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Knaster-Tarski Revisited

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Abstract. The concept "complete partial order" is generalized to the concept "functionally complete partial order." The correctness of a corresponding generalization of the Knaster-Tarski fixpoint theorem is proved. The theory is applied to yield a fixpoint mapping theorem.

1. Introduction

In [Hes90, Dij92, DiT93] certain generalizations of the Knaster-Tarski fixpoint theorem [Tar55] are proved. In a recent "Call for Proofs" [Dij95] readers are invited to submit their most elegant proof of yet another related generalization. In this note I intend to meet a slightly different challenge, namely to find a still more general version. The approach to this problem is suggested by the structure of the proof in [DiT93]. In fact, large parts of that proof carry over to the proof of the present generalization.

The outline of this paper is as follows. Section 2 introduces the basic terminology. Section 3 sketches the challenge and Section 4 sketches the outline of the proof. In Sections 5, 6 and 7 the challenge is met and the proof is given. Section 8 contains some specific consequences. Section 9 describes the precise extent of the generalization. Section 10 contains a modest application to fixpoint mapping (also known as "fixpoint fusion"). We conclude in Section 11.

2. Terminology

We assume the reader to be familiar with the notions of poset, upper bound, supremum, complete lattice and monotonicity.

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In the rest of this section, $(P, \sqsubseteq)$ is a fixed poset and f is a fixed endofunction on P. The identity function is denoted by *Id*. Whenever no explicit type is mentioned for some set, it is understood to be a subset of the universe P.
The irreflexive counterpart of $\sqsubseteq$ is denoted as $\sqsubset$.
If set S has a supremum, we denote it by $\sqcup S$.
Set S is called *linear* iff: $p \sqsubseteq q \;\vee\; q \sqsubseteq p$, for all $p, q \in S$.
We call the partial order $(P, \sqsubseteq)$ a *complete partial order (cpo)* iff: every linear subset of P has a supremum in P.
Set S is called *f-closed* iff: $f.p \in S$, for all $p \in S$.
Element p is called a *fixpoint of f* iff: $f.p = p$.
Element p is called a *prefixpoint of f* iff: $f.p \sqsubseteq p$.
Element p is called a *maximal element of* S iff: $p \in S \;\wedge\; \neg(\exists q \in S :: \; p \sqsubset q)$.
Element p is called the *maximum of* S iff: $p \in S \;\wedge\; (\forall q \in S :: \; q \sqsubseteq p)$.
Endofunction f is called *expanding on* set S iff: $p \sqsubseteq f.p$ for all $p \in S$. Also, in this case we say that set S is *f-expanding*.

We will have use for the following fresh notions.
Set $S \subseteq P$ *inherits the supremum* of a set $T \subseteq S$ iff:
T has a supremum in $P \;\Rightarrow\; \sqcup T \in S$.
Endofunction f on P is called *pseudo-monotonic* iff:
$p \sqsubseteq q \;\wedge\; f.p \sqsubseteq q \;\Rightarrow\; f.p \sqsubseteq f.q$, for all $p, q \in P$.
Set S is called *f-narrow* iff:
$p \sqsubseteq q \;\vee\; f.q \sqsubseteq p$, for all $p, q \in S$.
Note that a monotonic endofunction is certainly pseudo-monotonic. Also, an expanding function is certainly pseudo-monotonic. Finally, a set is linear if and only if it is *Id*-narrow.

3. The Challenge

Tarski's Lattice-theoretical fixpoint theorem in [Tar55] states that the fixpoints of a monotonic endofunction on a complete lattice constitute a complete lattice by themselves. In the course of the proof it is first shown that a monotonic endofunction on a complete lattice has a least fixpoint. In Computing Theory this is the most interesting aspect of the theorem. So much so, that quite often the intermediate result is also called Knaster-Tarski's theorem. We will follow suit.

In order to introduce the various versions of the Knaster-Tarski fixpoint theorem we need the following concepts.

First a set property *prop* is identified, i.e. a boolean function on subsets of P. Next we define: poset $(P, \sqsubseteq)$ is *prop-complete* iff every set T satisfying *prop* has a supremum in P. Finally, we define: set S in *prop*-complete poset $(P, \sqsubseteq)$ is a *prop-complete subset* iff S inherits the supremum of every set $T \subseteq S$ that satisfies *prop*. Note that if S is a *prop*-complete subset of a *prop*-complete universe and *prop* does not depend on the universe, then $(S, \sqsubseteq)$ is a *prop*-complete poset in itself (with the obvious restriction of $\sqsubseteq$ to S).

The generic theorem is now stated as:

Let f be a monotonic endofunction on a *prop*-complete poset $(P, \sqsubseteq)$. Then f has a least fixpoint, denoted μf, and in addition:

(i) μf satisfies the equation in p : $(\forall x : \; f.x \sqsubseteq x : \; p \sqsubseteq x)$,

(ii) μf is contained in every f-closed *prop*-complete subset of P.

By definition, in a complete lattice *any* set has a supremum. So the original theorem of Knaster-Tarski corresponds roughly to the choice of *prop* given by $prop.S = true$, for any S. Further, the theorem of Knaster-Tarski in [Hes90, Dij92, DiT93] in a cpo corresponds to the choice of *prop* given by $prop.S =$ "S is linear", for any S. Without proof we state that the theorem of Knaster-Tarski in [Dij95] corresponds to the choice of *prop* given by $prop.S =$ "S is the f-image of a linear set", for any S.

It is clearly the case that the stronger *prop* is, the weaker the notion of *prop*-completeness will be and the stronger the resulting theorem. So we strive to have *prop* as strong as feasible.

4. The Map

Our choice of *prop* is suggested by the steps of the proof in [DiT93]. There, *prop* is given by $prop.S =$ "S is linear", for all S. We summarize the landmarks of that proof:

(a) there is a smallest f-closed *prop*-complete subset, say M,
(b) if f is expanding or monotonic, $prop.M$ holds, therefore $\sqcup M$ exists and is contained in every *prop*-complete subset; $\sqcup M$ is the unique fixpoint of f in M,
(c) if f is monotonic, $\sqcup M$ is the least prefixpoint of f in P.

Interestingly, in the course of the proof it is also shown that M is f-expanding and f-narrow. So M satisfies a property stronger than linearity and we are led to consider the possibility that the same type of proof still holds if we choose $prop.S$ as: S is f-closed, f-expanding and f-narrow. In a further attempt to strengthen *prop* while retaining the proof, I observed that M contains $\perp$. Thereupon, Wim H. Hesselink observed that M inherits the suprema of *all* of its linear subsets and suggested to strengthen *prop* accordingly. Also, it turns out that the condition "f is expanding or monotonic" in landmark (b) can be weakened to "f is pseudo-monotonic". And lastly, the theorem can be generalized to cover not only the least fixpoint, but other fixpoints as well.

In the following section we fill in the details of the resulting definitions and travel to landmark (a). Our specific choice of *prop* will be expressed as "being a trail".

5. First Steps

Throughout this section, we assume that $(P, \sqsubseteq)$ is a partial order and f an endofunction on P. The domains P and $Pow(P)$ will be implicit.

We introduce the following notions:

(1) Definition. A set S is an *f-stretch* (or *stretch* for short, if function f is clear from the context) iff:

(a) S is f-expanding,
(b) S is f-narrow, and
(c) S is f-closed.

(2) Definition. A set T is an *f-trail* (or *trail* for short, if function f is clear from the context) iff:

(a) T is f-expanding,
(b) T is f-narrow,
(c) T is f-closed, and
(d) T inherits suprema of all subsets $S \subseteq T$.

Observe that, if an f-trail has a supremum, that supremum is contained in it because of (2.d), and is a fixpoint of f because of (2.a) and (2.c).

(3) Definition. The triple $(P, \sqsubseteq, f)$, is a *functionally complete partial order (fcpo)* if and only if every f-trail has a supremum.

(4) Definition. Let $(P, \sqsubseteq, f)$ be an fcpo. A subset Q of P is a *sub-fcpo* iff:

(a) Q is f-closed, and
(b) Q inherits suprema of all f-stretches $S \subseteq Q$.

For examples of stretches, trails and fcpo's we refer the reader to section 9.

Instead of the defining property (4.b) the reader may have expected the weaker: "Q inherits suprema of all f-trails $T \subseteq Q$". With that choice however, not every sub-fcpo would necessarily correspond to an fcpo. The source of this complication is the introduction of $\sqcup$-inheritance as one of the defining properties of an f-trail. As a consequence, "being an f-trail" depends implicitly on the universe.

Evidently, if $(P, \sqsubseteq, f)$ is an fcpo, then P is a sub-fcpo. Also, any f-trail is both an f-stretch and a sub-fcpo. Note further, that all defining properties are preserved by intersection. It follows that the intersection of an arbitrary collection of f-stretches (resp. f-trails, sub-fcpo's) is itself an f-stretch (resp. f-trail, sub-fcpo).

On inspection of the definitions (3) and (2) we find that, if $(P, \sqsubseteq)$ is a cpo and f an endofunction on P, then $(P, \sqsubseteq, f)$ is an fcpo. The converse for the special endofunction *Id* is Corollary (31) in Section 8. In Section 9 we will exhibit an fcpo $(P, \sqsubseteq, f)$ such that $(P, \sqsubseteq)$ is *not* a cpo.

We now prove a small collection of useful properties concerning existence and inheritance of suprema. Notably, we will show that every sub-fcpo corresponds to an fcpo.

Any subset of an f-stretch is both f-expanding and f-narrow. Therefore the following property is sometimes useful.

(5) Property. Let set S be f-expanding and f-narrow. Then S is linear and f is monotonic on S.

Proof: That S is linear is easily proved. In order to prove that f is monotonic on S we assume $p, q \in S$ and observe:

$$
\begin{array}{ll}
 & f.p \sqsubseteq f.q \\
\Leftarrow & \quad \{ S \text{ is } f\text{-expanding} \} \\
 & f.p \sqsubseteq q \\
\Leftarrow & \quad \{ S \text{ is } f\text{-narrow} \} \\
 & \neg(q \sqsubseteq p) \\
= & \quad \{ S \text{ is linear}, \sqsubseteq \text{ is antisymmetric} \} \\
 & p \sqsubseteq q \ \wedge \ p \neq q
\end{array}
$$

Additionally, we observe $p = q \;\Rightarrow\; f.p \sqsubseteq f.q$. □

Next we prove two properties of stretches in sub-fcpo's.

(6) PROPERTY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let Q be a sub-fcpo. Let S be an f-stretch in Q. Let U be a subset of S. Let set T be defined by

(∗) $p \in T \;\equiv\; p \in S \;\wedge\; (\exists q \in U :: \; p \sqsubseteq q)$

Then we have:

(i) r is an upper bound of $T \;\equiv\; r$ is an upper bound of U, for every r

(ii) T is an f-stretch $\;\vee\;$ U has a maximum

Proof: In order to prove (6.i) we observe:

$$
\begin{array}{ll}
 & r \text{ is an upper bound of } T \\
\Rightarrow & \quad \{ \sqsubseteq \text{ reflexive, so } U \subseteq T \} \\
 & r \text{ is an upper bound of } U \\
\Rightarrow & \quad \{ (6.*), \text{ definition of upper bound} \} \\
 & (\forall p \in T :: \; (\exists q \in U :: \; p \sqsubseteq q \;\wedge\; q \sqsubseteq r)) \\
\Rightarrow & \quad \{ \sqsubseteq \text{ transitive} \} \\
 & r \text{ is an upper bound of } T
\end{array}
$$

In order to prove (6.ii) we assume that U has no maximum and prove that T is an f-stretch. Because T is a subset of stretch S, it suffices to prove that T is f-closed. Thus we assume that p is any element of T and have to prove $f.p \in T$. From (6.∗) it follows that there exists a $q \in U$ such that $p \sqsubseteq q$. Now we note, using (5), that from $U \subseteq S$ it follows that U is linear. From the case premise it then follows that U does not have a maximal element. So there exists an $r \in U$ such that $q \sqsubset r$. And now we observe:

$$
\begin{array}{ll}
 & f.p \in T \\
\Leftarrow & \quad \{ \text{instantiation of } (6.*) \} \\
 & f.p \in S \;\wedge\; f.p \sqsubseteq r \\
= & \quad \{ S \text{ is } f\text{-closed} \} \\
 & f.p \sqsubseteq r \\
\Leftarrow & \quad \{ p, r \in U,\; U \subseteq S,\; S \text{ is } f\text{-narrow} \} \\
 & \neg(r \sqsubseteq p) \\
= & \quad \{ p \sqsubseteq q,\; q \sqsubset r \} \\
 & \mathit{true}
\end{array}
$$

□

(7) PROPERTY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let Q be a sub-fcpo. Let S be an f-stretch in Q. Let U be a subset of S. Then Q inherits the supremum of U.

Proof: In the case that U has a maximum, the property is trivial. For the rest of the proof we assume that U has no maximum. We let T be defined by (6.∗). From (6.ii) it follows that T is an f-stretch.

Now if p is the supremum of U, then by (6.i), it is also the supremum of the stretch T. Therefore, since Q is a sub-fcpo that contains T, property (4) implies $p \in Q$. This shows that Q inherits the supremum of U. □

The following property is an extension of the defining property (3) of an fcpo.

(8) PROPERTY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let Q be an f-trail. Let U be a subset of Q. Then U has a supremum.

Proof: The proof is by contradiction. We assume that U does not have a supremum. It suffices to show that U has a supremum. We let T be defined by:

(9) $\quad p \in T \quad \equiv \quad p \in Q \ \wedge \ (\exists q \in U :: \ p \sqsubseteq q)$

Because Q is an f-trail, Q is both a sub-fcpo and an f-stretch. From (6.ii) we find that T is an f-stretch.

And now we prove that T is an f-trail by proving that stretch T inherits the supremum of every set $V \subseteq T$. Thus, let V be any subset of T, and let V have a supremum. It follows from (7) with $S := Q$ and $U := V$ that $\sqcup V \in Q$. Then we observe:

$$
\begin{array}{ll}
 & \sqcup V \notin T \\
= & \quad \{\,(9) \text{ with } p := \sqcup V,\ \sqcup V \in Q\,\} \\
 & (\forall q \in U :: \ \neg(\sqcup V \sqsubseteq q)) \\
\Rightarrow & \quad \{\,\sqcup V \in Q,\ U \subseteq Q;\ Q \text{ linear}\,\} \\
 & (\forall q \in U :: \ q \sqsubseteq \sqcup V) \\
= & \quad \{\,(7) \text{ with } S, U := Q, V\,\} \\
 & \sqcup V \text{ is an upper bound of } U \\
= & \quad \{\,(6.\text{i}) \text{ with } S := Q\,\} \\
 & \sqcup V \text{ is an upper bound of } T \\
\Rightarrow & \quad \{\,V \subseteq T\,\} \\
 & \sqcup V \text{ is the supremum of } T \\
= & \quad \{\,(6.\text{i}) \text{ with } S := Q\,\} \\
 & \sqcup V \text{ is the supremum of } U \\
= & \quad \{\,\text{by assumption, } U \text{ has no supremum}\,\} \\
 & \textit{false}
\end{array}
$$

This proves, that T is an f-trail in fcpo $(P, \sqsubseteq, f)$. It follows that T has a supremum and, by (6.i), that U has a supremum. $\square$

The following property justifies the name sub-fcpo.

(10) PROPERTY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let Q be a sub-fcpo. Then $(Q, \sqsubseteq, f)$ is an fcpo (with $\sqsubseteq$ and f restricted to Q).

Proof: Two subtleties are involved: property (4.b) mentions stretches, not subsets as in (2.d), and the supremum of a set (and consequently, being a trail) depends implicitly on the universe.

With $\sqsubseteq$ and f restricted to Q, it is evident that $(Q, \sqsubseteq)$ is a partial order and f an endofunction. The remaining obligation is to prove that every trail in $(Q, \sqsubseteq, f)$ has a supremum. Thus, let T be any trail in $(Q, \sqsubseteq, f)$.

Now T is certainly a stretch, so by (7), sub-fcpo Q inherits the supremum in $(P, \sqsubseteq, f)$ of every subset of T. Because inheritance is transitive and trail T inherits the supremum in $(Q, \sqsubseteq)$ of every subset of T, T inherits the supremum in $(P, \sqsubseteq)$ of every subset of T. From definition (2) it then follows, that trail T in $(Q, \sqsubseteq, f)$ is also a trail in fcpo $(P, \sqsubseteq, f)$. Therefore T contains its supremum with respect to $(P, \sqsubseteq)$, i.e. T has a maximum. That maximum is also the supremum of T in $(Q, \sqsubseteq)$. $\square$

The following property gives a formally weaker, but (in an fcpo) equivalent characterization of f-trails.

(11) PROPERTY. Let $(P, \sqsubseteq, f)$ be an fcpo. Then we have:

$$Q \text{ is an } f\text{-expanding, } f\text{-narrow sub-fcpo} \quad \equiv \quad Q \text{ is an } f\text{-trail}$$

Proof: The implication ($\Leftarrow$) follows directly from the definitions. In order to prove the converse, we assume that Q is any f-expanding, f-narrow sub-fcpo. From definitions (1) and (4) it follows that Q is a stretch, and by definition (2) it suffices to show that Q inherits the supremum of every set $U \subseteq Q$. This follows directly from (7) with $S := Q$. $\square$

And now we are ready for landmark (a).

(12) THEOREM. Let $(P, \sqsubseteq, f)$ be an fcpo and U a subset of P. Then there exists a smallest sub-fcpo (with respect to set inclusion) that contains U.

Proof: The intersection of all sub-fcpo's that contain U is a sub-fcpo that contains U, and it is the smallest one. $\square$

Thus, in view of (10), landmark (a) has been reached with the choice: $prop.S =$ "S is an f-trail," for all S.

6. Carrying On

Landmark (b) requires a somewhat longer journey. It is subsumed by the following theorem.

(13) THEOREM. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be pseudo-monotonic. Let f be expanding on subset X. Let M be the smallest sub-fcpo that contains X. Finally, suppose that the following holds:

(*) $(\forall p \in X,\ q \in M :: \ p \sqsubseteq q \quad \vee \quad f.q \sqsubseteq p)$

Then we have:

(i) M is an f-trail,
(ii) $\sqcup M$ is the unique fixpoint of f among the upper bounds of X in M.

6.1. Proof of Theorem (13)

Note that subset X is f-expanding and f-narrow, but not necessarily f-closed. Concerning the proof of (13.i) we observe that M is a sub-fcpo. From property (11) we conclude that it suffices to prove

(a) M is f-expanding, and
(b) M is f-narrow.

Concerning the proof of (13.ii) we observe that, on condition that (13.i) has been proved, it follows, because $(P, \sqsubseteq, f)$ is an fcpo, that the supremum of M exists and is a fixpoint of f. As X is a subset of M, the supremum of M is an upper bound of X. So it suffices to show:

(c) $p = f.p \ \Rightarrow \ p = \sqcup M$ for any $p \in M$ such that $(\forall q \in X :: \ q \sqsubseteq p)$.

We proceed and fulfil these three proof obligations sequentially.

6.1.1. Proof of (a)

The proof of (a) amounts to the proof that $M \subseteq A$, where the set A is defined by

(14) $p \in A \quad \equiv \quad p \sqsubseteq f.p, \quad$ for all p

Because M is the smallest sub-fcpo that contains X, it suffices to show that A is a sub-fcpo that contains X.
Because X is f-expanding, it follows directly from (14) that A contains X. By definition (4) the proof that A is a sub-fcpo amounts to the proof that A is f-closed and inherits suprema of stretches.
We prove that A is f-closed by observing for any p:

$$
\begin{array}{ll}
 & f.p \in A \\
= & \quad \{ (14) \} \\
 & f.p \sqsubseteq f.(f.p) \\
\Leftarrow & \quad \{ f \text{ is pseudo-monotonic} \} \\
 & p \sqsubseteq f.p \;\wedge\; f.p \sqsubseteq f.p \\
= & \quad \{ (14),\ \sqsubseteq \text{ reflexive} \} \\
 & p \in A
\end{array}
$$

We prove that A inherits suprema of stretches by observing for any stretch $S \subseteq A$ that has a supremum:

$$
\begin{array}{ll}
 & \sqcup S \in A \\
= & \quad \{ (14) \} \\
 & \sqcup S \sqsubseteq f.\sqcup S \\
= & \quad \{ \text{property of } \sqcup \} \\
 & (\forall p : p \in S : p \sqsubseteq f.\sqcup S) \\
\Leftarrow & \quad \{ \sqsubseteq \text{ transitive} \} \\
 & (\forall p : p \in S : p \sqsubseteq f.p \;\wedge\; f.p \sqsubseteq f.\sqcup S) \\
= & \quad \{ S \subseteq A,\ (14) \} \\
 & (\forall p : p \in S : f.p \sqsubseteq f.\sqcup S) \\
\Leftarrow & \quad \{ f \text{ pseudo-monotonic} \} \\
 & (\forall p : p \in S : p \sqsubseteq \sqcup S \;\wedge\; f.p \sqsubseteq \sqcup S) \\
= & \quad \{ S \text{ is } f\text{-closed} \} \\
 & (\forall p : p \in S : p \sqsubseteq \sqcup S) \\
= & \quad \{ \text{property of } \sqcup \} \\
 & \mathit{true}
\end{array}
$$

6.1.2. Proof of (b)

A central role in the proof of (b) will be played by the function $E : P \to Pow(P)$ that is closely related to f-narrowness and defined by:

(15) $p \in E.q \;\equiv\; p \sqsubseteq q \;\vee\; f.q \sqsubseteq p,$ for any $p, q \in P$.

This function E enjoys the following weak f-closure property:

(16) PROPERTY. Let f be pseudo-monotonic on P. Then we have for any $p, q \in P$:

$$p \in E.q \;\wedge\; q \in E.p \;\Rightarrow\; f.p \in E.q \;.$$

Proof: For any $p, q \in P$ we have:

$$
\begin{array}{ll}
 & p \in E.q \;\wedge\; q \in E.p \\
= & \quad \{\text{definition (15)}\} \\
 & (\, p \sqsubseteq q \;\vee\; f.q \sqsubseteq p\,) \;\wedge\; (\, q \sqsubseteq p \;\vee\; f.p \sqsubseteq q\,) \\
\Rightarrow & \quad \{\text{distribution, weakening, ordering, } \sqsubseteq \text{ is anti-symmetric}\} \\
 & f.p \sqsubseteq q \;\vee\; p = q \;\vee\; (\, q \sqsubseteq p \;\wedge\; f.q \sqsubseteq p\,) \\
\Rightarrow & \quad \{\text{Leibniz, } \sqsubseteq \text{ is reflexive, } f \text{ is pseudo-monotonic}\} \\
 & f.p \sqsubseteq q \;\vee\; f.q \sqsubseteq f.p \;\vee\; f.q \sqsubseteq f.p \\
= & \quad \{\; \vee \text{ idempotent, definition (15)}\} \\
 & f.p \in E.q
\end{array}
$$

□

The second useful property of E is:

(17) PROPERTY. For any q, the set $E.q$ inherits suprema of arbitrary subsets.

Proof: Let S be any subset of P that has a supremum in P. We prove $S \subseteq E.q \;\Rightarrow\; \sqcup S \in E.q$ by starting from the consequent:

$$
\begin{array}{ll}
 & \sqcup S \in E.q \\
= & \quad \{\text{definition (15)}\} \\
 & \sqcup S \sqsubseteq q \;\vee\; f.q \sqsubseteq \sqcup S \\
= & \quad \{\text{property of } \sqcup\} \\
 & (\forall p : p \in S : p \sqsubseteq q) \;\vee\; f.q \sqsubseteq \sqcup S \\
= & \quad \{\text{distribution}\} \\
 & (\forall p : p \in S : p \sqsubseteq q \;\vee\; f.q \sqsubseteq \sqcup S) \\
\Leftarrow & \quad \{p \in S \;\Rightarrow\; p \sqsubseteq \sqcup S, \text{ transitivity of } \sqsubseteq\} \\
 & (\forall p : p \in S : p \sqsubseteq q \;\vee\; f.q \sqsubseteq p) \\
= & \quad \{\text{definition (15)}\} \\
 & S \subseteq E.q
\end{array}
$$

□

Still for the purpose of a compact formulation of the proof of (b) we define the set B by:

(18) $p \in B \;\equiv\; (\forall q \in M :: p \in E.q)$

From this and (15) we find immediately:

(19) PROPERTY. Premise (13.∗) is equivalent to $(\forall q \in M :: \; X \subseteq E.q)$ as well as to $X \subseteq B$.

After this introduction of E and B we now turn to the proof of (b), i.e. the proof that M is f-narrow. We observe:

$$
\begin{array}{ll}
 & M \text{ is } f\text{-narrow} \\
= & \quad \{ \text{definition of } f\text{-narrow} \} \\
 & (\forall p, q \in M :: \; p \sqsubseteq q \quad \vee \quad f.q \sqsubseteq p) \\
= & \quad \{ \text{definition of } E.q, B \text{ and set calculus} \} \\
 & M \subseteq B \cap M \\
\Leftarrow & \quad \{ \text{definition of } M \} \\
 & B \cap M \text{ is a sub-fcpo containing } X
\end{array}
$$

That $B \cap M$ contains X is immediate from the definition of M and (19). That $B \cap M$ inherits suprema of stretches follows from the fact that M does and (17). Thus, we only have f-closedness left. In order to prove that $B \cap M$ is f-closed we assume that p is any element of $B \cap M$ and prove $f.p \;\in\; B \cap M$ by first observing:

$$
\begin{array}{ll}
 & f.p \;\in\; B \cap M \\
= & \quad \{ p \in M, \; M \text{ is } f\text{-closed} \} \\
 & f.p \;\in\; B \\
= & \quad \{ (18) \} \\
 & (\forall q \in M :: \; f.p \in E.q) \\
\Leftarrow & \quad \{ f \text{ is pseudo-monotonic, (16)} \} \\
 & (\forall q \in M :: \; p \in E.q \;\wedge\; q \in E.p) \\
= & \quad \{ p \in B, \text{ definition (18)} \} \\
 & (\forall q \in M :: \; q \in E.p) \\
= & \quad \{ \text{set calculus } \} \\
 & M \;\subseteq\; E.p \cap M \\
\Leftarrow & \quad \{ M \text{ is the minimal sub-fcpo that contains } X \} \\
 & E.p \cap M \text{ is a sub-fcpo that contains } X
\end{array}
$$

That $E.p \cap M$ contains X is immediate from the definition of M and (19). That $E.p \cap M$ inherits suprema of stretches follows from the fact that M does and (17). Thus, again we only have f-closedness left. We prove that $E.p \cap M$ is f-closed by observing for any $q \in M$:

$$
\begin{array}{ll}
 & f.q \in E.p \cap M \\
= & \quad \{ M \text{ is } f\text{-closed} \} \\
 & f.q \in E.p \\
\Leftarrow & \quad \{ f \text{ is pseudo-monotonic, (16)} \} \\
 & q \in E.p \;\wedge\; p \in E.q \\
= & \quad \{ p \in B, \; q \in M, \; (18) \} \\
 & q \in E.p
\end{array}
$$

This concludes the proof of (b) and also the proof of (13.i), i.e. M is a trail.

6.1.3. Proof of (c)

In order to prove (c), we let $p \in M$ be such that

(20) $p = f.p$, and

(21) $(\forall q \in X :: q \sqsubseteq p)$

We have to prove $p = \sqcup M$. First we observe:

$$
\begin{array}{ll}
 & p = \sqcup M \\
= & \quad \{ p \in M \} \\
 & \sqcup M \sqsubseteq p \\
= & \quad \{ \text{property of } \sqcup \} \\
 & (\forall q \in M :: q \sqsubseteq p)
\end{array}
$$

In order to prove the last line, we rewrite it as

(22) $M \subseteq C.p \cap M$

where the set $C.p$ is defined by

(23) $q \in C.p \equiv q \sqsubseteq p$

Because M is the smallest sub-fcpo that contains X it suffices to prove that $C.p \cap M$ is a sub-fcpo that contains X, i.e. that $C.p \cap M$ is f-closed, inherits suprema of stretches, and contains X.

In order to prove that $C.p \cap M$ is f-closed we assume $q \in C.p \cap M$ and observe:

$$
\begin{array}{ll}
 & f.q \in C.p \cap M \\
= & \quad \{ q \in M, M \text{ is } f\text{-closed} \} \\
 & f.q \in C.p \\
= & \quad \{ (23) \} \\
 & f.q \sqsubseteq p \\
= & \quad \{ (20) \} \\
 & f.q \sqsubseteq f.p \\
\Leftarrow & \quad \{ q \in M, p \in M, M \text{ is a trail, (2), (5)} \} \\
 & q \sqsubseteq p \\
= & \quad \{ (23), q \in C.p \} \\
 & \mathit{true}
\end{array}
$$

In order to prove that $C.p \cap M$ inherits suprema of stretches we assume that S is a stretch in $C.p \cap M$ and that S has a supremum. We observe:

$$
\begin{array}{ll}
 & \sqcup S \in C.p \cap M \\
= & \quad \{ M \text{ is a sub-fcpo} \} \\
 & \sqcup S \in C.p \\
= & \quad \{ (23) \} \\
 & \sqcup S \sqsubseteq p \\
= & \quad \{ \text{property of } \sqcup \} \\
 & (\forall q : q \in S : q \sqsubseteq p) \\
= & \quad \{ (23) \} \\
 & (\forall q : q \in S : q \in C.p) \\
= & \quad \{ S \subseteq C.p \} \\
 & \mathit{true}
\end{array}
$$

In order to prove that $C.p \cap M$ contains X we observe for any $q \in X$:

$$
\begin{array}{ll}
 & q \;\in\; C.p \cap M \\
= & \quad \{\, q \in X,\ M \text{ contains } X \,\} \\
 & q \;\in\; C.p \\
= & \quad \{\, (23) \,\} \\
 & q \sqsubseteq p \\
= & \quad \{\, q \in X,\ (21) \,\} \\
 & \mathit{true}
\end{array}
$$

7. Arrival

Landmark (c) requires some further traveling, but the result is expressed in:

(24) Theorem. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be monotonic on P. Let L be a linear set of fixpoints of f. Let M be the smallest sub-fcpo that contains L. Then we have:

- (i) M is an f-trail,
- (ii) $\sqcup M$ is the least prefixpoint of f among the upper bounds of L.

Note that, because $(P, \sqsubseteq, f)$ is an fcpo, it follows immediately from (24.i) that $\sqcup M$ exists and is a fixpoint of f. From (24.ii) it then follows that $\sqcup M$ is also the least fixpoint of f among the upper bounds of L.

7.1. Proof of (24.i)

Because any monotonic function is pseudo-monotonic, and a set of fixpoints of f is certainly f-expanding, (i) follows from theorem (13) if we succeed in proving premise (13.$*$) for $X := L$. We observe:

$$
\begin{array}{ll}
 & (13.*) \text{ with } X := L \\
= & \quad \{\ \} \\
 & (\forall p \in L,\ q \in M :: \ p \sqsubseteq q \ \ \vee \ \ f.q \sqsubseteq p) \\
= & \quad \{\, \text{definition of } L \,\} \\
 & (\forall p \in L,\ q \in M :: \ f.p \sqsubseteq q \ \ \vee \ \ f.q \sqsubseteq f.p) \\
\Leftarrow & \quad \{\, f \text{ is monotonic} \,\} \\
 & (\forall p \in L,\ q \in M :: \ f.p \sqsubseteq q \ \ \vee \ \ q \sqsubseteq p) \\
= & \quad \{\, \text{define } E \text{ as in (15), set calculus} \,\} \\
 & (\forall p \in L :: \ M \subseteq E.p)
\end{array}
$$

Because M is the smallest sub-fcpo containing L, it suffices to prove that for any $p \in L$, $E.p$ is a sub-fcpo containing L, i.e. that $E.p$ inherits suprema of stretches, is f-closed and contains L. Thus, let p be any element of L.
From (17) it follows directly that $E.p$ inherits suprema of stretches.
We show that $E.p$ is f-closed by observing for any q:

$$
\begin{array}{ll}
 & f.q \in E.p \\
= & \quad \{\, (15) \,\} \\
 & f.q \sqsubseteq p \ \ \vee \ \ f.p \sqsubseteq f.q \\
= & \quad \{\, p \text{ is a fixpoint of } f \,\}
\end{array}
$$

$$
\begin{array}{ll}
 & f.q \sqsubseteq f.p \ \vee \ f.p \sqsubseteq f.q \\
\Leftarrow & \quad \{ f \text{ is monotonic} \} \\
 & q \sqsubseteq p \ \vee \ p \sqsubseteq q \\
= & \quad \{ p \text{ is a fixpoint of } f \} \\
 & q \sqsubseteq p \ \vee \ f.p \sqsubseteq q \\
= & \quad \{ (15) \} \\
 & q \in E.p
\end{array}
$$

We show that $E.p$ contains L by observing for any $q \in L$:

$$
\begin{array}{ll}
 & q \in E.p \\
= & \quad \{ (15) \} \\
 & q \sqsubseteq p \ \vee \ f.p \sqsubseteq q \\
= & \quad \{ p \text{ is a fixpoint of } f \} \\
 & q \sqsubseteq p \ \vee \ p \sqsubseteq q \\
= & \quad \{ p, q \in L, \ L \text{ is linear} \} \\
 & \mathit{true}
\end{array}
$$

This concludes the proof of (24.i).

7.2. Proof of (24.ii)

Let U be the set of upper bounds of L. From (24.i) and the premise $L \subseteq M$ we conclude that $\sqcup M$ is a fixpoint of f in U. In order to prove (24.ii) we define the set D by

(25) $p \in D \ \equiv \ (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, p \sqsubseteq q)$.

We observe

$$
\begin{array}{ll}
 & \sqcup M \text{ is the least prefixpoint of } f \text{ in } U \\
\Leftarrow & \quad \{ \sqcup M \text{ is a (pre)fixpoint of } f \text{ in } U \} \\
 & \sqcup M \text{ is a lower bound of the prefixpoints of } f \text{ in } U \\
= & \quad \{ (25) \} \\
 & \sqcup M \in D \\
\Leftarrow & \quad \{ \text{(i), so } \sqcup M \in M; \ M \text{ is the smallest sub-fcpo containing } L \} \\
 & D \text{ is a sub-fcpo containing } L
\end{array}
$$

We prove the last line by proving that D is f-closed, inherits suprema of stretches and contains L. In order to prove that D is f-closed we observe for any p:

$$
\begin{array}{ll}
 & f.p \in D \\
= & \quad \{ (25) \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, f.p \sqsubseteq q) \\
\Leftarrow & \quad \{ \sqsubseteq \text{ is transitive} \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, f.p \sqsubseteq f.q) \\
\Leftarrow & \quad \{ f \text{ is monotonic } \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, p \sqsubseteq q) \\
= & \quad \{ (25) \} \\
 & p \in D
\end{array}
$$

In order to prove that D inherits suprema of stretches we observe for any set S that has a supremum in P:

$$
\begin{array}{ll}
 & \sqcup S \in D \\
= & \quad \{ (25) \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, \sqcup S \sqsubseteq q) \\
= & \quad \{ \text{property of } \sqcup \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, (\forall p \in S \, :: \, p \sqsubseteq q)) \\
= & \quad \{ \text{interchange} \} \\
 & (\forall p \in S \, :: \, (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, p \sqsubseteq q)) \\
= & \quad \{ (25) \} \\
 & (\forall p \in S \, :: \, p \in D)) \\
= & \quad \{ \ \} \\
 & S \subseteq D
\end{array}
$$

In order to prove that D contains L we observe for any p:

$$
\begin{array}{ll}
 & p \in D \\
= & \quad \{ (25) \} \\
 & (\forall q \in U \, : \, f.q \sqsubseteq q \, : \, p \sqsubseteq q) \\
\Leftarrow & \quad \{ U \text{ is the set of upper bounds of } L \} \\
 & p \in L
\end{array}
$$

This concludes the proof of (24.ii).

8. Some Consequences

We mention some consequences of theorems (13) and (24).
First, in theorem (13) we take X to be the empty set. Because any element is an upper bound of the empty set, we find:

(26) COROLLARY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be pseudo-monotonic. Let M be the smallest sub-fcpo. Then we have:

- (i) M is a trail,
- (ii) $\sqcup M$ is the unique fixpoint of f in M.

Next, in theorem (24) we take L to be the empty set. We find:

(27) COROLLARY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be monotonic. Let M be the smallest sub-fcpo. Then we have:

- (i) M is a trail,
- (ii) $\sqcup M$ is the least prefixpoint of f in P.

In the context of corollary (27) we note that $\sqcup M$ is a fixpoint of f. So an immediate consequence of corollary (27) is:

(28) COROLLARY. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be monotonic. Let M be the smallest sub-fcpo. Then we have:

- (i) f has a least fixpoint, μf,
- (ii) μf is the least prefixpoint of f,
- (iii) $\mu f \; = \; \sqcup M$.

Another consequence of theorem (24) is:

(29) Corollary. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be monotonic. Let Q be the set of prefixpoints of f. Then $(Q, \sqsubseteq, f)$ is an fcpo in itself (with $\sqsubseteq$ and f restricted to Q).

Proof: Note that set Q does not necessarily inherit suprema of stretches, so Q is not necessarily a sub-fcpo. For monotonic f, the set of prefixpoints of f is certainly f-closed. So, with $\sqsubseteq$ and f restricted to Q, $(Q, \sqsubseteq)$ is a partial order and f a monotonic endofunction on Q. We have to prove that every f-trail in $(Q, \sqsubseteq, f)$ has a supremum in Q. Thus, let T be any f-trail in $(Q, \sqsubseteq, f)$. Then, because $\sqsubseteq$ is anti-symmetric, T is a linear set of fixpoints of f. Let M be the smallest sub-fcpo in $(P, \sqsubseteq, f)$ that contains T. From (24.ii) it follows that $\sqcup M$ is the supremum of T in Q. □

Because the supremum of an f-trail is a fixpoint of f, another immediate consequence of theorem (24) is:

(30) Corollary. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be monotonic on P. Let F be the set of fixpoints of f. Then $(F, \sqsubseteq)$ is a cpo in itself (with $\sqsubseteq$ restricted to F).

Next we take f to be the identity function *Id*, and get the converse of an observation in Section 5:

(31) Corollary. Let $(P, \sqsubseteq, Id)$ be an fcpo. Then $(P, \sqsubseteq)$ is a cpo.

Finally, we remark that theorem (24) can be used to prove the 'Lattice-theoretical fixpoint theorem' of [Tar55]. We omit the proof here, because it is hardly shorter, and because the result is not new.

9. Is this New Territory?

One may rightfully ask whether the definition of an fcpo really is a generalization of the definition of a cpo. We will presently show that the answer is 'yes'. Another relevant question is, whether there exists a formally stronger characterization of an fcpo. We tackle the latter question first. To that end we prove two properties.

(32) Property. Let $(P, \sqsubseteq, f)$ be an fcpo. Let f be expanding. Then P has a maximal element.

Proof: The proof is by contradiction. Thus we assume that P has no maximal element. From the axiom of choice it follows that there is an endofunction *maj* on P such that $(\forall p \in P :: p \sqsubset maj.p)$. Next we construct a function g according to:

(33) $g.p \;=\; (\text{if } p = f.p \text{ then } maj.p \text{ else } f.p \text{ fi})$

We first prove that $(P, \sqsubseteq, g)$ is an fcpo. It suffices to prove that every g-trail has a supremum. Thus, let T be any g-trail. We have to prove that T has a supremum. It suffices to prove that T is an f-trail. Thereto we observe separately:

$$\begin{array}{ll} & T \text{ is } f\text{-expanding} \\ = & \quad \{ f \text{ is expanding on } P \} \\ & \textit{true} \end{array}$$

and

$$
\begin{array}{cl}
 & T \text{ is } f\text{-narrow} \\
= & \quad \{ \text{definition} \} \\
 & (\forall p, q \in P :: \ p \sqsubseteq q \ \vee \ f.q \sqsubseteq p) \\
\Leftarrow & \quad \{ \text{from (33) we have: } f.q \sqsubseteq g.q, \text{ for every } q \} \\
 & (\forall p, q \in P :: \ p \sqsubseteq q \ \vee \ g.q \sqsubseteq p) \\
= & \quad \{ T \text{ is } g\text{-narrow} \} \\
 & \mathit{true}
\end{array}
$$

and

$$
\begin{array}{cl}
 & T \text{ is } f\text{-closed} \\
= & \quad \{ \text{definition} \} \\
 & (\forall p \in T :: \ f.p \in T) \\
= & \quad \{ \ \} \\
 & (\forall p \in T : \ p \neq f.p : \ f.p \in T) \\
\Leftarrow & \quad \{ (33) \} \\
 & (\forall p \in T :: \ g.p \in T) \\
= & \quad \{ T \text{ is } g\text{-closed} \} \\
 & \mathit{true}
\end{array}
$$

and

$$
\begin{array}{cl}
 & T \text{ inherits the supremum of every subset of } T \\
= & \quad \{ T \text{ is a } g\text{-trail} \} \\
 & \mathit{true}
\end{array}
$$

This completes the proof that $(P, \sqsubseteq, g)$ is an fcpo. From definition (33) we conclude that function g is expanding on P and has no fixpoint. So g is certainly pseudo-monotonic and from corollary (26) with $f := g$, we find that g has a fixpoint. This is a contradiction. $\square$

(34) Property. Let $(P, \sqsubseteq, f)$ be an fcpo. Let U be an f-expanding, linear sub-fcpo. Then U has a maximum.

Proof: From (10) and (32) it follows that U has a maximal element in $(U, \sqsubseteq)$. So, because U is linear, U has a maximum. $\square$

From the last property it follows that there is an alternative characterization of an fcpo in terms of a weaker notion of f-trail. The weakened definition of an f-trail is as follows.

(35) Definition. A set T is an f-trail* iff:

(a) T is f-expanding,
(b) T is linear,
(c) T is f-closed, and
(d) T inherits the supremum of every stretch in T.

The alternative characterization of an fcpo is given by the following theorem.

(36) Theorem. Let $(P, \sqsubseteq)$ be a partial order. Let f be an endofunction on P. The triple $(P, \sqsubseteq, f)$ is an fcpo if and only if every f-trail* has a supremum.

Omission of any of the four requirements (a–d) in (35) amounts to a weakening of the notion of f-trail* and a formal strengthening of the corresponding notion of an fcpo, i.e. "every f-trail* has a supremum." That it amounts to a strict strengthening of this property is shown by the following four examples. They are numbered (a–d) according to the requirement of (35) that is omitted. In each example we exhibit an fcpo that contains an f-trail** (yet weaker) without supremum. In three of the cases the particular f-trail** is linear. This shows that the definition of an fcpo is a proper generalization of the definition of a cpo.

(a) Let universe P be the set of natural numbers (including 0) and let P be linearly ordered by $\leqslant$. Let endofunction f on P be given by $f.0 = 0$, and $f.x = x - 1$ if $x > 0$. Function f is monotonic. The empty stretch has supremum 0. It then follows (using expandingness) that $T = \{0\}$ is the only trail. Since T has supremum 0, $(P, \leqslant, f)$ is an fcpo. As particular f-trail** we take the universe P. It is linear and f-closed and inherits suprema of all subsets. But f is not expanding on P, and P has no upper bound, let alone a supremum. Also, $(P, \leqslant)$ is not a cpo.

(b) Let universe P be the set $\{a, b, c\}$. The order $\sqsubseteq$ is given by $x \sqsubseteq y \;\equiv\; x = a \;\vee\; x = y$. Let f be the identity function on P. The f-trails are $\emptyset, \{a\}, \{a, b\}, \{a, c\}$. Clearly, $(P, \sqsubseteq, f)$ is an fcpo. As particular f-trail** we take the universe P. It is f-expanding, f-closed and inherits the supremum of every subset of P, but it is not linear and has no least upper bound.

(c) Let universe P be the set of natural numbers (including 0) and let the partial order $\sqsubseteq$ be given by

$$i \sqsubseteq j \quad \equiv \quad i \leqslant j \;\wedge\; (i = 1 \;\Rightarrow\; j = 1)$$

Let endofunction f on P be given by $f.0 = 1$, and $f.x = x$ if $x > 0$. Function f is expanding. The empty stretch has supremum 0. So every trail contains 0 and, since it is f-closed, also 1. It then follows (using narrowness) that $T = \{0, 1\}$ is the only trail. Since 1 is the maximum of T, it is the supremum of T. Therefore, $(P, \sqsubseteq, f)$ is an fcpo.
As particular f-trail**, let Q be the set of numbers $\neq 1$. Set Q is linear, f-expanding and inherits the supremum of every subset of Q, but Q is not f-closed. Obviously, Q does not have an upper bound, let alone a supremum, so $(P, \sqsubseteq)$ is not a cpo.

(d) In the context of the previous example and as particular f-trail**, let R be the set of numbers > 1. Set R is f-expanding, linear and f-closed, but it does not inherit the supremum of its empty subset and it has no upper bound, let alone a supremum.

10. Least Fixpoint Mapping

Suppose we are given two partially ordered universes P and Q, a mapping f from P to Q, and monotonic endofunctions g, h on P, Q respectively. One may be interested to know sufficient conditions in order that f maps the least fixpoint of g to the least fixpoint of h. The following theorem gives such conditions.
For convenience, we use the same symbol $\sqsubseteq$ for both partial orders. The g-image of set S is denoted as $g{\scriptstyle\bullet}S$. Function f is called $\sqcup$-junctive for set S iff:
S has a supremum $\Rightarrow$ ($f{\scriptstyle\bullet}S$ has a supremum and $f.\sqcup S = \sqcup(f{\scriptstyle\bullet}S)$).

(37) THEOREM. Let $(P, \sqsubseteq, g)$ and $(Q, \sqsubseteq, h)$ be fcpo's. Let g and h be monotonic. Let f be a function from P to Q. Then we have:

(i) $(h \circ f).\mu g \sqsubseteq (f \circ g).\mu g \;\Rightarrow\; \mu h \sqsubseteq f.\mu g$,
(ii) if f is $\sqcup$-junctive for all g-stretches then:
$(\forall p \in P :: (f \circ g).p \sqsubseteq (h \circ f).p) \;\Rightarrow\; f.\mu g \sqsubseteq \mu h.$

Proof: It follows from corollary (28) that the least fixpoints of g and h exist. We prove (37.i) by starting from the antecedent.

$$
\begin{array}{ll}
 & (h \circ f).\mu g \sqsubseteq (f \circ g).\mu g \\
= & \quad \{\text{property of fixpoint}\} \\
 & h.(f.\mu g) \sqsubseteq f.\mu g \\
\Rightarrow & \quad \{\text{corollary (28.ii) for } (Q, \sqsubseteq, h)\} \\
 & \mu h \sqsubseteq f.\mu g
\end{array}
$$

In order to prove (37.ii) we assume:

(38) f is $\sqcup$-junctive for all g-stretches, and

(39) $(\forall p \in P :: (f \circ g).p \sqsubseteq (h \circ f).p)$

We also define M to be the smallest sub-fcpo in $(P, \sqsubseteq, g)$.
We have to prove the consequent, which we rewrite:

$$
\begin{array}{ll}
 & f.\mu g \sqsubseteq \mu h \\
= & \quad \{\text{corollary (28.iii) for } (P, \sqsubseteq, g)\} \\
 & f.\sqcup M \sqsubseteq \mu h \\
= & \quad \{\text{corollary (27.i), (2), (38)}\} \\
 & \sqcup(f{\bullet}M) \sqsubseteq \mu h \\
= & \quad \{\text{property } \sqcup\} \\
 & (\forall p \in M :: f.p \sqsubseteq \mu h)
\end{array}
$$

Because M is the smallest sub-fcpo in P, the last line can be proved by showing that the set F is a sub-fcpo in P, where F is defined by

(40) $p \in F \;\equiv\; f.p \sqsubseteq \mu h$

Thus it remains to be proved that F is g-closed and inherits suprema of g-stretches.
In order to prove that F is g-closed we observe for any p:

$$
\begin{array}{ll}
 & g.p \in F \\
= & \quad \{(40)\} \\
 & f.(g.p) \sqsubseteq \mu h \\
\Leftarrow & \quad \{(39),\ \sqsubseteq \text{ is transitive, } \mu h \text{ is a fixpoint}\} \\
 & h.(f.p) \sqsubseteq h.\mu h \\
\Leftarrow & \quad \{h \text{ monotonic}\} \\
 & f.p \sqsubseteq \mu h \\
= & \quad \{(40)\} \\
 & p \in F
\end{array}
$$

In order to prove that F inherits suprema of g-stretches we assume that S is any g-stretch in F with a supremum in P. We prove $\sqcup S \in F$ by observing:

$$
\begin{array}{ll}
 & \sqcup S \in F \\
= & \quad \{(40)\}
\end{array}
$$

$$
\begin{array}{ll}
 & f.\sqcup S \sqsubseteq \mu h \\
= & \quad \{\, S \text{ is a stretch, (38)} \,\} \\
 & \sqcup f\bullet S \sqsubseteq \mu h \\
= & \quad \{\, \text{property } \sqcup \,\} \\
 & (\forall p \in S :: \ f.p \sqsubseteq \mu h) \\
= & \quad \{\, S \subseteq F, \text{(40)} \,\} \\
 & \mathit{true}
\end{array}
$$

□

11. Conclusion

We have given general conditions that guarantee the existence of a least fixpoint for a monotonic endofunction on a partially ordered set. A generalized version of the theorem of Knaster-Tarski is shown to hold in so-called "functionally complete partial orders". These are characterized by the property, that every so-called "f-trail" contained in them has a supremum. We have taken care to choose the defining properties of an fcpo and a sub-fcpo as weak as feasible, and the defining properties of an f-trail as strong as feasible. Moreover, from a range of equivalent characterizations of an fcpo, we have given both the weakest and the strongest form that we could achieve.

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