Project games

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Abstract This paper studies situations in which a project consisting of several activities is not executed as planned. It is divided into three parts. The first part analyzes the case where the activities may be delayed, this possibly induces a delay on the project as a whole with additional costs. Associated delayed project games are defined and are shown to have a nonempty core. The second part considers the case where the activities may be expedited, this possibly induces an expedition of the project as a whole creating profits. Corresponding expedited project games are introduced and are shown to be convex. The third and last part studies situations where some activities may be delayed and some activities may be expedited. Related project games are defined and shown to have a nonempty core.

Keywords Project planning · Delay · Expedition · Cooperative game · Convexity

JEL Classification C71

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1 Introduction

A project consists of a set of activities, for which the interconnections are known, being completed over a period of time and intended to achieve a particular aim. Before the project is realized, the time needed to execute each of the activities is estimated and thus, in particular, a planned duration of the project can be determined. In practice, the estimated duration and the duration after realization (or real duration) of an activity may differ and as result, the real duration of the project may not coincide with its planned duration. In many real-life situations, if a project is delayed, some additional costs arise. Moreover, if the project is expedited, some extra reward may be obtained. How to allocate the costs (rewards) due to the delay (expedition) of the project among the activities that have caused this difference in duration? Moreover, even if the real duration of the project is as planned, those activities that are delayed might compensate those that have been expedited in avoiding a delay of the project.

In the literature, the focus has been on projects where activities may be delayed but not expedited. Bergantiños and Sánchez (2002) propose two rules to allocate the total delay of the project. They propose to allocate first the total delay among the paths in the project and then, in a second step, the delay assigned to each path is attributed to the activities in the path. Branzêi et al. (2002) analyze the problem of sharing the total delay of a project within the framework of taxation. Their proposal is to consider an associated taxation problem (and associated rules) where the total delay of the project (total tax) has to be allocated among the different activities where the maximal ability to pay for each activity is given by that activity's delay. Here, the delay of an activity is calculated taking into account the slack of the paths in which it is involved. This slack is given only to the activities that are the last activity of a path while for the others, the delay is the real delay of the activity. Following the same ideas, Branzêi et al. (2004) analyze the problem of sharing the total delay or reward of a project in which the activities have not been executed according to plan. For this, they make use of generalized bankruptcy problems. Besides, Branzêi et al. (2002) propose a two-step allocation rule: first, by allocating the total delay of the project among the paths based on the aggregated delays; second, the delay assigned to a path is shared among the activities in the path proportionally to their delay. Finally, Castro and Tejada (2005), also in the same delayed project setting, propose a parameterized family of rules stemming from the cost sharing literature. A common aspect in these three papers is that game theoretical aspects are only indirectly present in analyzing the allocation problem related to project situations.

In this article we do not only provide a direct approach based on cooperative game theory to analyze allocation issues in delayed project situations, but we also consider the opposite setting where activities may be expedited but not delayed. Moreover, we also analyze the mixed case where some activities may be delayed and some activities expedited. Throughout we assume that the associated reward and cost functions are linear with respect to the difference between the planned and real project times. For a better understanding of the

rather technical general problem where some activities are delayed and some are expedited, we separately study the situations where all activities are either delayed or expedited. Moreover, in case activities cannot be delayed but only expedited, stronger results can be obtained. Another reason to treat delayed project problems separately is the direct connection to the usual setting in the literature. Hence, three different models are studied: delayed project problems (activities are possibly delayed), expedited project problems (activities are possibly expedited), and project problems (activities may be delayed or expedited). It is shown that (general) project games have a nonempty core, while expedited project games are convex.

One may note that an expedited project becomes a delayed project by interchanging the planned and the real durations of each activity. Hence, expedited project problems might be studied by analyzing delayed project problems and applying duality. We have decided to analyze both problems separately due to the following intrinsic difference among the cost and reward sharing problems related to project situations. While the delay of only one path (or even of only one activity) can cause a delay on the whole project, the expedition of at least all critical paths is needed to obtain an expedition on the whole project.

Section 2 introduces the definitions and terminology on projects. Section 3 studies delayed project problems. An associated delayed project game is introduced where the worth of a coalition measures the maximum contribution of the coalition to the total delay of the project caused by those paths in which the coalition is involved. It is shown that delayed project games have a nonempty core.

In Sect. 4 we study expedited project problems. First, we note that the total expedition can be divided in several parts depending on the slack of the paths of the planned project. Besides, we distinguish between several levels of expedition that can be claimed by a specific set of paths. Hence, the total expedition can be decomposed in several parts to be allocated among a particular set of paths. Using such a "peeling-off" approach, we define expedited project games by applying ideas from bankruptcy games recursively to the various levels of the total expedition in an interrelated way. Although expedited project games are not necessarily (strategically equivalent to) bankruptcy games, they turn out to be convex.
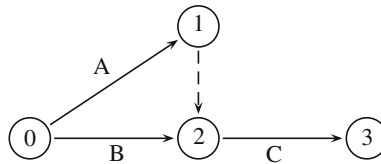
Section 5 studies general project problems. We define an associated project game where the underlying ideas of Sects. 3 and 4 are combined. It is shown that project games have a nonempty core.

2 Project situations

A project consists of a set of activities for which the inter-connections are known. These activities are completed over a period of time and intended to achieve a particular aim. Let N denote the set of activities of a project. Given an activity $i \in N$, let P_i denote the set of *predecessors* of i, i.e. the set of activities that have to be processed before i can start. Analogously, let F_i be the

Table 1 Predecessors of activities in Example 2.1

Activity	Predecessors
A	
B	
C	A, B

Fig. 1 Representation of the project given in Table 1

set of *followers* of i, i.e. the set of activities that need i to be completed before starting. A *project* is defined as a collection of *ordered* subsets of N or paths, $\{N_1, \ldots, N_m\}$, where a bijection $\sigma_t : \{1, \ldots, |N_t|\} \rightarrow N_t$ describes the order in $N_t, t \in \{1, \ldots, m\}$, satisfying the following conditions:

1. $N = \bigcup_{t=1}^{m} N_t$;
2. $F_{\sigma_t(|N_t|)} = \emptyset$, $P_{\sigma_t(1)} = \emptyset$, and $P_{\sigma_t(r)} = \{\sigma_t(1), \ldots, \sigma_t(r-1)\}$ for every $t \in \{1, \ldots, m\}$ and every $r \in \{2, \ldots, |N_t|\}$;
3. for $t, u \in \{1, \ldots, m\}$, if $i, j \in N_t \cap N_u$ with $\sigma_t^{-1}(i) < \sigma_t^{-1}(j)$, then $\sigma_u^{-1}(i) < \sigma_u^{-1}(j)$.

A project is called a *parallel project* if $\{N_1, \ldots, N_m\}$ is a partition of N. Throughout there is no specific need to explicitly keep track of the ordering. Therefore, $\sigma_1, \ldots, \sigma_m$ are suppressed from the notation.

Note that a project can be represented by a directed graph where the set of arcs corresponds to the set of activities. In order to avoid multiple arcs, dummy activities are introduced in the graph (a dummy activity is an activity that consumes neither time nor resources). Dummy activities are represented by a dashed arc.

Example 2.1 Table 1 gives the set of activities of a project with their corresponding predecessors.

Here, the set of activities is $N = \{A, B, C\}$ and the collection of paths is $\{N_1, N_2\}$, with $N_1 = \{A, C\}$, and $N_2 = \{B, C\}$. The graphical representation of this project is given in Fig. 1.

Associated to a project $\{N_1, \ldots, N_m\}$ there is a function $l : N \rightarrow [0, \infty)$ with $l(i)$ denoting the length or *duration* of activity $i \in N$. Given a project $\{N_1, \ldots, N_m\}$ and a duration function l, we define the *duration of a path N_t*, as the sum of the duration of its activities, i.e. as $\sum_{i \in N_t} l(i)$. The *duration of the project* according to l, $D(l)$, is the maximum duration of its paths, i.e. $D(l) = \max_{1 \leq t \leq m} \{\sum_{i \in N_t} l(i)\}$. The *slack of N_t* is the maximum time that the activities of N_t can be delayed without altering the duration of the project, i.e. $\text{slack}(N_t, l) = D(l) - \sum_{i \in N_t} l(i)$. We say that a path is *critical* if it has slack zero.

Table 2 Duration and slack of the paths in Example 2.2

N_t	Duration	slack (N_t)
AC	5	2
BC	7	0

Example 2.2 Consider the project given in Example 2.1 and let $l : N \to [0, \infty)$ be given by $l(A) = 3$, $l(B) = 5$, and $l(C) = 2$. Table 2 summarizes the duration and slack of the paths. Note that $D(l) = 7$.

Throughout we use a fixed notation for two specific duration functions. We denote by $p : N \to [0, \infty)$ the function representing the *planned* or estimated *time* of the activities, and by $r : N \to [0, \infty)$ the function giving the *real time* of the activities after the realization of the project. We define the *delay function* $d : N \to [0, \infty)$ as $d(i) = (r(i) - p(i))_+ (:= \max\{r(i) - p(i), 0\})$, i.e. $d(i)$ represents the delay of activity i. Analogously, we define the *expedition function* $e : N \to [0, \infty)$ as $e(i) = (p(i) - r(i))_+$, i.e. $e(i)$ represents the expedition of activity i.

In the following sections we study three kind of situations. Section 3 is devoted to delayed project problems where $r \geq p$, Sect. 4 to expedited project problems, where $r \leq p$, and Sect. 5 to the general situation.

3 Delayed project games

In this section we study those project situations where activities may be delayed but not expedited, which possibly causes the real duration of the project to be larger than the planned duration. A cost is associated to the delay of the project which is assumed linear w.r.t. the total delay of the project. Due to the linearity of the cost function, we can identify the total cost with the total delay of the project. We analyze the corresponding allocation problem with techniques from cooperative TU games.

We recall some basic concepts from game theory. A cooperative *cost game* in characteristic function form is an ordered pair (N, c) where N is a finite set (the set of players) and $c : 2^N \to \mathbb{R}$ represents the maximum cost chargeable to the different coalitions (or subsets of players) satisfying $c(\emptyset) = 0$. The *core* of a cost game (N, c) is defined by

$$\text{Core}(c) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N \right\},$$

i.e. the core is the set of allocations of $c(N)$ to which no coalition can reasonably object. An important subclass of cost games with nonempty core is the class of concave games. A game (N, c) is *concave* if

$$c(T) + c(S) \geq c(T \cup S) + c(T \cap S) \tag{3.1}$$

for every $S, T \subset N$.

Table 3 Duration of the paths in Example 3.1

N_t	Duration w.r.t. p	Duration w.r.t. r
AC	5	9
BC	7	12

We define *delayed project problems* as those project problems where the planned time of the activities was underestimated, i.e. the real time of an activity is at least its planned time. Hence, a delayed project problem can be described by $(\{N_1, \ldots, N_m\}, p, r)$ with $p \le r$. To a delayed project problem we associate a delayed project game where the set of players is the set of activities and the cost of a coalition is the maximal contribution of the coalition to the delay of the project caused by those paths where members of S are involved. Formally, given a delayed project problem $(\{N_1, \ldots, N_m\}, p, r)$, we define the associated cost game (N, c) by

$$
c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \max \left\{ \sum_{i \in N_t} r(i), D(p) \right\} - \max \left\{ \sum_{i \in N_t \setminus S} r(i) + \sum_{i \in N_t \cap S} p(i), D(p) \right\} \right\}
$$

for an $S \subset N$, where $\mathcal{P}(S) = \{t \in \{1, \ldots, m\} \mid N_t \cap S \ne \emptyset\}$ represents the set of paths in which S is involved. Note that $c(N)$ equals the total delay of the project.

Example 3.1 Consider the project given in Example 2.1 and let $p : N \to [0, \infty)$ be given by $p(A) = 3$, $p(B) = 5$, and $p(C) = 2$ and $r : N \to [0, \infty)$ by $r(A) = 6$, $r(B) = 9$, and $r(C) = 3$. Table 3 gives the duration of the paths according to the planned and real times.

Hence, the planned duration of the project is $D(p) = 7$ while the real duration is $D(r) = 12$. Therefore, there is a total delay of 5. The value of the associated delayed project game, (N, c), for coalition $\{A, B\}$ is obtained as follows: $\mathcal{P}(\{A, B\}) = \{1, 2\}$, the contribution of $\{A, B\}$ to the delay caused by N_1 is $\max\{r(A) + r(C), D(p)\} - \max\{p(A) + r(C), D(p)\} = \max\{6 + 3, 7\} - \max\{3 + 3, 7\} = 2$ and the contribution of $\{A, B\}$ to the delay caused by N_2 is $\max\{r(B) + r(C), D(p)\} - \max\{p(B) + r(C), D(p)\} = \max\{9 + 3, 7\} - \max\{5 + 3, 7\} = 4$, then $c(\{A, B\}) = \max\{2, 4\} = 4$. All values of the game are: $c(\{A\}) = 2$, $c(\{B\}) = 4$, $c(\{C\}) = 1$, $c(\{A, B\}) = 4$, $c(\{A, C\}) = 2$, $c(\{B, C\}) = 5$, $c(N) = 5$.

Next, we give an alternative expression of the coalitional values in a delayed project game.

Lemma 3.1 *Let* $(\{N_1, \ldots, N_m\}, p, r)$ *be a delayed project problem and let* (N, c) *be the associated delayed project game. Then,*

$$
c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \mathrm{slack}(N_t, p) \right)_+ \right\} \right\}
$$

for every $S \subset N$.

Proof Let $S \subset N$. Then,

$$
c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \max \left\{ \sum_{i \in N_t} r(i), D(p) \right\} - \max \left\{ \sum_{i \in N_t \backslash S} r(i) + \sum_{i \in N_t \cap S} p(i), D(p) \right\} \right\}
$$

$$
= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} r(i) - D(p) \right)_+ - \left(\sum_{i \in N_t \backslash S} r(i) + \sum_{i \in N_t \cap S} p(i) - D(p) \right)_+ \right\}
$$

$$
= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} (r(i) - p(i)) + \sum_{i \in N_t} p(i) - D(p) \right)_+ \right.
$$
$$
\left. - \left(\sum_{i \in N_t \backslash S} (r(i) - p(i)) + \sum_{i \in N_t} p(i) - D(p) \right)_+ \right\}
$$

$$
= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ - \left(\sum_{i \in N_t \backslash S} d(i) - \text{slack}(N_t, p) \right)_+ \right\}
$$

$$
= \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\} \right\}
$$

The first four equalities are straightforward. The last equality is proven below. Take $t \in \mathcal{P}(S)$, we show that

$$
\left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ - \left(\sum_{i \in N_t \backslash S} d(i) - \text{slack}(N_t, p) \right)_+
$$
$$
= \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\}.
$$

Recall that d is a non-negative function. We distinguish between three cases:

Case 1 $\left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ = 0$. Then, $\left(\sum_{i \in N_t \backslash S} d(i) - \text{slack}(N_t, p) \right)_+ = 0$ and

$$
\left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ - \left(\sum_{i \in N_t \backslash S} d(i) - \text{slack}(N_t, p) \right)_+ = 0
$$
$$
= \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\}.
$$

Case 2 $\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ > 0$, $\left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+ = 0$.
Then, $\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p) \leq 0$ and hence $\sum_{i\in N_t} d(i) - \text{slack}(N_t,p) \leq \sum_{i\in N_t\cap S} d(i)$. Therefore,

$$
\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ - \left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+
$$
$$
= \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+
$$
$$
= \min\left\{\sum_{i\in N_t\cap S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\}.
$$

Case 3 $\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ > 0$, $\left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+ > 0$.
Then, $\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p) > 0$ and hence $\sum_{i\in N_t} d(i) - \text{slack}(N_t,p) > \sum_{i\in N_t\cap S} d(i)$. Therefore,

$$
\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ - \left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+
$$
$$
= \sum_{i\in N_t\cap S} d(i)
$$
$$
= \min\left\{\sum_{i\in N_t\cap S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\}. \qquad \square
$$

Next, we show that delayed project games have a nonempty core. To do this, we first study delayed parallel project games. The following result states that subgames obtained by restricting delayed parallel project games to the paths of the project are concave.
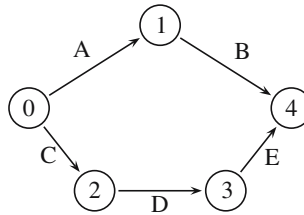
Lemma 3.2 *Let* $(\{N_1,\ldots,N_m\},p,r)$ *be a delayed parallel project problem and let* (N,c) *be the associated delayed project game. Then,* $(N_t,c_{|N_t})$ *is concave for any* $t \in \{1,\ldots,m\}$.

Proof Let $t \in \{1,\ldots,m\}$. According to Lemma 3.1, $c_{|N_t}(S) = \min\left\{\sum_{i\in S} d_i, \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\}$ for $S \subset N_t$. Let $S,T \subset N_t$. We show that

$$
c_{|N_t}(S) + c_{|N_t}(T) \geq c_{|N_t}(S\cup T) + c_{|N_t}(S\cap T).
$$

We distinguish between two cases.

Case 1 $c_{|N_t}(S) = \sum_{i \in S} d_i$ and $c_{|N_t}(T) = \sum_{i \in T} d_i$. Hence, $c_{|N_t}(S \cap T) = \sum_{i \in S \cap T} d_i$ and

$$c_{|N_t}(S) + c_{|N_t}(T) = \sum_{i \in S} d_i + \sum_{i \in T} d_i = \sum_{i \in S \cup T} d_i + \sum_{i \in S \cap T} d_i$$

$$\geq \min \left\{ \sum_{i \in S \cup T} d_i, \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\} + \sum_{i \in S \cap T} d_i$$

$$= c_{|N_t}(S \cup T) + c_{|N_t}(S \cap T).$$

Case 2 $c_{|N_t}(S) = \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+$. Hence, $c_{|N_t}(S \cup T) = \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+$ and

$$c_{|N_t}(S) + c_{|N_t}(T) = \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+$$

$$+ \min \left\{ \sum_{i \in T} d_i, \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\}$$

$$\geq \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+$$

$$+ \min \left\{ \sum_{i \in S \cap T} d_i, \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\} \qquad \square$$

$$= c_{|N_t}(S \cup T) + c_{|N_t}(S \cap T).$$

The next example illustrates that games arising from delayed parallel project problems need not be concave.

Example 3.2 Consider the delayed parallel project problem $(\{N_1, N_2\}, p, r)$ with $N_1 = \{A, B\}$, $N_2 = \{C, D, E\}$; $p = (3, 5, 2, 1, 3)$; and $r = (4, 7, 5, 4, 5)$. The project is represented in Fig. 2. Let (N, c) be the associated delayed parallel project game. Let $S = \{A, B, D\}$ and $T = \{A, B, C\}$. Then, $c(S) + c(T) = 3 + 3 = 6 < 9 = 6 + 3 = c(S \cup T) + c(S \cap T)$ and the game is not concave.

Lemma 3.3 *Delayed project games associated to delayed parallel project problems have a nonempty core.*

Proof Let $(\{N_1,\ldots,N_m\},p,r)$ be a delayed parallel project problem and let (N,c) be the associated game. Let N_t be a critical path in the realization. Hence, $c(N_t) = \sum_{i \in N_t} r(i) - D(p) = D(r) - D(p) = c(N)$. By Lemma 3.2, there exists $y \in \text{Core}(c_{|N_t})$. Let $x \in \mathbb{R}^N$ defined as

$$x_i = \begin{cases} y_i & \text{if } i \in N_t \\ 0 & \text{if } i \in N \setminus N_t \end{cases}$$

for every $i \in N$. We show that $x \in \text{Core}(c)$. First, we show efficiency.

$$\sum_{i \in N} x_i = \sum_{i \in N_t} y_i = c(N_t) = c(N)$$

where the second equality holds because $y \in \text{Core}(c_{|N_t})$ and $c(N_t) = c_{|N_t}(N_t)$, and the third one because N_t is critical in the realization.

Next, we show stability. If $S \subset N \setminus N_t$, then $\sum_{i \in S} x_i = 0 \leq c(S)$ because c is non-negative. Let $S \subset N, S \cap N_t \neq \emptyset$. Hence,

$$\sum_{i \in S} x_i = \sum_{i \in S \cap N_t} y_i \leq c_{|N_t}(S \cap N_t) = c(S \cap N_t) \leq c(S)$$

where the first inequality holds because $y \in \text{Core}(c_{|N_t})$ and the second because $c(S) = \max_{u \in \mathcal{P}(S)}\{c(S \cap N_u)\}$. □
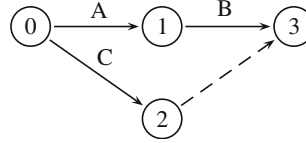
Theorem 3.4 *Delayed project games have a nonempty core.*

Proof Let $(\{N_1,\ldots,N_m\},p,r)$ be a delayed project problem and let (N,c) be the associated delayed project game. Let $(\{N_1^*,\ldots,N_m^*\},p^*,r^*)$ be the delayed project problem defined as follows: $N_t^* = \{i^t| \ i \in N_t\}$, with $p^*(i^t) = p(i)$ and $r^*(i^t) = r(i)$ for all $i \in N$. Note that $(\{N_1^*,\ldots,N_m^*\},p^*,r^*)$ is a delayed parallel project problem with $N^* = \bigcup_{t=1}^m N_t^*$. Let (N^*,c^*) be the associated delayed project game. One readily verifies that, $c(S) = c^*(S^*)$ for every $S \subset N$ with $S^* := \bigcup_{t=1}^m \{i^t| \ i \in N_t \cap S\} \subset N^*$.

By Lemma 3.3, there exists $y \in \text{Core}(c^*)$. Let $x \in \mathbb{R}^N$ defined as $x_i = \sum_{t=1}^m y_{i^t}$. We show that $x \in \text{Core}(c)$. Efficiency holds by construction of x, because $y \in \text{Core}(c^*)$ and $c(N) = c^*(N^*)$. Next, we show stability. Let $S \subset N$. Then,

$$\sum_{i \in S} x_i = \sum_{i \in S} \sum_{t=1}^m y_{i^t} = \sum_{i \in S^*} y_i \leq c^*(S^*) = c(S)$$ □

4 Expedited project games

This section analyzes project situations in which activities may be expedited but not delayed. Consequently, the duration of the project after realization may be shorter than the planned duration. A reward is associated to the expedition of the project which is assumed to be linear w.r.t. the total expedition of the project. Again, due to the linearity of the reward function, we identify the total reward with the total expedition of the project.

Contrary to the previous section we are now in a reward setting. For this reason we briefly overview concepts from reward games. A cooperative *reward game* in characteristic function form is an ordered pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is the function representing the worth of each coalition, which satisfies $v(\emptyset) = 0$. The *core* of a game (N, v) is defined by

$$\text{Core}(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\},$$

i.e. the core is the set of efficient allocations of $v(N)$ to which no coalition can reasonably object. An important subclass of games with nonempty core is the class of convex games (see Shapley 1971). A game (N, v) is *convex* if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)^1 \tag{4.1}$$

for every $i \in N$ and every $S \subset T \subset N \setminus \{i\}$.
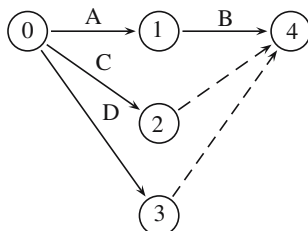
We define *expedited project problems* as those project situations where the planned time of the activities was overestimated, i.e. the real time of an activity is at most its planned time. Hence, an expedited project problem can be described by a three-tuple $(\{N_1, \ldots, N_m\}, p, r)$ with $p \geq r$. At this stage, one may think about using the duality between delayed and expedited project problems: an expedited project problem can be viewed as a delayed project problem by interchanging the planned and real time vectors. The following example illustrates however that this approach is inadequate to solve the corresponding problem.

Example 4.1 Consider the expedited project problem $(\{N_1, N_2\}, p, r)$ with $N_1 = \{A,B\}$ and $N_2 = \{C\}$, $p = (2, 2, 3)$, and $r = (1, 1, 2)$. This project is represented in Fig. 3.

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[1] This is equivalent to the dual of Eq. (3.1): $v(T) + v(S) \leq v(T \cup S) + v(T \cap S)$ for every $S, T \subset N$.

Table 4 Coalitional values of the delayed project game associated to $(\{N_1, N_2\}, \tilde{p}, \tilde{r})$ in Example 4.1

S	A	B	C	AB	AC	BC	ABC
$c(S)$	1	1	1	2	1	1	2

Fig. 4 Representation of the project in Example 4.2

Table 5 Duration and slack of the paths according to the planned times in Example 4.2

N_t	Duration	slack(N_t, p)
AB	10	0
C	9	1
D	7	3

The total duration of the planned project is 4, while the real duration of the project after realization is 2. Hence, the total expedition of the project is 2.

By interchanging the planned and real time vectors the expedited project problem becomes a delayed project problem. Let consider the delayed project problem $(\{N_1, N_2\}, \tilde{p}, \tilde{r})$ with $\tilde{p} = r$ and $\tilde{r} = p$. The coalitional values of the associated delayed project game are given in Table 4. It is left to the reader to check that the core of the "dual" delayed project game has a unique point: $\text{Core}(c) = \{(1, 1, 0)\}$. Note that the expedition of activity C is needed in order to obtain an expedition of 2 in the project (otherwise the total expedition would be 1). Still, C is not compensated according to an efficient and stable allocation.

Next, we give an example to explain the ideas behind our approach to solving the reward sharing problem associated to an expedited project problem.

Example 4.2 Consider the expedited project problem $(\{N_1, N_2, N_3\}, p, r)$ with $N_1 = \{A, B\}$, $N_2 = \{C\}$, and $N_3 = \{D\}$; $p = (6, 4, 9, 7)$; and $r = (3, 2, 4, 6)$. This project is represented in Fig. 4.

The total duration of the planned project is 10, while the real duration of the project after realization is 6. Hence, the total expedition of the project is 4. Table 5 gives the duration and slack of the paths according to the plan.

Note that the project cannot be expedited if the critical path N_1 according to plan is not expedited. Next, we analyze how this total expedition is obtained. First, suppose that only the activities in N_1 are executed according to the realization while the activities in paths N_2 and N_3 are executed according to the plan. Then, the project has an expedition of just 1, while path N_2 becomes critical, and path N_3 has a slack of 2 in the new situation. Hence, path N_1 is responsible by itself of one unit of time of the total expedition. Second, suppose that the

Table 6 Contribution of the paths to the total expedition of the project in Example 4.2

	Phase 1	Phase 2	Phase 3
N_1	1	2	1
N_2	0	2	1
N_3	0	0	1

activities both in N_1 and N_2 are executed according to realization while the activities in N_3 are executed according to plan. Then, N_3 becomes critical and there is an additional expedition of 2. Hence, both paths N_1 and N_2 are needed for and responsible of two units of time of the total expedition. Finally, suppose that all the activities are executed according to realization, then there is an additional expedition of 1 for which all paths N_1, N_2, and N_3 are responsible. The contribution of the paths to the total expedition of the project during the different phases is summarized in Table 6. Note that the sum of the first row gives the total expedition of the project. This kind of "peeling-off" into levels of expedition play a prominent role in the definition of expedited project games below.

Let $(\{N_1, \ldots, N_m\}, p, r)$ be an expedited project problem. We denote by I_1 the set (of indices) of critical paths according to the planned time. Formally,

$$I_1 = \big\{t \in \{1, \ldots, m\} |\ \text{slack}(N_t, p) = 0\big\}.$$

Recursively, we define for $k \geq 2$,

$$I_k = \bigg\{t \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} I_l |\ \text{slack}(N_t, p) \leq \text{slack}(N_u, p)$$

$$\text{for all } u \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} I_l \bigg\},$$

i.e. I_k corresponds to all paths that would be critical in the (sub)project if all the paths in $I_1, \ldots, I_{k-1}$ were not present. We denote by $\text{slack}(I_k)$ the slack of the paths in I_k according to the planned time, i.e. $\text{slack}(I_k) = \text{slack}(N_t, p)$ for each $t \in I_k$. Let h be such that $\text{slack}(I_h) < D(p) - D(r) \leq \text{slack}(I_{h+1})$. For $k = 1, \ldots, h$, we define E^k as the level of expedition for which all paths in $I_1, \ldots, I_k$ are needed:

$$E^k = \begin{cases} \text{slack}(I_{k+1}) - \text{slack}(I_k) & \text{if } 1 \leq k < h; \\ D(p) - D(r) - \text{slack}(I_h) & \text{if } k = h. \end{cases}$$

Note that $\sum_{k=1}^{h} E^k = D(p) - D(r)$.

To an expedited project problem we associate an *expedited project game*. The set of players is the set of activities. The worth of a coalition is the sum

over all $k \in \{1, \ldots, h\}$ of those specific parts of the level of expedition E^k for which the activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} I_l$ cannot be held responsible for anymore at that phase. Formally, given an expedited project problem $(\{N_1, \ldots, N_m\}, p, r)$ and an expedition level $k \in \{1, \ldots, h\}$, we first recursively define w^k by

$$w^k(S) = \min \left\{ \sum_{i \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} e(i) - \sum_{l=1}^{k-1} w^l(S), \ E^k \right\}, \tag{4.2}$$

for all $S \subset N$, where $N_{I_l} = \bigcup_{t \in I_l} N_t$. Here, $w^k(S)$ represents the part of the level of expedition E^k that players in S maximally would have to concede to players in the paths corresponding to $\bigcup_{l=1}^{k} I_l$ outside S, taking into account earlier concessions from the previous phases. Note that w^k is non-negative. Subsequently, we define the associated game (N, v) by

$$v(S) = \sum_{k=1}^{h} \left(E^k - w^k(S) \right) \tag{4.3}$$

for every $S \subset N$. Note that $v(N)$ equals the total expedition of the project because $w^k(N) = 0$ for any $k \in \{1, \ldots, h\}$.

Example 4.3 Consider the expedited project problem given in Example 4.2. Here, $D(p) = 10$ and $D(r) = 6$. Hence, the total expedition is $D(p) - D(r) = 4$. In this case, $e = (3, 2, 5, 1)$; $I_1 = \{1\}, I_2 = \{2\}$, and $I_3 = \{3\}$; and $E^1 = \text{slack}(I_2) - \text{slack}(I_1) = 1, E^2 = \text{slack}(I_3) - \text{slack}(I_2) = 2$, and $E^3 = D(p) - D(r) - \text{slack}(I_3) = 1$. Consequently, for coalition $\{A, C\}$:

$$w^1(\{A, C\}) = \min \left\{ e(B), \ E^1 \right\} = \min \{2, 1\} = 1,$$

$$w^2(\{A, C\}) = \min \left\{ e(B) - w^1(\{A, C\}), \ E^2 \right\} = \min \{2 - 1, 2\} = 1,$$

$$w^3(\{A, C\}) = \min \left\{ e(B) + e(D) - w^1(\{A, C\}) - w^2(\{A, C\}), \ E^3 \right\}$$
$$= \min \{2 + 1 - 1 - 1, 1\} = 1.$$

Let (N, v) be the associated expedited project game. Then,

$$v(\{A, C\}) = \left(E^1 - w^1(\{A, C\}) \right) + \left(E^2 - w^2(\{A, C\}) \right) + \left(E^3 - w^3(\{A, C\}) \right)$$
$$= (1 - 1) + (2 - 1) + (1 - 1) = 1.$$

Coalitional worths are given by: $v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{D\}) = 0$, $v(\{A, B\}) = v(\{A, C\}) = 1, v(\{A, D\}) = v(\{B, C\}) = v(\{B, D\}) = v(\{C, D\}) = 0$, $v(\{A, B, C\}) = 3, v(\{A, B, D\}) = 1, v(\{A, C, D\}) = 2, v(\{B, C, D\}) = 1, v(N) = 4$.

For $R \subset N$, $\mathcal{A}(R)$ denotes the set of "active levels of expedition" of R, i.e.

$$\mathcal{A}(R) := \left\{ k \in \{1, \ldots, h\} \mid w^k(R) < E^k \right\}.$$

and $A(R)$ is the highest active level of expedition of R, i.e.

$$A(R) := \begin{cases} \max \mathcal{A}(R) & \text{if } \mathcal{A}(R) \neq \emptyset; \\ 0 & \text{if } \mathcal{A}(R) = \emptyset. \end{cases}$$

For $j \in N$, $k(j)$ is the first level of expedition in which j is involved, i.e.

$$k(j) := \min \left\{ k \in \{1, \ldots, h\} \mid j \in N_{I_k} \right\}.$$

For $R \subset N$ and $j \in N$, $\mathcal{A}(j, R)$ is the set of all active levels of expedition for R in which j is also involved, i.e.

$$\mathcal{A}(j, R) := \left\{ k \in \{1, \ldots, h\} \mid k \geq k(j), \ k \in \mathcal{A}(R) \right\}$$

and $a(j, R)$ is the first active level of expedition of R in which j is also involved, i.e.

$$a(j, R) := \begin{cases} \min \mathcal{A}(j, R) & \text{if } \mathcal{A}(j, R) \neq \emptyset; \\ h + 1 & \text{if } \mathcal{A}(j, R) = \emptyset. \end{cases}$$

The following example illustrates the notation above.

Example 4.4 Consider the expedited project problem given in Example 4.2. Let $R = \{A, C\}$ and $i = B$. Recall that $N_1 = \{A, B\}$, $N_2 = \{C\}$, and $N_3 = \{D\}$. Furthermore, $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$ and $E^1 = 1$, $E^2 = 2$, and $E^3 = 1$. Moreover,

$$\begin{aligned} w^1(\{A, C\}) &= 1 = E^1, \\ w^2(\{A, C\}) &= 1 < E^2, \\ w^3(\{A, C\}) &= 1 = E^3. \end{aligned}$$

Hence, $\mathcal{A}(R) = \{2\}$, $A(R) = 2$, $k(B) = 1$, $\mathcal{A}(i, R) = \{2\}$ and $a(i, R) = 2$.

The following lemma says that, at each level of expedition, the concession of a smaller coalition exceeds the concession of a larger coalition.

Lemma 4.1 *Let $(\{N_1, \ldots, N_m\}, p, r)$ be an expedited project problem. Let $k \in \{1, \ldots, h\}$. Then, for every $S \subset T \subset N$,*

(i) $$\sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \setminus S} e(i) - \sum_{l=1}^{k-1} w^l(S) \geq \sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \setminus T} e(i) - \sum_{l=1}^{k-1} w^l(T),$$

(ii) $w^k(S) \geq w^k(T)$,

(iii) $A(S) \subset A(T)$.

Proof Obviously, (ii) is a direct consequence of (i) together with the definition of w^k and (iii) follows immediately from (ii).

Let $S \subset T \subset N$. We show i by induction on k. For $k = 1$, it is obvious since $e(i) \geq 0$ for every i. Let $k = 2$. Then

$$\sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash S} e(i) - w^1(S) = \sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash S} e(i) - \min \left\{ \sum_{i \in N_{I_1} \backslash S} e(i), \, E^1 \right\}$$

$$= \max \left\{ \sum_{i \in N_{I_2} \backslash (N_{I_1} \cup S)} e(i), \, \sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash S} e(i) - E^1 \right\}$$

$$\geq \max \left\{ \sum_{i \in N_{I_2} \backslash (N_{I_1} \cup T)} e(i), \, \sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash T} e(i) - E^1 \right\}$$

$$= \sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash T} e(i) - \min \left\{ \sum_{i \in N_{I_1} \backslash T} e(i), \, E^1 \right\}$$

$$= \sum_{i \in (N_{I_1} \cup N_{I_2}) \backslash T} e(i) - w^1(T)$$

where the inequality holds because $S \subset T$.

Now, suppose that $k \geq 3$ and suppose (i) is true for $k - 1$. Then,

$$\sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \backslash S} e(i) - \sum_{l=1}^{k-1} w^l(S) = \sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \backslash S} e(i) - \sum_{l=1}^{k-2} w^l(S)$$

$$- \min \left\{ \sum_{i \in \left(\bigcup_{l=1}^{k-1} N_{I_l} \right) \backslash S} e(i) - \sum_{l=1}^{k-2} w^l(S), \, E^{k-1} \right\}$$

$$= \max \left\{ \sum_{i \in N_{I_k} \backslash \left(\bigcup_{l=1}^{k-1} N_{I_l} \cup S \right)} e(i), \, \sum_{i \in N_{I_k} \backslash \left(\bigcup_{l=1}^{k-1} N_{I_l} \cup S \right)} e(i) \right.$$

$$\left. + \sum_{i \in \left(\bigcup_{l=1}^{k-1} N_{I_l} \right) \backslash S} e(i) - \sum_{l=1}^{k-2} w^l(S) - E^{k-1} \right\}$$

$$\geq \max \left\{ \sum_{i \in N_{I_k} \backslash \left(\bigcup_{l=1}^{k-1} N_{I_l} \cup T \right)} e(i), \, \sum_{i \in N_{I_k} \backslash \left(\bigcup_{l=1}^{k-1} N_{I_l} \cup T \right)} e(i) \right.$$

$$+ \sum_{i \in \left(\bigcup_{l=1}^{k-1} N_{I_l} \right) \setminus T} e(i) - \sum_{l=1}^{k-2} w^l(T) - E^{k-1} \Bigg\}$$

$$= \sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \setminus T} e(i) - \sum_{l=1}^{k-2} w^l(T)$$

$$- \min \left\{ \sum_{i \in \left(\bigcup_{l=1}^{k-1} N_{I_l} \right) \setminus T} e(i) - \sum_{l=1}^{k-2} w^l(T), \ E^{k-1} \right\}$$

$$= \sum_{i \in \left(\bigcup_{l=1}^{k} N_{I_l} \right) \setminus T} e(i) - \sum_{l=1}^{k-1} w^l(T)$$

where the inequality holds by induction together with $S \subset T$. $\qquad\square$

In the proof of the main convexity theorem, we refer to some technical lemmas that can be found in the Appendix.

Theorem 4.2 *Expedited project games are convex.*

Proof Let $i \in N$ and $S \subset T \subset N \setminus \{i\}$. It suffices to show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

By Eq. (4.3),

$$v(S \cup \{i\}) - v(S) = \sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] \tag{4.4}$$

and

$$v(T \cup \{i\}) - v(T) = \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\}) \right] \tag{4.5}$$

We distinguish between three cases.

Case 1 $\mathcal{A}(i, S \cup \{i\}) = \emptyset$. Then,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = 0 \leq \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\}) \right]$$

where the equality holds by Lemma A.3 and the inequality holds by Lemma 4.1(ii).

Case 2 $\mathcal{A}(i, S) \neq \emptyset$. Then, by Lemma 4.1(iii), $\mathcal{A}(i, T) \neq \emptyset$. Using Lemma A.3,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i) = \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\}) \right]$$

Case 3 $\mathcal{A}(i, S) = \emptyset$ and $\mathcal{A}(i, S \cup \{i\}) \neq \emptyset$. By Lemma 4.1(iii), $\mathcal{A}(i, T \cup \{i\}) \neq \emptyset$. We distinguish between two subcases.

Subcase 3.1 $\mathcal{A}(i, T) = \emptyset$. By Lemma A.1(i), it holds $w^l(S) = w^l(S \cup \{i\})$ and $w^l(T) = w^l(T \cup \{i\})$ for every $l < k(i)$. Moreover, $w^l(S) = w^l(T) = E^l$ for every $l \geq k(i)$ since $\mathcal{A}(i, S) = \mathcal{A}(i, T) = \emptyset$. Then,

$$\begin{aligned}
v(S \cup \{i\}) - v(S) &= \sum_{l=1}^{h} \left[E^l - w^l(S \cup \{i\}) \right] - \sum_{l=1}^{h} \left[E^l - w^l(S) \right] \\
&= \sum_{l=1}^{h} \left[E^l - w^l(S \cup \{i\}) \right] - \sum_{l=1}^{k(i)-1} \left[E^l - w^l(S) \right] \\
&= \sum_{l=k(i)}^{h} \left[E^l - w^l(S \cup \{i\}) \right] \\
&\leq \sum_{l=k(i)}^{h} \left[E^l - w^l(T \cup \{i\}) \right] \\
&= \sum_{l=1}^{h} \left[E^l - w^l(T \cup \{i\}) \right] - \sum_{l=1}^{k(i)-1} \left[E^l - w^l(T) \right] \\
&= \sum_{l=1}^{h} \left[E^l - w^l(T \cup \{i\}) \right] - \sum_{l=1}^{h} \left[E^l - w^l(T) \right] = v(T \cup \{i\}) - v(T)
\end{aligned}$$

where the inequality holds by Lemma 4.1(ii).

Subcase 3.2 $\mathcal{A}(i, T) \neq \emptyset$. By Lemma A.3,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i) - \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l} \right) \setminus S} e(j) + \sum_{l=1}^{A(S \cup \{i\})} w^l(S)$$

$$\leq e(i) - w^{A(S \cup \{i\})}(S) \leq e(i) = \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\}) \right]$$

where the first inequality holds by Eq. (4.2) and the second because w^k is non-negative for all $k \in \{1, \ldots, h\}$. □

5 Project games

In this section we study general project situations in which some activities may have suffered a delay with respect to the planned time while other may have been expedited. The basis of analysis is rewards where costs are considered to be negative rewards. We assume that the reward function is linear in the difference between real duration and planned duration.

Let $\mathcal{D} = \{i \in N |\ d(i) > 0\}$ and $\mathcal{E} = \{i \in N |\ e(i) > 0\}$ denote the sets of delayed activities and expedited activities, respectively. Associated to a (general) project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define a project game where the set of players is the set of activities and the worth of a coalition combines the underlying ideas from Sects. 3 and 4. In determining the worth of a coalition we pessimistically assume that all delayed activities have indeed been executed according to realization and that all expedited activities outside the coalition have been executed according to plan. Then, if the expedition given by the expedited activities in the coalition itself is not enough to expedite the project, the worth of the coalition is negative and is determined along the lines of delayed project games. Otherwise, the worth of the coalition is positive and is determined along the lines of expedited project games. Formally, given a project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define the associated game (N, u), where u is defined by

$$u(S) = \begin{cases} -c(S) & \text{if } D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p); \\ v(S) & \text{if } D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p). \end{cases} \tag{5.1}$$

for every $S \subset N$.

Let $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p)$. Then, $c(S)$ reflects the maximum delay a coalition can be held responsible for. Given a path N_t, coalition S cannot be held responsible for more than its (positive) net delay nor for more than the net delay of the path as a consequence of the delay of activities in the path and the expedition of the activities within the coalition. Formally,

$$c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \left(\sum_{i \in N_t \cap S} d(i) - \sum_{i \in N_t \cap S} e(i) \right)_+, \right. \right.$$
$$\left. \left. \left(\sum_{i \in N_t} d(i) - \sum_{i \in N_t \cap S} e(i) - \text{slack}(N_t, p) \right)_+ \right\} \right\}. \tag{5.2}$$

Note that Eq. (5.2) applied to a coalition S with $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p)$ gives $c(S) = 0$. Moreover, if $D(r) \geq D(p)$, then $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p)$ for every $S \subset N$, and hence $u(S) = -c(S)$ for every $S \subset N$.

Next, consider the case $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p)$. In order to define $v(S)$, we introduce some notation. We denote by $\mathrm{rslack}(N_t, p, r)$ the amount of *remaining slack* of a path w.r.t. the planned duration if only its delayed activities are executed according to realization, i.e. $\mathrm{rslack}(N_t, p, r) = \mathrm{slack}(N_t, p) - \sum_{i \in N_t} d(i)$. Note that $\mathrm{rslack}(N_t, p, r)$ can be negative, meaning that the delayed activities have consumed all the initial slack and would produce a delay on the project as a whole of $-\mathrm{rslack}(N_t, p, r)$ if the expedited activities had been executed according to plan. We denote by J_1 the set of indexes of paths with remaining slack less than or equal to zero:

$$J_1 = \big\{ t \in \{1, \ldots, m\} | \ \mathrm{rslack}(N_t, p, r) \leq 0 \big\}.$$

Recursively, we define for $k \geq 2$

$$J_k = \left\{ t \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l | \ \mathrm{rslack}(N_t, p, r) \leq \mathrm{rslack}(N_u, p, r) \right.$$

$$\left. \text{for all } u \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l \right\},$$

i.e. J_k contains all paths that would have the smallest remaining slack if the paths in $J_1, \ldots, J_{k-1}$ were not present.

Set $\mathrm{rslack}(J_1) := 0$ and let $\mathrm{rslack}(J_k)$ denote the remaining slack of the paths in J_k for $k \geq 2$, i.e. $\mathrm{rslack}(J_k) = \mathrm{rslack}(N_t, p, r)$ for each $t \in J_k$, $k \geq 2$. Let g be such that $\mathrm{rslack}(J_g) < D(p) - D(r) \leq \mathrm{rslack}(J_{g+1})$ if $D(p) - D(r) > 0$ and $g = 0$ otherwise. For $k = 1, \ldots, g$, we define F^k as the level of expedition that the paths in $J_1, \ldots, J_k$ can obtain by acting according to realization, i.e.

$$F^k = \begin{cases} \mathrm{rslack}(J_{k+1}) - \mathrm{rslack}(J_k) & \text{if } 1 \leq k < g; \\ D(p) - D(r) - \mathrm{rslack}(J_g) & \text{if } k = g. \end{cases}$$
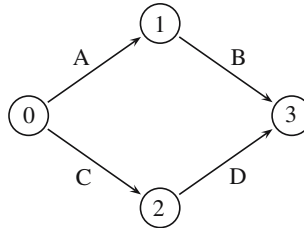
Note that $\sum_{k=1}^{g} F^k = D(p) - D(r)$.

Next, we define $v(S)$ as the sum over all $k = 1, \ldots, g$ of those specific parts of the corresponding level of expedition F^k for which expedited activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} J_l$ cannot be held responsible. Formally, given an expedition level $k \in \{1, \ldots, h\}$, we recursively define w^k by

$$w^k(S) = \min \left\{ \sum_{i \in \left(\bigcup_{l=1}^{k} N_{J_l} \right) \setminus S} e(i) - \sum_{l=1}^{k-1} w^l(S), \ F^k \right\}, \tag{5.3}$$

where $N_{J_l} = \bigcup_{t \in J_l} N_t$. Here, $w^k(S)$ represents the part of the level of expedition F^k that players in S maximally would have to concede to players in

$\bigcup_{l=1}^{k} J_l$ outside S, taking into account concessions from the previous phases. Subsequently, we define game v by

$$v(S) = \sum_{k=1}^{g} \left(F^k - w^k(S) \right) \tag{5.4}$$

for all $S \subset N$. Note that Eqs. (5.3) and (5.4) applied to a coalition S with $D(p_{|N \setminus (\mathcal{D} \cup (\mathcal{E} \cap S))}, r_{|\mathcal{D} \cup (\mathcal{E} \cap S)}) \geq D(p)$ give $v(S) = 0$.

Example 5.1 Consider the project problem $(\{N_1, N_2\}, p, r)$ with $N_1 = \{A,B\}$ and $N_2 = \{C,D\}$. Let $p = (2, 5, 3, 6)$ and $r = (5, 3, 4, 3)$. The project is represented in Fig. 5.

Here, $D(p) = 9$, slack$(N_1, p) = 2$, slack$(N_2, p) = 0$, $D(r) = 8$, $d = (3, 0, 1, 0)$, $e = (0, 2, 0, 3)$, $\mathcal{D} = \{A, C\}$, $\mathcal{E} = \{B, D\}$, rslack$(N_1, p, r) = -1$, rslack$(N_2, p, r) = -1$, $J_1 = \{1, 2\}$, $g = 1$ and $F^1 = 1$. Note that the project has an expedition of 1 and hence $u(N) = 1$. Below, we explain in detail how to compute the value of the associated project game for coalitions $\{A, C, D\}$ and $\{B, C, D\}$.

First, consider coalition $\{A, C, D\}$. Since $D(p_{|\{B\}}, r_{|\{A,C,D\}}) = 10 > 9 = D(p)$, we have that $u(\{A, C, D\}) = -c(\{A, C, D\})$. In this case, $\mathcal{P}(\{A, C, D\}) = \{1, 2\}$. The maximum amount chargeable w.r.t. path N_1 is

$$\min \left\{ (d(A) - e(A))_+, \ (d(A) + d(B) - e(A) - \text{slack}(N_1, p))_+ \right\}$$
$$= \min\{(3)_+, \ (3 - 2)_+\} = 1$$

and for path N_2 the corresponding amount is

$$\min \left\{ (d(C) + d(D) - e(C) - e(D))_+, (d(C) + d(D) - e(C) - e(D) - \text{slack}(N_2, p))_+ \right\}$$
$$= \min\{(1 - 3)_+, (1 - 3 - 0)_+\} = 0.$$

Hence, $c(\{A, C, D\}) = \max\{1, 0\} = 1$. Therefore,

$$u(\{A, C, D\}) = -c(\{A, C, D\})) = -1.$$

Second, consider coalition $\{B, C, D\}$. Since $D(r) = 8 < 9 = D(p)$ we have $u(\{B, C, D\}) = v(\{B, C, D\})$. In this case,

$$w^1(\{B, C, D\}) = \min \left\{ e(A), \ F^1 \right\} = \min \{0, 1\} = 0,$$

and therefore $v(\{B,C,D\}) = F^1 - w^1(\{B,C,D\}) = 1 - 0 = 1$ and

$$u(\{B,C,D\}) = 1.$$

The coalitional values of the project game are: $u(\{A\}) = -1$, $u(\{B\}) = 0$, $u(\{C\}) = -1, u(\{D\}) = 0, u(\{A,B\}) = 0, u(\{A,C\}) = u(\{A,D\}) = u(\{B,C\}) = -1$, $u(\{B,D\}) = 1$, $u(\{C,D\}) = 0$, $u(\{A,B,C\}) = -1$, $u(\{A,B,D\}) = 1$, $u(\{A,C,D\}) = -1$, $u(\{B,C,D\}) = 1$, $u(N) = 1$.

Theorem 5.1 *Project games have a nonempty core.*

Proof Let $(\{N_1,\ldots,N_m\},p,r)$ be a project problem and let (N,u) be the associated project game. By the remarks above, $u(S) = v(S) - c(S)$ for every $S \subset N$. Hence, it suffices to show that (N,v) and (N,c) have nonempty cores.

Note that the game (N,v) has a similar structure as an expedited project game (J_k is replaced by I_k, F^k is substituted by E^k, and g is replaced by h). Moreover, the explicit definition of I_k, E^k, and h is not relevant for the proof of Theorem 4.2. Therefore, it can be shown that (N,v) is convex by following the same line of reasoning. Hence, (N,v) has a nonempty core.

Next, we show that (N,c) has a nonempty core. Let $S \subset N$, then

$$c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \left(\sum_{i \in N_t \cap S} d(i) - \sum_{i \in N_t \cap S} e(i) \right)_+ , \right. \right.$$
$$\left. \left. \left(\sum_{i \in N_t} d(i) - \sum_{i \in N_t \cap S} e(i) - \text{slack}(N_t,p) \right)_+ \right\} \right\}$$

$$= \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \max \left\{ \sum_{i \in N_t \cap S} d(i), \sum_{i \in N_t \cap S} e(i) \right\} , \right. \right.$$
$$\left. \left. \max \left\{ \sum_{i \in N_t} d(i) - \text{slack}(N_t,p), \sum_{i \in N_t \cap S} e(i) \right\} \right\} - \sum_{i \in N_t \cap S} e(i) \right\}$$

$$= \max_{t \in \mathcal{P}(S)} \left\{ \max \left\{ 0, \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t,p) \right)_+ \right\} \right. \right.$$
$$\left. \left. - \sum_{i \in N_t \cap S} e(i) \right\} \right\}$$

First, suppose that the project is parallel. Then, for $t \in \{1,\ldots,m\}$ and $S \subset N_t$

$$c_{|N_t}(S) = \max \left\{ 0, \min \left\{ \sum_{i \in S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t,p) \right)_+ \right\} - \sum_{i \in S} e(i) \right\}$$

Hence, $c_{|N_t}$ is the maximum of the zero(-sub)game with (a game which is strategically equivalent to) a concave game according to Lemma 3.2. Therefore, $(N_t, c_{|N_t})$ has a nonempty core. Subsequently, following the same argument as in the proof of Lemma 3.3, we can explicitly provide a core element for delayed parallel project games. By applying the same translation technique as in the proof of Theorem 3.4, it follows that (N, c) has a nonempty core in general. $\square$

Appendix

Lemma A.1 *Let* $(\{N_1, \ldots, N_m\}, p, r)$ *be an expedited project problem. Let* $i \in N$ *and* $S \subset N \setminus \{i\}$. *Then,*

(i) $w^k(S \cup \{i\}) = w^k(S)$ *for all* $k < k(i)$.
(ii) *Let* $\mathcal{A}(i, S \cup \{i\}) = \emptyset$. *Then,* $w^k(S \cup \{i\}) = w^k(S)$ *for all* $k \geq k(i)$, *and consequently* $A(S \cup \{i\}) = A(S)$.
(iii) *Let* $\mathcal{A}(i, S) \neq \emptyset$. *Then* $w^k(S \cup \{i\}) = w^k(S)$ *for all* $k > a(i, S)$, *and consequently* $A(S \cup \{i\}) = A(S)$.

Proof (i) follows readily by definition of w^k and (ii) by definition of $\mathcal{A}(i, S \cup \{i\})$ and the fact that $\mathcal{A}(S) \subset \mathcal{A}(S \cup \{i\})$. Next, we show (iii).

Let $\mathcal{A}(i, S) \neq \emptyset$. It is sufficient to show that

$$w^k(S \cup \{i\}) = w^k(S) \quad \text{for all } k > a(i, S) \tag{A.1}$$

From the definition of "active level" and Eq. (4.2) we have

$$w^{a(i,S)}(S) = \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l}\right) \setminus S} e(j) - \sum_{l=1}^{a(i,S)-1} w^l(S),$$

or equivalently

$$\sum_{l=1}^{a(i,S)} w^l(S) = \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l}\right) \setminus S} e(j). \tag{A.2}$$

By Lemma 4.1(iii), $a(i, S) \in \mathcal{A}(S \cup \{i\})$. Then

$$w^{a(i,S)}(S \cup \{i\}) = \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l} \right) \setminus S \cup \{i\}} e(j) - \sum_{l=1}^{a(i,S)-1} w^l(S \cup \{i\})$$

$$= \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l} \right) \setminus S} e(j) - e(i) - \sum_{l=1}^{a(i,S)-1} w^l(S \cup \{i\}),$$

or equivalently

$$\sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}) = \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l} \right) \setminus S} e(j) - e(i). \tag{A.3}$$

We show (A.1) by induction on k. First,

$$w^{a(i,S)+1}(S \cup \{i\}) = \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)+1} N_{I_l} \right) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}), \ E^{a(i,S)+1} \right\}$$

$$= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)+1} N_{I_l} \right) \setminus S} e(j) - e(i) - \left(\sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l} \right) \setminus S} e(j) - e(i) \right), \right.$$

$$\left. E^{a(i,S)+1} \right\}$$

$$= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)+1} N_{I_l} \right) \setminus S} e(j) - \left(\sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l} \right) \setminus S} e(j) \right), E^{a(i,S)+1} \right\}$$

$$= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^{a(i,S)+1} N_{I_l} \right) \setminus S} e(j) - \sum_{l=1}^{a(i,S)} w^l(S), \ E^{a(i,S)+1} \right\}$$

$$= w^{a(i,S)+1}(S)$$

where the second equality holds by Eq. (A.3), and the fourth equality holds by Eq. (A.2).

Let $k > a(i,S)+1$ and suppose (A.1) holds for all levels from $a(i,S)+1$ to $k-1$,

$$
w^k(S \cup \{i\}) = \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^k N_{I_l}\right) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{k-1} w^l(S \cup \{i\}), \ E^k \right\}
$$

$$
= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^k N_{I_l}\right) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}) \right.
$$
$$
\left. - \sum_{l=a(i,S)+1}^{k-1} w^l(S \cup \{i\}), \ E^k \right\}
$$

$$
= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^k N_{I_l}\right) \setminus S} e(j) - e(i) - \left(\sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l}\right) \setminus S} e(j) - e(i) \right) \right.
$$
$$
\left. - \sum_{l=a(i,S)+1}^{k-1} w^l(S \cup \{i\}), \ E^k \right\}
$$

$$
= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^k N_{I_l}\right) \setminus S} e(j) - \left(\sum_{j \in \left(\bigcup_{l=1}^{a(i,S)} N_{I_l}\right) \setminus S} e(j) \right) - \sum_{l=a(i,S)+1}^{k-1} w^l(S), \ E^k \right\}
$$

$$
= \min \left\{ \sum_{j \in \left(\bigcup_{l=1}^k N_{I_l}\right) \setminus S} e(j) - \sum_{l=1}^{k-1} w^l(S), \ E^k \right\} = w^k(S)
$$

where the third equality holds by Eq. (A.3), the fourth equality holds by induction, and the fifth one by Eq. (A.2). □

The following result provides an explicit expression for the sum of all concessions for a coalition S.

Lemma A.2 *Let* $(\{N_1,\ldots,N_m\},p,r)$ *be an expedited project problem. Let* $S \subset N$. *Then,*

$$
\sum_{l=1}^h w^l(S) = \sum_{i \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right) \setminus S} e(i) + \sum_{l=A(S)+1}^h E^l
$$

Proof For $\mathcal{A}(S) = \emptyset$, the statement is obvious. Suppose $\mathcal{A}(S) \neq \emptyset$. Then

$$\sum_{l=1}^{h} w^l(S) = \sum_{l=1}^{A(S)-1} w^l(S) + w^{A(S)}(S) + \sum_{l=A(S)+1}^{h} w^l(S)$$

$$= \sum_{l=1}^{A(S)-1} w^l(S) + \sum_{i \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right)\setminus S} e(i) - \sum_{l=1}^{A(S)-1} w^l(S) + \sum_{l=A(S)+1}^{h} E^l$$

$$= \sum_{i \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right)\setminus S} e(i) + \sum_{l=A(S)+1}^{h} E^l$$

where the second equality holds because $A(S) \in \mathcal{A}(S)$ and by definition of $w^{A(S)}(S)$. $\square$

Lemma A.3 *Let $(\{N_1, \ldots, N_m\}, p, r)$ be an expedited project problem. Let $i \in N$ and $S \subset N \setminus \{i\}$. Then,*

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right]$$

$$= \begin{cases} 0 & \text{if } \mathcal{A}(i, S \cup \{i\}) = \emptyset; \\ e(i) & \text{if } \mathcal{A}(i, S) \neq \emptyset; \\ e(i) - \displaystyle\sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right)\setminus S} e(j) + \sum_{l=1}^{A(S \cup \{i\})} w^l(S) & \text{if } \mathcal{A}(i, S \cup \{i\}) \neq \emptyset \text{ and } \mathcal{A}(i, S) = \emptyset. \end{cases}$$

Proof (i) Let $\mathcal{A}(i, S \cup \{i\}) = \emptyset$. By Lemma 4.1(iii), $\mathcal{A}(i, S) = \emptyset$. Hence, by Lemma A.1(i) and A.1(ii) we find that $\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = 0$.
   (ii) Let $\mathcal{A}(i, S) \neq \emptyset$. Then

$$\sum_{l=1}^{h} w^l(S) = \sum_{j \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right)\setminus S} e(j) + \sum_{l=A(S)+1}^{h} E^l = \sum_{j \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right)\setminus S} e(j) + \sum_{l=A(S)+1}^{h} w^l(S)$$

$$\text{(A.4)}$$

where the first equality holds by Lemma A.2 and the second by definition of $A(S)$. Moreover

$$\sum_{l=1}^{h} w^l(S \cup \{i\}) = \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right) \setminus (S \cup \{i\})} e(j) + \sum_{l=A(S \cup \{i\})+1}^{h} E^l$$

$$= \sum_{j \in \left(\bigcup_{l=1}^{A(S)} N_{I_l}\right) \setminus S} e(j) - e(i) + \sum_{l=A(S)+1}^{h} w^l(S)$$

$$= \sum_{l=1}^{h} w^l(S) - e(i),$$

where the first equality holds by Lemma A.2, the second equality holds by Lemma A.1(iii) and the third one by Eq. (A.4). Hence,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i).$$

(iii) Let $\mathcal{A}(i, S \cup \{i\}) \neq \emptyset$ and $\mathcal{A}(i, S) = \emptyset$.[1] Then

$$\sum_{l=1}^{h} w^l(S \cup \{i\}) = \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right) \setminus (S \cup \{i\})} e(j) + \sum_{l=A(S \cup \{i\})+1}^{h} E^l$$

$$= \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right) \setminus S} e(j) - e(i) + \sum_{l=A(S \cup \{i\})+1}^{h} w^l(S).$$

where the first equality holds by Lemma A.2 and the second by the fact that $A(S) \leq A(S \cup \{i\})$. Hence,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i) - \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right) \setminus S} e(j) + \sum_{l=1}^{A(S \cup \{i\})} w^l(S).$$

$\square$

---

[1] In fact $\mathcal{A}(i, S) = \emptyset$ can be omitted here. In particular, part (ii) offers a further elaboration if $\mathcal{A}(i, S) \neq \emptyset$. In this case, $- \sum_{j \in \left(\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}\right) \setminus S} e(j) + \sum_{l=1}^{A(S \cup \{i\})} w^l(S) = 0$.

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