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Sequential share bargaining

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Abstract This paper presents a new extension of the Rubinstein-Ståhl bargaining model to the case with n players, called sequential share bargaining. The bargaining protocol is natural and has as its main feature that the players' shares in the surplus are determined sequentially rather than simultaneously. The protocol also assumes orderly voting, a restriction on the order in which players respond to a proposal. The bargaining protocol requires unanimous agreement for proposals to be implemented. Unlike all existing bargaining protocols with unanimous agreement, the resulting game has unique subgame perfect equilibrium utilities for any value of the discount factor. The result builds on the analysis of so-called one-dimensional bargaining problems. We show that also one-dimensional bargaining problems have unique subgame perfect equilibrium utilities for any value of the discount factor.

Keywords Noncooperative bargaining · Dynamic games · Subgame perfect equilibrium · Unanimous agreement

JEL Classification C78

1 Introduction

In many socioeconomic problems, parties can create a surplus by collaborating. Bargaining problems study the distribution of the surplus over the parties involved. In the strategic theory of bargaining, a detailed process of negotations concerning the surplus

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is described, which is then analyzed by the tools of game theory. A commonly studied negotiation process is the one of alternating offers bargaining, first studied by Ståhl (1972) under the assumption of an exogenous deadline, next extended by Rubinstein (1982) to the case of an infinite horizon.

In the Rubinstein-Ståhl bargaining model there are two players. The game has a unique subgame perfect equilibrium under weak assumptions. This analysis does not carry over to bargaining problems with n players. As reported in Osborne and Rubinstein (1990), a first extension to the n-person case is due to Shaked and consists of an example involving three players. In this example, player 1 starts by making a public proposal about splitting the surplus to the other two players. A proposal consists of specifying a share in the surplus for each of the players. The other players must accept or reject this proposal sequentially. If all agree, the proposal is implemented, otherwise it is rejected, one period of time elapses, and the next player makes a new proposal. Bargaining continues in this way. Herrero (1985) and Haller (1986) show that there is no unique subgame perfect equilibrium for the n-person case if the discount factor is sufficiently high. In particular, any feasible agreement is supported by a subgame perfect equilibrium, and equilibria with arbitrarily long delay exist.

Alternative extensions of the Rubinstein-Ståhl bargaining model are given by Jun (1987); Chae and Yang (1988); Yang (1992); Chae and Yang (1994); Krishna and Serrano (1996); Huang (2002), and Suh and Wen (2006). These authors consider games with partial agreement, also referred to as exit games. In an exit game, players need not agree unanimously to a proposal. In case of partial agreement, those players who have accepted the proposal may exit the game with the shares awarded by the proposer. These papers reproduce the basic results of the 2-player case for the n-player case. Under weak assumptions, a unique subgame perfect equilibrium exists, and in this equilibrium agreement is reached without delay.

This paper studies n-person bargaining problems where unanimous agreement of all players is needed before an agreement can be implemented. The case where the members of an n-person coalition together generate a surplus that has to be shared is naturally analyzed by a bargaining game with unanimous agreement, where it is not allowed for players to leave the bargaining table with only partial agreements of others. Indeed, coalitions are very often thought of as being based on unanimous consent, see for instance Hart and Kurz (1983, p. 1060). Another example concerns a firm that is unable to pay its debts and files with a federal bankruptcy court for protection under Chapter 11. When Chapter 11 results in the reorganization of the firm's assets, all the firm's creditors have to approve of the reorganization plan. As a final example, we would like to mention jury trials, where juries have to reach a unanimous verdict in order to convict a defendant.

Since allocation mechanisms that have many equilibria are undesirable, the question addressed in this paper is whether we can design bargaining protocols with unanimous agreement having unique subgame perfect equilibrium utilities. This paper answers this question affirmatively. This conclusion is strikingly different from all the existing multilateral bargaining protocols with unanimous agreement. The crucial distinguishing feature of the proposed class of bargaining protocols is that players' shares are not discussed simultaneously, but rather sequentially. A proposal does therefore not consist of the specification of the shares of all players simultaneously, but rather the specification of the share of a single player, referred to as the player on the agenda. Once this player's share has been approved of by all players remaining on the agenda, bargaining proceeds by determining the share of the next player on the agenda, and so on until all the shares have been decided upon. We refer to a bargaining protocol with this distinguishing feature as a sequential share bargaining protocol. Players whose share has not been decided yet are referred to as players remaining on the agenda.

The class of sequential share bargaining protocols allows for a wide variety of protocols. Apart from the feature that players' shares are discussed one at a time, the main requirement is the one of orderly voting. Orderly voting means that in each period the player whose share is discussed is either the proposer or the last player to respond to a proposal. In determining the share of player a, it is natural to think about player a as being involved in a bargaining situation involving two coalitions. One coalition consists of player a himself, the other coalition of the other players remaining on the agenda. All players in the second coalition share a preference for making the share awarded to player a as small as possible. The assumption of orderly voting can therefore be interpreted as the requirement that a proposer first seeks approval within his own coalition, followed by approval of all the players in the opposing coalition.

To solve the sequential share bargaining game, we study so-called one-dimensional bargaining problems. A one-dimensional bargaining problem consists of two coalitions S and T that bargain over the choice of x in an interval $[0, r]$. The utility functions of the players in S are identical and monotonically increasing in x. The utility functions of the players in T are identical too, but monotonically decreasing in x.

Although we need the one-dimensional bargaining problem as a building block to obtain results for sequential share bargaining, one-dimensional bargaining problems are worth studying in their own right. Many real-life bargaining situations can be approximated by the case where the bargaining space is one-dimensional, the players involved can be partitioned in two groups, with preferences within the group identical, and between groups diametrically opposed. Examples include the division of a fixed budget over two possible goals, the location of a public facility on a line, and negotiations between two firms (where a firm is viewed upon as a collection of agents with identical preferences) or between a firm and an individual about the price of a product or a service. One-dimensional bargaining problems are also studied in Banks and Duggan (2000); Imai and Salonen (2000); Cho and Duggan (2003); Cardona and Ponsatí (2007), and Herings and Predtetchinski (2010).

We show that one-dimensional bargaining leads to unique subgame perfect equilibrium utilities when the requirement of orderly voting is satisfied. In the context of one-dimensional bargaining, orderly voting means that the coalition members of the proposer respond before the members in the opposing coalition. We provide a linear system of characteristic equations that makes the computation of an equilibrium strategy profile an easy task. All existing uniqueness results in the one-dimensional bargaining literature (Imai and Salonen 2000; Cho and Duggan 2003; Cardona and Ponsatí 2007; Herings and Predtetchinski 2010) need the much stronger concept of subgame perfect equilibrium in stationary strategies. Surprisingly, in our model subgame perfection suffices to obtain unique predictions.

From our results on one-dimensional bargaining problems, we derive that the subgame perfect equilibrium utilities in sequential share bargaining are uniquely determined by means of the following induction argument. Clearly, sequential share bargaining problems where the set of players remaining on the agenda is a singleton have unique subgame perfect equilibrium utilities. Consider a sequential share bargaining game where the set of players remaining on the agenda has cardinality $n + 1$. For all its subgames where the set of players remaining on the agenda has cardinality n, one can substitute the corresponding subgame perfect equilibrium utilities. The resulting reduced sequential share bargaining game belongs to the class of one-dimensional bargaining problems, having unique subgame perfect equilibrium utilities. Moreover, the equilibria in sequential share bargaining games are characterized by absence of delay.

Section 2 introduces an illustrative example to highlight how the principle of determining shares one by one avoids the huge multiplicity of subgame perfect equilibria that is common in multilateral bargaining with unanimous agreement. Section 3 introduces the class of sequential share bargaining protocols and claims that for any such protocol subgame perfect equilibrium utilities are unique. Section 4 introduces the one-dimensional bargaining problem and Sect. 5 its characteristic equations. In Sect. 6 it is shown that the one-dimensional bargaining problem leads to unique subgame perfect equilibrium utilities. Building on the result of Sect. 6, it is shown in Sect. 7 that sequential share bargaining leads to unique subgame perfect equilibrium utilities. Section 8 concludes.

2 An example

As an illustrative example we consider the case where a set of three players, $N = \{1, 2, 3\}$, bargains over the division of a surplus of size 1. The three players rotate in making proposals, starting with player 1. A proposal $x \in \mathbb{R}^3_+$ satisfies that $x_1 + x_2 + x_3 = 1$. After a proposal by player i, player $i + 1$ responds by an acceptance or a rejection. In case of an acceptance by player $i + 1$, it is player $i + 2$'s turn to accept or reject the proposal by player i. If both responders accept the proposal, it is implemented. Otherwise, after the first rejection of the proposal by player i, player $i + 1$ makes a proposal at time $t + 1$, and so on, and so forth. A proposal x accepted in period t yields a utility of $\delta^t x_j$ to player $j = 1, 2, 3$, where $\delta \in (0, 1)$ is the common discount factor. As reported in Binmore et al. (1992), for every proposal x there exists a subgame perfect equilibrium in which x is accepted immediately whenever the discount factor is greater than or equal to $1/2$. This result stands in sharp contrast to the two-player case, where Rubinstein (1982) has shown the uniqueness of a subgame perfect equilibrium.

The bargaining game of the previous paragraph admits a unique subgame perfect equilibrium in stationary strategies. In this equilibrium, player 1 proposes

$$x = \left(\frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right),$$

and this proposal is accepted by both players 2 and 3.

We argue that the extreme multiplicity of equilibria can be avoided by choosing an appropriate bargaining procedure. In particular, a procedure where players' shares are discussed sequentially rather than simultaneously does not suffer from multiplicity of subgame perfect equilibrium utilities. Consider again the case where a set of three players $N = \{1, 2, 3\}$ has to bargain over the division of a surplus of size 1. The players' shares are now not discussed simultaneously, but rather one at a time. As an illustration, we consider the case where the players start bargaining over the share to be received by player 3, so player 3 is the first player on the agenda, followed by the share of player 2, so player 2 is the second player on the agenda. What is left can be claimed by player 1, the last player on the agenda. Therefore, initially the set of players remaining on the agenda is $R = \{1, 2, 3\}$, after player 3's share is agreed upon by all players in R, the set of players remaining on the agenda becomes $R' = \{1, 2\}$, and after player 2's share is agreed upon by all players in R', the set of players remaining on the agenda becomes $R'' = \{1\}$.

To determine the share of the player on the agenda, players rotate in making proposals. For the sake of the example, we assume that player 1 starts by making a proposal x_3^1 for the share player 3 should obtain. In case of a rejection by player 2 or player 3, player 2 is next to make a proposal x_3^2 for the share of player 3, followed by a proposal x_3^3 by player 3 in case of a rejection by player 1 or player 2. We assume that a proposal of player 1 is followed by a response by player 2, and if positive, by a response by player 3. Similarly, a proposal by player 2 is followed by a response by player 1, and if positive, by a response by player 3. Finally, after a proposal by player 3, player 1 responds first, followed by player 2. After the acceptance of the share x_3 for player 3 by all the players, players 1 and 2 continue bargaining over the share of player 2. Again, player 1 starts by making a proposal, followed by player 2 in case of a rejection, and so on, and so forth. After the acceptance of the share x_2 for player 2, player 1 makes a (rather trivial) proposal for his share x_1. One period of time elapses after every rejection.

The proposal of player i for the share of player a who is currently on the agenda is denoted x_a^i. The set of shares of player a approved of by player i when proposed by player j is denoted A_{aj}^i. For the moment we restrict attention to stationary strategies.

Tables 1 and 2 summarize a profile of strategies that is, as is shown in this paper, a subgame perfect equilibrium. Table 1 summarizes the players' strategies in the subgames where $R = N$. In such subgames it is player 3 who is on top of the agenda,

Table 1 Strategies in subgames with $R = N$

	$\bar{x}_3^i$	A_{31}^i	A_{32}^i	A_{33}^i
$i = 1$	$\frac{\delta^2}{1+\delta+\delta^2}$	$*$	$\left[0, \frac{1+\delta-\delta^3}{1+\delta+\delta^2}\right]$	$\left[0, \frac{1}{1+\delta+\delta^2}\right]$
$i = 2$	$\frac{\delta}{1+\delta+\delta^2}$	$\left[0, \frac{1+\delta^2-\delta^3}{1+\delta+\delta^2}\right]$	$*$	$\left[0, \frac{1}{1+\delta+\delta^2}\right]$
$i = 3$	$\frac{1}{1+\delta+\delta^2}$	$\left[\frac{\delta^2}{1+\delta+\delta^2}, 1\right]$	$\left[\frac{\delta}{1+\delta+\delta^2}, 1\right]$	$*$

Table 2 Strategies in subgames where $R = \{1, 2\}$ and player 3 has received x_3

	$\bar{x}_2^i$	A_{21}^i	A_{22}^i
$i = 1$	$\frac{\delta}{1+\delta}(1 - x_3)$	$*$	$\left[0, \frac{1}{1+\delta}(1 - x_3)\right]$
$i = 2$	$\frac{1}{1+\delta}(1 - x_3)$	$\left[\frac{\delta}{1+\delta}(1 - x_3), 1 - x_3\right]$	$*$

hence $a = 3$. The first entry in row i of the table lists player i's proposal $\bar{x}_3^i$ for the share of player 3. The next entries in row i list player i's acceptance sets A_{31}^i, A_{32}^i, and A_{33}^i for the shares of player 3 when proposed by player 1, 2, and 3, respectively. Since a proposer does not respond to his own proposal, a $*$ is used to denote the corresponding entry. Table 2 summarizes the players' strategies in subgames corresponding to R', where player 3 has received a share x_3. In such subgames it is player 2 who is on the agenda, so $a = 2$. The table lists player i's proposal $\bar{x}_2^i$ for the share of player 2, as well as player i's acceptance set A_{2j}^i for player 2's share when proposed by the other player.

In subgames corresponding to R'', player 3 has received a share x_3, and player 2 a share x_2, the proposal of player 1 for his own share is given by

$$\bar{x}_1^1 = 1 - x_2 - x_3.$$

In this equilibrium, first player 3 receives a share $x_3 = \bar{x}_3^1 = \delta^2/(1 + \delta + \delta^2)$, followed by player 2 receiving $x_2 = \bar{x}_2^1 = \delta(1 - \bar{x}_3^1)/(1 + \delta) = \delta/(1 + \delta + \delta^2)$, and finally player 1 receives $x_1 = \bar{x}_1^1 = 1 - \bar{x}_2^1 - \bar{x}_3^1 = 1/(1 + \delta + \delta^2)$.

Notice that the equilibrium strategy profile is stationary. Could there be other subgame perfect equilibria? We show in this paper that *all* subgame perfect equilibria of the sequential bargaining game lead to the same equilibrium utilities.

We present the main idea behind the uniqueness proof for the example. The proof is by induction on the number of players remaining on the agenda. Any subgame where the set of remaining players is $\{1, 2\}$ is the familiar 2-player game with alternating offers, with $1 - x_3 > 0$ being the size of the cake still available for players 1 and 2. The results of Rubinstein (1982) apply and show that any such subgame has the unique subgame perfect equilibrium described by Table 2. We now replace all such two-player subgames by their respective subgame perfect equilibrium payoffs. In the reduced game thus obtained if player 3 is allocated a share x_3 of the cake in period 0, the payoffs to the players 1, 2, and 3 are given by the vector

$$\left(\frac{1}{1 + \delta}(1 - x_3), \frac{\delta}{1 + \delta}(1 - x_3), x_3\right).$$

In the reduced game, players 1 and 2 have the same preferences over the outcomes of the game, the observation that is central to the proof of the uniqueness of subgame perfect equilibrium payoffs. Such a game can be thought of as one-dimensional bargaining game, played by the player 3 against the coalition $T = \{1, 2\}$.

Subgame perfect equilibrium utilities can be shown to be unique in all such one-dimensional bargaining games where the remainder of the surplus $1 - x_3$ is strictly positive. The formal argument is given in Theorem 6.1 and consists of an extension of proof strategy of Binmore (1987). Other subgames can be argued not to be reached on the equilibrium path.

The subgame perfect equilibrium utilities of the reduced game are computed as follows. In equilibrium, all proposals are unanimously accepted. This gives a system of equations

$$\begin{aligned} x_3^1 &= \delta x_3^2 \\ x_3^2 &= \delta x_3^3 \\ 1 - x_3^3 &= \delta(1 - x_3^1). \end{aligned}$$

The first of these equations is the condition that player 3 is indifferent between the acceptance and the rejection of the proposal x_3^1 of player 1. The second equation is the condition that player 3 is indifferent between the acceptance and the rejection of the proposal x_3^2 of player 2. The third equation is the condition that the players 1 and 2 are indifferent between the acceptance and the rejection of the proposal x_3^3 of player 3. The unique solution to the system is the vector $(\bar{x}_3^1, \bar{x}_3^2, \bar{x}_3^3)$ given by Table 1.

Though the procedure with less and less players remaining on the agenda might give the impression that the basic argument is identical to the one for procedures that do not require unanimous approval, such is not the case. Indeed, apart from the sequentiality of the procedure, the uniqueness result also depends crucially on the order in which players respond. The latter feature plays no role in procedures that do not insist on unanimous agreement.

Consider the same sequential share bargaining procedure as before, but with a different voting order in case $R = N$ and a proposal is made by player 1, where now player 3 responds before player 2. The subgame perfect equilibrium strategy profile of the original sequential share bargaining procedure is still a subgame perfect equilibrium here. However, now there are also subgame perfect equilibria giving rise to different equilibrium utilities. In fact, immediate agreement on any share $x_3 \in [\delta^2/(1 + \delta + \delta^2), (1 + \delta^2)/(1 + \delta + \delta^2)]$ for player 3 is supported by some subgame perfect equilibrium whenever $2\delta^3 \geq 1$, which is for instance the case if δ exceeds 0.8.[1]

3 Sequential share bargaining

We consider the problem of dividing a surplus of size X among a set of N players. The cardinality of N is denoted by n. In the standard modeling approach, players make proposals that consist of the specification of the entire allocation of the surplus. Here, on the contrary, players' shares are discussed one at a time, and a proposal consists of the specification of the share of the particular player who is currently on the agenda. The players whose share has not been determined yet are called players remaining

[1] A formal proof of this statement is available upon request from the authors.

on the agenda and are denoted by $R \subset N$. A player's share is determined once it is approved of by all players remaining on the agenda.

Potentially there are many ways regarding the order in which the players' shares could be discussed. One possibility would be a fixed sequence, another would be that the player whose share is going to be determined next is drawn according to a uniform probability distribution from the set of players remaining on the agenda. Both alternatives are captured by our specification that assigns to each potential set R of players remaining on the agenda a probability distribution $\rho(R)$ determining for each player in R the probability that his share is going to be discussed next. The probability that player $a \in R$ is the next one on the agenda is therefore equal to $\rho_a(R)$. We denote the tuple of probability distributions $\rho(R)$ for non-empty subsets R of N by ρ. The example where shares are discussed in a fixed sequence $i_1, \ldots, i_n$ is obtained by setting, for $j = 1, \ldots, n$, $\rho_{i_j}(\{i_j, \ldots, i_n\}) = 1$ and $\rho_i(\{i_j, \ldots, i_n\}) = 0$ for $i \neq i_j$. In this case, there are only n sets R that matter. The example where the next player on the agenda is selected according to a uniform probability distribution follows from setting $\rho_i(R) = 1/|R|$ for each $\emptyset \neq R \subset N$ and $i \in R$, where $|R|$ denotes the cardinality of R.

Given the choice of the player on the agenda, we use a standard bargaining process to determine his share. One of the players in R is chosen as proposer, all the other players respond sequentially. As soon as one of them disagrees, a new player in R is chosen to be the proposer. One standard bargaining protocol chooses proposers in a rotating order, as in Rubinstein (1982) or Haller (1986), another chooses proposers from the set of players remaining on the agenda according to some time-invariant probability distribution as in Binmore (1987) or Banks and Duggan (2000). Both protocols are special cases of our approach, where following Kalandrakis (2004) and Herings and Predtetchinski (2010) we assume that the selection of the proposer is determined by a Markov process with the set R of players remaining on the agenda as the state space. The transition probabilities of the process are given by the irreducible transition matrix $\pi(R)$, so $\pi_{ij}(R)$ is the probability that the next proposer is player j if the current proposer is player i. The probability distribution $\pi^0(R)$ determines the initial proposer. The tuple of transition matrices $\pi(R)$ for non-empty subsets R of N is denoted by π and the tuple of probability distributions $\pi^0(R)$ by π^0. The example where, given a set of players remaining on the agenda R with cardinality r, proposers are chosen in the rotating order $i_1, \ldots, i_r, i_1$ follows from specifying $\pi_{i_j i_{j+1}}(R) = 1$ for $j = 1, \ldots, r$, and $\pi_{ij}(R) = 0$ otherwise. When player $j \in R$ is selected with probability π_j according to a time-invariant probability distribution, we set $\pi_{ij}(R) = \pi_j$ for all $i, j \in R$.

We now describe the sequential share bargaining game $\Gamma = (N, X, \delta, \rho, \pi^0, \pi)$ more formally. We define a state as the tuple $s = (R, x_{-R}, t, a, i)$, where $\emptyset \neq R \subset N$ is the set of players remaining on the agenda, x_{-R} are the unanimously agreed upon shares of the players in $N \setminus R$, $t \in \mathbb{Z}_+$ is the time period, $a \in R$ is the player on the agenda, and $i \in R$ is the proposer.

At the beginning of the game, player $a \in N$ is chosen to be the first player on the agenda with probability $\rho_a(N)$ and $i \in N$ is chosen to be the first proposer with probability $\pi_i^0(N)$, so the first state is $(N, x_{-N}, 0, a, i)$ with probability $\rho_a(N)\pi_i^0(N)$.[2]

In each state (R, x_{-R}, t, a, i), player i proposes some share x_a of the surplus to player a. It should hold that $x_a \geq 0$ and $x_a \leq X - \sum_{j \in N \setminus R} x_j$. All players in $R \setminus \{i\}$ respond sequentially to the proposal. The voting order may depend on all the parameters describing the state and in fact on the entire history of play, as long as the property of orderly voting is satisfied.

Definition 3.1 A sequential share bargaining game satisfies orderly voting if at every state $s = (R, x_{-R}, t, a, i)$ it holds that either $i = a$ or $i \neq a$ and player i is the last one to respond.

Players can either accept or reject the proposal. As soon as the first rejection takes place, a transition to state $(R, x_{-R}, t+1, a, j)$ occurs with probability $\pi_{ij}(R)$. If no rejection occurs, so all players accept the proposal, player a receives the proposed share x_a, the set of players remaining on the agenda becomes $R' = R \setminus \{a\}$, and a transition to state $(R', x_{-R'}, t, a', j)$ occurs with probability $\rho_{a'}(R')\pi_j^0(R')$.

Notice that in the course of the game each state is visited at most once. The game terminates as soon as there are no more players on the agenda. The utility of a player a receiving share x_a is equal to $\delta^t x_a$, where t is the period in which the game terminates and $\delta \in (0, 1)$ is the common discount factor. In the case of perpetual disagreement, the utility is equal to zero for all players.

The main result of this paper is that sequential share bargaining games with orderly voting have unique subgame perfect equilibrium utilities. The gist of the argument is as follows. A subgame in state $s = (R, x_{-R}, t, a, i)$ starting with a proposal by player i is denoted by $\Gamma(s)$. All sequential share bargaining games $\Gamma(R, x_{-R}, t, a, a)$ such that the set R contains a single player are trivial and have trivial subgame perfect equilibrium utilities equal to $\delta^t x_{-R}$ and $\delta^t(X - \sum_{j \in N \setminus R} x_j)$. Subgames where the set R consists of more than one player and $X - \sum_{j \in N \setminus R} x_j = 0$ are less trivial, since there are no incentives for the remaining players in R to agree, causing a multiplicity of subgame perfect equilibrium utilities of the other players. We will argue that such subgames are not reached on the equilibrium path.

Suppose we have shown that all games $\Gamma(s)$ with orderly voting have unique subgame perfect equilibrium utilities whenever s is a state with $m - 1$ players remaining on the agenda and the remainder of the surplus is strictly positive.

Consider a game in $\Gamma(s)$ where $s = (R, x_{-R}, t, a, i)$ with R containing m players. For $0 \leq x_a \leq X - \sum_{j \in N \setminus R} x_j$ we replace all subgames in $\Gamma(R \setminus \{a\}, X - \sum_{j \in N \setminus R} x_j - x_a, t', a', i')$ of the game $\Gamma(R, x_{-R}, t, a, i)$ by their subgame perfect equilibrium utilities. For $x_a = X - \sum_{j \in N \setminus R} x_j$, we select subgame perfect equilibrium utilities arbitrarily. The resulting reduced game is denoted $\hat{\Gamma}(s)$. As we show in the proof of Theorem 7.1, the game $\hat{\Gamma}(s)$ is a so-called one-dimensional bargaining game with orderly voting, played by coalitions $S = \{a\}$ and $T = R \setminus \{a\}$. Subgame perfect equilibrium utilities of $\hat{\Gamma}(s)$ are therefore unique by Theorem 6.1, and are found

[2] We have included x_{-N} to avoid notational confusion.

using the characteristic equations that we develop for the one-dimensional bargaining game. It now follows that the subgame perfect equilibrium utilities of Γ are unique.

4 One-dimensional bargaining

This section studies one-dimensional bargaining with unanimous agreement. In a one-dimensional bargaining game $\hat{\Gamma}$, a finite set of players N has to agree on the choice of x in a non-degenerate interval $[0, r]$. The n players in N are partitioned in the non-empty coalitions S and T. All players in S have identical preferences that are monotonically increasing in x on $[0, r)$. All players in T have identical preferences that are monotonically decreasing in x on $[0, r]$.

The game $\hat{\Gamma}$ is a dynamic game of perfect information in discrete time. At each time period $t = 0, 1, \ldots$ a proposer is selected from the set N. The selected player makes a proposal, i.e. a choice for x in the interval $[0, r]$. We denote the proposal by player i by x^i. All the remaining players respond, sequentially, to the proposal. Either a proposal is unanimously accepted, it is implemented, and the game ends. Or some player rejects the proposal, period $t + 1$ begins, and nature selects a new proposer. An outcome of the game is either perpetual disagreement or a pair (t, x), i.e. x is agreed upon in period t.

The selection of the proposer is determined by a Markov process with state space N. The transition probabilities of the process are given by the irreducible transition matrix $\pi(N)$. Thus $\pi_{ij}(N)$ is the probability of a transition from state i to state j, i.e. if the last proposer has been player i, then with probability π_{ij} the next proposer is player j. The first proposer is a player $i^0 \in N$.

The players in coalition S have identical preferences over outcomes. The utility of a player $i \in S$ who receives outcome $x \in [0, r)$ in period t is $u_i(t, x) = \delta^t x$, where $\delta \in (0, 1)$ is the common discount factor. The utility of receiving outcome r in period t may depend on the entire history of play and might be any value less than or equal to $\delta^t r$. The utility of perpetual disagreement is 0. Similarly, the players in T have identical preferences. The utility of a player $i \in T$ who receives outcome $x \in [0, r]$ in period t is $u_i(t, x) = \delta^t (r - x)$. The utility of perpetual disagreement is 0.

The reason for the slight asymmetry in the description of the preferences of members of coalitions S and T is that we want to apply the results for the one-dimensional bargaining model to the sequential share bargaining model. When a player in the sequential share bargaining model has received the entire remaining surplus, his subgame perfect equilibrium utility is indeterminate as it depends on the timing of the acceptance by the remaining players, who are all indifferent. For one-dimensional bargaining this corresponds to the case where the outcome is r.

A one-dimensional bargaining game is $\hat{\Gamma} = (S, T, r, \delta, i^0, \pi(N))$. When S and T are both singletons, the game $\hat{\Gamma}$ is a two-player game. The model in Rubinstein (1982) with alternating offers is a special case.

To show that one-dimensional bargaining games have unique subgame perfect equilibria, we need the assumption of *orderly voting,* defined analogously to Definition 3.1. Suppose the proposer belongs to coalition S. Then orderly voting means that all the remaining players in S respond before all the players in T. Similarly, if the proposer belongs to coalition T, orderly voting means that all the remaining players in T respond before all the players in S. A straightforward interpretation of the assumption of orderly voting is that the proposer first consults the players in his own coalition, before making the proposal to the other players.

Since members within a given coalition have identical preferences, it is tempting to assume that they should adopt the same strategy. This reasoning is not correct, even if one restricts attention to stationary strategies. The reason is that though preferences are identical within a coalition, the transition probabilities $\pi(N)$ depend on the identity of the proposer, implying that different players of coalition S or T have different positions in the bargaining game. As a consequence, different members of coalition S or coalition T may find it optimal to make different proposals.

5 The characteristic equations for one-dimensional bargaining

In this section we derive a subgame perfect equilibrium of the game $\hat{\Gamma}$ from the solution to a linear system of characteristic equations. In this and the next section, we drop N from the notation $\pi(N)$.

For a player i in N and a coalition $C \in \{S, T\}$, the variable z_C^i denotes the continuation utility of a member of coalition C after the rejection of a proposal made by player i in period 0. All members of a given coalition have the same preferences, so receive the same utility in any outcome of the game. The *characteristic equations* of $\hat{\Gamma}$ describe a particular subgame perfect equilibrium in terms of the variables z_C^i. The characteristic equations are as follows:

$$z_T^i = \pi_{iT}\delta r + \delta \sum_{j \in S} \pi_{ij} z_T^j - \delta \sum_{j \in T} \pi_{ij} z_S^j, \quad i \in N, \tag{1}$$

$$z_S^i = \pi_{iS}\delta r - \delta \sum_{j \in S} \pi_{ij} z_T^j + \delta \sum_{j \in T} \pi_{ij} z_S^j, \quad i \in N, \tag{2}$$

where $\pi_{iC} = \sum_{j \in C} \pi_{ij}$. This is a system with number of equations and unknowns both equal to twice the cardinality of N. We will show that it has a unique solution and that the solution is strictly positive.

The idea behind system (1)–(2) is that a proposal of any member of coalition $C \in \{S, T\}$ leaves any member of the rival coalition indifferent between acceptance and rejection of the proposal. Thus, a player $j \in S$ makes a proposal $r - z_T^j$. Such a proposal makes all members of coalition T indifferent between acceptance and rejection, since either action results in utility z_T^j. Similarly, a player $j \in T$ makes a proposal z_S^j. Such a proposal makes each member of S indifferent between acceptance and rejection, since either action results in utility z_S^j.

Now suppose that the proposal of player i has been rejected. Then the continuation utility of any member of coalition S is

$$\sum_{j \in S} \pi_{ij} \delta (r - z_T^j) + \sum_{j \in T} \pi_{ij} \delta z_S^j.$$

Setting this expression equal to z^i_S gives Eq. 2. In a similar way, we find that the continuation utility of any member of T equals

$$\sum_{j\in S}\pi_{ij}\delta z^j_T + \sum_{j\in T}\pi_{ij}\delta(r - z^j_S).$$

Setting this expression equal to z^i_T gives Eq. 1.

Theorem 5.1 *The system of characteristic Eqs. 1–2 has a unique solution and the solution is strictly positive.*

Proof Adding up Eqs. 1 and 2 for fixed i we obtain the equation $z^i_T + z^i_S = \delta r$. We can therefore express each z^j_T as $\delta r - z^j_S$ and substitute this into Eq. 2. This yields

$$z^i_S = (1-\delta)\pi_{iS}\delta r + \delta\sum_{j\in N}\pi_{ij}z^j_S,\quad i\in N. \tag{3}$$

It is sufficient to show that system (3) has a unique solution. System (3) is a system with number of equations and unknowns both equal to the cardinality of N. It can be rewritten in vector-matrix notation as $z_S = \delta(1-\delta)r\pi_{\cdot S} + \delta\pi z_S$. Because π is a row-stochastic matrix, the spectral radius of π is at most 1. It follows that the matrix $I - \delta\pi$ is invertible, where I is the identity matrix. The solution $(I-\delta\pi)^{-1}\delta(1-\delta)r\pi_{\cdot S}$ is therefore unique.

Since the matrix π is irreducible and $0 < \delta < 1$, it also holds that $(I-\delta\pi)^{-1}$ is a positive matrix and $\pi_{iS} > 0$ for some $i \in N$. It follows that $\delta(1-\delta)r(I-\delta\pi)^{-1}\pi_{\cdot S}$ is a strictly positive vector, so z_S is a strictly positive vector. An analogous argument shows that z_T is a strictly positive vector. □

We define a profile of strategies $\hat{\sigma} = (\hat{\sigma}^i)_{i\in N}$ and verify that it is a subgame perfect equilibrium of $\hat{\Gamma}$. Let z be the solution to the system of characteristic Eqs. 1–2.

The behavioral strategy of a player $i \in S$ is defined as follows. Whenever player i is selected to make a proposal, he proposes $x^i = r - z^i_T$, and whenever player i responds to a proposal $x^j \in [0, r)$ of player $j \in N$, he accepts if and only if $x^j \geq z^j_S$. Whenever player i responds to a proposal r of player $j \in N$ in time period t, he accepts if and only if the utility obtained when all remaining players accept r is greater than or equal to $\delta^t z^j_S$.

The behavioral strategy of a player $i \in T$ is defined as follows. Whenever player i is selected to make a proposal, he proposes $x^i = z^i_S$, and whenever player i responds to a proposal $x^j \in [0, r]$ of player $j \in N$, he accepts if and only if $x^j \leq r - z^j_T$.

When players play according to strategy profile $\hat{\sigma}$, bargaining proceeds as follows. If player i^0 belongs to S, he proposes $x^{i^0} = r - z^{i^0}_T$. By Theorem 5.1 it holds that $x^{i^0} < r$. Since $z^{i^0}_S + z^{i^0}_T = \delta r$ and $\delta < 1$, it holds that $x^{i^0} > z^{i^0}_S$, so as a consequence all players in S accept. Players $i \in T$ accept if and only if $x^{i^0} \leq r - z^{i^0}_T$, an inequality that holds with equality, so all players in T accept.

If player i^0 belongs to T, bargaining proceeds in basically the same way. The proposal x^{i^0} equals $z^{i^0}_S$, which is subsequently accepted by all the other players.

Theorem 5.2 *The strategy profile $\hat{\sigma}$ is a subgame perfect equilibrium of the game $\hat{\Gamma}$.*

Proof Consider a subgame $\hat{\Gamma}(h)$ of $\hat{\Gamma}$ that starts after history h. Suppose a player $i \in N$ has a profitable deviation, which increases the subgame utility by $\varepsilon > 0$. Since the utility player i can get from a node that is t periods later than the initial node of the subgame is bounded by $\delta^t r$, player i has a profitable deviation $\bar{\sigma}^i$ that deviates from $\hat{\sigma}^i$ only at nodes corresponding to the first T periods, where T is the smallest natural number greater than $(\ln(\varepsilon) - \ln(r))/\ln(\delta)$.

Consider a node h' where player i, when playing according to $\bar{\sigma}^i$, deviates from $\hat{\sigma}^i$, and which is not succeeded by another node where i deviates from $\hat{\sigma}^i$. Consider the subsubgame $\hat{\Gamma}(h')$ starting at this node. Then either $\bar{\sigma}^i$ induces a profitable deviation in the subsubgame, or the strategy $\tilde{\sigma}^i$ that is equal to $\bar{\sigma}^i$, except at h', where $\tilde{\sigma}^i(h') = \hat{\sigma}^i(h')$, is a profitable deviation from $\hat{\sigma}^i$ in subgame $\hat{\Gamma}(h)$. Iterating this argument, we can show that there is a subgame $\hat{\Gamma}(h^0)$ of $\hat{\Gamma}$ such that player i acts at node h^0 and player i has a profitable deviation which only deviates from $\hat{\sigma}$ at h^0.

Consider the subgame $\hat{\Gamma}(h^0)$ and let t be the corresponding period. We complete the proof by showing that a one-shot deviation from $\hat{\sigma}^i$ cannot be profitable.

Suppose player $i \in S$ is a proposer in the first node of $\hat{\Gamma}(h^0)$. The use of strategy $\hat{\sigma}^i$ leads to a proposal $x^i = r - z^i_T$, which is unanimously accepted by all responders, and leads to utility $\delta^t(r - z^i_T)$ for player i.

Proposing $x > x^i$ leads to rejection by a player in T. The utility of player i is therefore equal to $\delta^t z^i_S$. From $\delta^t z^i_S + \delta^t z^i_T = \delta^{t+1} r$, it follows that $\delta^t z^i_S < \delta^t(r - z^i_T)$, and player i looses utility by proposing $x > x^i$.

Now consider a proposal x by i satisfying $x < x^i$. If this proposal is accepted, it leads to utility for i less than $\delta^t(r - z^i_T)$. If it is rejected, then it will lead to utility $\delta^t z^i_S$, which is less than $\delta^t(r - z^i_T)$.

Suppose player $i \in T$ is a proposer in the first node of $\hat{\Gamma}(h^0)$. Then a fully analogous argument shows that he does not have a profitable one-shot deviation.

Suppose player $i \in S$ is a responder in the first node of $\hat{\Gamma}(h^0)$ and responds to a proposal $x \in [0, r)$ by a player $j \in N$. If i is asked to respond to this proposal, then according to $\hat{\sigma}^i$ acceptance takes place if $x \geq z^j_S$, and results in utility equal to $\delta^t x$ if all players responding after player i accept x or to $\delta^t z^j_S$ otherwise. A deviation to rejection leads to utility $\delta^t z^j_S$, and is therefore not profitable. If $x < z^j_S$, then player i rejects the proposal when playing according to $\hat{\sigma}^i$, and obtains utility $\delta^t z^j_S$. A deviation from rejection to acceptance results in utility equal to $\delta^t x$ if all players responding after player i accept x or to $\delta^t z^j_S$ otherwise, and is therefore not profitable.

Suppose player $i \in S$ is a responder in the first node of $\hat{\Gamma}(h^0)$ and responds to proposal r by a player $j \in N$. If i is asked to respond to this proposal, then according to $\hat{\sigma}^i$ acceptance takes place if the utility obtained when all remaining players accept r is greater than or equal to $\delta^t z^j_S$, and results in that utility if all players responding after player i accept r or to $\delta^t z^j_S$ otherwise. A deviation to rejection leads to utility $\delta^t z^j_S$, and is therefore not profitable. If the utility obtained when all remaining players accept r is less than $\delta^t z^j_S$, then player i rejects the proposal when playing according to $\hat{\sigma}^i$, and obtains utility $\delta^t z^j_S$. A deviation from rejection to acceptance results in utility less than $\delta^t z_S^j$ if all players responding after player i accept r or to $\delta^t z_S^j$ otherwise, and is therefore not profitable.

Suppose player $i \in T$ is a responder in the first node $\hat{\Gamma}(h^0)$ and responds to a proposal $x \in [0, r]$ by a player $j \in N$. Then a fully analogous argument shows that he does not have a profitable one-shot deviation. □

Example 5.1 Suppose coalition S is a singleton consisting of player i_1, $T = \{i_2, \ldots, i_n\}$, and the identity of the proposer cycles within the player set: $i_1, \ldots, i_n, i_1$. The system of characteristic Eqs. 1–2 yields

$$\begin{aligned} z_T^{i_1} &= \delta r - \delta z_S^{i_2}, \\ z_S^{i_j} &= \delta z_S^{i_{j+1}}, \quad j = 2, \ldots, n-1, \\ z_S^{i_n} &= \delta r - \delta z_T^{i_1}. \end{aligned}$$

Solving it, we find

$$z_T^{i_1} = \frac{\delta - \delta^n}{1 - \delta^n} r,$$

so the equilibrium proposal of player i_1 equals

$$x^{i_1} = r - z_T^{i_1} = \frac{1 - \delta}{1 - \delta^n} r.$$

This proposal will be accepted by all players in equilibrium. The equilibrium utility of player i_1 is

$$\frac{1 - \delta}{1 - \delta^n} r$$

and the equilibrium utility of players in T is

$$\frac{\delta - \delta^n}{1 - \delta^n} r.$$

Since $1 - \delta$ does not necessarily exceed $\delta - \delta^n$, the equilibrium utility of the first mover, player i_1, may be lower than that of the players in T.

In subgames (that will not be reached in equilibrium) where a player i_k, $k = 2, \ldots, n$, has to make a proposal, i_k proposes

$$x^{i_k} = \delta^{n-k+1} \frac{1 - \delta}{1 - \delta^n} r,$$

which equals δ^{n-k+1} times the proposal of player i_1.

In a subgame where player i_2 makes a proposal, he may propose a low value of x, since a long time will elapse before a player belonging to the opposing coalition can make a proposal. In a subgame where player i_n makes a proposal, i_n knows that player

i_1 will be the next proposer, so i_n proposes a relatively high value of x. Observe that we obtain the Rubinstein (1982) result for the case where $n = 2$.

If, for fixed δ, the number of players n goes to infinity, then the equilibrium proposal of player i_1 converges to $(1-\delta)r$, the proposal of player i_2 to 0, and the proposal of player i_n to $\delta(1-\delta)r$. If, for a fixed number of players, δ converges to 1, then the equilibrium proposals of all players converge to r/n.

Example 5.2 Suppose the coalition S consists of k players and the coalition T of $n-k$ players. Each player is selected with probability $1/n$ as the proposer. For $i \in N$, the system of characteristic Eqs. 1–2 is then equal to

$$\begin{aligned} z_T^i &= \frac{n-k}{n}\delta r + \delta \sum_{j \in S} \frac{1}{n} z_T^j - \delta \sum_{j \in T} \frac{1}{n} z_S^j, \\ z_S^i &= \frac{k}{n}\delta r - \delta \sum_{j \in S} \frac{1}{n} z_T^j + \delta \sum_{j \in T} \frac{1}{n} z_S^j. \end{aligned}$$

Solving it, we find

$$\begin{aligned} z_S^i &= \delta \frac{k}{n} r, \\ z_T^i &= \delta \frac{n-k}{n} r. \end{aligned}$$

The equilibrium proposal of a player $i \in S$ equals

$$x^i = r - z_T^i = \delta \frac{k}{n} r + (1-\delta) r.$$

This proposal will be accepted by all players in equilibrium. The equilibrium proposal of a player $i \in T$ is given by

$$x^i = \delta \frac{k}{n} r.$$

The larger the ratio k/n, the higher the fraction of players belonging to coalition S, and the higher the proposed value of x.

If the initial proposer is also chosen randomly with equal probabilities, the expected utility of a player $i \in S$ is given by

$$\frac{k}{n}\left(\delta \frac{k}{n} r + (1-\delta) r\right) + \frac{n-k}{n} \delta \frac{k}{n} r = \frac{k}{n} r.$$

Also the expected proposal is equal to $(k/n)r$. The expected utility of a player $i \in T$ equals $((n-k)/n)r$.

If, for fixed δ and a fixed size of coalition S, the number of players in T goes to infinity, then the equilibrium proposal of players in S converges to $(1-\delta)r$, and the proposal of players in T converges to 0. If, for fixed δ and a fixed size of coalition T, the number of players in S goes to infinity, then the equilibrium proposal of players in S converges to r, and the proposal of players in T converges to δr. If, for a fixed number of players, δ converges to 1, then the equilibrium proposals of all players converge to $(k/n)r$. □

6 Uniqueness of equilibrium in one-dimensional bargaining games

In this section we show that subgame perfect equilibrium utilities are unique, and therefore correspond to the ones following from $\hat{\sigma}$.

For $i \in N$, for $t \in \{0, 1, \ldots\}$, let $\hat{\Gamma}^{it}$ denote the class of subgames of the game $\hat{\Gamma}$ starting in period t with player i in the role of the proposer. For $i \in N$, $t \in \{0, 1, \ldots\}$, and $C \in \{S, T\}$, let $\underline{u}_C^{it}$ ($\overline{u}_C^{it}$) be the infimum (supremum) of the utilities to coalition C over all subgame perfect equilibria of games in $\hat{\Gamma}^{it}$. Let $\underline{z}_C^{it}$ ($\overline{z}_C^{it}$) be the infimum (supremum) of the continuation utilities to coalition C following the rejection of a proposal by player i in period t over all subgame perfect equilibria of games in $\hat{\Gamma}^{it}$.

The following result asserts that, for $i \in N$, for $C \in \{S, T\}$, $\underline{u}_C^{i0} = \bar{u}_C^{i0}$, and $\underline{z}_C^{i0} = \bar{z}_C^{i0}$. This result implies that subgame perfect equilibrium utilities are unique.

Theorem 6.1 *Let $\hat{\Gamma}$ be a one-dimensional bargaining game with orderly voting. For any $\hat{\Gamma}$, for $i \in N$ and $C \in \{S, T\}$, $\underline{u}_C^{i0} = \bar{u}_C^{i0}$ and $\underline{z}_C^{i0} = \bar{z}_C^{i0}$. In any subgame perfect equilibrium, agreement is reached without delay.*

Proof First we establish the following inequalities. For $t \in \{0, 1, \ldots\}$ it holds that

$$\begin{aligned} &\underline{z}_S^{it} \leq \underline{u}_S^{it} \leq \overline{u}_S^{it} \leq \overline{z}_S^{it}, && i \in T, \\ &\delta^t r - \overline{z}_S^{it} \leq \underline{u}_T^{it} \leq \overline{u}_T^{it} \leq \delta^t r - \underline{z}_S^{it}, && i \in T, \end{aligned}$$

and

$$\begin{aligned} &\underline{z}_T^{it} \leq \underline{u}_T^{it} \leq \overline{u}_T^{it} \leq \overline{z}_T^{it}, && i \in S, \\ &\delta^t r - \overline{z}_T^{it} \leq \underline{u}_S^{it} \leq \overline{u}_S^{it} \leq \delta^t r - \underline{z}_T^{it}, && i \in S. \end{aligned}$$

The inequality $\underline{z}_S^{it} \leq \underline{u}_S^{it}$ follows from the fact that when a player of coalition S rejects a proposal of player $i \in T$ in period t, he obtains a utility of at least $\underline{z}_S^{it}$. His subgame perfect equilibrium utility can therefore not be less than $\underline{z}_S^{it}$. By a similar argument it follows that $\underline{z}_T^{it} \leq \underline{u}_T^{it}$ for $i \in S$.

Let (v_S^{it}, v_T^{it}) be subgame perfect equilibrium utilities of a game in $\hat{\Gamma}^{it}$. Then $v_S^{it} \leq \delta^t r - v_T^{it}$, where the inequality comes from the fact that there might be delay before an agreement is reached. As a consequence, $\overline{u}_S^{it} \leq \delta^t r - \underline{u}_T^{it}$, which yields the inequality $\overline{u}_S^{it} \leq \delta^t r - \underline{z}_T^{it}$ for $i \in S$.

Now suppose that in period t player $i \in T$ makes a proposal $x \in (\overline{z}_S^{it}/\delta^t, r)$. Notice that obviously $\overline{z}_S^{it} \leq \delta^{t+1} r$, so x can indeed be chosen in this way. We will argue that in a subgame perfect equilibrium this leads to utility of at least $\delta^t(r - x)$ to i, and therefore to all players in T. The assumption of orderly voting implies that first the players in T respond, say in the order $i_1, \ldots, i_k$, next the players in S, say in the sequence $j_1, \ldots, j_\ell$. If player j_ℓ is given the option to respond, it means that all other players have accepted the proposal. If player j_ℓ accepts, his utility is $\delta^t x$, otherwise it is at most $\overline{z}_S^{it} < \delta^t x$. Player j_ℓ will accept therefore. By a backwards induction argument it follows that all players $j_1, \ldots, j_\ell$ will accept the proposal. Consider next player i_k. Acceptance by player i_k leads to utility $\delta^t(r - x)$, rejection will therefore only occur if it leads to utility at least equal to $\delta^t(r - x)$, so in any case the utility to any player in T is at least $\delta^t(r - x)$. By a backwards induction argument it follows that player i_1, the first to respond, can ensure a utility of at least $\delta^t(r - x)$ by accepting x. Thus, player i can guarantee himself a utility of at least $\delta^t(r - x)$ for any $x \in (\overline{z}_S^{it}/\delta^t, r)$ by proposing x. This shows that $\underline{u}_T^{it} \geq \delta^t(r - \overline{z}_S^{it}/\delta^t) = \delta^t r - \overline{z}_S^{it}$. The inequality $\overline{u}_S^{it} \leq \delta^t r - \underline{u}_T^{it} \leq \overline{z}_S^{it}$ for $i \in T$ follows.

All the remaining inequalities follow by a symmetric argument.

The above inequalities imply that for all $t \in \{0, 1, \ldots\}$, $C \in \{S, T\}$, and $i \in C$,

$$\begin{aligned}
&\overline{u}_{N\backslash C}^{it} - \underline{u}_{N\backslash C}^{it} \leq \overline{z}_{N\backslash C}^{it} - \underline{z}_{N\backslash C}^{it},\\
&\overline{u}_{C}^{it} - \underline{u}_{C}^{it} \leq \delta^t r - \underline{z}_{N\backslash C}^{it} - \delta^t r + \overline{z}_{N\backslash C}^{it} = \overline{z}_{N\backslash C}^{it} - \underline{z}_{N\backslash C}^{it}.
\end{aligned}$$

For $t \in \{0, 1, \ldots\}$, we define

$$\Delta^t = \max\{\overline{z}_C^{it} - \underline{z}_C^{it} : i \in N,\ C \in \{S, T\}\}.$$

Since rejection of a proposal by player $i \in N$ in period t leads to a subgame in $\hat{\Gamma}^{j,t+1}$ with probability π_{ij}, it holds for $C \in \{S, T\}$ that

$$\underline{z}_C^{it} \geq \sum_{j \in N} \pi_{ij} \underline{u}_C^{j,t+1}.$$

Similarly, it can be derived that

$$\overline{z}_C^{it} \leq \sum_{j \in N} \pi_{ij} \overline{u}_C^{j,t+1}.$$

It follows that

$$\overline{z}_C^{it} - \underline{z}_C^{it} \leq \sum_{j \in N} \pi_{ij} \Delta^{t+1} = \Delta^{t+1}$$

and therefore that $\Delta^t \leq \Delta^{t+1}$. Iterating this inequality we obtain $\Delta^t \leq \Delta^{t+m}$ for every $m \in \mathbb{N}$. On the other hand, $\Delta^t \leq \delta^t r$, so in particular, $\Delta^t \leq \delta^{t+m} r$ for every $m \in \mathbb{N}$. It follows that $\Delta^t = 0$.

We have shown that the subgame perfect equilibrium utility levels of all players are unique. By Theorems 5.1 and 5.2, the unique subgame perfect equilibrium utility levels are strictly positive and sum up to r, so agreement is reached without delay in all subgame perfect equilibria. $\square$

In general, $\hat{\Gamma}$ may have more than one subgame perfect equilibrium, but the multiplicity is inessential in the following sense. Assume for instance that coalition S consists of two players, i_1 and i_2, and suppose that the players respond to proposals in this order. If a player j in T makes a proposal smaller than z_S^j, the players in S will reject this proposal in any subgame perfect equilibrium. It is completely irrelevant, however, whether this proposal will be rejected by i_1, or whether i_1 accepts this proposal and has it rejected by i_2. In fact, even a member of T different from j may reject the, from his perspective, very favorable proposal, anticipating that some player in S will reject it anyway. What matters is not the responses by individual players, but how the coalition $S \cup T \setminus \{j\}$ reacts to proposals.

Let H be the set of all decision nodes of the extensive form game $\hat{\Gamma}$. We denote the decision nodes where some player has to make a proposal by H^{p}, and the decision nodes immediately following nodes in H^{p}, so nodes where the first player responds, by H^{r}. Given a strategy profile σ, for $h \in H^{\mathrm{p}}$, $\sigma^{\mathrm{p}}(h)$ denotes the proposal made at decision node h, and for $h \in H^{\mathrm{r}}$, $\sigma^{\mathrm{r}}(h) = 1$ if the proposal is accepted by all players, and $\sigma^{\mathrm{r}}(h) = 0$, otherwise.

Let σ be a subgame perfect equilibrium strategy profile. It is not hard to show that σ is essentially equivalent to $\hat{\sigma}$ in the following sense. For any $h \in H^{\mathrm{p}}$, $\sigma^{\mathrm{p}}(h) = \hat{\sigma}^{\mathrm{p}}(h)$. For any $h \in H^{\mathrm{r}}$ where the proposal under discussion is x by player $j \in S$, if $x \neq z_S^j$, then $\sigma^{\mathrm{r}}(h) = \hat{\sigma}^{\mathrm{r}}(h)$. For any $h \in H^{\mathrm{r}}$ where the proposal under discussion is x by player $j \in T$, if $x \neq r - z_T^j$, then $\sigma^{\mathrm{r}}(h) = \hat{\sigma}^{\mathrm{r}}(h)$. Subgame perfect equilibrium proposals are therefore unique. Only in subgames where by mistake a player $j \in S$ proposes $x^j = z_S^j$, so leaving players in his own coalition indifferent between accepting and rejecting, or in subgames where by mistake a player $j \in T$ proposes $x^j = r - z_T^j$, so leaving players in his own coalition indifferent between accepting and rejecting, could there be a difference in response behavior at the coalition level.

Let $\hat{\Gamma}(r)$ be a one-dimensional bargaining game with surplus size r. Consider any subgame perfect equilibrium of $\hat{\Gamma}(r)$. The continuation utility of a member of coalition C after rejecting a proposal by player i in period 0 is $z_C^i(r)$. The variable $u_C^i(r)$ denotes the expected utility of a member of coalition C in a subgame starting with a proposal by player i in period 0. The next result specifies how these variables can be computed, and claims that all utilities are linear functions of r. The latter property is crucial to derive our results for sequential share bargaining games.

Theorem 6.2 *For $r > 0$, let $\hat{\Gamma}(r)$ be a one-dimensional bargaining game with orderly voting. Consider any subgame perfect equilibrium of $\hat{\Gamma}(r)$. Then $(z_S^i(r), z_T^i(r))_{i \in N}$ are given by the solution to (1)–(2). Moreover, we have that*

$$u_S^i(r) = r - z_T^i(r),\ u_T^i(r) = z_T^i(r), \quad i \in S,$$
$$u_S^i(r) = z_S^i(r),\ u_T^i(r) = r - z_S^i(r), \quad i \in T.$$

For $i \in N$ and $C \in \{S, T\}$, the utility $u_C^i(r)$ is linear in r and it holds that $u_C^i(r) > 0$.

Proof The expressions above follow in a straightforward way from Theorems 5.2 and 6.1. It follows from the proof of Theorem 5.1 that the solution to system (3) is given by $z_S(r) = \delta(1-\delta)r(I-\delta\pi)^{-1}\pi_{\cdot S}$. It follows that $z_S(r)$ is linear in r. Since, for $i \in N$, $z_T^i(r) = r - z_S^i(r)$, it holds that $z_T^i(r)$ is linear in r. Linearity of $u_S(r)$ and $u_T(r)$ is now immediate.

Since for all $i \in N$ and $C \in \{S, T\}$ obviously $z_C^i(r) \leq \delta r$, and $z_C^i(r) > 0$ by Theorem 5.1, the above formulae imply that $u_C^i(r) > 0$. □

7 Uniqueness of equilibrium in sequential share bargaining games

In this section we prove that sequential share bargaining games have unique subgame perfect equilibrium utilities.

A subgame perfect equilibrium strategy profile σ^* of Γ is defined as follows. Consider a subgame with state (R, x_{-R}, t, a, i). Set $S = \{a\}$, $T = R\setminus\{a\}$, $r = X - \sum_{j\in N\setminus R} x_j$. Let z_C^i for $i \in N$ and $C \in \{S, T\}$ denote the solution to the system (1)–(2) of characteristic equations. Whenever the state is (R, x_{-R}, t, a, a), player a makes the proposal $x_a = r - z_T^a$ and each player $j \in T$ accepts a proposal x of player a if and only if $x \leq r - z_T^a$. Whenever the state is (R, x_{-R}, t, a, i) for $i \in R\setminus\{a\}$, player i makes the proposal $x_a = z_S^i$, each player $j \in T$ accepts a proposal x of player i if and only if $x \leq r - z_T^i$, and player a accepts a proposal x of player i if and only if $x \geq z_S^i$. In particular, equilibrium proposals are unanimously accepted.

Theorem 7.1 *Let Γ be a sequential share bargaining game with orderly voting. Then the strategy profile σ^* is a subgame perfect equilibrium of the game Γ. The subgame perfect equilibrium utilities of Γ are unique. In any subgame perfect equilibrium, agreement is reached without delay.*

Proof We show first that the subgame perfect equilibrium utilities of Γ are uniquely determined. The proof is by induction on the number of players remaining on the agenda in the subgames. We use the notation $u(s)$ for a subgame perfect equilibrium utility vector of a game in $\Gamma(s)$. Subgames for which there is still some surplus left to be divided, i.e. $\sum_{j\in N\setminus R} x_j < X$ will have unique subgame perfect equilibrium utilities. In other subgames, the players in R receive equilibrium utility 0, but are indifferent between accepting and rejecting proposals, causing a multiplicity of subgame perfect equilibrium utilities for the players $j \in N\setminus R$. The latter utilities are obviously bounded from above by $\delta^t x_j$. Subgames without a positive remaining surplus are never reached on the equilibrium path.

Let $s = (\{i\}, x_{-\{i\}}, t, i, i)$ and consider a game in $\Gamma(s)$. It obviously holds that $u_i(s) = \delta^t(X - \sum_{j\in N\setminus\{i\}} x_j)$.

Consider a non-empty $R \subset N$ with cardinality $m < n$ and $a, i \in R$. Assume we have shown that for each $k \in R$ there exists a constant $c_{R,a,i,k} > 0$ such that for all x_{-R} with $\sum_{j\in N\setminus R} x_j < X$ and for all $t \in \{0, 1, \ldots\}$, $u_k(R, x_{-R}, t, a, i) = \delta^t(X - \sum_{j\in N\setminus R} x_j)c_{R,a,i,k}$, and for $j \in N\setminus R$ it holds that $u_j(R, x_{-R}, t, a, i) = \delta^t x_j$. The constant $c_{R,a,i,k}$ is equal to player k's subgame perfect equilibrium utility of a sequential share bargaining subgame in $\Gamma(R, x_{-R}, 0, a, i)$ with a remaining surplus of size 1. Though not important for the remainder, it therefore holds that $\sum_{k\in R} c_{R,a,i,k} = 1$.

Consider now a non-empty $R \subset N$ with cardinality $m+1$ and $a, i \in R$. We analyze the subgame perfect equilibrium utilities of a game in $\Gamma(s)$ with $s = (R, x_{-R}, t, a, i)$.

We define $R' = R\setminus\{a\}$. If the players unanimously agree to allocate x_a of the surplus to player a in period $t' \geq t$, then the players enter a game in $\Gamma(s')$, where $s' = (R', x_{-R'}, t', a', i')$, with probability $\rho_{a'}(R')\pi^0_{i'}(R')$. Replacing every subgame of $\Gamma(s)$ following the acceptance of a proposal x_a in period t' by the weighted average of subgame perfect equilibrium utility vectors of the games in $\Gamma(s')$ with weights $\rho_{a'}(R')\pi^0_{i'}(R')$ yields a reduced game denoted by $\hat{\Gamma}(s)$. We argue that $\hat{\Gamma}(s)$ is strategically equivalent to a one-dimensional bargaining game whenever $\sum_{j\in N\setminus R} x_j < X$. To show this, we only have to verify that the preferences over terminal nodes of the reduced game $\hat{\Gamma}(s)$ are affine transformation of those of a one-dimensional bargaining game.

Define $r = X - \sum_{j\in N\setminus R} x_j$ and assume $r > 0$. If in the reduced game $\hat{\Gamma}(s)$ the share $x_a \in [0, r)$ is allocated in period t' to player a, then we use the induction hypothesis and find the expected utility to player a is

$$\sum_{a'\in R'}\sum_{i'\in R'} \rho_{a'}(R')\pi^0_{i'}(R')\delta^{t'} x_a = \delta^{t'} x_a,$$

and the expected utility to player $k \in R\setminus\{a\}$ is

$$\begin{aligned}&\sum_{a'\in R'}\sum_{i'\in R'} \rho_{a'}(R')\pi^0_{i'}(R')c_{R',a',i',k}\delta^{t'}(r - x_a)\\ &\quad = \delta^{t'}(r - x_a)\sum_{a'\in R'}\sum_{i'\in R'} \rho_{a'}(R')\pi^0_{i'}(R')c_{R',a',i',k}.\end{aligned}$$

If in the reduced game $\hat{\Gamma}(s)$ the share $x_a = r$ is allocated in period t' to player a, then the expected utility to player a is less than or equal to $\delta^{t'} x_a$ and the expected utility to player $k \in R\setminus\{a\}$ is 0. The factor $\sum_{a'\in R'}\sum_{i'\in R'} \rho_{a'}(R')\pi^0_{i'}(R')c_{R',a',i',k}$ is positive by the induction hypothesis. The utility to player a is therefore proportional to $\delta^{t'-t}x_a$ for $x_a \in [0, r)$ and the utility to a player $k \in R\setminus\{a\}$ is proportional to $\delta^{t'-t}(r - x_a)$ for $x_a \in [0, r]$.

We conclude that the game $\hat{\Gamma}(s)$ is a one-dimensional bargaining game as defined in Sect. 4, with coalitions $S = \{a\}$ and $T = R\setminus\{a\}$. Thus Theorems 6.1 and 6.2 apply to show that the reduced game $\hat{\Gamma}(s)$ and therefore the game $\Gamma(s)$ has unique subgame perfect equilibrium utilities. For each $k \in R$ there exists a constant $c_{R,a,i,k} > 0$ such that for all x_{-R} with $\sum_{j\in N\setminus R} x_j < X$ and for all $t \in \{0, 1, \ldots\}$, $u_k(R, x_{-R}, t, a, i) = \delta^t(X - \sum_{j\in N\setminus R} x_j)c_{R,a,i,k}$, and for $j \in N\setminus R$ it holds that $u_j(R, x_{-R}, t, a, i) = \delta^t x_j$. This completes the induction hypothesis and shows that the game Γ has uniquely determined subgame perfect equilibrium utilities and in any subgame perfect equilibrium, agreement is reached without delay. □

In general, Γ may have more than one subgame perfect equilibrium. Since each subgame of Γ can be reduced to a one-dimensional bargaining game, the multiplicity is inessential in exactly the same way as for one-dimensional bargaining games. When the set of players remaining on the agenda is N and a proposal is made which is unacceptable to some of the players, it does not matter which player in N rejects the proposal. The collective response to a proposal is uniquely determined, except in states (R, x_{-R}, t, a, i) with $|R| \geq 3$ and $i \in T = R \setminus \{a\}$, when player i proposes $x = X - \sum_{j \in N \setminus R} x_j - z_T^i$, so leaving players in his own coalition indifferent between accepting and rejecting x. The other case where aggregate behavior was not uniquely determined in one-dimensional bargaining, where in the state (R, x_{-R}, t, a, a) by mistake player a proposes $x = z_S^a$, does not occur in sequential share bargaining, since player a does not have any coalition members. Such a proposal would be accepted in all subgame perfect equilibria by the members of T.

Example 7.1 Consider a sequential share bargaining game Γ where $N = \{1, \ldots, n\}$. We assume that the order in which players appear on the agenda is deterministic and is given by the sequence $1, \ldots, n$. Moreover, we assume that when the set of players remaining on the agenda is $\{k, \ldots, n\}$, player k is the first player to make a proposal. Thereafter the identity of the proposer cycles clockwise within the player set: $k, \ldots, n, k$. Thus whenever the set of players remaining on the agenda is $R = \{k, \ldots, n\}$, $\rho_k(R) = 1$ and $\pi_k^0(R) = 1$.

We know that all equilibrium proposals are immediately accepted. This means that the equilibrium proposal x^k of player k, when player k is on the agenda, is also player k's equilibrium utility.

Consider the subgame $\Gamma(R_k, x_{-R_k}, 0, k, k)$ reached on the equilibrium path, where the set of players remaining on the agenda is $R_k = \{k, \ldots, n\}$. Let $n_k = n - k + 1$ be the cardinality of the set R_k and let $r_k = X - \sum_{j \in N \setminus R_k} x_j$ denote the remainder of the surplus. The reduced game $\hat{\Gamma}(R_k, x_{-R_k}, 0, k, k)$ is the one-dimensional bargaining game of Example 5.1 with $S = \{k\}$ and $T = \{k + 1, \ldots, n\}$. It follows from Example 5.1 that the equilibrium proposal of player k is

$$x^k = \frac{1 - \delta}{1 - \delta^{n_k}} r_k.$$

When player k receives his share the surplus size shrinks to

$$r_{k+1} = r_k - x^k = \frac{\delta - \delta^{n_k}}{1 - \delta^{n_k}} r_k.$$

Solving this system of recursive equations gives

$$r_k = \frac{\delta^{k-1} - \delta^n}{1 - \delta^n} X.$$

Thus the equilibrium proposal of player k, which is also the share of the surplus received by him, is

$$x^k = \frac{\delta^{k-1} - \delta^k}{1 - \delta^n} X.$$

Example 7.2 Consider the sequential share bargaining game Γ, where $N = \{1, \ldots, n\}$. As in the previous example the sequence in which the players' shares are determined is $1, \dots, n$. The proposer is chosen randomly from the set of players remaining on the agenda according to the uniform probability distribution.

Consider a subgame $\Gamma(R_k, x_{-R_k}, 0, k, i)$ reached on the equilibrium path, where the set of players remaining on the agenda is $R_k = \{k, \dots, n\}$. Let $n_k = n - k + 1$ be the cardinality of the set R_k and let $r_k = X - \sum_{j \in N \setminus R_k} x_j$ denote the remainder of the surplus. The reduced game $\hat{\Gamma}(R_k, x_{-R_k}, 0, k, i)$ is the one-dimensional bargaining game of Example 5.2 with $S = \{k\}$ and $T = \{k+1, \dots, n\}$. It follows from Example 5.2 that the equilibrium proposal of player i is

$$x^i = \begin{cases} \delta(r_k/n_k) + (1-\delta)r_k, & \text{if } i = k, \\ \delta(r_k/n_k), & \text{if } i \in \{k+1, \dots, \bar{n}\}. \end{cases}$$

Since by assumption the proposer is chosen according to the uniform probability distribution, the expected share of the surplus received by player k is

$$(1/n_k) \sum_{i \in R_k} x^i = r_k/n_k.$$

It follows that the expected size of the surplus left for the players in the set R_{k+1}, conditional on the current remainder being r_k, is given by

$$\mathbb{E}(r_{k+1} \mid r_k) = r_k - r_k/n_k = \frac{n-k}{n-k+1} r_k,$$

from which it follows that

$$\mathbb{E}(r_{k+1}) = \frac{n-k}{n-k+1} \mathbb{E}(r_k).$$

We now have a system of recursive equations which together with the initial condition $r_1 = X$ gives $\mathbb{E}(r_k) = (n_k/n)X$. The expected share of the surplus received by player k is therefore X/n. Since all equilibrium proposals are accepted, this is also the expected utility of player k. □

8 Conclusions

The existing results on n-player bargaining problems with unanimous agreement point towards a large multiplicity of subgame perfect equilibria. There are, however, several ways to extend the Rubinstein-Ståhl bargaining model for 2 players to the case with n players. This paper considers the case where the shares of the players are not determined simultaneously, but sequentially, thereby removing a potential source of multiplicity of equilibria.

The paper obtains unique subgame perfect equilibrium utilities for this bargaining protocol. In equilibrium, proposals are accepted without delay. Our results for n players are qualitatively the same as the results for the two player case. The paper also studies a related class of bargaining problems, called one-dimensional bargaining problems, and obtains a uniqueness result there as well.

Our results imply that the choice of the bargaining protocol is important in obtaining desirable bargaining outcomes. The idea of determining the players' shares sequentially is natural, and avoids the coordination problem that occurs when all shares are determined at the same time.

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References

Banks J, Duggan J (2000) A bargaining model of collective choice. Am Polit Sci Rev 94:73–88
Binmore K (1987) Perfect equilibria in bargaining models. In Binmore K, Dasgupta P (eds) The economics of bargaining. Basil Blackwell, Oxford, UK, pp 77–105
Binmore K, Osborne MJ, Rubinstein A (1992) Non-cooperative models of bargaining. In: Aumann RJ, Hart S (eds) Handbook of game theory, vol 1. North Holland, Amsterdam, pp 179–225
Cardona D, Ponsatí C (2007) Bargaining one-dimensional social choices. J Econ Theory 137:627–651
Chae S, Yang J-A (1988) The unique perfect equilibrium of an N-Person bargaining game. Econ Lett 28:221–223
Chae S, Yang J-A (1994) An N-Person pure bargaining game. J Econ Theory 62:86–102
Cho SJ, Duggan J (2003) Uniqueness of stationary equilibria in a one-dimensional model of bargaining. J Econ Theory 113:118–130
Haller H (1986) Non-cooperative bargaining of $N \geq 3$ players. Econ Lett 22:11–13
Hart S, Kurz M (1983) Endogenous formation of coalitions. Econometrica 51:1047–1064
Herings PJJ, Predtetchinski A (2010) One-dimensional bargaining with Markov recognition probabilities. J Econ Theory 145:189–215
Herrero M (1985) A strategic bargaining approach to market Institutions. PhD Thesis, London School of Economics, London
Huang C-Y (2002) Multilateral bargaining: conditional and unconditional offers. Econ Theory 20:401–412
Imai H, Salonen H (2000) The representative nash solution for two-sided bargaining problems. Math Social Sci 39:349–365
Jun BH (1987) A strategic model of 3-person bargaining. Unpublished paper. State University of New York at Stony Brook
Kalandrakis T (2004) Equilibria in sequential bargaining games as solutions to systems of equations. Econ Lett 84:407–411
Krishna V, Serrano R (1996) Multilateral bargaining. Rev Econ Stud 63:61–80
Osborne MJ, Rubinstein A (1990) Bargaining and markets. Academic Press, San Diego
Rubinstein A (1982) Perfect equilibrium in a bargaining model. Econometrica 50:97–109
Ståhl I (1972) Bargaining theory. Economics Research Institute, Stockholm School of Economics, Stockholm
Suh S-C, Wen Q (2006) Multi-agent bilateral bargaining and the Nash bargaining solution. J Math Econ 42:61–73
Yang J-A (1992) Another n-Person bargaining game with a unique perfect equilibrium. Econ Lett 38:275–277