



# On tests for symmetry and radial symmetry of bivariate copulas towards testing for ellipticity

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## Abstract

Very simple non-parametric tests are proposed to detect symmetry and radial symmetry in the dependence structure of bivariate copula data. The performance of the proposed tests is illustrated in an intensive simulation study and compared to the one of similar more advanced tests, which do not require known margins. Further, a powerful non-parametric testing procedure to decide whether the dependence structure of the underlying bivariate copula data may be captured by an elliptical copula is provided. The testing procedure makes use of intrinsic properties of bivariate elliptical copulas such as symmetry, radial symmetry, and equality of Kendall's tau and Blomqvist's beta. The proposed tests as well as the testing procedure are very simple to use in applications. For an illustration of the testing procedure for ellipticity, financial and insurance data is analyzed.

**Keywords** Asymptotic normality · Elliptical copulas · Goodness-of-fit test · Kendall's tau · Non-parametric tests · U-statistics

## 1 Introduction

Since Embrechts et al. (2003), Frees and Valdez (1998), and Li (2000), copulas were widely used in economics, finance, and risk management to capture the dependence of multivariate data. Bivariate parametric copulas are usually the basis of many multivariate copula constructions [see, e.g., Aas et al. (2009) or Fischer et al. (2009)]. Therefore, the choice of a parametric bivariate copula family is very crucial to accurately capture the multivariate dependence. For large and huge sample sizes, carrying out known goodness-of-fit tests is very time consuming. Graphical tools like scatter plots can

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significantly reduce the amount of copulas to be considered but may lead to erroneous decisions. In this paper, we fill this existing gap and propose simple statistical tests to detect symmetry or radial symmetry of the underlying bivariate copula data.

The existing tests for symmetry and radial symmetry of bivariate copulas by Genest et al. (2012), Genest and Nešlehová (2014), Li and Genton (2013), and Quesy (2016) assume unknown marginal distributions and take into account their non-parametric estimation. Therefore, the asymptotic distribution of their test statistics is of complex nature and derived using the weak convergence of empirical copula processes. In applications, bootstrap techniques are needed for the computation of  $p$ -values, and this is computationally expensive for huge sample sizes.

Assuming given copula data, we propose simpler non-parametric tests for symmetry and radial symmetry of bivariate copulas. We manipulate the underlying copula data without changing its dependence structure to create two bivariate samples. Our test statistics are then based on the difference between the empirical Kendall's tau of both samples. The limiting distributions of the test statistics can be derived using the classical theory of  $U$ -statistics. Therefore, our non-parametric tests are related to asymptotic normal distributions and are very simple at work. Our tests are based only on a sample characteristic of the bivariate copula data. Therefore, they are easy to implement and computationally very fast. In times of Big Data, this nice feature of our tests is very useful in the analysis of data sets with huge sample sizes.

In Jaser et al. (2017), we proposed a goodness-of-fit test for elliptical copulas under the assumption of given copula data. It utilizes the known equality of Kendall's tau and Blomqvist's beta for elliptical copulas (see Schmid and Schmidt 2007). Therefore, this test may illustrate poor performance in finite samples if Kendall's tau and Blomqvist's beta are very close for a particular copula family. In this paper, we propose a multiple testing procedure for ellipticity of copula data, which combines our simple non-parametric tests for symmetry, radial symmetry, and the equality of Kendall's tau and Blomqvist's beta. Thus, the proposed multiple testing procedure utilizes the most common properties of elliptical copulas, which should make it powerful to detect a non-elliptical dependence structure in bivariate copula data.

This paper is organized as follows. In Sect. 2, copulas, the general properties of symmetry, radial symmetry, and ellipticity, as well as the concordance measure Kendall's tau are introduced. Simple non-parametric tests for symmetry and radial symmetry are proposed in Sect. 3. Section 4 presents a Monte Carlo simulation study to evaluate the finite-sample performance. In Sect. 5, a simple and powerful non-parametric testing procedure is proposed to decide whether the dependence structure of underlying bivariate copula data may be captured by an elliptical copula. Applications to financial and insurance data are reported in Sect. 6 to illustrate the testing procedure at work. Finally, Sect. 7 concludes, and the Appendix contains one technical derivation and the main proof.

## 2 Preliminaries

Here and in the sequel, we consider bivariate distribution functions with continuous univariate marginal distribution functions. Let  $H$  be a bivariate distribution function

with continuous margins  $F$  and  $G$ . According to Sklar (1959), there exists a unique copula  $C : [0, 1]^2 \rightarrow [0, 1]$  such that  $H$  can be represented at each  $(x, y) \in \mathbb{R}^2$  as

$$H(x, y) = C(F(x), G(y)). \quad (1)$$

By virtue of Eq. (1), the copula  $C(u, v)$  of  $H$ , for any  $u, v \in [0, 1]$ , is then given by

$$C(u, v) = H(F^-(u), G^-(v)),$$

where  $F^-$  and  $G^-$  are the generalized inverses of  $F$  and  $G$ , respectively.

A bivariate copula  $C$  is symmetric if and only if  $C(u, v) = C(v, u)$ , for all  $(u, v) \in [0, 1]^2$ . If  $C$  is symmetric and the distribution function of a random vector  $(U, V)$ , then the dependence structure between  $U$  and  $V$  is symmetric and, hence, we have

$$(U, V) \stackrel{d}{=} (V, U). \quad (2)$$

A test for the hypothesis that the unknown copula  $C$  is symmetric, that is

$$H_0^s : C(u, v) = C(v, u), \text{ for all } (u, v) \in [0, 1]^2,$$

against the alternative

$$H_1^s : \exists(u, v) \in [0, 1]^2, \text{ such that } C(u, v) \neq C(v, u),$$

is proposed in this paper.

The bivariate copula  $C$  is radially symmetric if  $(U - 0.5, V - 0.5) \stackrel{d}{=} (0.5 - U, 0.5 - V)$  or, equivalently,  $(U, V) \stackrel{d}{=} (1 - U, 1 - V)$ . Since the survival copula  $\bar{C}$  is the distribution function of  $(1 - U, 1 - V)$ , a bivariate copula  $C$  is radially symmetric if and only if it coincides with its own survival copula, that is  $C = \bar{C}$ . The null hypothesis and the alternative to test whether the unknown copula  $C$  is radially symmetric are given by

$$H_0^r : C = \bar{C} \text{ versus } H_1^r : C \neq \bar{C}.$$

Our test statistics for the null hypotheses  $H_0^s$  and  $H_0^r$  are based on Kendall's tau, which contemplates one of the most popular rank-based dependence measures. However, any non-parametric bivariate measure of ordinal dependence, e.g. Spearman's rho or Blomqvist's beta, could be used instead. Let  $(U_1, V_1)$  and  $(U_2, V_2)$  be independent copies of the random vector  $(U, V)$  whose distribution function is the copula  $C$ . Kendall's tau is defined by

$$\tau_{UV} := \mathbb{E}[\text{sgn}(U_1 - U_2)\text{sgn}(V_1 - V_2)],$$

where  $\text{sgn}$  denotes the sign function. Since Kendall's tau is completely determined by the underlying copula  $C$ , we denote  $\tau_C := \tau_{UV}$ . Given a random sample  $(U_1, V_1), \dots,$

$(U_n, V_n)$  of size  $n$  from the random vector  $(U, V)$ , Kendall's tau can be empirically estimated by

$$\hat{\tau}_{C,n} := \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \text{sgn}(U_i - U_j) \text{sgn}(V_i - V_j).$$

The asymptotic distribution of this estimator for Kendall's tau is well investigated (see Höfding 1947) and independent of the knowledge of the true marginal distributions.

In Jaser et al. (2017), we designed a goodness-of-fit test for elliptical copulas based on the equality of Kendall's tau  $\tau_C$  and Blomqvist's beta  $\beta_C$ , that is the null hypothesis

$$H_0^e : \tau_C = \beta_C \text{ is tested against the alternative } H_1^e : \tau_C \neq \beta_C.$$

Now, our proposed tests for symmetry and radial symmetry are combined with this test in order to develop a powerful and simple statistical procedure to test whether the dependence structure of a bivariate random vector with uniform margins is captured by an elliptical copula. Let  $C$  be the unknown bivariate copula of the given bivariate random vector with uniform margins and  $\mathcal{C}^{ellipt}$  the class of elliptical copulas. Then, the null hypothesis and the alternative of the testing procedure are given by

$$H_0 : C \in \mathcal{C}^{ellipt} \text{ versus } H_1 : C \notin \mathcal{C}^{ellipt}.$$

### 3 Simple non-parametric tests for symmetry and radial symmetry

In this section, we derive our two statistical tests for symmetry and radial symmetry for bivariate copulas. We assume that we are given a copula sample and neglect unknown marginal distributions and their estimation. In practical applications, one usually estimates marginal distribution functions non-parametrically to avoid misspecification. For the following subsections, let  $(U_1, V_1), \dots, (U_n, V_n) \in [0, 1]^2$  be a sample from the statistical model  $(([0, 1]^2)^n, \mathcal{B}([0, 1]^2)^{\otimes n}, P^{\otimes n})$ , where  $P$  is a distribution with copula  $C$  and uniform margins.

#### 3.1 Test for symmetry

Let  $(U, V)$  be distributed according to the symmetric copula  $C$ , that is  $(U, V) \stackrel{d}{=} (V, U)$ . Further, we assume that  $\mathbb{P}(U=V) = 0$ . For a given sample realization from  $C$ , the scatter plot displays symmetry with respect to the main diagonal. By interchanging the coordinates, any two observations, one below and one above the diagonal, can be mirrored to the opposite side of the diagonal. The modified data set can still be considered as a realization from the given copula  $C$ . Therefore, a sample realization from the copula  $C$  can be generated just using all observations either above or below the diagonal.

The complementary events that  $(U, V)$  is below or above the diagonal, that is

$$B^s := \{\omega : U - V > 0\} \quad \text{and} \quad \overline{B^s} := \{\omega : U - V < 0\}, \quad (3)$$

have equal probabilities of 0.5. The law of total probability now implies that the symmetric copula  $C$  can be represented as a mixture of two conditional distribution functions given by

$$C(u, v) = 0.5 \cdot F_{U,V|B^s}(u, v) + 0.5 \cdot F_{V,U|\overline{B^s}}(u, v), \quad (4)$$

or

$$C(u, v) = 0.5 \cdot F_{U,V|\overline{B^s}}(u, v) + 0.5 \cdot F_{V,U|B^s}(u, v). \quad (5)$$

Here,  $F_{X,Y|A}$  denotes the conditional distribution function of  $(X, Y)$  given  $(X, Y) \in A$ . Details on the derivation of Eqs. (4) and (5) are provided in the Appendix.

According to Eq. (4) and (5), the symmetric copula  $C$  can be represented either as a mixture of two conditional distribution functions given the event that  $(U, V)$  is below the diagonal or as a mixture of two conditional distribution functions given the event that  $(U, V)$  is above the diagonal. This constitutes the key idea of our testing procedure for symmetric copulas pursued to produce two i.i.d. random samples out of a given i.i.d. random sample from  $C$ .

Let  $(U_1, V_1), \dots, (U_n, V_n)$  be an i.i.d. random sample from the symmetric copula  $C$ . First, we consider the sub-sample  $(U_1^{B^s}, V_1^{B^s}), \dots, (U_{N_{B^s}}^{B^s}, V_{N_{B^s}}^{B^s})$  for which  $U_i^{B^s} - V_i^{B^s} > 0$  holds, that is, whose realizations are below the diagonal. By virtue of Eq. (4), a new sample from  $C$  can be obtained by choosing either  $(U_i^{B^s}, V_i^{B^s})$  with probability 0.5 or  $(V_i^{B^s}, U_i^{B^s})$  also with probability 0.5, for  $i \in \{1, \dots, N_{B^s}\}$ . The resulting random sample is denoted by

$$(\tilde{U}_1^{B^s}, \tilde{V}_1^{B^s}), \dots, (\tilde{U}_{N_{B^s}}^{B^s}, \tilde{V}_{N_{B^s}}^{B^s}). \quad (6)$$

Similarly, we proceed with the sub-sample  $(U_1^{\overline{B^s}}, V_1^{\overline{B^s}}), \dots, (U_{N_{\overline{B^s}}}^{\overline{B^s}}, V_{N_{\overline{B^s}}}^{\overline{B^s}})$  for which  $U_i^{\overline{B^s}} - V_i^{\overline{B^s}} < 0$  holds, that is, whose realizations are above the diagonal, and create a second random sample

$$(\tilde{U}_1^{\overline{B^s}}, \tilde{V}_1^{\overline{B^s}}), \dots, (\tilde{U}_{N_{\overline{B^s}}}^{\overline{B^s}}, \tilde{V}_{N_{\overline{B^s}}}^{\overline{B^s}}). \quad (7)$$

It should be mentioned that the sampling algorithm can be generalized for  $0 < \mathbb{P}(U=V) < 1$  by discarding observations with  $U_i = V_i$ .

Note that the sample size  $N_{B^s}$  is a binomially distributed random variable with size  $n$  and success probability 0.5. From the law of large numbers, it follows that  $N_{B^s}/n$  converges to 0.5 in probability as  $n$  tends to infinity. The same conclusions can be drawn for the sample size  $N_{\overline{B^s}}$  since the relation  $N_{\overline{B^s}} = n - N_{B^s}$  holds. Defining the sequence of random variables  $N_n^s := \min(N_{B^s}, N_{\overline{B^s}})$ , it follows that  $N_n^s/n$  similarly

converges to 0.5 in probability as  $n$  tends to infinity. Choosing the first  $N_n^s$  realizations from (6) and (7) yields random samples of equal sample size  $N_n^s$  given by

$$(\tilde{U}_1^{B^s}, \tilde{V}_1^{B^s}), \dots, (\tilde{U}_{N_n^s}^{B^s}, \tilde{V}_{N_n^s}^{B^s}) \quad \text{and} \quad (\tilde{U}_1^{\overline{B^s}}, \tilde{V}_1^{\overline{B^s}}), \dots, (\tilde{U}_{N_n^s}^{\overline{B^s}}, \tilde{V}_{N_n^s}^{\overline{B^s}}). \quad (8)$$

Under the null hypothesis  $H_0^s$  of  $C$  being symmetric, the two newly generated random samples have the same underlying copula  $C$  and, hence, Kendall's tau. Therefore, the empirically estimated Kendall's tau for both random samples should be of the same magnitude. Now, we base our test on the difference

$$S_{N_n^s} := \hat{\tau}_{C, N_n^s}^{B^s} - \hat{\tau}_{C, N_n^s}^{\overline{B^s}},$$

where  $\hat{\tau}_{C, N_n^s}^{B^s}$  and  $\hat{\tau}_{C, N_n^s}^{\overline{B^s}}$  denote the empirically estimated Kendall's taus based on the two samples from (8).

It is clear that

$$\frac{N_{B^s}}{n} \xrightarrow{\mathbb{P}} 0.5 \quad \text{and} \quad \frac{N_{\overline{B^s}}}{n} \xrightarrow{\mathbb{P}} 0.5.$$

For  $n \geq 2$ , the above sampling algorithm can be slightly modified to ensure that  $N_{B^s}$  and  $N_{\overline{B^s}}$  are positive random variables. Therefore,  $N_n^s$  is a sequence of positive integer-valued random variables with

$$\frac{N_n^s}{n} \xrightarrow{\mathbb{P}} 0.5. \quad (9)$$

To state the asymptotic distribution of the test statistic  $S_{N_n^s}$  in Theorem 1, we define  $\tilde{h}_1((U_1, V_1)) := \mathbb{E}[\text{sgn}(U_1 - U_2) \text{sgn}(V_1 - V_2) \mid U_1, V_1]$ .

**Theorem 1** *Let  $(U_1, V_1), \dots, (U_n, V_n)$  be an i.i.d. random sample from a bivariate random vector  $(U, V)$  with  $\mathbb{P}(U=V) = 0$ , whose distribution function is a symmetric copula  $C$ . Further, let (9) hold. Then,*

$$\sqrt{n^*} \cdot S_{N_n^s} \xrightarrow{d} N(0, 2\sigma^2),$$

where  $n^* = n/2$  and  $\sigma^2 = \mathbb{V}\text{ar}(2\tilde{h}_1((U_1, V_1)))$ .

The proof of Theorem 1 is given in the Appendix and relies on Anscombe (1952), who showed sufficient conditions to preserve convergence in distribution for a sequence of random variables indexed by a proper sequence of random variables. The test statistic  $S_{N_n^s}$  is the difference of two  $U$ -statistics with random sample sizes, whose asymptotic distributions were derived by Sproule (1974).

In practical applications, the unknown variance  $\sigma^2$  in Theorem 1 should be consistently estimated. The following remark describes a possible consistent estimation procedure for  $\sigma^2$ .

**Remark 1** The function  $\tilde{h}_1$  has the representation (see e.g. Theorem 4.3 in Dengler (2010))

$$\tilde{h}_1((U, V)) = 1 - 2U - 2V + 4C(U, V).$$

Subsequently, the asymptotic variance of  $S_{N_n^s}$  can be consistently estimated in the framework of Jaser et al. (2017). Using the whole random sample  $(U_1, V_1), \dots, (U_n, V_n)$ ,  $\tilde{h}_1((U_i, V_i))$  is estimated non-parametrically by

$$\hat{h}_1((U_i, V_i)) = 1 - 2U_i - 2V_i + 4C_n(U_i, V_i), i \in \{1, \dots, n\},$$

where  $C_n$  denotes the empirical copula given by

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^n I\{U_i \leq u, V_i \leq v\},$$

with  $I\{\cdot, \cdot\}$  denoting the indicator function. Now,  $\sigma^2$  is consistently estimated by the sample variance  $\hat{\sigma}_n^2$  of

$$2\hat{h}_1((U_1, V_1)), \dots, 2\hat{h}_1((U_n, V_n)).$$

For details see Jaser et al. (2017).

Based on Theorem 1, we propose the test function

$$\delta^s(U_1, \dots, U_n) = I\left\{|\sqrt{n^\star} \cdot S_{N_n^s} / \hat{\sigma}_n| > z_{1-\alpha/2}\right\}$$

to test  $H_0^s$  against  $H_1^s$  at the significance level  $\alpha$ , where  $z_\alpha$  denotes the  $\alpha$ -quantile of the standard normal distribution.

### 3.2 Test for radial symmetry

Let  $(U, V)$  be distributed according to the radially symmetric copula  $C$ . Hence,  $C$  coincides with its survival copula  $\bar{C}$ , and it holds that  $(U, V) \stackrel{d}{=} (1 - U, 1 - V)$ . Further, we assume that  $\mathbb{P}(U+V=1) = 0$ . For sample realizations from  $C$ , scatter plots show symmetry with respect to the the point  $(0.5, 0.5)$ . Now, we split a given data set with respect to the counter-diagonal into two sub-sets: one below and the other above the counter-diagonal. By reflecting any two observations from different sub-sets with respect to the point  $(0.5, 0.5)$ , the copula of the resulting sample is not changed. Therefore, a sample from the copula  $C$  can be generated just using all observations either below or above the counter-diagonal.

More precisely, note that the complementary events

$$B^r := \{\omega : U + V < 1\} \quad \text{and} \quad \bar{B}^r := \{\omega : U + V > 1\}$$

have equal probabilities of 0.5. We follow the idea of our test for symmetry and use two mixture representations conditioned on the events that  $(U, V)$  is below and above the counter-diagonal, respectively, in order to generate two i.i.d. random samples of size  $N_{B^r}$  and  $N_{\overline{B^r}}$  out of one given i.i.d. random sample from  $C$ .

Similarly to Sect. 3.1, the corresponding test statistic is given by

$$R_{N_n^r} := \widehat{\tau}_{C, N_n^r}^{B^r} - \widehat{\tau}_{C, N_n^r}^{\overline{B^r}},$$

where  $\widehat{\tau}_{C, N_n^r}^{B^r}$  and  $\widehat{\tau}_{C, N_n^r}^{\overline{B^r}}$  denote the empirically estimated Kendall's taus based on the two samples, and  $N_n^r := \min(N_{B^r}, N_{\overline{B^r}})$ . As before,  $N_n^r$  can be assumed to be a sequence of positive integer-valued random variables with

$$\frac{N_n^r}{n} \xrightarrow{\mathbb{P}} 0.5. \quad (10)$$

The asymptotic distribution of the test statistic  $R_{N_n^r}$  is given in the following theorem.

**Theorem 2** *Let  $(U_1, V_1), \dots, (U_n, V_n)$  be an i.i.d. random sample from a bivariate random vector  $(U, V)$  with  $\mathbb{P}(U+V=1) = 0$ , whose distribution function is a radially symmetric copula  $C$ . Further, let (10) hold. Then,*

$$\sqrt{n^*} \cdot R_{N_n^r} \xrightarrow{d} N(0, 2\sigma^2),$$

where  $n^* = n/2$  and  $\sigma^2 = \mathbb{V}\text{ar}\left(2\tilde{h}_1((U_1, V_1))\right)$ .

The proof of Theorem 2 is similar to the proof of Theorem 1 and, therefore, omitted. Note that the asymptotic variance  $\sigma^2$  is the same as in Theorem 1. Hence, Remark 1 yields a consistent estimation procedure for the asymptotic variance of  $R_{N_n^r}$  and the test function  $\delta^r$  is constructed similarly to  $\delta^s$ .

## 4 Simulation study

In order to assess the finite-sample performance of our proposed tests for symmetry and radial symmetry, a Monte Carlo study was conducted for the test problems  $H_0^s$  and  $H_0^r$ . First, we would like to point out that the tests are based on a random sampling algorithm. Therefore, the value of the test statistic inherits some variability. The upcoming simulation study shows that the randomness of the test statistic does not affect the empirical level of the tests and the tests still provide good empirical power.

As a benchmark, we use the more advanced tests by Genest et al. (2012) and Genest and Nešlehová (2014), respectively, which are available in the R-package `copula` (see `exchTest` and `radSymTest` in Hofert et al. (2018)). Note that our proposed tests rely on the assumption of known marginal distributions, while the tests by Genest et al. (2012) and Genest and Nešlehová (2014) take into account their non-parametric



estimation. Further, their tests compare the whole copulas while our proposed tests are based on two sample characteristics of the bivariate copula. We assume that this fact is mainly responsible for the differences between our and their numerical results.

The mixture representations for symmetric or radial symmetric copulas may not hold if marginal distributions are estimated. Therefore, it is not straightforward for us to extend the proposed tests for unknown margins. Further, if marginal distributions are estimated non-parametrically, the two newly generated samples may contain ties. Our Monte Carlo study empirically assesses the influence of non-parametrically estimated marginal distributions on the level and power of our proposed tests. For this, each copula sample  $(U_1, V_1), \dots, (U_n, V_n)$  is replaced by the corresponding bivariate pseudo-observations  $(\widehat{U}_1, \widehat{V}_1), \dots, (\widehat{U}_n, \widehat{V}_n)$ , where

$$(\widehat{U}_i, \widehat{V}_i) = \frac{1}{n+1} \left( \text{rank of } U_i \text{ in } U_1, \dots, U_n, \text{rank of } V_i \text{ in } V_1, \dots, V_n \right),$$

for  $i \in \{1, \dots, n\}$ .

#### 4.1 Setup

First of all, the number of Monte Carlo replications was set to  $N = 1000$ , and all tests were performed at a significance level of  $\alpha = 0.05$ . To determine the empirical level and power of the tests, the simulation study was carried out for different sample sizes, levels of dependence measured in terms of Kendall's tau and types of dependence expressed in terms of copula families.

More precisely, random samples of size  $n \in \{100, 250, 500, 1000\}$  were considered for all tests throughout the study. In addition, the influence of the strength of dependence was investigated by choosing five different levels of dependence in terms of Kendall's tau given by  $\tau \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$ . Finally, the type of dependence is determined through the choice of a specific copula family. For this, some of the most popular copula families and some derived special cases were considered in the simulation study. The performance of all tests was studied for samples from the Gaussian,  $t$ , Frank, Clayton, and Gumbel copula families. The Gaussian and the  $t$  copula are elliptical copulas and, thus, also symmetric and radially symmetric. Further, the Frank, Clayton, and Gumbel copula are symmetric Archimedean copulas. In addition, the Frank copula is also radially symmetric.

Since all listed copulas are symmetric, asymmetrized versions of the Gaussian, Clayton, and Gumbel copula families were additionally used to assess the power of the test for symmetry. Regarding the asymmetrization, we followed the procedure in Genest et al. (2012) and used Khoudraji's device (see Khoudraji 1995). The asymmetrized copulas are given in terms of an asymmetrization parameter  $\delta \in (0, 1)$ . Maximum asymmetry is observed for  $\delta = 0.5$  and, hence, we also chose  $\delta \in \{0.25, 0.5, 0.75\}$ . Since there is only little asymmetry for small values of  $\tau$ , we analyzed the performance of the test for symmetry for  $\tau \in \{0.5, 0.75, 0.9\}$  in this context. Following Genest and Nešlehová (2014), a Skewed- $t$  copula with 4 degrees of freedom and skewness parameter  $\gamma = (1, 1)$  was chosen to study the power of the test for radial symmetry.

## 4.2 Test for symmetry

In this section, the finite-sample performance of the test of  $H_0^s$  for symmetry based on the test statistics  $S_{N_n^s}$  is analyzed. To study the level of the test, random samples from the Gaussian,  $t$ , Frank, Clayton, and Gumbel copula were considered. Table 1 reports the empirical level of our test (in Column JMS), of our test for pseudo-observations (in Column JMSP), and of the test by Genest et al. (2012) (in Column GNQ).

First, note that our test holds its nominal level across all copula models, sample sizes, and values of Kendall's tau. Compared to the more advanced test by Genest et al. (2012), our test seems to hold its nominal level a little better. For pseudo-observations, our test is generally rather conservative and its empirical level is decreasing with increasing sample size. Surprisingly, this does not influence the empirical power negatively.

Random samples from the asymmetrized versions of the Gaussian, Clayton, and Gumbel copula families were used to investigate the power of the test for symmetry. Table 2 displays the empirical power of our test (in Column JMS), of our test for pseudo-observations (in Column JMSP), and of the test by Genest et al. (2012) (in Column GNQ). Even if the results vary noticeably across the different combinations of factors, our test generally achieves sufficient power. As expected, the rejection rates increase with the sample size as well as with the strength of dependence. In terms of the asymmetrization parameter  $\delta$ , the largest power is mostly observed for  $\delta = 0.5$ . Since maximum asymmetry occurs near  $\delta = 0.5$ , this is also expected.

Compared to the test by Genest et al. (2012), our test has slightly lower power and needs higher sample sizes to achieve similar power. The empirical power of our test for pseudo-observations is in most cases comparable to the one for the copula samples. Moreover, across all different combinations of factors, there are several scenarios with higher empirical power for the pseudo-observations even though the empirical level for them is lower than for copula data.

Our test for symmetry is computationally less intensive than the more advanced test by Genest et al. (2012), where bootstrap methods are applied. Table 3 illustrates the running times of the tests (in Row JMS and GNQ, respectively) for samples of size  $n = 10^3$ ,  $10^4$ , and  $10^5$ . For one sample of size  $n = 10^4$ , the running time of our test is about 2 seconds in comparison to more than 2 minutes for the corresponding test by Genest et al. (2012). For  $n = 10^5$ , it was not possible to conduct the test for symmetry of Genest et al. (2012) using the R-package *copula*, while our test runs in a bit more than 3 minutes. Thus, our test for symmetry is up to 75 times faster and can especially be recommended for huge samples.

## 4.3 Test for radial symmetry

In this section, the finite-sample performance of the test of  $H_0^r$  for radial symmetry based on the test statistic  $R_{N_n^r}$  is analyzed. Random samples from the Gaussian,  $t$ , and Frank copula were considered in order to examine the empirical level. Table 4 presents the empirical level of our test (in Column JMR), of our test for pseudo-observations (in Column JMRP), and of the test by Genest and Nešlehová (2014) (in Column GN).

**Table 1** Empirical level of our test for symmetry (JMS), our test for pseudo-observations (JMSP), and the test by Genest et al. (2012) (GNQ) with significance level  $\alpha = 0.05$ : rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  with Kendall's tau  $\tau_C$ 

$C$ $\tau_C$	$n = 100$			$n = 250$			$n = 500$			$n = 1000$		
	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ
Gauss												
0.25	0.066	0.012	0.022	0.055	0.016	0.039	0.055	0.012	0.048	0.054	0.007	0.044
0.50	0.060	0.012	0.015	0.061	0.006	0.027	0.045	0.010	0.020	0.051	0.006	0.032
0.75	0.041	0.024	0.011	0.052	0.021	0.011	0.046	0.016	0.004	0.050	0.006	0.013
$t_{\nu=5}$												
0.25	0.059	0.014	0.033	0.043	0.016	0.046	0.052	0.023	0.035	0.059	0.013	0.035
0.50	0.047	0.018	0.014	0.058	0.016	0.035	0.049	0.009	0.031	0.062	0.006	0.046
0.75	0.034	0.027	0.022	0.055	0.019	0.014	0.057	0.010	0.013	0.051	0.008	0.019
Frank												
0.25	0.054	0.014	0.031	0.052	0.014	0.038	0.051	0.010	0.043	0.045	0.009	0.032
0.50	0.057	0.024	0.015	0.060	0.013	0.025	0.052	0.005	0.038	0.061	0.007	0.035
0.75	0.036	0.017	0.016	0.032	0.009	0.011	0.042	0.010	0.006	0.048	0.009	0.016
Clayton												
0.25	0.060	0.024	0.033	0.062	0.018	0.040	0.052	0.010	0.032	0.051	0.009	0.043
0.50	0.063	0.029	0.031	0.050	0.013	0.029	0.059	0.003	0.029	0.045	0.004	0.035
0.75	0.059	0.051	0.021	0.059	0.028	0.015	0.056	0.015	0.021	0.049	0.009	0.027
Gumbel												
0.25	0.063	0.022	0.036	0.049	0.020	0.038	0.061	0.011	0.035	0.049	0.013	0.042
0.50	0.057	0.015	0.027	0.050	0.010	0.026	0.053	0.004	0.024	0.049	0.006	0.039
0.75	0.053	0.034	0.017	0.051	0.021	0.013	0.050	0.019	0.008	0.049	0.003	0.028

**Table 2** Empirical power of our test for symmetry (JMS), our test for pseudo-observations (JMSP), and the test by Genest et al. (2012) (GNQ) with significance level  $\alpha = 0.05$ ; rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  asymmetricized with parameter  $\delta$  and with Kendall's tau  $\tau_C$

$C$	$n = 100$			$n = 250$			$n = 500$			$n = 1000$		
$\tau_C$	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ
Gauss, $\delta = 0.25$												
0.50	0.095	0.067	0.082	0.157	0.140	0.233	0.285	0.242	0.466	0.480	0.530	0.803
0.75	0.434	0.553	0.618	0.849	0.954	0.995	0.991	1.000	1.000	1.000	1.000	1.000
0.90	0.963	0.983	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gauss, $\delta = 0.5$												
0.50	0.155	0.145	0.199	0.296	0.312	0.499	0.564	0.645	0.851	0.846	0.936	0.989
0.75	0.672	0.798	0.907	0.976	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	0.973	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gauss, $\delta = 0.75$												
0.50	0.158	0.134	0.168	0.298	0.289	0.393	0.542	0.604	0.764	0.846	0.889	0.968
0.75	0.521	0.576	0.626	0.896	0.961	0.988	0.996	1.000	1.000	1.000	1.000	1.000
0.90	0.657	0.744	0.844	0.979	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
Clayton, $\delta = 0.25$												
0.50	0.177	0.149	0.093	0.361	0.343	0.260	0.586	0.633	0.548	0.885	0.954	0.909
0.75	0.678	0.763	0.779	0.958	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	0.892	0.981	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Clayton, $\delta = 0.5$												
0.50	0.148	0.115	0.111	0.307	0.264	0.339	0.509	0.578	0.715	0.813	0.901	0.965
0.75	0.463	0.550	0.834	0.871	0.953	1.000	0.991	1.000	1.000	1.000	1.000	1.000
0.90	0.817	0.920	0.999	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2 continued

$C$ $\tau_C$	$n = 100$			$n = 250$			$n = 500$			$n = 1000$		
	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ	JMS	JMSP	GNQ
Clayton, $\delta = 0.75$												
0.50	0.092	0.058	0.072	0.132	0.110	0.169	0.197	0.173	0.295	0.370	0.365	0.586
0.75	0.190	0.189	0.366	0.440	0.469	0.814	0.751	0.832	0.988	0.961	0.990	1.000
0.90	0.460	0.515	0.764	0.827	0.919	0.997	0.989	0.998	1.000	1.000	1.000	1.000
Gumbel, $\delta = 0.25$												
0.50	0.142	0.149	0.110	0.263	0.268	0.275	0.515	0.573	0.637	0.744	0.855	0.916
0.75	0.592	0.743	0.679	0.944	0.982	0.997	0.998	1.000	1.000	1.000	1.000	1.000
0.90	0.990	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gumbel, $\delta = 0.5$												
0.50	0.285	0.305	0.272	0.599	0.669	0.704	0.895	0.963	0.974	0.992	0.999	1.000
0.75	0.862	0.950	0.970	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	0.987	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gumbel, $\delta = 0.75$												
0.50	0.273	0.284	0.284	0.638	0.690	0.690	0.888	0.966	0.963	0.990	1.000	1.000
0.75	0.619	0.693	0.752	0.951	0.985	0.993	0.999	1.000	1.000	1.000	1.000	1.000
0.90	0.722	0.799	0.893	0.987	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000

**Table 3** Running times in seconds for our tests (JMS/JMR) and the tests by Genest et al. (2012)/ Genest and Nešlehová (2014) (GNQ/GN) for samples of size  $n$

	$n = 10^3$	$n = 10^4$	$n = 10^5$
JMS	0.02	1.77	197.70
GNQ	1.34	134.03	–
JMR	0.04	3.55	399.91
GN	7.60	655.05	63,982.67 (17.77 h)

In general, our test and the test by Genest and Nešlehová (2014) hold their nominal level. For pseudo-observations, our test also holds its nominal level in most cases. One exception is the Frank copula for  $\tau_C = 0.75$ . Further analysis showed that increasing the sample size does not reduce the problem of inflated rejection rates as the empirical levels oscillate around 0.119. Hence, our test for radial symmetry is systematically too liberal in this setting.

To assess the empirical power, random samples from the Clayton, Gumbel, and Skewed- $t_4$  copula were used. Table 5 reports the empirical power of our test (in Column JMR), of our test for pseudo-observations (in Column JMRP), and of the test by Genest and Nešlehová (2014) (in Column GN). First, note that the results differ considerably for the various combinations of factors. For all copulas, the power increases with the sample size, which is expected. Further, for the Clayton and the Gumbel copula, the power also increases with the degree of dependence, whereas for the Skewed- $t_4$  copula, the power decreases with increasing  $\tau_C$ . Lastly, note that the rejection rates are slightly lower for the Gumbel copula.

Our test overall achieves satisfactory empirical power against the various alternatives. Compared to the test by Genest and Nešlehová (2014), it is in many cases somewhat less powerful. However, it achieves equal or even slightly higher power especially in scenarios where the more advanced test has difficulties to detect the radial asymmetry. Examples are given by the Gumbel copula and the Skewed- $t_4$  copula for  $n = 100$  and  $n = 250$  in combination with  $\tau_C = 0.75$ . The empirical power of our test for pseudo-observations is overall slightly higher than the one for copula samples, which might be caused by possible high empirical levels.

Table 3 illustrates the running times for our test (in Row JMR) and the test by Genest and Nešlehová (2014) (in Row GN) for samples of size  $n = 10^3$ ,  $10^4$ , and  $10^5$ . For one sample of size  $n = 10^4$ , the running time of our test is less than 4 seconds in comparison to almost 11 minutes for the corresponding test by Genest and Nešlehová (2014). For one sample of size  $n = 10^5$ , it runs in less than 7 minutes, while the test by Genest and Nešlehová (2014) requires almost 18 hours. Thus, it is up to 190 times faster and, similarly to our test for symmetry, it can especially be recommended for huge samples.

**Table 4** Empirical level of our test for radial symmetry (JMR), our test for pseudo-observations (JMRP), and the test by Genest and Neslehová (2014) (GN) with significance level  $\alpha = 0.05$ : rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  with Kendall's tau  $\tau_C$

$C$	$\tau_C$	$n = 100$			$n = 250$			$n = 500$			$n = 1000$		
		JMR	JMRP	GN	JMR	JMRP	GN	JMR	JMRP	GN	JMR	JMRP	GN
Gauss													
	0.25	0.049	0.049	0.041	0.058	0.029	0.047	0.054	0.030	0.042	0.042	0.030	0.049
	0.50	0.055	0.059	0.037	0.053	0.053	0.044	0.060	0.055	0.059	0.047	0.045	0.044
	0.75	0.052	0.073	0.042	0.046	0.077	0.051	0.039	0.060	0.052	0.056	0.079	0.051
$t_{\nu=5}$													
	0.25	0.063	0.038	0.050	0.065	0.040	0.052	0.054	0.034	0.039	0.056	0.036	0.049
	0.50	0.041	0.040	0.031	0.053	0.042	0.043	0.056	0.056	0.057	0.049	0.039	0.040
	0.75	0.052	0.054	0.029	0.053	0.064	0.051	0.047	0.049	0.036	0.051	0.052	0.052
Frank													
	0.25	0.061	0.033	0.039	0.053	0.034	0.045	0.052	0.032	0.044	0.042	0.036	0.043
	0.50	0.044	0.067	0.052	0.046	0.067	0.049	0.045	0.066	0.052	0.061	0.068	0.048
	0.75	0.032	0.111	0.037	0.044	0.116	0.040	0.050	0.119	0.036	0.037	0.125	0.052

**Table 5** Empirical power of our test for radial symmetry (JMR), our test for pseudo-observations (JMRP), and the test by Genest and Nešlehová (2014) (GN) with significance level  $\alpha = 0.05$ : rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  with Kendall's tau  $\tau_C$

$C$	$\tau_C$	$n = 100$			$n = 250$			$n = 500$			$n = 1000$		
		JMR	JMRP	GN	JMR	JMRP	GN	JMR	JMRP	GN	JMR	JMRP	GN
Clayton													
	0.25	0.256	0.229	0.377	0.517	0.506	0.730	0.851	0.858	0.959	0.983	0.993	1.000
	0.50	0.625	0.640	0.811	0.955	0.965	0.997	1.000	1.000	1.000	1.000	1.000	1.000
	0.75	0.775	0.884	0.921	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gumbel													
	0.25	0.119	0.123	0.092	0.207	0.215	0.246	0.343	0.342	0.491	0.615	0.638	0.800
	0.50	0.166	0.193	0.161	0.413	0.447	0.458	0.703	0.722	0.814	0.934	0.948	0.987
	0.75	0.166	0.234	0.132	0.516	0.575	0.495	0.814	0.823	0.828	0.981	0.985	0.992
Skewed $t_{\nu=4}$													
	0.25	0.470	0.493	0.514	0.885	0.905	0.951	0.996	0.998	1.000	1.000	1.000	1.000
	0.50	0.331	0.395	0.336	0.713	0.734	0.770	0.965	0.963	0.991	1.000	1.000	1.000
	0.75	0.152	0.230	0.113	0.497	0.575	0.436	0.834	0.878	0.843	0.991	0.993	0.997



## 5 Testing procedure for ellipticity

This section presents a powerful and simple non-parametric statistical procedure to test whether the dependence structure of a bivariate random vector with uniform margins is captured by an elliptical copula.

### 5.1 The testing procedure

The testing procedure consists of the following three steps. First, the hypothesis that the unknown copula  $C$  is symmetric, that is  $H_0^s$  is tested against the alternative  $H_1^s$ . If the hypothesis  $H_0^s$  cannot be rejected, we test the hypothesis that the unknown copula  $C$  is radially symmetric, that is  $H_0^r$  against the alternative  $H_1^r$ . In the third step of our testing procedure, the equality of Kendall's tau and Blomqvist's beta is tested, that is  $H_0^e$  is tested against the alternative  $H_1^e$ . If any of the three hypotheses is rejected, we also reject our original null hypothesis  $H_0$  that  $C$  belongs to the class of elliptical copulas. If none of the three hypotheses can be rejected, we cannot reject the null hypothesis  $H_0$  of  $C$  being elliptical. To assess the effect of non-parametrically estimated marginal distribution functions on the proposed testing procedure, the following simulation study is also conducted for pseudo-observations.

### 5.2 Simulation study

In this section, the finite-sample performance of the proposed testing procedure is analyzed. The corresponding Monte Carlo study was set up similarly to Sect. 4. Note that our testing procedure for ellipticity consists of a multiple test problem with three sub-hypotheses. In order to maintain the global level  $\alpha = 0.05$ , we made use of the standard Bonferroni procedure (see, e.g., Miller and Rupert 1981). For this, the three null hypotheses  $H_0^s$ ,  $H_0^r$ , and  $H_0^e$  were tested sequentially and separately at the significance level  $\alpha/3$ . Finally, the null hypothesis  $H_0 : C \in \mathcal{C}^{ellipt}$  was rejected if any of the considered sub-hypotheses was rejected.

Table 6 reports the empirical level of the testing procedure (in Column JMT) and of the testing procedure for pseudo-observations (in Column JMTP) based on random samples from the Gaussian and the  $t$  copula. The testing procedure appears to hold its nominal level for copula data as well as for pseudo-observations across all combinations of factors.

To study the power of the testing procedure, random samples of the Frank, Clayton, and Gumbel copula were considered. Table 7 shows the empirical power of the testing procedure (in Column JMT) and of the testing procedure for pseudo-observations (in Column JMTP). As already observed for all individual tests, the rejection rates vary clearly across copula families, levels of dependence, and sample sizes. As expected, the power increases with the sample size and with the level of dependence. The lowest rejection rates are observed for the Frank copula. However, it is still sufficiently good in detecting the lack of ellipticity if the sample size is large enough and the level of dependence is not too close to independence. For the Clayton copula, the testing procedure performs best in detecting the non-ellipticity, even in very small samples of

**Table 6** Empirical level of our testing procedure for ellipticity (JMT) and of our testing procedure for pseudo-observations (JMTP) with significance level  $\alpha = 0.05$ : rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  with Kendall's tau  $\tau_C$ 

$C$ $\tau_C$	$n = 100$		$n = 250$		$n = 500$		$n = 1000$	
	JMT	JMTP	JMT	JMTP	JMT	JMTP	JMT	JMTP
Gauss								
0.25	0.039	0.034	0.050	0.035	0.053	0.026	0.047	0.023
0.50	0.061	0.041	0.071	0.038	0.053	0.037	0.052	0.032
0.75	0.049	0.044	0.057	0.046	0.054	0.038	0.057	0.048
$t_{\nu=5}$								
0.25	0.062	0.036	0.054	0.032	0.051	0.035	0.047	0.031
0.50	0.046	0.037	0.052	0.037	0.051	0.034	0.052	0.029
0.75	0.039	0.038	0.056	0.046	0.062	0.043	0.055	0.040

**Table 7** Empirical power of our testing procedure for ellipticity (JMT) and of our testing procedure for pseudo-observations (JMTP) with significance level  $\alpha = 0.05$ : rate of rejecting  $H_0$  as observed in 1000 random samples of size  $n$  from copula family  $C$  with Kendall's tau  $\tau_C$ 

$C$ $\tau_C$	$n = 100$		$n = 250$		$n = 500$		$n = 1000$	
	JMT	JMTP	JMT	JMTP	JMT	JMTP	JMT	JMTP
Frank								
0.25	0.074	0.045	0.071	0.055	0.112	0.099	0.190	0.169
0.50	0.085	0.087	0.175	0.165	0.331	0.316	0.619	0.597
0.75	0.098	0.167	0.226	0.238	0.459	0.444	0.747	0.749
Clayton								
0.25	0.177	0.138	0.375	0.338	0.729	0.729	0.959	0.977
0.50	0.455	0.462	0.896	0.915	0.998	1.000	1.000	1.000
0.75	0.591	0.779	0.993	0.999	1.000	1.000	1.000	1.000
Gumbel								
0.25	0.096	0.091	0.140	0.126	0.219	0.218	0.467	0.441
0.50	0.107	0.095	0.270	0.275	0.557	0.551	0.863	0.874
0.75	0.112	0.156	0.340	0.415	0.680	0.714	0.949	0.947

size  $n = 100$ . The results for the Gumbel copula are only slightly worse than for the Clayton copula and, hence, the testing procedure is still powerful. Similar observations can be made for the empirical power of the testing procedure for pseudo-observations. As for the individual tests, there are scenarios with higher empirical power for the pseudo-observations than for copula data.

Compared to the results of our test based on the equality of Kendall's tau and Blomqvist's beta in Jaser et al. (2017), the testing procedure performs much better for the Clayton and the Gumbel copula. It is now possible to distinguish non-elliptical copulas with very close Kendall's tau and Blomqvist's beta if they are not symmetric or not radially symmetric. Furthermore, the testing procedure is still able to detect the

non-ellipticity of the Frank copula, which is symmetric and radially symmetric. All in all, the idea of using the most known properties of elliptical copulas in the testing procedure shows clear advantages.

## 6 Empirical analysis

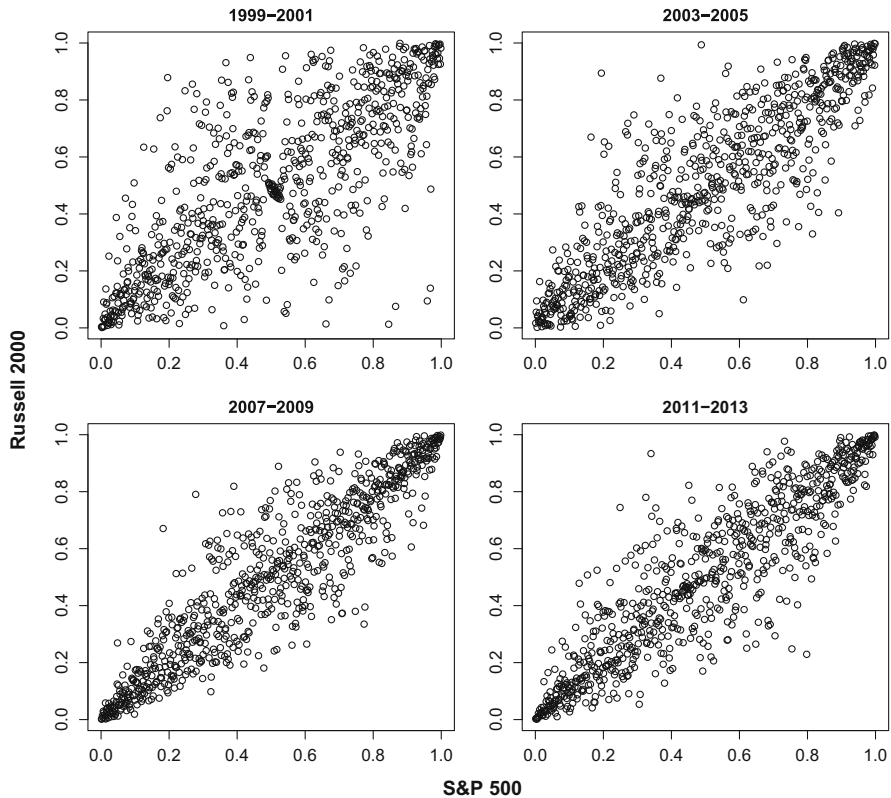
The main aim of this section is to illustrate our testing procedure for ellipticity in practice using financial and insurance data. For this, the results of the three building tests are reported. Since our tests for symmetry and radial symmetry are based on a random sampling algorithm (see Sects. 3.1 and 3.2), we performed the tests within the testing procedure for ellipticity 1000 times and consider the averages of the resulting  $p$ -values.

In the sequel, the testing procedure is applied to six different data sets in total. For the majority of these data sets, the decision resulting from the testing procedure is the same for all 1000 replications. For two datasets, we get a different decision from 2 and 1 out of 1000 replications, respectively, than we get from the testing procedure using the average of the  $p$ -values. Hence, the number of cases with a different decision seems to be negligible. We recommend to perform the testing procedure more than 2 times, if the decision to accept or reject the null hypothesis is very close.

### 6.1 Financial data

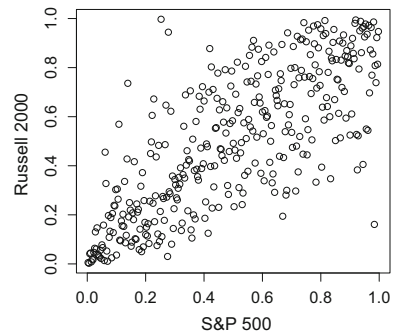
As a first illustration, our testing procedure is applied to financial data from the US stock market. Two major US stock price indices are selected: the Standard & Poor's 500 (S&P 500), as one of the most popular indices of large-cap US equities, and the Russell 2000, as one of the most popular small-cap US indices. In order to get data sets of large sample sizes, daily returns of the two indices over different periods of three years are considered. It is well known that the dependence structure of financial data for crisis and non-crisis periods differs. Therefore, the following analysis is based on the daily log-returns of the S&P 500 and the Russell 2000 indices for the crisis periods from 1999 to 2001 and 2007 to 2009, as well as for the non-crisis periods from 2003 to 2005 and from 2011 to 2013. Furthermore, we are also interested in the dependence structure between monthly returns of the two indices, which is of more interest from a macroeconomic point of view. For this, monthly returns are considered for the period of the last 30 years from 1988 to 2017.

To remove temporal dependencies, ARMA-GARCH time series models are fitted to each series of log-returns. The choice of the final model is done using the BIC (see, e.g., Schwarz 1978). The resulting standardized residuals are transformed non-parametrically by using the empirical cumulative distribution functions to achieve approximate i.i.d. uniform margins. Figures 1 and 2 display the scatter plots of the underlying copula data for the different data sets comprised of daily and monthly returns of the S&P 500 and the Russell 2000 for the selected time periods, respectively. In Fig. 1, an elliptical shape is visually observable for the non-crisis periods from 2003 to 2005 and from 2011 to 2013, whereas the shape of the data for the crisis periods



**Fig. 1** Daily data: Scatter plots of the non-parametrically transformed standardized residuals of the ARMA-GARCH models for the log-returns of the S&P 500 and the Russell 2000 indices for different time periods

**Fig. 2** Monthly data: Scatter plot of the non-parametrically transformed standardized residuals of the ARMA-GARCH model for the log-returns of the S&P 500 and the Russell 2000 indices for the time period from 1988 to 2017



from 1999 to 2001 and from 2007 to 2009 might be non-elliptical from the visual impression. The shape of the copula data in Fig. 2 is clearly non-elliptical.

Table 8 presents  $p$ -values for the above discussed data sets. For the two non-crisis periods, the null hypothesis  $H_0$  of the elliptical dependence structure cannot be rejected at the considered significance level of 5%. In contrast, the testing procedure rejects

**Table 8**  $p$ -values of our tests for symmetry, radial symmetry, and equality of Kendall's tau and Blomqvist's beta for the dependence structure of the financial data (S&P 500 and Russell 2000) for different time periods

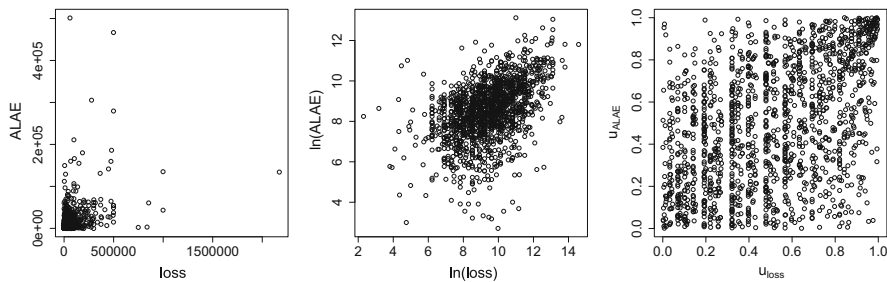
Data	Time period	Symmetry	Radial symmetry	Equality
Daily data (crisis)	1999–2001	0.154	0.124	0.004
Daily data (non-crisis)	2003–2005	0.943	0.352	0.705
Daily data (crisis)	2007–2009	0.857	0.043	0.030
Daily data (non-crisis)	2011–2013	0.346	0.345	0.965
Monthly data	1988–2017	0.417	0.002	0.380

$H_0$  for the crisis period from 1999 to 2001 due to a very low  $p$ -value of the test for equality. For the crisis period from 2007 to 2009,  $H_0$  cannot be rejected at the considered significance level of 5%. However, the  $p$ -values of 0.042 and 0.030 for the test for radial symmetry and the test for equality, respectively, are quite low and provide some indication against  $H_0$ . Note that the test for radial symmetry leads to a rejection of  $H_0$  for 2 out of the 1000 replications. Hence, also for the crisis period from 2007 to 2009, elliptical copulas cannot be recommended to model the dependence structure of the underlying data. The same applies for the data set comprised of the monthly log-returns. Due to the very low  $p$ -value of the test for radial symmetry, the null hypothesis of ellipticity is rejected at the considered significance level of 5%. All in all, the results are in accordance with our expectations and the visual observations from Figs. 1 and 2.

## 6.2 Insurance data

One famous example for a bivariate data set from the insurance sector is given by losses and corresponding allocated loss adjustment expenses (short ALAE) of insurance claims. The US Insurance Services Office has collected data on 1500 general liability claims randomly chosen from late settlement lags. Each claim contains an indemnity payment (loss) and an allocated loss adjustment expense (ALAE). A detailed description of the data set can be found in Frees and Valdez (1998). The modeling of the joint distribution of losses and ALAEs has also been analyzed in Genest and Ghoudi (January 1998), Klugman and Parsa (1999), Denuit et al. (2006), Chen and Fan (2005), and Zhang et al. (2016), among others. In Fig. 3, scatter plots of the observations (left) and of the logarithm of the observations (middle) are displayed.

To achieve the approximate i.i.d. uniform margins, data is transformed non-parametrically by using the marginal empirical cumulative distribution functions. A scatter plot of the resulting transformed loss and ALAE is presented in Fig. 3 (right). Applying our testing procedure for ellipticity then leads to  $p$ -values of 0.391 for symmetry, 0.034 for radial symmetry, and 0.118 for the equality of Blomqvist's beta and Kendall's tau. At the considered significance level of 5%, our testing procedure cannot reject the null hypothesis  $H_0$ . However, the low  $p$ -value of 0.034 for the radial symmetry provides some indication against  $H_0$ . Note that the test for radial symmetry leads to a rejection of  $H_0$  for 1 out of the 1000 replications. Hence, we would not



**Fig. 3** Scatter plots for data of loss and ALAE (left), logarithms of loss and ALAE (middle), non-parametrically transformed data of loss and ALAE (right). Sample size  $n = 1500$

recommend elliptical copulas to model the dependence structure of the underlying loss ALAE data.

The different parametric and semiparametric model selection procedures in Frees and Valdez (1998), Genest and Ghoudi (January 1998), Denuit et al. (2006), Chen and Fan (2005), and Zhang et al. (2016) all resulted in the Gumbel copula as the preferred model for the given loss ALAE data set. In the scatter plot of the copula data (Fig. 3, right), positive upper-tail dependence but no lower-tail dependence can be observed between the two variables. This is expected by actuaries, since large losses are often accompanied by large ALAEs, and in line with the tail dependence properties of the Gumbel copula, which exhibits only upper-tail dependence. The choice of the Gumbel copula is therefore not surprising.

## 7 Conclusion

In this paper, we derive very simple non-parametric tests for symmetry and radial symmetry for bivariate copula data, which are computationally very fast. An extensive simulation study is conducted to investigate the finite-sample performance and to compare the proposed tests to the already existing more advanced tests for symmetry and radial symmetry by Genest et al. (2012) and Genest and Nešlehová (2014), respectively, which do not require copula data and are applicable on the original scale of the observations. The results of the Monte Carlo simulation show that the proposed tests for symmetry and radial symmetry overall achieve sufficient empirical power against the various alternatives. In comparison to the more advanced tests with non-parametrically estimated margins, they are slightly less powerful and equally powerful starting from a sample size of 1000. It should be mentioned that the proposed tests are simpler and computationally less expensive and, hence, attractive for huge samples. However, the proposed tests are not consistent and may fail to detect the asymmetry if the two samples resulting from our algorithm have similar Kendall's taus.

Our next contribution is the construction of a powerful non-parametric goodness-of-fit testing procedure for elliptical copulas by combining our proposed tests for symmetry and radial symmetry with our test for copula data in Jaser et al. (2017). Hence, the most common intrinsic properties of bivariate elliptical copulas, namely symmetry, radial symmetry, and the equality of Kendall's tau and Blomqvist's beta are

utilized. The corresponding Monte Carlo simulation study shows that the proposed testing procedure is more powerful than the test in Jaser et al. (2017) for samples from non-symmetric or non-radially symmetric copula families.

Elliptical copulas are very popular in applied sciences. However, their application should be treated with caution. To illustrate the testing procedure for ellipticity in practice, it is applied to financial and insurance data. The first empirical application to data from the US stock market highlights that the dependence structure of two major US stock price indices is not always captured by an elliptical copula. The second application to the loss and ALAE insurance data set indicates that an elliptical copula might not be the right choice to model the corresponding dependence structure.

Our tests for symmetry and radial symmetry can be combined with variance reduction techniques (see, e.g., Korn et al. 2010). To this end, the considered sub-samples are reflected with respect to the main diagonal or the point  $(0.5, 0.5)$ , respectively. Thus, the two sub-samples with realizations below and above the main or counter diagonal are expanded by their reflected counterparts. The empirical estimator of Kendall's tau is then based on the enlarged random samples with dependent sample points. The derivation of the statistical tests for symmetry and radial symmetry based on the reflected random samples as well as the development of the testing procedure for ellipticity in higher dimensions are subject of our future research.

Finally, note that the proposed tests can be based on any bivariate non-parametric measure of ordinal association. Our tests with Kendall's tau outperform the tests with Blomqvist's beta while our tests with Spearman's rho show comparable performance as with Kendall's tau.

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## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

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## Appendix: Details on the test for symmetry

### Mixture representations of a symmetric copula $C$

Let  $C$  be a symmetric copula and the distribution function of a random vector  $(U, V)$ . Using (2), it follows for the events in (3) that

$$\mathbb{P}(B^s) = \mathbb{P}(U - V > 0) = \mathbb{P}(U - V < 0) = \mathbb{P}(\overline{B^s}) = 0.5. \quad (11)$$

The representations of  $C$  given in (4) and (5) can be derived using the law of total probability as well as (2) and (11). Thus, it follows that

$$\begin{aligned} C(u, v) &= \mathbb{P}(U \leq u, V \leq v) \\ &= \mathbb{P}(U \leq u, V \leq v \mid B^s) \cdot \mathbb{P}(B^s) + \mathbb{P}(U \leq u, V \leq v \mid \overline{B^s}) \cdot \mathbb{P}(\overline{B^s}) \\ &= 0.5 \cdot \mathbb{P}(U \leq u, V \leq v \mid U - V > 0) + 0.5 \cdot \mathbb{P}(V \leq u, U \leq v \mid V - U < 0) \\ &= 0.5 \cdot F_{U,V \mid B^s}(u, v) + 0.5 \cdot F_{V,U \mid B^s}(u, v). \end{aligned}$$

Similarly, it holds that

$$\begin{aligned} C(u, v) &= 0.5 \cdot \mathbb{P}(V \leq u, U \leq v \mid V - U > 0) + 0.5 \cdot \mathbb{P}(U \leq u, V \leq v \mid U - V < 0) \\ &= 0.5 \cdot F_{V,U \mid \overline{B^s}}(u, v) + 0.5 \cdot F_{U,V \mid \overline{B^s}}(u, v). \end{aligned}$$

### Proof of Theorem 1

Let  $(U_1, V_1), \dots, (U_n, V_n) \in [0, 1]^2$  be a sample from the statistical model

$$\left( ([0, 1]^2)^n, \mathcal{B}([0, 1]^2)^{\otimes n}, P^{\otimes n} \right),$$

where  $P$  is a distribution with symmetric copula  $C$  and uniform margins. The samples given in (8) can then be derived and the test statistic  $S_{N_n^s}$  is given by the difference of the corresponding empirical estimators  $\widehat{\tau}_{C, N_n^s}^{B^s}$  and  $\widehat{\tau}_{C, N_n^s}^{\overline{B^s}}$  of  $\tau_C$ .

For the random sample size  $N_n^s$ , it holds that  $N_n^s/n$  converges to 0.5 in probability as  $n$  tends to infinity. It follows for  $n \rightarrow \infty$  that

$$\frac{N_n^s}{\lfloor n/2 \rfloor} \xrightarrow{\mathbb{P}} 1,$$

where  $\lfloor x \rfloor$ ,  $x \in \mathbb{R}$ , denotes the integer part of  $x$ . Thus, the assumption of Theorem 1 from Anscombe (1952) is satisfied, and it is sufficient to show that the difference  $\widehat{\tau}_{C,n}^{B^s} - \widehat{\tau}_{C,n}^{\overline{B^s}}$  satisfies the conditions (C1) and (C2) of Anscombe (1952).

From the theory of  $U$ -statistics (see Hoeffding 1947), it holds that  $\sqrt{n}(\widehat{\tau}_{C,n} - \tau_C)$  converges in distribution to a centered normal distribution with variance  $\sigma^2 =$



$\text{Var} \left( 2\tilde{h}_1((U_1, V_1)) \right)$ . The independence of  $\hat{\tau}_{C,n}^{B^s}$  and  $\hat{\tau}_{C,n}^{\overline{B^s}}$ , and the Delta method imply

$$\sqrt{n} \left( \hat{\tau}_{C,n}^{B^s} - \hat{\tau}_{C,n}^{\overline{B^s}} \right) \xrightarrow{d} N \left( 0, 2\sigma^2 \right).$$

Thus, the difference  $\hat{\tau}_{C,n}^{B^s} - \hat{\tau}_{C,n}^{\overline{B^s}}$  satisfies condition (C1) with  $w_n = 1/\sqrt{n}$ .

A sequence of random variables  $\{Y_n\}$  satisfies condition (C2) of Anscombe (1952) if, given  $\epsilon > 0$  and  $\eta > 0$ , there exists a large  $\nu_{\epsilon,\eta}$  and a small  $c > 0$  such that for all  $n > \nu_{\epsilon,\eta}$  it holds that

$$\mathbb{P} \left( \sup_{n': |n' - n| < cn} \sqrt{n} |Y_{n'} - Y_n| \geq \epsilon \right) < \eta.$$

Further, the proof of Theorem 6 in Sproule (1974) yields that  $\hat{\tau}_{C,n}^{B^s}$  and  $\hat{\tau}_{C,n}^{\overline{B^s}}$  satisfy condition (C2). Therefore, the difference  $\hat{\tau}_{C,n}^{B^s} - \hat{\tau}_{C,n}^{\overline{B^s}}$  also satisfies condition (C2). Finally, Theorem 1 of Anscombe (1952) implies the desired asymptotic convergence

$$\sqrt{n^\star} \left( \hat{\tau}_{C,N_n^s}^{B^s} - \hat{\tau}_{C,N_n^s}^{\overline{B^s}} \right) \xrightarrow{d} N \left( 0, 2\sigma^2 \right).$$

□

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