$.

(Why? (\boxtimes_1) follows from the choice of $\eta_\varepsilon, q_\varepsilon$, (\boxtimes_2) is a consequence of $(\otimes)_Y$.) Hence, $\{\rho_\varepsilon : \varepsilon \in Y\}$ is an uncountable $km + m^*$ -selector from \mathcal{F} , a contradiction. \square

Claim 1.5.4. *For some $q \in \mathbb{Q}$ we have*

$$q \Vdash_{\mathbb{Q}} \text{ “ } \mathbb{P} \text{ does not satisfy the ccc ”.}$$

Proof of the claim. As we stated before, if $q \in \mathbb{Q}$ and $\alpha, \beta \in q$ are distinct then the conditions p_α, p_β are incompatible in \mathbb{P} . Since, by 1.5.2, the forcing notion \mathbb{Q} is ccc, there is a condition $q \in \mathbb{Q}$ such that

$$q \Vdash_{\mathbb{Q}} \text{ “ } \{\alpha < \omega_1 : \{\alpha\} \in \Gamma_{\mathbb{Q}}\} \text{ is uncountable ”.}$$

It should be clear now that the condition q forces (in \mathbb{Q}) that \mathbb{P} fails the ccc. \square

Let $q \in \mathbb{Q}$ be as guaranteed by 1.5.4 and let $\mathbb{P}^{\mathcal{F}}$ be the \mathbb{Q} restricted to elements stronger than that q . It should be clear that $\mathbb{P}^{\mathcal{F}}$ is as required in the proposition. \square

Conclusion 1.6. If \mathcal{F} is an \mathcal{S} -family of good graphs and \mathbb{P} is a forcing notion with the Knaster property, then

$$\Vdash_{\mathbb{P}} \text{ “ } \mathcal{F} \text{ is an } \mathcal{S}\text{-family ”.}$$

2. WHERE ARE OUR \mathcal{S} -FAMILIES FROM?

It follows from 1.4 that to make sure that $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ fails for all $\mathbb{P} \in \mathcal{K}$ it is enough to have an \mathcal{S} -family of good graphs such that for every $\mathbb{P} \in \mathcal{K}$ some $\mathcal{G} \in \mathcal{F}$ is densely representable by $\perp_{\mathbb{P}}$. In this section we will (almost) show how the respective \mathcal{S} -family is created in our model. Basically, it will come from Cohen reals, but interpreted in a special way.

Let $\mathbb{P} \in \mathcal{K}$ and let p be a condition in \mathbb{P} . Considering all possible cases (of 0.4) we define a countable basis $\mathcal{U}(\mathbb{P})$ of a topology on $X(\mathbb{P}) \stackrel{\text{def}}{=} \mathbb{P}$, a set $E(\mathbb{P}) \subseteq [X(\mathbb{P}) \times \omega_1]^2$ and a forcing notion $\mathbb{C}(\mathbb{P}, p)$. (In the last case we will assume additionally that the condition p has some special form, which will restrict us to a dense subset of \mathbb{P} .)

CASE 1: $\mathbb{P} = \mathbb{Q}_1^{\text{tree}}(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary 2-big typical tree-creating pair for a function $\mathbf{H} \in \mathcal{H}(\aleph_1)$.

For $q = \langle t_\eta^q : \eta \in T^q \rangle \in \mathbb{P}$ and $N < \omega$ let $U(q, N)$ be the family of all $q' \in \mathbb{P}$ such that

$$\text{root}(q') = \text{root}(q) \quad \text{and} \quad (\forall \eta \in T^{q'}) (\ell g(\eta) < N \Rightarrow (\eta \in T^q \ \& \ t_\eta^q = t_\eta^{q'})).$$

Let $\mathcal{U}(\mathbb{P})$ consist of all nonempty sets $U(q, N)$ (for $q \in \mathbb{P}$, $N < \omega$). Clearly $\mathcal{U}(\mathbb{P})$ is a countable basis of a topology on \mathbb{P} (which is the natural product topology). Next we define

$$E(\mathbb{P}) = \{ \{ (q_0, \alpha_0), (q_1, \alpha_1) \} \in [\mathbb{P} \times \omega_1]^2 : \text{for each } N < \omega \\ \text{dcl}(T^{q_0}) \cap \prod_{i < N} \mathbf{H}(i) \cap \text{dcl}(T^{q_1}) \neq \emptyset \}$$

(where $\text{dcl}(T)$ is the downward closure of a quasi tree T ; see [17, Def. 1.3.1]). The forcing notion $\mathbb{C}(\mathbb{P}, p)$ is defined as follows:

a condition r in $\mathbb{C}(\mathbb{P}, p)$ is a finite system $r = \langle s_\eta^r : \eta \in \hat{S}^r \rangle$ such that $S^r \subseteq T^p$ is a (finite) quasi tree, $\text{root}(S^r) = \text{root}(T^p)$,

$$(\forall \eta \in \hat{S}^r)(s_\eta^r \in \Sigma(t_\eta^p) \ \& \ \mathbf{nor}[s_\eta^r] \geq \mathbf{nor}[t_\eta^p] - 2 \ \& \ \text{pos}(s_\eta^r) = \text{succ}_{S^r}(\eta)),$$

and if $\mathbf{nor}[t_\eta^p] \leq 4$, $\eta \in \hat{S}^r$ then $s_\eta^r = t_\eta^p$;

the order of $\mathbb{C}(\mathbb{P}, p)$ is the end extension, i.e., $r_0 \leq r_1$ if and only if $S^{r_0} \subseteq S^{r_1}$ and $(\forall \eta \in \hat{S}^{r_0})(s_\eta^{r_0} = s_\eta^{r_1})$.

It should be clear that $\mathbb{C}(\mathbb{P}, p)$ is a countable atomless (remember 0.3) forcing notion, so it is equivalent to the Cohen forcing. Moreover, the forcing with $\mathbb{C}(\mathbb{P}, p)$ adds a condition $p^* = \bigcup \Gamma_{\mathbb{C}(\mathbb{P}, p)} \in \mathbb{P}$ stronger than p .

CASE 2: $\mathbb{P} = \mathbb{Q}_1^{\text{tree}}(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical t-omittory tree-creating pair for a function $\mathbf{H} \in \mathcal{H}(\aleph_1)$.

$\mathcal{U}(\mathbb{P})$ and $E(\mathbb{P})$ are defined like in the Case 1. The forcing notion $\mathbb{C}(\mathbb{P}, p)$ is defined similarly too, though we now make an advantage from ‘‘t-omittory’’:

a condition r in $\mathbb{C}(\mathbb{P}, p)$ is a finite system $r = \langle s_\eta^r : \eta \in \hat{S}^r \rangle$ such that $S^r \subseteq T^p$ is a quasi tree, $\text{root}(S^r) = \text{root}(T^p)$, and for each $\eta \in \hat{S}^r$

- there is a (finite) quasi tree $T_\eta^* \subseteq T^p$ such that $\text{root}(T_\eta^*) = \eta$ and $s_\eta^r \in \Sigma(t_\nu^p) : \nu \in \hat{T}_\eta^*$,
- $\mathbf{nor}[s_\eta^r] \geq \min\{\mathbf{nor}[t_\nu^p] - 2 : \nu \in T_\eta^*\}$,

and if $\mathbf{nor}[t_\eta^p] \leq 4$, $\eta \in \hat{S}^r$ then $s_\eta^r = t_\eta^p$;

the order of $\mathbb{C}(\mathbb{P}, p)$ is the end extension, so $r_0 \leq r_1$ if and only if $S^{r_0} \subseteq S^{r_1}$ and $(\forall \eta \in \hat{S}^{r_0})(s_\eta^{r_0} = s_\eta^{r_1})$.

Again, $\mathbb{C}(\mathbb{P}, p)$ is a countable atomless forcing notion. The forcing with $\mathbb{C}(\mathbb{P}, p)$ adds a condition $p^* = \bigcup \Gamma_{\mathbb{C}(\mathbb{P}, p)} \in \mathbb{P}$ stronger than p .

CASE 3: $\mathbb{P} = \mathbb{Q}_f^*(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical creating pair for $\mathbf{H} \in \mathcal{H}(\aleph_1)$, $f : \omega \times \omega \rightarrow \omega$ is an \mathbf{H} -fast function, (K, Σ) is $\bar{2}$ -big, has the Halving Property and is either simple or gluing.

For a condition $q = (w^q, t_0^q, t_1^q, \dots) \in \mathbb{P}$ and $N < \omega$ we let $U(q, N)$ be the collection of all conditions $q' \in \mathbb{P}$ such that $w^q = w^{q'}$ and $t_n^q = t_n^{q'}$ for all $n < N$. Next, $\mathcal{U}(\mathbb{P})$ is the family of all non-empty sets $U(q, N)$ (for $q \in \mathbb{P}$ and $N < \omega$). Like before, $\mathcal{U}(\mathbb{P})$ is a countable basis of the standard (product) topology on \mathbb{P} . We define

$$E(\mathbb{P}) = \{ \{ (q_0, \alpha_0), (q_1, \alpha_1) \} \in [\mathbb{P} \times \omega_1]^2 : \text{for each } N < \omega \\ \text{dcl}(\text{pos}(w^{q_0}, t_0^{q_0}, \dots, t_N^{q_0})) \cap \prod_{i < N} \mathbf{H}(i) \cap \text{dcl}(\text{pos}(w^{q_1}, t_0^{q_1}, \dots, t_N^{q_1})) \neq \emptyset \}.$$

The forcing notion $\mathbb{C}(\mathbb{P}, p)$ is defined so that

a condition r in $\mathbb{C}(\mathbb{P}, p)$ is a finite sequence $r = (w^r, s_0^r, \dots, s_{n_r}^r) \in \text{FC}(K, \Sigma)$ such that $w^r = w^p$, $s_k^r \in \Sigma(t_k^p)$, and

- if $\mathbf{nor}[t_k^p] > f(n, m_{\text{dn}}^{t_k^p})$, $n \geq 4$ then $\mathbf{nor}[s_k^r] > f(n-2, m_{\text{dn}}^{s_k^r})$,
- if $\mathbf{nor}[t_k^p] \leq f(4, m_{\text{dn}}^{t_k^p})$ then $s_k^r = t_k^p$;

the order of $\mathbb{C}(\mathbb{P}, p)$ is the extension, i.e., $r_0 \leq r_1$ if and only if $n_{r_0} \leq n_{r_1}$ and $s_k^{r_0} = s_k^{r_1}$ for all $k \leq n_{r_0}$.

Clearly $\mathbb{C}(\mathbb{P}, p)$ is countable and atomless (remember 0.3). It adds a condition $\dot{p}^* = \bigcup \Gamma_{\mathbb{C}(\mathbb{P}, p)} \in \mathbb{P}$ stronger than p .

CASE 4: $\mathbb{P} = \mathbb{Q}_{w\infty}^*(K, \Sigma)$, where $(K, \Sigma) \in \mathcal{H}(\aleph_1)$ is a strongly finitary typical creating pair for $\mathbf{H} \in \mathcal{H}(\aleph_1)$, (K, Σ) captures singletons.

Both $\mathcal{U}(\mathbb{P})$ and $E(\mathbb{P})$ are defined like in Case 3. As we said before, here we will require that p is of special form. Namely, we demand that $p = (w^p, t_0^p, t_2^p, \dots) \in \mathbb{P}$ is such that for some strictly increasing sequence $\langle m_k : k < \omega \rangle \subseteq \omega$, $m_0 = 0$ and for each $k \in \omega$:

- $\mathbf{nor}[t_{m_k}^p] \geq 4 + k$, and
- if $m_k + 1 < m_{k+1}$ then for some (equivalently: all) $u \in \text{pos}(w^p, t_0^p, \dots, t_{m_k}^p)$ we have $|\text{pos}(u, t_{m_k+1}^p, \dots, t_{m_{k+1}-1}^p)| = 1$.

(Since (K, Σ) captures singletons the conditions of this form are dense in \mathbb{P} .) Now, the forcing notion $\mathbb{C}(\mathbb{P}, p)$ is defined so that

a condition r in $\mathbb{C}(\mathbb{P}, p)$ is a finite sequence $r = (w^r, s_0^r, \dots, s_{n_r}^r) \in \text{FC}(K, \Sigma)$ such that $w^r = w^p$, $s_\ell^r \in \Sigma(t_\ell^p)$, and

$$(\forall k < \omega)(m_{2k} \leq n_r \Rightarrow \mathbf{nor}[s_{m_{2k}}^r] \geq \mathbf{nor}[t_{m_{2k}}^p] - 2;$$

the order of $\mathbb{C}(\mathbb{P}, p)$ is the extension, i.e., $r_0 \leq r_1$ if and only if $n_{r_0} \leq n_{r_1}$ and $s_k^{r_0} = s_k^{r_1}$ for all $k \leq n_{r_0}$.

Again, $\mathbb{C}(\mathbb{P}, p)$ is countable and atomless, and it adds a condition $\dot{p}^* = \bigcup \Gamma_{\mathbb{C}(\mathbb{P}, p)} \in \mathbb{P}$ stronger than p .

Lemma 2.1. *Suppose $\mathbb{P} \in \mathcal{K}$ and $p \in \mathbb{P}$ is such that $\mathbb{C}(\mathbb{P}, p)$ is defined. Let $r \in \mathbb{C}(\mathbb{P}, p)$. Then there are two conditions $r_0, r_1 \in \mathbb{C}(\mathbb{P}, p)$ stronger than r and basic open sets $U_0, U_1, U \in \mathcal{U}(\mathbb{P})$ such that*

- $U_0 \cap U_1 = \emptyset$, $p \in U$,
- if $p' \in U$ and $\mathbb{C}(\mathbb{P}, p')$ is defined then $r, r_0, r_1 \in \mathbb{C}(\mathbb{P}, p')$, r_0, r_1 are stronger than r (in $\mathbb{C}(\mathbb{P}, p')$) and $r_0 \Vdash_{\mathbb{C}(\mathbb{P}, p')} \dot{p}^* \in U_0$, $r_1 \Vdash_{\mathbb{C}(\mathbb{P}, p')} \dot{p}^* \in U_1$,
- $(\forall q_0 \in U_0)(\forall q_1 \in U_1)(\forall \alpha_0, \alpha_1 < \omega_1)(\{(q_0, \alpha_0), (q_1, \alpha_1)\} \notin E(\mathbb{P}))$.

(Note that these formulas are absolute.)

Proof. In Cases 1 and 3 (of 0.4) use 0.3; in other cases use directly the assumption that (K, Σ) is typical. \square

The \mathcal{S} -families in our model will be created by choosing sets $A \subseteq X(\mathbb{P}) \times \omega_1$ (for each $\mathbb{P} \in \mathcal{K}$) so that in each pair $(q, \alpha) \in A$ the first coordinate q is added generically by the forcing notion $\mathbb{C}(\mathbb{P}, p)$ (for some condition $p \in \mathbb{P}$). For this we will use the finite support product of \aleph_1 copies of $\mathbb{C}(\mathbb{P}, p)$, denoted by $\mathbb{C}^{\omega_1}(\mathbb{P}, p) = \prod_{\delta < \omega_1} \mathbb{C}(\mathbb{P}, p)$

(so a condition in $\mathbb{C}^{\omega_1}(\mathbb{P}, p)$ is a finite function $c : \text{dom}(c) \rightarrow \mathbb{C}(\mathbb{P}, p)$ and the order is the natural one). The forcing with $\mathbb{C}^{\omega_1}(\mathbb{P}, p)$ adds the set

$$\dot{Z}_p^{\mathbb{P}} = \{(q, \alpha) : \alpha < \omega_1 \ \& \ q = \bigcup \{r : (\alpha, r) \in \Gamma_{\mathbb{C}^{\omega_1}(\mathbb{P}, p)}\}\}.$$

(Sets of these form will be used to build a good graph \mathcal{G} densely representable by $\perp_{\mathbb{P}}$.)

Definition 2.2. Suppose that $\mathbb{P} \in \mathcal{K}$, $p \in \mathbb{P}$ and $\mathbb{C}(\mathbb{P}, p)$ is defined. We say that two conditions $\bar{c}_0, \bar{c}_1 \in \mathbb{C}^{\omega_1}(\mathbb{P}, p)$ are *isomorphic* (and then we write $\bar{c}_0 \sim \bar{c}_1$) if $|\text{dom}(\bar{c}_0)| = |\text{dom}(\bar{c}_1)|$ and if $H : \text{dom}(\bar{c}_0) \rightarrow \text{dom}(\bar{c}_1)$ is the order preserving bijection then $\bar{c}_1(H(\alpha)) = \bar{c}_0(\alpha)$ for each $\alpha \in \text{dom}(\bar{c}_0)$.

(Note that there are countably many isomorphism types of conditions in $\mathbb{C}^{\omega_1}(\mathbb{P}, p)$.)

The main technical advantage of using the forcing notions $\mathbb{C}^{\omega_1}(\mathbb{P}, p)$ to create our \mathcal{S} -families is presented by the following lemma.

Lemma 2.3. Let $\mathbb{P}^0, \dots, \mathbb{P}^k \in \mathcal{K}$. Suppose that $\langle \mathbb{P}_\xi, \dot{Q}_\xi : \xi < \gamma \rangle$ is a finite support iteration of ccc forcing notions, γ is a limit ordinal. Furthermore assume that for some disjoint sets $I_0, \dots, I_k \subseteq \gamma$ we have

- (α) if $\xi \in I_\ell$ then $\Vdash_{\mathbb{P}_\xi} \dot{Q}_\xi = \mathbb{C}^{\omega_1}(\mathbb{P}^\ell, p^\xi)$ for some $p^\xi \in \mathbb{P}^\ell$,
- (β) for $\zeta < \gamma$, \dot{A}_ζ^ℓ is the \mathbb{P}_ζ -name for the set $\bigcup \{ \dot{Z}_{p^\xi}^{\mathbb{P}^\ell} : \xi \in I_\ell \cap \zeta \} \subseteq \mathbb{P}^\ell \times \omega_1$,
- (γ) for each $\zeta < \gamma$

$\Vdash_{\mathbb{P}_\zeta} \{ (\dot{A}_\zeta^\ell, \mathcal{U}(\mathbb{P}^\ell), E(\mathbb{P}^\ell) \cap [\dot{A}_\zeta^\ell]^2) : \ell \leq k \}$ is an \mathcal{S} -family of good graphs ”.

Then

$\Vdash_{\mathbb{P}_\gamma} \{ (\dot{A}_\gamma^\ell, \mathcal{U}(\mathbb{P}^\ell), E(\mathbb{P}^\ell) \cap [\dot{A}_\gamma^\ell]^2) : \ell \leq k \}$ is an \mathcal{S} -family of good graphs. ”

Proof. For a condition $q_0 \in \mathbb{P}_\gamma$ we may find a stronger condition $q \in \mathbb{P}_\gamma$ with the following property

(*) $_q$ for each $\xi \in I_\ell \cap \text{dom}(q)$, $\ell \leq k$ there are $\bar{c}(q, \xi)$, $\bar{c}_0(q, \xi)$, $\bar{c}_1(q, \xi)$, and $U(q, \xi)$, $\bar{U}_0(q, \xi)$, $\bar{U}_1(q, \xi)$ (objects, not names) such that the condition $q \upharpoonright \xi$ forces the following:

- $U(q, \xi) \in \mathcal{U}(\mathbb{P}^\ell)$, $p_\xi \in U(q, \xi)$, $q(\xi) = \bar{c}(q, \xi) \in \mathbb{C}^{\omega_1}(\mathbb{P}^\ell, p_\xi)$,
- $\bar{U}_0(q, \xi), \bar{U}_1(q, \xi) : \text{dom}(\bar{c}(q, \xi)) \rightarrow \mathcal{U}(\mathbb{P}^\ell)$, $\bar{U}_0(q, \xi)(\varepsilon) \cap \bar{U}_1(q, \xi)(\varepsilon) = \emptyset$ for each $\varepsilon \in \text{dom}(\bar{c}(q, \xi))$,
- for each $p' \in U(q, \xi)$ such that $\mathbb{C}(\mathbb{P}^\ell, p')$ is defined: $\bar{c}(q, \xi), \bar{c}_0(q, \xi), \bar{c}_1(q, \xi)$ are in $\mathbb{C}^{\omega_1}(\mathbb{P}^\ell, p')$, the conditions $\bar{c}_0(q, \xi), \bar{c}_1(q, \xi)$ are stronger than $\bar{c}(q, \xi)$ and $\text{dom}(\bar{c}_0(q, \xi)) = \text{dom}(\bar{c}_1(q, \xi)) = \text{dom}(\bar{c}(q, \xi))$, and if $i < 2$, $\varepsilon \in \text{dom}(\bar{c}(q, \xi))$, and p_ε^* is the $\mathbb{C}^{\omega_1}(\mathbb{P}^\ell, p')$ -name for the ε^{th} generic real (i.e., $\bigcup \{ r : (\alpha, r) \in \Gamma_{\mathbb{C}^{\omega_1}(\mathbb{P}^\ell, p')} \}$) then

$$\bar{c}_i(q, \xi) \Vdash_{\mathbb{C}^{\omega_1}(\mathbb{P}, p')} \text{“ } p_\varepsilon^* \in \bar{U}_i(q, \xi)(\varepsilon) \text{”},$$

- for each $\varepsilon \in \text{dom}(\bar{c}(q, \xi))$ and $q_0 \in \bar{U}_0(q, \xi)(\varepsilon)$, $q_1 \in \bar{U}_1(q, \xi)(\varepsilon)$ we have

$$(\forall \varepsilon_0, \varepsilon_1 < \omega_1) (\{ (q_0, \varepsilon_0), (q_1, \varepsilon_1) \} \notin E(\mathbb{P}^\ell)).$$

[Why? Just apply 2.1 (and remember that supports are finite).] From now on we will restrict ourselves to conditions $q \in \mathbb{P}_\gamma$ with the property (*) $_q$ (what is allowed as they are dense in \mathbb{P}_γ). So we will assume that for each condition q under considerations and $\xi \in I_\ell \cap \text{dom}(q)$, $\ell \leq k$, the objects (not names!) $\bar{c}(q, \xi)$, $\bar{c}_0(q, \xi)$, $\bar{c}_1(q, \xi)$, $U(q, \xi)$, $\bar{U}_0(q, \xi)$, $\bar{U}_1(q, \xi)$ are defined and have the respective properties.

Note that, in $\mathbf{V}^{\mathbb{P}_\gamma}$, if $\ell \leq k$, $\xi_0, \xi_1 \in I_\ell$, $(q, \alpha_0) \in \dot{Z}_{p^{\xi_0}}^{\mathbb{P}^\ell}$ and $(q, \alpha_1) \in \dot{Z}_{p^{\xi_1}}^{\mathbb{P}^\ell}$ then $\alpha_0 = \alpha_1$ and $\xi_0 = \xi_1$ (remember 2.1). Therefore, we may label elements of \dot{A}_γ^ℓ by pairs from $I_\ell \times \omega_1$ and allow ourselves small abuse of notation identifying $(\xi, \alpha) \in I_\ell \times \omega_1$ with the respective $(q, \alpha) \in \dot{Z}_{p^\xi}^{\mathbb{P}^\ell}$. Next let $E_\ell = E(\mathbb{P}^\ell) \cap [\dot{A}_\gamma^\ell]^2$.

Now, suppose that some condition $q' \in \mathbb{P}_\gamma$ forces that

$$\{ (\dot{A}_\gamma^\ell, \mathcal{U}(\mathbb{P}^\ell), E_\ell) : \ell \leq k \} \text{ is not an } \mathcal{S}\text{-family.}$$

Then we may find a condition $q \in \mathbb{P}_\gamma$, an integer $m < \omega$, $\ell_0, \dots, \ell_{m-1} \leq k$ (not necessarily distinct) and \mathbb{P}_γ -names $\dot{\nu}_\alpha$ (for $\alpha < \omega_1$) of sequences of length m such that

$$q \Vdash_{\mathbb{P}_\gamma} \text{ “ } (\forall \alpha < \beta < \omega_1)(\dot{\nu}_\alpha \neq \dot{\nu}_\beta) \ \& \ (\forall \alpha < \omega_1)(\forall i < m)(\dot{\nu}_\alpha(i) \in I_{\ell_i} \times \omega_1) \\ (\forall \alpha < \beta < \omega_1)(\exists i < m)(\{\dot{\nu}_\alpha(i), \dot{\nu}_\beta(i)\} \in E_{\ell_i}) \text{ ”}.$$

For each $\alpha < \omega_1$ choose a condition $q_\alpha \in \mathbb{P}_\gamma$ (satisfying $(*)_{q_\alpha}$ and) stronger than q and a sequence $\nu_\alpha \in \prod_{i < m} (I_{\ell_i} \times \omega_1)$ such that $q_\alpha \Vdash \dot{\nu}_\alpha = \nu_\alpha$ and

$$(\forall i < m)(\nu_\alpha(i) = (\xi, \varepsilon) \Rightarrow \xi \in \text{dom}(q_\alpha) \ \& \ \varepsilon \in \text{dom}(q_\alpha(\xi))).$$

Now we consider two cases.

CASE A: $\text{cf}(\gamma) \neq \omega_1$.

Then for some $\zeta < \gamma$, for uncountably many $\alpha < \omega_1$, $\text{dom}(q_\alpha) \subseteq \zeta$. Let $G \subseteq \mathbb{P}_\gamma$ be a generic over \mathbf{V} and work in $\mathbf{V}[G \cap \mathbb{P}_\zeta]$. Because of the ccc of \mathbb{P}_ζ , the set $\{\alpha < \omega_1 : q_\alpha \in G \cap \mathbb{P}_\zeta\}$ is uncountable, so we get an uncountable m -selector from $\{(\dot{A}_\zeta^\ell, \mathcal{U}(\mathbb{P}^\ell), E(\mathbb{P}^\ell) \cap [\dot{A}_\zeta^\ell]^2)^{G \cap \mathbb{P}_\zeta} : \ell \leq k\}$ (in $\mathbf{V}[G \cap \mathbb{P}_\zeta]$), contradicting the assumption (γ) .

CASE B: $\text{cf}(\gamma) = \omega_1$.

If for some $\zeta < \gamma$ the set $\{\alpha < \omega_1 : \text{dom}(q_\alpha) \subseteq \zeta\}$ is uncountable then we may repeat the arguments of Case A. So assume that $\{\alpha < \omega_1 : \text{dom}(q_\alpha) \subseteq \zeta\}$ is countable for each $\zeta < \gamma$.

Applying “standard cleaning procedure” and passing to an uncountable subsequence (and possibly increasing our conditions) we may assume that $|\text{dom}(q_\alpha)| = N$ for each $\alpha < \omega_1$ and, letting $\{\xi_0^\alpha, \dots, \xi_{N-1}^\alpha\}$ be the increasing enumeration of $\text{dom}(q_\alpha)$:

1. $\{\text{dom}(q_\alpha) : \alpha < \omega_1\}$ forms a Δ -system with heart u^* ,
2. for some $n^* < N$ and $\zeta^* < \gamma$, we have $(\forall \alpha < \omega_1)(\forall j < n^*)(\xi_j^\alpha < \zeta^*)$ and $(\forall \alpha < \beta < \omega_1)(\zeta^* < \xi_{n^*}^\alpha \leq \xi_{N-1}^\alpha < \xi_{n^*}^\beta)$ (so necessarily $u^* \subseteq \zeta^*$),
3. $\sup\{\xi_{n^*}^\alpha : \alpha < \omega_1\} = \gamma$,
4. $(\forall \alpha, \beta < \omega_1)(\forall \ell \leq k)(\forall j < N)(\xi_j^\alpha \in I_\ell \Leftrightarrow \xi_j^\beta \in I_\ell)$,
5. if $\alpha, \beta < \omega_1$, $\ell \leq k$, $j < N$ and $\xi_j^\alpha \in I_\ell$ then $U(q_\alpha, \xi_j^\alpha) = U(q_\beta, \xi_j^\beta)$, $\bar{c}(q_\alpha, \xi_j^\alpha) \sim \bar{c}(q_\beta, \xi_j^\beta)$, $\bar{c}_0(q_\alpha, \xi_j^\alpha) \sim \bar{c}_0(q_\beta, \xi_j^\beta)$, $\bar{c}_1(q_\alpha, \xi_j^\alpha) \sim \bar{c}_1(q_\beta, \xi_j^\beta)$ (see 2.2), and
 (*) if $H : \text{dom}(\bar{c}(q_\alpha, \xi_j^\alpha)) \rightarrow \text{dom}(\bar{c}(q_\beta, \xi_j^\beta))$ is the order preserving bijection then H is the identity on $\text{dom}(\bar{c}(q_\alpha, \xi_j^\alpha)) \cap \text{dom}(\bar{c}(q_\beta, \xi_j^\beta))$ and for each $\varepsilon \in \bar{c}(q_\alpha, \xi_j^\alpha)$

$$\bar{U}_0(q_\alpha, \xi_j^\alpha)(\varepsilon) = \bar{U}_0(q_\beta, \xi_j^\beta)(H(\varepsilon)), \quad \bar{U}_1(q_\alpha, \xi_j^\alpha)(\varepsilon) = \bar{U}_1(q_\beta, \xi_j^\beta)(H(\varepsilon)) \quad \text{and} \\ (\forall i < m)(\nu_\alpha(i) = (\xi_j^\alpha, \varepsilon) \Leftrightarrow \nu_\beta(i) = (\xi_j^\beta, H(\varepsilon))).$$

Let w^* be the set of these $i < m$ that for some (equivalently: all) $\alpha < \omega_1$ we have $\nu_\alpha(i) \in \zeta^* \times \omega_1$.

Claim 2.3.1. *There are $q^* \in \mathbb{P}_{\zeta^*}$ and $\alpha < \beta < \omega_1$ such that q^* is stronger than both $q_\alpha \restriction \zeta^*$ and $q_\beta \restriction \zeta^*$ and $(\forall i \in w^*)(\{\nu_\alpha(i), \nu_\beta(i)\} \notin E_{\ell_i})$.*

Proof of the claim. Let $G_{\zeta^*} \subseteq \mathbb{P}_{\zeta^*}$ be a generic filter over \mathbf{V} . Work in $\mathbf{V}[G_{\zeta^*}]$. By the ccc of \mathbb{P}_{ζ^*} , the set $\{\alpha < \omega_1 : q_\alpha \restriction \zeta^* \in G_{\zeta^*}\}$ is uncountable. Look at the

sequence $\langle \nu_\alpha \upharpoonright w^* : q_\alpha \upharpoonright \zeta^* \in G_{\zeta^*} \rangle$. By assumption (γ) of the lemma, it cannot be a $|w^*|$ -selector, so there are $\alpha < \beta < \omega_1$ such that

$$q_\alpha \upharpoonright \zeta^* \ \& \ q_\beta \upharpoonright \zeta^* \ \& \ (\forall i \in w^*)(\{\nu_\alpha(i), \nu_\beta(i)\} \notin E_{\ell_i}).$$

Now, going back to \mathbf{V} , we easily find a condition $q^* \in \mathbb{P}_{\zeta^*}$ such that q^*, α, β are as required. \square

Let q^*, α, β be as guaranteed by 2.3.1. For $j < N$ such that $\xi_j^\alpha \in I_\ell$, $\ell \leq k$, let $H_j : \text{dom}(\bar{c}(q_\alpha, \xi_j^\alpha)) \rightarrow \text{dom}(\bar{c}(q_\beta, \xi_j^\beta))$ be the order preserving bijection (see clause (5) above). We define a condition $q^+ \in \mathbb{P}_\gamma$ as follows:

$\text{dom}(q^+) = \text{dom}(q^*) \cup \text{dom}(q_\alpha) \cup \text{dom}(q_\beta)$ and

- $q^+ \upharpoonright \zeta^* = q^*$,
- if $\xi_j^\alpha \in I_\ell$, $n^* \leq j < N$, $\ell \leq k$ then

$$q^+(\xi_j^\alpha) = \bar{c}_0(q_\alpha, \xi_j^\alpha) \quad \text{and} \quad q^+(\xi_j^\beta) = \bar{c}_0(q_\beta, \xi_j^\beta),$$

- if $n^* \leq j < N$, $\xi_j^\alpha \notin \bigcup_{\ell \leq k} I_\ell$ then $q^+(\xi_j^\alpha) = q_\alpha(\xi_j^\alpha)$, $q^+(\xi_j^\beta) = q_\beta(\xi_j^\beta)$.

It should be clear that $q^+ \in \mathbb{P}_\gamma$ is a condition stronger than both q_α and q_β . If $i < m$ and $\varepsilon < \omega_1$ then

$$\nu_\alpha(i) = (\xi_j^\alpha, \varepsilon) \Leftrightarrow \nu_\beta(i) = (\xi_j^\beta, H_j(\varepsilon)).$$

If $i \in m \setminus w^*$, $\nu_\alpha(i) = (\xi_j^\alpha, \varepsilon)$, $n^* \leq j < N$ and $\dot{p}_{\varepsilon, \xi_j^\alpha}^*$, $\dot{p}_{H(\varepsilon), \xi_j^\beta}^*$ are the names for ε^{th} ($H(\varepsilon)^{\text{th}}$ respectively) generic reals added by $\dot{\mathbb{Q}}_{\xi_j^\alpha}$ ($\dot{\mathbb{Q}}_{\xi_j^\beta}$, resp.) then

$$q^+ \Vdash_{\mathbb{P}_\gamma} \quad \text{“ } \dot{p}_{\varepsilon, \xi_j^\alpha}^* \in \bar{U}_0(q_\alpha, \xi_j^\alpha)(\varepsilon) = \bar{U}_0(q_\beta, \xi_j^\beta)(H(\varepsilon)) \quad \text{and} \\ \dot{p}_{H(\varepsilon), \xi_j^\beta}^* \in \bar{U}_1(q_\beta, \xi_j^\beta)(H(\varepsilon)) = \bar{U}_1(q_\alpha, \xi_j^\alpha)(\varepsilon) \text{”}.$$

If $i \in w^*$ then look at the choice of q^*, α, β (see 2.3.1). Putting everything together we conclude that

$$q^+ \Vdash \text{“ } (\forall i < m)(\{\dot{\nu}_\alpha(i), \dot{\nu}_\beta(i)\} \notin E_{\ell_i}) \text{”},$$

a contradiction \square

3. PROOF OF THEOREM 0.6

Let κ be regular cardinal such that $\kappa = \kappa^{<\kappa} \geq \aleph_2$. By induction on $\xi \leq \kappa$ we build a finite support iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \kappa \rangle$ and sequences $\langle \dot{\mathbb{P}}^\xi, I_\xi : \xi < \kappa \rangle$, $\langle \dot{A}_\xi^\xi : \xi < \zeta \leq \kappa \rangle$ and $\langle \dot{p}^\zeta : \zeta \in I_\xi \rangle$ such that for each $\xi, \xi_0, \xi_1 < \kappa$

- (i) $\dot{\mathbb{Q}}_\xi$ is a \mathbb{P}_ξ -name for a ccc forcing notion on a bounded subset of κ ,
- (ii) $I_\xi \in [\{2 \cdot \alpha : \xi < \alpha < \kappa\}]^\kappa$, $\dot{\mathbb{P}}^\xi$ is a \mathbb{P}_ξ -name for an element of \mathcal{K} , and if $\xi_0 \neq \xi_1$ then $I_{\xi_0} \cap I_{\xi_1} = \emptyset$,
- (iii) for $\zeta \in I_\xi$, \dot{p}^ζ is a \mathbb{P}_ζ -name for a condition in $\dot{\mathbb{P}}^\xi$ for which $\mathbb{C}(\dot{\mathbb{P}}^\xi, \dot{p}^\zeta)$ is defined,
- (iv) if $\zeta \in I_\xi$ then $\dot{\mathbb{Q}}_\zeta$ is (equivalent to) $\mathbb{C}^{\omega_1}(\dot{\mathbb{P}}^\xi, \dot{p}^\zeta)$,
- (v) if $\xi < \zeta \leq \kappa$ then \dot{A}_ζ^ξ is the \mathbb{P}_ζ -name for the set $\bigcup \{ \dot{Z}_{\dot{p}^\varepsilon}^{\dot{\mathbb{P}}^\xi} : \varepsilon \in I_\xi \cap \zeta \} \subseteq \dot{\mathbb{P}}^\xi \times \omega_1$ (where $\dot{Z}_{\dot{p}^\varepsilon}^{\dot{\mathbb{P}}^\xi}$ is the generic object added by $\dot{\mathbb{Q}}_\zeta$; compare 2.3),
- (vi) for each $\zeta \leq \kappa$

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \{(\dot{A}_\zeta^\xi, \mathcal{U}(\dot{\mathbb{P}}^\xi), E(\dot{\mathbb{P}}^\xi) \cap [\dot{A}_\zeta^\xi]^2) : \varepsilon < \zeta\} \text{ is an } \mathcal{S}\text{-family of good graphs”},$$

(vii) if \dot{Q} is a \mathbb{P}_κ -name for a ccc forcing notion on a bounded subset of κ then

$$|\{\zeta < \kappa : \Vdash_{\mathbb{P}_\zeta} \text{“} \dot{Q}_\zeta = \dot{Q} \text{”}\}| = \kappa,$$

(viii) if \dot{P} is a \mathbb{P}_κ -name for an element of \mathcal{K} then then for some $\varepsilon < \kappa$ we have

$$\Vdash_{\mathbb{P}_\varepsilon} \text{“} \dot{P}^\varepsilon = \dot{P} \text{”} \quad \text{and} \quad \Vdash_{\mathbb{P}_\kappa} \text{“} \text{the set } \{\dot{p}^\zeta : \zeta \in I_\varepsilon\} \text{ is dense in } \dot{P} \text{”}.$$

We use the standard bookkeeping arguments to choose the lists $\langle \dot{P}^\xi, I_\xi : \xi < \kappa \rangle$, $\langle \dot{p}^\zeta : \zeta \in I_\xi \rangle$ so that clauses (ii), (iii) and (viii) are satisfied. Similarly we choose a list $\langle \dot{Q}'_\xi : \xi < \kappa \rangle$ of all \mathbb{P}_κ -names for partial orders on bounded subsets of κ so that each name appears κ many times in the list, and \dot{Q}'_ξ is a \mathbb{P}_ξ -name (for $\xi < \kappa$) (this list will be used to take care of clauses (i), (vii)).

Now we have to be more specific. So suppose that for some $\xi < \kappa$ we have already defined the iteration $\langle \mathbb{P}_\zeta, \dot{Q}_\zeta : \zeta < \xi \rangle$. If ξ is a limit ordinal, before we go further we should argue that the clause (vi) is satisfied by the limit \mathbb{P}_ξ , i.e.,

$$\Vdash_{\mathbb{P}_\xi} \text{“} \mathcal{F}_\xi \stackrel{\text{def}}{=} \{(\dot{A}_\zeta^\varepsilon, \mathcal{U}(\dot{P}^\varepsilon), E(\dot{P}^\varepsilon) \cap [\dot{A}_\zeta^\varepsilon]^2) : \varepsilon < \zeta\} \text{ is an } \mathcal{S}\text{-family of good graphs”}.$$

But this is immediate by 2.3 — if a problem occurs than it is caused by a finite subfamily of \mathcal{F} and we may assume that the respective forcing notions $\mathbb{P}^{\varepsilon_\ell}$ are from the ground model.

Suppose $\xi = 2 \cdot \alpha + 1$. Then we look at the \mathbb{P}_α -name \dot{Q}'_α and we ask if, in $\mathbf{V}^{\mathbb{P}^\xi}$, it is a ccc forcing notion. If not that we let \mathbb{Q}_ξ be the Cohen real forcing. If yes, then we we ask if (in $\mathbf{V}^{\mathbb{P}^\xi}$) it forces that \mathcal{F}_ξ remains an \mathcal{S} -family. If again yes, then we let \dot{Q}_ξ be \dot{Q}'_α ; otherwise $\dot{Q}_\xi = \mathbb{P}^{\mathcal{F}_\xi}$ (see 1.5). In any case we are sure that the relevant instances of clauses (i)–(viii) are satisfied (remember 1.6).

Assume now that $\xi = 2 \cdot \alpha \in I_\zeta$, $\zeta < \kappa$. Then clause (iv) determines \dot{Q}_ξ . We should show that the clause (vi) holds true. Suppose that we may find a condition $q \in \mathbb{P}_{\xi+1}$, an integer $m < \omega$, $\varepsilon_0, \dots, \varepsilon_{m-1} \leq \xi$ and $\mathbb{P}_{\xi+1}$ -names $\dot{\nu}_\alpha$ (for $\alpha < \omega_1$) of sequences of length m such that

$$q \Vdash_{\mathbb{P}_{\xi+1}} \text{“} (\forall \alpha < \beta < \omega_1)(\dot{\nu}_\alpha \neq \dot{\nu}_\beta) \ \& \ (\forall \alpha < \omega_1)(\forall i < m)(\dot{\nu}_\alpha(i) \in \dot{A}_{\xi+1}^{\varepsilon_i}) \\ (\forall \alpha < \beta < \omega_1)(\exists i < m)(\{\dot{\nu}_\alpha(i), \dot{\nu}_\beta(i)\} \in E(\dot{P}^{\varepsilon_i})) \text{”}.$$

We may additionally demand that for some $k < m$ we have

$$q \Vdash_{\mathbb{P}_{\xi+1}} \text{“} (\forall \alpha < \omega_1)(\forall i < k)(\dot{\nu}_\alpha(i) \in \dot{A}_\xi^{\varepsilon_i}) \quad \text{and} \\ (\forall \alpha < \omega_1)(\forall i \in [k, m])(\varepsilon_i = \xi \ \& \ \dot{\nu}_\alpha(i) \in \dot{Z}_{\dot{p}^\xi}^{\dot{P}^\xi}) \text{”}.$$

Claim 3.0.2. *Suppose that $\mathbb{P} \in \mathcal{K}$, $p \in \mathbb{P}$ (and $\mathbb{C}(\mathbb{P}, p)$ is defined). Then*

$$\Vdash_{\mathbb{C}^{\omega_1}(\mathbb{P}, p)} \text{“} (\forall s_0, s_1 \in \dot{Z}_p^{\mathbb{P}})(\{s_0, s_1\} \notin E(\mathbb{P})) \text{”}.$$

Proof of the claim. Like 2.1. □

It follows from 3.0.2 that

$$q \Vdash_{\mathbb{P}_{\xi+1}} \text{“} (\forall \alpha < \beta < \omega_1)(\exists i < k)(\{\dot{\nu}_\alpha(i), \dot{\nu}_\beta(i)\} \in E(\dot{P}^{\varepsilon_i})) \text{”}.$$

But, by 1.6, we have

$$\Vdash_{\mathbb{P}_{\xi+1}} \text{“} \mathcal{F}_\xi \text{ is an } \mathcal{S}\text{-family of good graphs”},$$

so we get an immediate contradiction.

Finally if $\xi = 2 \cdot \alpha \notin \bigcup_{\zeta < \kappa} I_\zeta$ then we let \mathbb{Q}_ξ be the Cohen real forcing (again all clauses are preserved).

The construction is complete. We claim that the limit forcing notion \mathbb{P}_κ is as required in 0.6. Clearly it satisfies the ccc and (a dense subset of it) is of size κ . Clause (vii) guarantees that $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \kappa \ \& \ \mathbf{MA}$. It follows from the clause (vi) and 2.3 that

$\Vdash_{\mathbb{P}_\kappa} \mathcal{F}_\kappa \stackrel{\text{def}}{=} \{(\dot{A}_\zeta^\varepsilon, \mathcal{U}(\dot{\mathbb{P}}^\varepsilon), E(\dot{\mathbb{P}}^\varepsilon) \cap [\dot{A}_\zeta^\varepsilon]^2) : \varepsilon < \kappa\}$ is an \mathcal{S} -family of good graphs".

Clauses (v)+(viii) (and the definition of $E(\dot{\mathbb{P}}^\varepsilon)$) imply that for each $\varepsilon < \kappa$

$\Vdash_{\mathbb{P}_\kappa} (\dot{A}_\zeta^\varepsilon, \mathcal{U}(\dot{\mathbb{P}}^\varepsilon), E(\dot{\mathbb{P}}^\varepsilon) \cap [\dot{A}_\zeta^\varepsilon]^2)$ is densely representably by $\perp_{\dot{\mathbb{P}}^\varepsilon}$ ".

Consequently, by 1.4 and clause (viii) we get

$$\Vdash_{\mathbb{P}_\kappa} (\forall \mathbb{P} \in \mathcal{K})(\neg \mathbf{FA}_{\aleph_1}(\mathbb{P})),$$

finishing the proof of 0.6.

Corollary 3.1. *It is consistent with $\mathbf{MA} + \neg \text{CH}$ that any forcing notion $\mathbb{P} \in \mathcal{K}$ collapses \mathfrak{c} to ω_1 (and thus is not ω_1 -proper).*

4. OPEN PROBLEMS

The model constructed in the previous section provides $(\forall \mathbb{P} \in \mathcal{K})(\neg \mathbf{FA}_{\aleph_1}(\mathbb{P}))$ by dealing with each forcing $\mathbb{P} \in \mathcal{K}$ separately. We would like to have one common reason for $\neg \mathbf{FA}_{\aleph_1}(\mathbb{P})$ for all forcing notions $\mathbb{P} \in \mathcal{K}$, i.e., a combinatorial principle \mathcal{P} which is consistent with $\mathbf{MA} + \neg \text{CH}$ and which implies $(\forall \mathbb{P} \in \mathcal{K})(\neg \mathbf{FA}_{\aleph_1}(\mathbb{P}))$. A possible candidate for a principle like that was already pointed in [6, §2]. As we stated in the Introduction, our method is a slight generalization of that of Steprans. Steprans' method in turn was based on the proof of Abraham, Rubin and Shelah [1] that it is consistent with $\mathbf{MA} + \neg \text{CH}$ that there are two non-isomorphic \aleph_1 -dense sets of reals. In the latter proof, a 2-entangled set of reals was used. This leads us to the following question

Problem 4.1 (Compare [6, Question 2.4]). Does the existence of a 2-entangled set of reals of size \aleph_1 imply $(\forall \mathbb{P} \in \mathcal{K})(\neg \mathbf{FA}_{\aleph_1}(\mathbb{P}))$?

If one tries to repeat the proof of [14, Theorem 2.1] for elements of \mathcal{K} then one gets into some problems in cases 1,3 of Definition 0.4. Possible reason for it is that a proof as in [14, Theorem 2.1] would give a property that seems to be stronger than $\neg \mathbf{FA}_{\aleph_1}(\mathbb{P})$.

Definition 4.2. Let \mathbb{P} be a forcing notion of size \mathfrak{c} , $\bar{p} = \langle p_i : i < \mathfrak{c} \rangle \subseteq \mathbb{P}$. We say that \bar{p} is an JMSH-sequence if

$(\oplus)_{\text{JMSH}}$ given $\langle F_\alpha : \alpha < \omega_1 \rangle$ pairwise disjoint finite subsets of \mathfrak{c} , there exist $\alpha < \beta < \omega_1$ such that

$$(\forall i \in F_\alpha)(\forall j \in F_\beta)(p_i \perp_{\mathbb{P}} p_j).$$

Proposition 4.3. *Suppose that \mathbb{P} is a forcing notion of size \mathfrak{c} such that*

1. *above any condition in \mathbb{P} , there is an antichain (in \mathbb{P}) of size \mathfrak{c} , and*
2. *there is an JMSH-sequence $\bar{p} \subseteq \mathbb{P}$ which is dense in \mathbb{P} .*

Then $\neg \mathbf{FA}_{\aleph_1}(\mathbb{P})$.

Problem 4.4. 1. Is the existence of dense JMSH-sequences in \mathbb{P} really stronger than $\neg \mathbf{FA}_{\aleph_1}(\mathbb{P})$ (for \mathbb{P} of size \mathfrak{c} satisfying the assumption 4.3(1)) ?

2. Is it consistent with $\mathbf{MA} + \neg\text{CH}$ that for each $\mathbb{P} \in \mathcal{K}$ there is a dense JMSH-sequence in \mathbb{P} ?

On the other hand, Judah, Miller and Shelah [14] and Goldstern, Johnson and Spinas [11] showed that $\mathbf{MA}_{\omega_1}(\text{ccc})$ implies the forcing axiom for the Miller and Laver forcing notions. This gives a strong expectation that $\mathbf{MA}_{\omega_1}(\text{ccc})$ implies forcing axioms for most of forcing notions (with norms) adding unbounded reals. Brendle [5, Proposition 5.1] showed that \mathbf{MA} implies that the Laver forcing, the Mathias forcing, the Miller forcing and the Blass-Shelah forcing are α -proper for all $\alpha < \mathfrak{c}$. Again, one expects that this could be generalized further.

Problem 4.5. Let \mathcal{K}^\perp be the class of the forcing notions of [17] which are not in \mathcal{K} for nontrivial reasons.

1. Does $\mathbf{MA} + \neg\text{CH}$ imply $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{K}^\perp$?
2. Does $\mathbf{MA} + \neg\text{CH}$ imply that all $\mathbb{P} \in \mathcal{K}^\perp$ are α -proper (for all $\alpha < \mathfrak{c}$) ?

Finally, possible further generalizations of the present paper could go into the direction of nep/snep forcing notions of Shelah [18], [19].

- Problem 4.6.**
1. Is $\mathbf{MA} + \neg\text{CH}$ consistent with the failure of $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ for all snep ω^ω -bounding forcing notions \mathbb{P} which do not have ccc above any condition?
 2. Does $\mathbf{MA} + \neg\text{CH}$ imply $\mathbf{FA}_{\aleph_1}(\mathbb{P})$ for all snep forcing notions \mathbb{P} adding unbounded reals?

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