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Non-symmetric discrete General Lotto games

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Abstract We study a non-symmetric variant of General Lotto games introduced in Hart (Int J Game Theory 36:441–460, 2008). We provide a complete characterization of optimal strategies for both players in non-symmetric discrete General Lotto games, where one of the players has an advantage over the other. By this we complete the characterization given in Hart (Int J Game Theory 36:441–460, 2008), where the strategies for symmetric case were fully characterized and some of the optimal strategies for the non-symmetric case were obtained. We find a group of completely new atomic strategies, which are used as building components for the optimal strategies. Our results are applicable to discrete variants of all-pay auctions.

Keywords General Lotto · Allocation games · All-pay auctions

JEL Classification C72 · C02 · D44

1 Introduction

General Lotto games are allocation games introduced in Hart (2008) as a technical tool for studying *Colonel Blotto games*. These are allocation games, where two (or more) players engage in a 'winner-takes-all' conflicts over several fronts, or 'battlefields', allocating their limited resources to them. Games of this kind have numerous applications in the areas such as R&D races, presidential elections, auctions, tournaments

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as well as anti-terrorism efforts or information systems security (for an overview of research on allocation games, see Kovenock and Roberson 2010).

In the case of General Lotto games, the battlefields are assumed to be indistinguishable, and, instead of deciding how many resources to assign to which battlefield, the players decide which fraction of the battlefields different amounts of the resources will be assigned to. Thus, budget constraints of the players are expressed in terms of expected values rather then in terms of total amounts of the resources. The strategies of the players are probability distributions over possible amounts of the resources. If these amounts are allowed to take any (non-negative) real value, then these probability distributions are over non-negative real numbers and the game is called *continuous*. If these amounts take integer values (e.g. because there exists some minimum unit of exchange), then the game is called *discrete*. Additionally, if all players face the same budget constraints, then the game is called *symmetric*, and it is called *non-symmetric* otherwise.

Myerson (1993) introduces this formulation of competitive resource allocation in a model of electoral competition in which each candidate makes promises involving how the budget will be distributed among the electorate and each voter votes for the candidate who promises the highest transfer. They are modelled by *offer distributions* which are probability distributions over non-negative real numbers. An offer distribution, represented by a cumulative distribution function F, specifies what fractions of the electorate are promised values from different intervals. Thus the mass of an interval $(x, x+\varepsilon)$ is the fraction of voters to whom a value from $(x, x+\varepsilon)$ is promised. Budget constraints of each candidate are expressed as constraints on the average offer per voter that a candidate can promise. Budget constraints of all the candidates are assumed to be equal and this model results in a multi-player symmetric General Lotto game.

Sahuguet and Persico (2006) extend this model to a situation where there are only two candidates but with unequal budgets constraints. This leads to a non-symmetric General Lotto game. A recent paper by Kovenock and Roberson (2009) extends this model further, to allow the parties to vary in the efficiency with which they are able to target transfers to different groups of voters.

Also related is Dekel et al. (2008) who use a sequential variant of the General Lotto game to investigate *vote buying games* where two parties, alternately, make one of the two possible offers: up-front payments or campaign promises. Two games with these two different types of offers are studied with the assumption of a minimal unit of exchange (i.e. offers can be made in multiples of a smallest money unit $\varepsilon > 0$ only). The model considered is essentially a sequential variant of General Lotto game. It is also closely related to dollar auction game of Shubik (1971).

Sahuguet and Persico (2006) connect the non-symmetric General Lotto games to complete information all-pay auctions, as studied by Baye et al. (1996). In this kind of auctions equilibria in pure strategies do not exist. The mixed strategies are probability distributions over possible bids. As shown by Sahuguet and Persico (2006), there is a correspondence between the budget constraints in the model of political competition they study and the bidders valuations in all-pay auctions, as well as between the equilibrium mixed strategies in the auctions game and the political promises game.

A symmetric continuous variant of General Lotto games was solved by Bell and Cover (1980), while Sahuguet and Persico (2006) provided the solution for the non-symmetric case of these games. The main feature of these results is that in both, the symmetric and the non-symmetric variants of the game, there exists unique Nash equilibrium. If the game is symmetric, then in equilibrium each player uses a uniform distribution over the interval $[0, 2a]$ (where a denotes the budget constraints of each player). If the game is non-symmetric, then the advantaged player sticks to the uniform distribution, while the disadvantaged one plays a mixed distribution, by giving up and playing 0 with some probability, and distributing the remaining probability uniformly over the interval $(0, 2a]$. The proof in Sahuguet and Persico (2006) uses a reduction to "all-pay-auctions", thus providing a link between the multi-object auctions and General Lotto games. A significantly easier proof based on first principles was given by Hart (2008).

A discrete variant of General Lotto games was solved by Hart (2008). Apart from providing the value of these games, Hart (2008) obtained full characterization of optimal strategies of the players in the cases where the game is symmetric or very close to symmetric. The interesting feature of the strategies found is that they show how the uniform distributions that are optimal in the continuous variant are "approximated" by the discrete distributions. The optimal strategies found are convex combinations of discrete uniform distributions over odd and even numbers within the interval $[0, 2a]$. Thus if the game is discrete, then a player may, but does not have to, use a uniform distribution over integer numbers within the interval.

If the game is non-symmetric, then Hart (2008) provides full characterization of optimal strategies only for the case where both a and b are not integers and $\lfloor a \rfloor = \lfloor b \rfloor$. In the remaining cases it was shown that one of the players always has a unique optimal strategy (it may be player A or B, depending on the values of budget constraints), and the optimal strategy was provided for all the cases. In the case of the other player, only a subset of his optimal strategies was provided. Additionally, bounds on the maximal value obtaining non-zero probability and bounds on the probability of playing zero were given for these optimal strategies.

It is interesting to note that the connection between the General Lotto games and "all-pay-auctions" extends to the discrete variant as well. The examples of optimal strategies found by Cohen and Sela (2007) for discrete "all-pay-auctions" display features similar to the optimal strategies found by Hart (2008) for discrete General Lotto games. The players use uniform distributions on odd or even numbers over the interval determined by their valuations, with disadvantaged player giving up and playing 0 with some probability.

In this paper we fill in the missing cases by providing complete characterization of the optimal strategies in discrete General Lotto games. Such a characterization is useful for the following reasons. Firstly, as we discussed above, General Lotto games are of interest on their own, due to their connection to political economics and multi-object auctions. In these applications a continuous variant was mostly used, however in most cases it should be considered as a simplification, as usually there exists a minimal unit of exchange, and agents cannot propose any real number (e.g. as their promises to electorate). Thus it is important to see how the analysis under the continuity assumption corresponds to the discrete case and if the interesting features of the results obtained are not significantly affected by this simplification. Moreover, using the optimal strategies for discrete General Lotto games one could try to solve the unsolved variants of the discrete Colonel Blotto games (see Hart 2008 for the cases solved so far).

The paper is structured as follows. In Sect. 2 we formally define the continuous General Lotto game and provide the associated results. In Sect. 3 we formally define the discrete variant of the game. In Sect. 4 we give the complete characterization of the optimal strategies for the game. In Sect. 5 we describe the connection with Colonel Blotto game and we conclude in Sect. 6. The Appendix contains a more technical part of the proofs.

2 Continuous General Lotto games

There are two players, A and B, who simultaneously choose probability distributions over non-negative real numbers. The distributions are restricted by two positive numbers $a, b > 0$, so that the expectations under the distributions proposed must be a and b for players A and B, respectively. Throughout the paper we will identify random variables with their distributions. Saying that a player proposes a random variable X, we will mean that he proposes a distribution of a random variable which we denote by X.

Let X and Y be the random variables proposed by A and B, respectively. The payoff of player A is given by

$$H(X, Y) := \mathbf{P}(X > Y) - \mathbf{P}(X < Y). \tag{1}$$

while the payoff of player B is $-H(X, Y)$. Hence the game is a zero sum game. This defines a *Continuous Colonel Lotto* game $\Lambda(a, b)$. The game is called *symmetric* if $a = b$ and it is called *non-symmetric* otherwise.

The main result, providing the solution and full characterization of Nash equilibrium, obtained by Bell and Cover (1980) and Sahuguet and Persico (2006) is as follows.

Theorem 1 *Let $a, b > 0$. The value of Continuous General Lotto game $\Lambda(a, b)$ is*

$$\operatorname{val} \Lambda(a, b) = 1 - \frac{b}{a},$$

and the unique optimal strategies are $X^* = U(0, 2a)$ for Player A and $Y^* = (1 - b/a)\mathbf{1}_0 + (b/a)U(0, 2a)$ for Player B.

$U(x, y)$ is used to denote the uniform distribution over the interval $[x, y]$, while $\mathbf{1}_x$ denotes a distribution where x is assigned probability 1. If the game is symmetric, then the unique optimal strategy for each of the players is to propose uniform distribution over the interval $[0, 2a]$. If the game is non-symmetric, then the unique optimal strategy for the advantaged player, A, is still to propose uniform distribution over the interval $[0, 2a]$. The situation for the disadvantaged player changes, however, and his unique optimal strategy is a mixed distribution, that puts probability $1 - b/a$ on 0 and distributes b/a uniformly on the interval $(0, 2a]$.[1]

3 Discrete General Lotto games

In the discrete variant of the General Lotto Games the strategies of the players are restricted, so that each of them can propose a discrete probability distribution over non-negative integer numbers. Thus the set of strategies of a player $C \in \{\mathrm{A}, \mathrm{B}\}$ is

$$S_C = \left\{ p \in [0,1]^{\mathbb{N}_{\geq 0}} : \sum_{i=0}^{+\infty} p_i = 1 \text{ and } \sum_{i=0}^{+\infty} i p_i = c \right\},$$

where $c = a$, if $C = \mathrm{A}$ and $c = b$, if $C = \mathrm{B}$. Every such strategy can be represented by $\sum_{i=0}^{+\infty} p_i \mathbf{1}_i$, where $p_i = \mathbf{P}(X = i)$. Given $a, b > 0$, we will denote the associated Discrete General Lotto game by $\Gamma(a, b)$.

In the next section we give the complete characterization of the optimal strategies in Discrete General Lotto games.

4 Solution of the discrete General Lotto game

All random variables considered from now on are non-negative and integer-valued. As we mentioned above, every random variable X is $\sum_{i=0}^{+\infty} p_i \mathbf{1}_i$, where $p_i = \mathbf{P}(X = i)$ and $\mathbf{1}_i$ denotes Dirac's measure which puts probability 1 on i. Also $\mathbf{E}(X) = \sum_{i=1}^{+\infty} i\mathbf{P}(X = i) = \sum_{i=1}^{+\infty} \mathbf{P}(X \geq i)$. Expected payoff of player A from using strategy X against strategy Y of player B is:

$$\begin{aligned} H(X,Y) &= \sum_{i=0}^{+\infty} p_i[\mathbf{P}(i > Y) - \mathbf{P}(i < Y)] = \sum_{i=0}^{+\infty} p_i\,[1 - \mathbf{P}(Y \geq i) - \mathbf{P}(Y \geq i+1)] \\ &= 1 - \sum_{i=0}^{+\infty} p_i[\mathbf{P}(Y \geq i) + \mathbf{P}(Y \geq i+1)]. \end{aligned}$$

Notice that H satisfies the following properties:

$$H(X,Y) = -H(Y,X), \tag{2}$$

$$H(\alpha X_1 + \beta X_2, Y) = \alpha H(X_1, Y) + \beta H(X_2, Y). \tag{3}$$

The following two distributions were crucial for players strategies discovered in Hart (2008):

[1] The cumulative distribution function for Y^* is then $F_{Y^*}(z) = P(Y^* \geq z) = \frac{b}{2a^2} z + 1 - \frac{b}{a}$.

$$U_{\mathrm{O}}^m := U(\{1, 3, \ldots, 2m-1\}) = \sum_{i=1}^{m} \left(\frac{1}{m}\right) \mathbf{1}_{2i-1}, \text{ and}$$
$$U_{\mathrm{E}}^m := U(\{0, 2, \ldots, 2m\}) = \sum_{i=0}^{m} \left(\frac{1}{m+1}\right) \mathbf{1}_{2i}.$$

Distributions U_{O}^m and U_{E}^m can be thought of as "uniform on odd numbers" and "uniform on even numbers", respectively. We will use

$$\mathscr{U}^m = \{U_{\mathrm{E}}^m, U_{\mathrm{O}}^m\}$$

to denote the set of these distributions. We will also use $\vec{u}_{\mathrm{O}}^m$ and $\vec{u}_{\mathrm{E}}^m$ to denote stochastic vectors representing these distributions.

As was shown in Hart (2008), for every Y it holds that

$$H(U_{\mathrm{O}}^m, Y) = 1 - \left(\frac{1}{m}\right) \sum_{i=1}^{2m} \mathbf{P}(Y \geq i) \geq 1 - \frac{\mathbf{E}(Y)}{m}, \tag{4}$$

with equality if and only if $\sum_{j=2m+1}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m$. For every Y it also holds that

$$H(U_{\mathrm{E}}^m, Y) = 1 - \left(\frac{1}{m+1}\right)\left(1 + \sum_{i=1}^{2m+1} \mathbf{P}(Y \geq i)\right) \geq 1 - \frac{\mathbf{E}(Y)+1}{m+1}, \tag{5}$$

with equality if and only if $\sum_{j=2m+2}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m+1$.

We extend this repertoire with the following distributions: W_j^m (with $1 \leq j \leq m-1$), defined for $m \geq 2$, and V_j^m (with $1 \leq j \leq m$) defined for $m \geq 1$, represented by stochastic vectors:

$$\vec{w}_j^m := \frac{1}{2m}[1, \underbrace{0, 2, \ldots, 0, 2}_{2(j-1)}, 0, 1, \underbrace{2, 0, \ldots, 2, 0}_{2(m-j)}]^T,$$
$$\vec{v}_j^m := \frac{1}{2m+1}[\underbrace{0, 2, \ldots, 0, 2}_{2(j-1)}, 0, 1, 2\, \underbrace{0, 2, \ldots, 0, 2}_{2(m-j)}]^T.$$

Distribution W_j^m could be thought of as distribution U_{O}^m distorted at the first $2j+1$ positions with a sort of 2-moving average, so that $\mathbf{P}(W_j^m = i) = (\mathbf{P}(U_{\mathrm{O}}^j = i-1) + \mathbf{P}(U_{\mathrm{O}}^j = i+1))/2$, for $0 \leq i \leq 2j$ (where $\mathbf{P}(U_{\mathrm{O}}^j = -1) = 0$). Similarly, distribution V_j^m could be thought of as distribution U_{E}^m distorted at the first $2j$ positions with a sort of 2-moving average, so that $\mathbf{P}(V_j^m = i) = (\mathbf{P}(U_{\mathrm{E}}^{j-1} = i-1) + \mathbf{P}(U_{\mathrm{E}}^{j-1} = i+1))/2$, for $0 \leq i \leq 2j-1$ (where $\mathbf{P}(U_{\mathrm{E}}^{j-1} = -1) = 0$). It could be also thought of as distribution W_j^{m+1} 'shifted to the left' by one position.

We will also use

$$\mathscr{W}^m = \{W_1^m, \dots, W_{m-1}^m\}$$

to denote the set of distributions W_j^m, as well as

$$\mathscr{V}^m = \{V_1^m, \dots, V_m^m\}$$

to denote the set of distributions V_j^m. These sets are defined for $m \geq 0$. In the case of $m < 2$, we assume that $\mathscr{W}^m = \varnothing$. Similarly, in the case of $m < 1$ we assume that $\mathscr{V}^m = \varnothing$.

Additionally, will consider the following distribution, defined for $m \geq 1$:

$$U_{\mathrm{O}\uparrow 1}^m := U(\{2, 4, \dots, 2m-2\}) = \sum_{i=1}^{m-1} \left(\frac{1}{m-1}\right) \mathbf{1}_{2i},$$

which could be thought of as uniform on even numbers from 2 to $2m-2$, or as the distribution U_{O}^{m-1} 'shifted to the right' by one position. We will also use $\vec{u}_{\mathrm{O}\uparrow 1}^m$ to denote stochastic vector associated with this distribution.

Notice that the distribution that could be thought of as W_m^m, represented by stochastic vector

$$\vec{w}_m^m := \frac{1}{2m}[1, \underbrace{0, 2, \dots, 0, 2}_{2(m-1)}, 0, 1]^T,$$

is a convex combination of distributions $U_{\mathrm{O}\uparrow 1}^m$ and U_{E}^m, with coefficients $\frac{m-1}{2m}$ and $\frac{m+1}{2m}$, respectively.

Let $p_i = \mathbf{P}(Y = i)$. For every Y it holds that

$$\begin{aligned}
H(W_j^m, Y) = 1 - \left(\frac{1}{2m}\right) \Bigg(& \mathbf{P}(Y \geq 0) + \mathbf{P}(Y \geq 1) + 2\sum_{i=1}^{j-1} [\mathbf{P}(Y \geq 2i) \\
& + \mathbf{P}(Y \geq 2i+1)] + \mathbf{P}(Y \geq 2j) + \mathbf{P}(Y \geq 2j+1) \\
& + 2\sum_{i=j+1}^{m} [\mathbf{P}(Y \geq 2i-1) + \mathbf{P}(Y \geq 2i)] \Bigg) \\
= 1 - \left(\frac{1}{2m}\right) & \left(p_0 - p_{2j} + 2\sum_{i=1}^{2m} \mathbf{P}(Y \geq i) \right) \geq 1 - \frac{\mathbf{E}(Y)}{m} + \frac{p_{2j} - p_0}{2m}
\end{aligned} \tag{6}$$

with equality if and only if $\sum_{j=2m+1}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m$.

$$\begin{aligned}
H(V_j^m, Y) = 1 - \left(\frac{1}{2m+1}\right) \Bigg(& 2\sum_{i=1}^{j-1} [\mathbf{P}(Y \geq 2i-1) + \mathbf{P}(Y \geq 2i)] \\
& + \mathbf{P}(Y \geq 2j-1) + \mathbf{P}(Y \geq 2j)
\end{aligned}$$

$$+2\sum_{i=j}^{m}[\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)]\Bigg)$$

$$= 1 - \left(\frac{1}{2m+1}\right)\left(-p_{2j-1} + 2\sum_{i=1}^{2m+1}\mathbf{P}(Y \geq i)\right)$$

$$\geq 1 - \frac{2\mathbf{E}(Y)}{2m+1} + \frac{p_{2j-1}}{2m+1} \tag{7}$$

with equality if and only if $\sum_{j=2m+2}^{+\infty}\mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m+1$.

$$H(U^m_{\mathrm{O}\uparrow 1}, Y) = 1 - \left(\frac{1}{m-1}\right)\left(\sum_{i=1}^{m-1}[\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)]\right)$$

$$= 1 - \left(\frac{1}{m-1}\right)\left(-\mathbf{P}(Y \geq 1) + \sum_{i=1}^{2m-1}\mathbf{P}(Y \geq i)\right)$$

$$= 1 - \left(\frac{1}{m-1}\right)\left(p_0 - 1 + \sum_{i=1}^{2m-1}\mathbf{P}(Y \geq i)\right) \geq 1 - \frac{\mathbf{E}(Y)-1}{m-1} - \frac{p_0}{m-1} \tag{8}$$

with equality if and only if $\sum_{j=2m}^{+\infty}\mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m-1$.

Before starting the analysis, we introduce some additional notation that will be used. Given a distribution X, a set of distributions $\mathscr{Y}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we will use

$$\lambda_1 X + \lambda_2 \mathscr{Y} = \{\lambda_1 X + \lambda_2 Y : Y \in \mathscr{Y}\}$$

to denote the set of distributions that can be obtained by linearly combining X and the distributions from $\mathscr{Y}$ with coefficients λ_1 and λ_2, respectively. Similarly, given two sets of distributions $\mathscr{X}$ and $\mathscr{Y}$ as well as $\lambda_1, \lambda_2 \in \mathbb{R}$, we will use

$$\lambda_1 \mathscr{X} + \lambda_2 \mathscr{Y} = \{\lambda_1 X + \lambda_2 Y : X \in \mathscr{X} \text{ and } Y \in \mathscr{Y}\}$$

to denote the set of distributions that can be obtained by linearly combining distributions from $\mathscr{X}$ and $\mathscr{Y}$ with coefficients λ_1 and λ_2, respectively. Given a set of distributions $\mathscr{X}$ we will use $\operatorname{conv}(\mathscr{X})$ to denote the set of all convex combinations of distributions from $\mathscr{X}$.

Hart (2008) provided full characterization of optimal strategies for both players in the cases where $a = b$ are both integers and where $\lfloor a \rfloor = \lfloor b \rfloor$ and neither a nor b are integers. The cases left incomplete in Hart (2008) are

- a is an integer and $b < a$,
- a is not an integer and $b \leq \lfloor a \rfloor$.

4.1 The case of integer a and $b < a$

We start the analysis with the first case, where a is an integer and $b < a$. The following theorem characterizing this case was shown in Hart (2008).

Theorem 2 (Hart) *Let $a > b > 0$, where a is an integer. Then the value of General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val} \Gamma(a, b) = \frac{a-b}{a} = 1 - \frac{b}{a}.$$

The optimal strategies are as follows:

(i) *Strategy U_{O}^{a} is the unique optimal strategy of Player* A.
(ii) *The strategies $(1-b/a)\mathbf{1}_0+(b/a)Z$ with $Z \in \operatorname{conv}(\mathscr{U}^a)$ are optimal strategies of Player* B.
(iii) *Every optimal strategy Y of Player* B *satisfies $Y \leq 2a$ and*

$$1 - \frac{b}{a} \leq \mathbf{P}(Y = 0) \leq 1 - \frac{b}{a+1}.$$

Thus, apart from establishing the value of the game, the theorem above provides the unique optimal strategy for the advantaged player A. It also gives examples of optimal strategies for player B and provides bounds on the probability player B puts on 0 and maximal value that can get non-zero probability. What is missing in this case is the complete characterization of the optimal strategies for the disadvantaged player B. We characterize them in two theorems covering the case where $b \leq a - 1$ first and then the case where $a - 1 < b < a$.

Theorem 3 *Let $a - 1 \geq b > 0$, where a is an integer. The strategy Y is optimal for Player* B *if and only if*

$$Y = \left(1 - \frac{b}{a}\right)\mathbf{1}_0 + \left(\frac{b}{a}\right)Z, \text{ with } Z \in \operatorname{conv}\left(\mathscr{U}^a \cup \mathscr{W}^a \cup \{U_{\mathrm{O}\uparrow 1}^{a}\}\right).$$

Proof Suppose that Y is an optimal strategy for Player B. By Theorem 2 we have $\mathbf{P}(Y = 0) \geq 1 - (a/b)$, hence Y can be written as

$$Y = \left(1 - \frac{b}{a}\right)\mathbf{1}_0 + \left(\frac{b}{a}\right)Z.$$

Since Y is optimal and, by Theorem 2, $\operatorname{val} \Gamma(a, b) = 1 - b/a$ so for any X with $\mathbf{E}(X) = a$ it must hold that

$$\begin{aligned} 1 - \frac{b}{a} \geq H(X, Y) &= \left(1 - \frac{b}{a}\right) H(X, \mathbf{1}_0) + \left(\frac{b}{a}\right) H(X, Z) \\ &= \left(1 - \frac{b}{a}\right) \mathbf{P}(X > 0) + \left(\frac{b}{a}\right) H(X, Z). \end{aligned} \tag{9}$$

Thus Z is optimal (i.e. such that Y is optimal) if and only if

$$H(Z, X) \geq -\left(\frac{a-b}{b}\right)(1 - \mathbf{P}(X > 0)) = -\left(\frac{a-b}{b}\right) p_0, \tag{10}$$

where $p_0 = \mathbf{P}(X = 0)$.

Consider distributions $T^a_{i,j} = \lambda \mathbf{1}_i + (1-\lambda)\mathbf{1}_j$ with $\mathbf{E}(T^a_{i,j}) = a$ (i.e. with $\lambda = (j-a)/(j-i)$). Take any $T^a_{i,j}$ with $0 < i \leq a \leq j$. For optimal Z, from (10), we have:

$$H(Z, T^a_{i,j}) = \lambda H(Z, \mathbf{1}_i) + (1-\lambda) H(Z, \mathbf{1}_j) \geq 0.$$

Let $w_i = H(Z, \mathbf{1}_i)$. Then we have

$$(j-a)w_i + (a-i)w_j \geq 0. \tag{11}$$

Since U^a_{O} puts strictly positive mass on all positive and odd $j \leq 2a-1$, so for any odd i and j such that $1 \leq i \leq a \leq j \leq 2a-1$ we have $U^a_{\mathrm{O}} = \tau T^a_{i,j} + (1-\tau)W$ for some $0 < \tau < 1$ and $W \geq 0$ with $\mathbf{E}(W) = a$. Since U^a_{O} is optimal and $\mathbf{P}(U^a_{\mathrm{O}} = 0) = 0$, so for optimal Z it must be that $H(Z, U^a_{\mathrm{O}}) = 0$. Thus $\tau H(Z, T^a_{i,j}) + (1-\tau)H(Z, W) = 0$ and since, by optimality of Z, $H(Z, T^a_{i,j}) \geq 0$ and $H(Z, W) \geq 0$, so $H(Z, T^a_{i,j}) = 0$. Hence for i and j odd and such that $1 \leq i \leq a \leq j \leq 2a-1$, (11) becomes equality.

Suppose that a is even. Taking $i = a-1$ from (11) we get $w_j \geq (a-j)w_{a-1}$ (with equality for positive and odd $j \leq 2a-1$). In particular, this yields $w_{a-1} = -w_{a+1}$. On the other hand, taking $j = a+1$ from (11) we get $w_i \geq (i-a)w_{a+1}$ and, further, $w_i \geq (a-i)w_{a-1}$. Hence for all $i > 0$ it holds that $w_i \geq (a-i)w_{a-1}$ (with equality for positive and odd $i \leq 2a-1$). For odd $1 \leq i \leq 2a-1$ this implies

$$w_i - w_{i+1} \leq w_{a-1}. \tag{12}$$

On the other hand, for even $2 \leq i \leq 2a-2$ this implies

$$w_i - w_{i+1} \geq w_{a-1}. \tag{13}$$

Let $q_i = \mathbf{P}(Z = i)$. Then $w_i - w_{i+1} = q_i + q_{i+1}$ and, from (12)–(13) we get $q_i + q_{i+1} \leq w_{a-1}$ (for all odd $1 \leq i \leq 2a-1$) and $q_i + q_{i+1} \geq w_{a-1}$ (for all even $2 \leq i \leq 2a-2$). Hence for all odd $1 \leq i \leq 2a-3$ we have $q_i + q_{i+1} \leq q_{i+1} + q_{i+2}$ and for all even $2 \leq i \leq 2a-2$ we have $q_i + q_{i+1} \geq q_{i+1} + q_{i+2}$. Thus there exist $d_i \geq 0$ (with $1 \leq i \leq 2a-2$) such that

$$q_i - q_{i+2} + d_i = 0, \text{ for odd } 1 \leq i \leq 2a-2 \tag{14}$$

$$-q_i + q_{i+2} + d_i = 0, \text{ for even } 1 \leq i \leq 2a-2. \tag{15}$$

In the case of odd $1 \leq i \leq 2a-1$, (11) becomes equality and it yields $w_i = (a-i)w_{a-1}$. Thus $w_i - w_{i+2} = 2w_{a-1}$ (for odd $1 \leq i \leq 2a-3$) and so $w_i - w_{i+2} =$

$w_{i+2} - w_{i+4}$ (for odd $1 \le i \le 2a-5$). Since $w_i - w_{i+2} = q_i + 2q_{i+1} + q_{i+2}$, so this implies

$$q_i + 2q_{i+1} - 2q_{i+3} - q_{i+4} = 0, \text{ for odd } 1 \le i \le 2a-5. \tag{16}$$

Moreover, since $w_{2a-1} - w_{2a+1} \le 2w_{a-1}$ (as $w_{2a-1} = -(a-1)w_{a-1}$ and $w_{2a+1} \ge -(a+1)w_{a-1}$), so in the case of $i = 2a-3$ we have inequality $w_{2a-3} - w_{2a-1} \ge w_{2a-1} - w_{2a+1}$. Thus there exist $d_{2a-1} \ge 0$ such that

$$q_{2a-3} + 2q_{2a-2} - 2q_{2a} - d_{2a-1} = 0 \tag{17}$$

(recall that, by Theorem 2, $q_{2a+1} = 0$). Equations 14–17 can be obtained for odd a as well, taking $i = a-2$, $j = a-2$ and noticing that $w_{a-2} = w_{a+2}$.

Observe also that since $\sum_{i=0}^{+\infty} q_i = 1$ and $\sum_{i=0}^{+\infty} iq_i = a$, so $\sum_{i=0}^{+\infty}(i-a)q_i = 0$. Since $Z \le 2a$, so in this case

$$\sum_{i=0}^{2a}(i-a)q_i = 0. \tag{18}$$

Lemma 1 *The set of solutions of the system of Eqs. 14–18 with additional constraints:*

$$0 \le q_i \le 1, \textit{ for all } 0 \le i \le 2a, \tag{19}$$
$$d_i \ge 0, \textit{ for all } 1 \le i \le 2a-1, \tag{20}$$
$$q_0 + \cdots + q_{2a} = 1, \tag{21}$$

is $\mathrm{conv}\left(\left\{\begin{bmatrix}\vec{z}_{-1}\\ \vec{d}_{-1}\end{bmatrix}, \ldots, \begin{bmatrix}\vec{z}_a\\ \vec{d}_a\end{bmatrix}\right\}\right)$, *where*

$$\vec{z}_{-1} = \vec{u}^a_{\mathrm{O}}, \quad \vec{z}_0 = \vec{u}^a_{\mathrm{E}}, \quad \vec{z}_i = \vec{w}^a_i, \textit{ for } 1 \le i \le a-2,$$
$$\vec{z}_{a-1} = \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1}, \quad \vec{z}_a = \vec{u}^a_{\mathrm{O}\uparrow 1},$$

and $\vec{d}_i$, $-1 \le i \le a$, *satisfy Constraints* (20).

(Proof of Lemma 1 is moved to the Appendix).

If Z is optimal, then it must satisfy Eqs. 14–18 with Constraints (19)–(21). Hence, by Lemma 1, it must be that

$$Z = \lambda_{\mathrm{O}} U^a_{\mathrm{O}} + \lambda_{\mathrm{E}} U^a_{\mathrm{E}} + \sum_{j=1}^{a-2} \lambda_j W^a_j + \lambda_{a-1}\left(\frac{2a}{a+1} W^a_{a-1} - \frac{a-1}{a+1} U^a_{\mathrm{O}\uparrow 1}\right) + \lambda_{\mathrm{O}\uparrow 1} U^a_{\mathrm{O}\uparrow 1}, \tag{22}$$

with $\lambda_{\mathrm{O}}+\lambda_{\mathrm{E}}+\sum_{j=1}^{a-1}\lambda_j+\lambda_{\mathrm{O}\uparrow 1}=1$ and $\lambda_{\mathrm{O}}, \lambda_{\mathrm{E}}, \lambda_{\mathrm{O}\uparrow 1}, \lambda_i \geq 0$, for all $1 \leq i \leq a-1$. Consider any distribution $T^a_{i,2a}$ with odd $1 \leq i < a$. Then, by (3), (4)–(6)

$$\begin{aligned} H(Z, T^a_{i,2a}) =& \lambda_{\mathrm{O}}\left(1-\frac{\mathbf{E}(T^a_{i,2a})}{a}\right)+\lambda_{\mathrm{E}}\left(1-\frac{\mathbf{E}(T^a_{i,2a})+1}{a+1}\right) \\ &+\sum_{j=1}^{a-2}\lambda_j\left(1-\frac{\mathbf{E}(T^a_{i,2a})}{a}-\frac{p_{2j}-p_0}{2a}\right) \\ &+\lambda_{a-1}\left(\left(1-\frac{\mathbf{E}(T^a_{i,2a})}{a}-\frac{p_{2a-2}-p_0}{2a}\right)-\frac{a-1}{a+1}H(U^a_{\mathrm{O}\uparrow 1}, T^a_{i,2a})\right) \\ &+\lambda_{\mathrm{O}\uparrow 1}H(U^a_{\mathrm{O}\uparrow 1}, T^a_{i,2a}), \end{aligned}$$

where $p_k = \mathbf{P}(T^a_{i,2a} = k)$. Since $\mathbf{E}(T^a_{i,2a}) = a$ and $p_{2j} = 0$, for $0 \leq j \leq a-1$ (as $1 \leq i < a$ is odd), so this reduces to

$$H(Z, T^a_{i,2a}) = -\lambda_{a-1}\left(\frac{a-1}{a+1}\right)H(U^a_{\mathrm{O}\uparrow 1}, T^a_{i,2a})+\lambda_{\mathrm{O}\uparrow 1}H(U^a_{\mathrm{O}\uparrow 1}, T^a_{i,2a}).$$

By (8),

$$\begin{aligned} H(U^a_{\mathrm{O}\uparrow 1}, T^a_{i,2a}) &= 1-\left(\frac{1}{a-1}\right)\left(p_0-1+\sum_{j=1}^{2a-1}\mathbf{P}(T^a_{i,2a}\geq j)\right) \\ &= 1-\left(\frac{1}{a-1}\right)\left(p_0-1-p_{2a}+\sum_{j=1}^{2a}\mathbf{P}(T^a_{i,2a}\geq j)\right) \\ &= 1-\left(\frac{1}{a-1}\right)\left(\mathbf{E}(T^a_{i,2a})-1+p_0-p_{2a}\right) = \frac{p_{2a}-p_0}{a-1} = \frac{p_{2a}}{a-1}. \end{aligned}$$

(notice that $p_{2a} > 0$). Inserting this into the equation above we get

$$H(Z, T^a_{i,2a}) = -\left(\frac{\lambda_{a-1}}{a+1}-\frac{\lambda_{\mathrm{O}\uparrow 1}}{a-1}\right)p_{2a}. \tag{23}$$

On the other hand, by (10), it must be that $H(Z, T^a_{i,2a}) \geq 0$. Thus it must be that $\lambda_{\mathrm{O}\uparrow 1} \geq \frac{a-1}{a+1}\lambda_{a-1}$. Hence any optimal Z can be represented as

$$Z = \lambda_{\mathrm{O}}U^a_{\mathrm{O}}+\lambda_{\mathrm{E}}U^a_{\mathrm{E}}+\sum_{i=1}^{a-1}\lambda_i W^a_i+\lambda'_{\mathrm{O}\uparrow 1}U^a_{\mathrm{O}\uparrow 1},$$

where $\lambda'_{\mathrm{O}\uparrow 1} = \lambda_{\mathrm{O}\uparrow 1}-\frac{a-1}{a+1}\lambda_{a-1} \geq 0$ and $\lambda_{\mathrm{O}}+\lambda_{\mathrm{E}}+\sum_{i=1}^{a-1}\lambda_i+\lambda'_{\mathrm{O}\uparrow 1} = 1$. Therefore $Z \in \mathrm{conv}\left(\mathscr{U}^a \cup \mathscr{W}^a \cup \{U^a_{\mathrm{O}\uparrow 1}\}\right)$.

On the other hand it can be easily checked that for any $Z \in \mathscr{U}^a \cup \mathscr{W}^a \cup \{U^a_{O\uparrow 1}\}$, $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, if $b \leq a-1$. Hence $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, for any $Z \in \operatorname{conv}\left(\mathscr{U}^a \cup \mathscr{W}^a \cup \{U^a_{O\uparrow 1}\}\right)$. Thus Z satisfies (10), which implies that Z is optimal (i.e. such that Y is optimal). □

In the case of $b < a$ with integer a and b close to a the structure of optimal strategies for B is like in the case of $b \leq a-1$, but not every Z from Theorem 3 leads to an optimal strategy. Theorem 4 below characterize completely all the Z that do, thus providing complete characterization of optimal strategies for B in this case as well.

Theorem 4 *Let $a = m$ and $b = m - \beta$, where $m \geq 1$ is an integer and $0 < \beta < 1$. The strategy Y is optimal for Player* B *if and only if*

$$Y = \left(\frac{\beta}{m}\right) \mathbf{1}_0 + \left(1 - \frac{\beta}{m}\right) Z, \text{ with } Z \in \operatorname{conv}\left(\mathscr{U}^m \cup \mathscr{Y}^{m,\beta}\right), \text{ where}$$

– $\mathscr{Y}^{m,\beta} = \varnothing$, *if* $m = 1$,
– $\mathscr{Y}^{m,\beta} = (\beta\sigma \mathscr{W}^m + (1-\beta\sigma)\, \mathscr{U}^m) \cup \left(\beta\delta U^m_{O\uparrow 1} + (1-\beta\delta)\, \mathscr{U}^m\right)$, *if* $m \geq 2$ *and* $0 < \beta \leq \frac{m}{2m+1}$,
– $\mathscr{Y}^{m,\beta} = \mathscr{W}^m \cup \left(\beta\delta U^m_{O\uparrow 1} + (1-\beta\delta)\, \mathscr{U}^m\right) \cup \left((1-(1-\beta)\sigma\rho)\, U^m_{O\uparrow 1} + (1-\beta)\sigma\rho \mathscr{W}^m\right)$, *if* $m \geq 2$ *and* $\frac{m}{2m+1} < \beta < 1$, *where*

$$\delta = \frac{m-1}{m-\beta}, \quad \sigma = \frac{2m}{m-\beta}, \quad \rho = \frac{m}{m+1}.$$

Proof It is easy to check that for any $Z \in \mathscr{U}^m \cup \mathscr{Y}^{m,\beta}$ and any X with $\mathbf{E}(X) = m$, $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, in the two cases given above. Hence $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, for any $\vec{z} \in \operatorname{conv}\left(\mathscr{U}^m \cup \mathscr{Y}^{m,\beta}\right)$. Thus Z satisfies Ineq. 10, which, as we observed in proof of Theorem 3, means that Z is optimal (i.e. such that Y is optimal).

What remains to be shown is the left to right implication, i.e. that if Z is optimal, then $Z \in \operatorname{conv}\left(\mathscr{U}^m \cup \mathscr{Y}^{m,\beta}\right)$. Consider the case with $m = 1$ first. By Theorem 2, $Z \leq 2$ in this case and we need to find the values of q_0, q_1 and q_2, where $q_i = \mathbf{P}(Z = i)$. From $q_0 + q_1 + q_2 = 1$ and $q_1 + 2q_2 = 1$ (as $\mathbf{E}(Z) = 2$), we get $q_0 = q_2$. Hence any Z must be a convex combination of $\mathscr{U}^1$, which completes the proof of this case.

Suppose now that $m \geq 2$. As was already shown in proof of Theorem 3, if Z is optimal, then it must be that $Z \in \operatorname{conv}\left(\mathscr{U}^m \cup \mathscr{W}^m \cup \{U^m_{O\uparrow 1}\}\right)$. Thus any optimal Z can be represented as

$$Z = \lambda_{\mathscr{U}} U^m + \lambda_{\mathscr{W}} W^m + \lambda_{O\uparrow 1} U^m_{O\uparrow 1}, \tag{24}$$

where $U^m \in \operatorname{conv}(\mathscr{U}^m)$, $W^m \in \operatorname{conv}(\mathscr{W}^m)$, $\lambda_{\mathscr{U}} + \lambda_{\mathscr{W}} + \lambda_{O\uparrow 1} = 1$ and $0 \leq \lambda_{\mathscr{U}}, \lambda_{\mathscr{W}}, \lambda_{O\uparrow 1} \leq 1$. Consider a strategy $T^m_{0,j}$ (as defined in proof of Theorem 3), with odd j such that $m + 1 \leq j \leq 2m - 1$. Then, by (4)–(6) and (8):

$$H(Z, T^m_{0,j}) = -p_0\left(\frac{\lambda_{\mathscr{W}}}{2m}\right) - p_0\left(\frac{\lambda_{O\uparrow 1}}{m-1}\right). \tag{25}$$

Since Z is optimal, so it must satisfy (10) for any X with $\mathbf{E}(X) = 1$. This, together with (25), implies

$$\lambda_{\mathscr{W}}\frac{\delta}{\sigma} + \lambda_{O\uparrow 1} \leq \beta\delta. \tag{26}$$

where $\delta = \frac{m-1}{m-\beta}$ and $\sigma = \frac{2m}{m-\beta}$.

Suppose that $0 < \beta \leq \frac{m}{2m+1}$ (in which case $0 \leq \beta\sigma \leq 1$). Inequality 26 implies that $\lambda_{\mathscr{W}} \leq \beta\sigma$ and $\lambda_{O\uparrow 1} \leq \beta\delta$. Hence $\lambda_{\mathscr{W}}$ and $\lambda_{O\uparrow 1}$ can be represented as $\alpha_1\beta\sigma$ and $\alpha_2\beta\delta$, respectively, with $0 \leq \alpha_1, \alpha_2 \leq 1$. From this and from (26) we also get $\alpha_1 + \alpha_2 \leq 1$. Now, Eq. 24 can be rewritten as:

$$Z = \alpha_3 U^m + \alpha_1((1-\beta\sigma)U^m + \beta\sigma W^m) + \alpha_2((1-\beta\delta)U^m + \beta\delta U^m_{O\uparrow 1}),$$

where $\alpha_3 = \lambda_{\mathscr{U}} - \alpha_1(1-\beta\sigma) - \alpha_2(1-\beta\delta) = \lambda_{\mathscr{U}} + \lambda_{\mathscr{W}} + \lambda_{O\uparrow 1} - (\alpha_1+\alpha_2) = 1 - (\alpha_1+\alpha_2)$. This shows that any optimal Z can be represented as a convex combination of vectors in $\mathscr{U}^m \cup (\beta\sigma\mathscr{W}^m + (1-\beta\sigma)\,\mathscr{U}^m) \cup \left(\beta\delta U^m_{O\uparrow 1} + (1-\beta\delta)\,\mathscr{U}^m\right)$.

Suppose that $\frac{m}{2m+1} < \beta < 1$ (in which case $0 < (1-\beta)\sigma\rho < 1$). By (26), $\lambda_{O\uparrow 1} \leq \beta\delta - \lambda_{\mathscr{W}}\frac{\delta}{\sigma}$. Hence $\lambda_{O\uparrow 1}$ can be represented as $\alpha\left(\beta\delta - \lambda_{\mathscr{W}}\frac{\delta}{\sigma}\right)$, where $0 \leq \alpha \leq 1$. We rewrite Eq. 24 as

$$\begin{aligned} Z = {} & \alpha_1 U^m + \alpha_2 W^m + \alpha_3\left((1-\beta\delta)\,U^m + \beta\delta U^m_{O\uparrow 1}\right) \\ & + \alpha_4\left((1-\beta)\,\sigma\rho W^m + (1-(1-\beta)\,\sigma\rho)\,U^m_{O\uparrow 1}\right) \end{aligned}$$

$\alpha_1 = \lambda_{\mathscr{U}} - \alpha_3(1-\beta\delta), \alpha_2 = \lambda_{\mathscr{W}} - \alpha_4(1-\beta)\sigma\rho = \lambda_{\mathscr{W}}(1-\alpha), \alpha_3 = \alpha\frac{1-\beta\delta-\lambda_{\mathscr{W}}(\sigma-\delta)/\sigma}{1-\beta\delta}$ and $\alpha_4 = \lambda_{\mathscr{W}}\alpha\frac{\delta(1-\beta)(\beta\sigma-1)}{(1-\beta\delta)(1-(1-\beta)\sigma\rho)} = \lambda_{\mathscr{W}}\alpha\frac{(m-\beta)(m+1)}{2m^2(1-\beta)}$. It is easy to check that $\alpha_3\beta\delta + \alpha_4(1-(1-\beta)\sigma\rho) = \lambda_{O\uparrow 1}$ and, consequently, that $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$. It is also easy to see that $\alpha_2, \alpha_4 \geq 0$. By the fact that $\beta > \frac{m}{2m+1}$ we also have $\alpha_3 \geq 0$. For α_1 notice that adding $\frac{\sigma-\delta}{\sigma}\lambda_{\mathscr{W}}$ to both sides of (26) and using the fact that $\lambda_{\mathscr{W}} + \lambda_{O\uparrow 1} = 1 - \lambda_{\mathscr{U}}$, from (26) we get

$$\lambda_{\mathscr{U}} \geq 1 - \beta\delta - \lambda_{\mathscr{W}}\frac{\sigma-\delta}{\sigma}. \tag{27}$$

From this it follows that $\alpha_1 \geq 0$. This shows that if $\frac{m}{2m+1} < \beta < 1$, then any optimal Z can be represented as a convex combination of vectors in

$$\mathscr{U}^m \cup \mathscr{W}^m \cup \left(\beta\delta U^m_{O\uparrow 1} + (1-\beta\delta)\,\mathscr{U}^m\right) \cup \left((1-(1-\beta)\sigma\rho)\,U^m_{O\uparrow 1} + (1-\beta)\sigma\rho\mathscr{W}^m\right).$$

$\square$

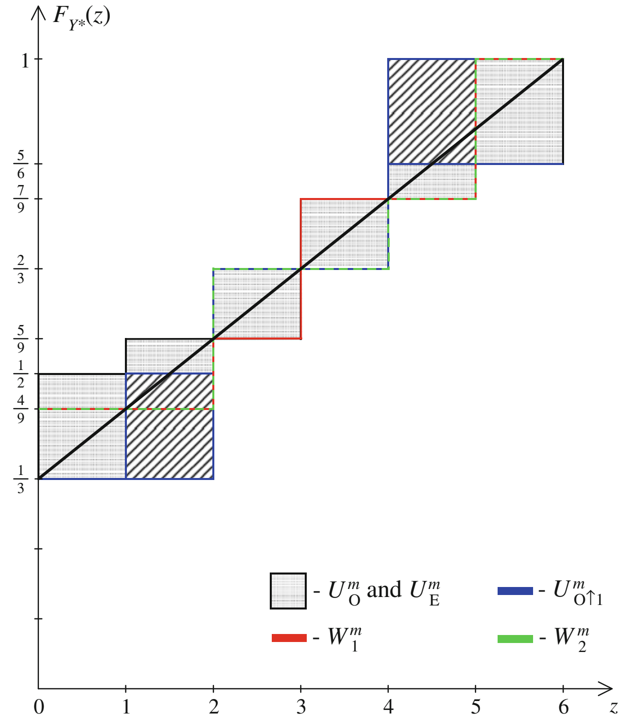

Fig. 1 Cumulative distribution functions for (extreme) optimal strategies of player B in discrete General Lotto game $\Gamma(3, 2)$

Let us illustrate the results above with two examples. First, consider a game $\Gamma(4, 1)$. Then the value of the game is $3/4$ and the strategy $Y^0 = (25/32)\mathbf{1}_0 + (1/16)\mathbf{1}_2 + (1/32)\mathbf{1}_4 + (1/16)\mathbf{1}_5 + (1/16)\mathbf{1}_7$, given as an example in Hart (2008) of one of the strategies not captured by Theorem 2, is $(3/4)\mathbf{1}_0 + (1/4)W_2^4$.

Second, consider a game $\Gamma(3, 2)$. The value of the game is $1/3$ in this case. Any optimal strategy of player B is in this case a convex hull of the strategies $\frac{1}{3}\mathbf{1}_0 + \frac{2}{3}\{U^3_{\mathrm{O}}, U^3_{\mathrm{E}}, W^3_1, W^3_2, U^3_{\mathrm{O}\uparrow 1}\}$, where

$$U^3_{\mathrm{O}} = \frac{1}{3}\mathbf{1}_1 + \frac{1}{3}\mathbf{1}_3 + \frac{1}{3}\mathbf{1}_5, \qquad U^3_{\mathrm{E}} = \frac{1}{4}\mathbf{1}_0 + \frac{1}{4}\mathbf{1}_2 + \frac{1}{4}\mathbf{1}_4 + \frac{1}{4}\mathbf{1}_6$$

$$W^3_1 = \frac{1}{6}\mathbf{1}_0 + \frac{1}{6}\mathbf{1}_2 + \frac{1}{3}\mathbf{1}_3 + \frac{1}{3}\mathbf{1}_5, \qquad W^3_2 = \frac{1}{6}\mathbf{1}_0 + \frac{1}{3}\mathbf{1}_2 + \frac{1}{6}\mathbf{1}_4 + \frac{1}{3}\mathbf{1}_5$$

$$U^3_{\mathrm{O}\uparrow 1} = \frac{1}{2}\mathbf{1}_2 + \frac{1}{2}\mathbf{1}_4.$$

Cumulative distribution functions for these strategies are presented in Fig. 1. This figure allows us to examine how the new strategies add to the bounds of the region where the (known) optimal strategies lie. The grey region represents the area identified by Hart (2008), where the strategies being convex combinations of U^3_{O} and U^3_{E} lie.

The extension in terms of bounds comes from the strategy $U^3_{\mathrm{O}\uparrow 1}$. Of course not every strategy lying within the region is an optimal strategy for player B and even though the strategies W^3_1 and W^3_2 do not extend the bounds, they allow for obtaining new strategies which would not be obtainable by combining U^3_{O}, U^3_{E} and $U^3_{\mathrm{O}\uparrow 1}$ only.

It is also interesting to compare the optimal strategies of the disadvantaged player in the continuous and discrete variants of General Lotto games. The cumulative distribution function for the continuous variant is the black line in Fig. 1, depicting the function $\frac{1}{3} + \frac{z}{9}$. Any optimal strategy in the discrete variant can be represented as $\frac{1}{3}\mathbf{1}_0 + \frac{2}{3}\mathrm{conv}\Big(\{U^3_{\mathrm{O}}, U^3_{\mathrm{E}}, W^3_1, W^3_2, U^3_{\mathrm{O}\uparrow 1}\}\Big)$, and the second part of this expression illustrates how the optimal strategies in the discrete variant approximate the uniform distribution in the continuous variant. As could be already concluded from the result obtained in Hart (2008), an optimal strategy of the disadvantaged player in the discrete variant may, but does not have to, be a uniform discrete distribution over the set $\{1, \dots, 6\}$. The full characterization given in Theorem 3 allows us to see that such an optimal distribution may be even further away from the uniform distribution (as for example the $U^3_{\mathrm{O}\uparrow 1}$ extreme) and does not even have to be a combination of uniform distributions on any subset of $\{1, \dots, 6\}$ (as it involves W^3_1 and W^3_2).

4.2 The case of non-integer a and $b \leq \lfloor a \rfloor$

Now we move to the case of $b \leq \lfloor a \rfloor$. The following theorem characterizing this case was shown in Hart (2008).

Theorem 5 (Hart) *Let $a = m + \alpha$ and $b \leq m$, where $m \geq 1$ is an integer and $0 < \alpha < 1$. Then the value of General Lotto game $\Gamma(a, b)$ is*

$$\mathrm{val}\,\Gamma(a, b) = (1-\alpha)\frac{\lfloor a \rfloor - b}{\lfloor a \rfloor} + \alpha\frac{\lceil a \rceil - b}{\lceil a \rceil} = 1 - \frac{(1-\alpha)b}{m} - \frac{\alpha b}{m+1}.$$

The optimal strategies are as follows:

(i) *Strategy $Y^* = (1 - b/m)\mathbf{1}_0 + (b/m)U^m_{\mathrm{E}}$ is the unique optimal strategy of Player* B.

(ii) *The strategy $X^* = (1-\alpha)U^m_{\mathrm{O}} + \alpha U^{m+1}_{\mathrm{O}}$ is an optimal strategy of Player* A *and, when $b = m$, so are $(1-\alpha)V + \alpha U^{m+1}_{\mathrm{O}}$ for all $v \in \mathrm{conv}(\mathscr{U}^m)$.*

(iii) *Every optimal strategy X of Player* A *satisfies $Y \leq 2m+1$; moreover, it also satisfies $X \geq 1$, when $b < m$, and*

$$\mathbf{P}(X = 0) \leq \frac{1-\alpha}{m+1},$$

when $b = m$.

Thus apart from providing the value of the game, this theorem gives the unique optimal strategy for the disadvantaged player B. It also gives examples of optimal strategies for player A and provides bounds on the probability player A puts on 0 and on the maximal value that can obtain non-zero probability. What is missing is the complete characterization of optimal strategies for the advantaged player A. We give this characterization in the theorem below.

Theorem 6 *Let $a = m + \alpha$ and $b \leq m$, where $m \geq 1$ is an integer and $0 < \alpha < 1$. The strategy X is optimal for Player* A *if and only if*

$$X \in \operatorname{conv}\left(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha}\right), \textit{ where}$$

- $\mathscr{U}^{m,\alpha} = (1-\alpha)\mathscr{U}^m + \alpha U_{\mathrm{O}}^{m+1}$, *if* $b = m$,
- $\mathscr{U}^{m,\alpha} = \left\{(1-\alpha)U_{\mathrm{O}}^m + \alpha U_{\mathrm{O}}^{m+1}\right\}$, *if* $b < m$

and

- $\mathscr{X}^{m,\alpha} = \alpha\delta\mathscr{V}^m + (1-\alpha\delta)\,\mathscr{U}^m$, *if* $0 < \alpha \leq \frac{m+1}{2m+1}$ *and* $b = m$,
- $\mathscr{X}^{m,\alpha} = \alpha\delta\mathscr{V}^m + (1-\alpha\delta)\,U_{\mathrm{O}}^m$, *if* $0 < \alpha \leq \frac{m+1}{2m+1}$ *and* $b < m$,
- $\mathscr{X}^{m,\alpha} = (1-\alpha)\sigma\mathscr{V}^m + (1-(1-\alpha)\sigma)\,U_{\mathrm{O}}^{m+1}$, *if* $\frac{m+1}{2m+1} < \alpha < 1$, *where*

$$\delta = \frac{2m+1}{m+1}, \quad \sigma = \frac{2m+1}{m}.$$

Proof Suppose that X is an optimal strategy for player A. Consider any strategy Y of player B of the form $(1 - b/m)\,\mathbf{1}_0 + (b/m)Z$, where $\mathbf{E}(Z) = m$. Then

$$\begin{aligned} H(X, Y) &= \left(1 - \frac{b}{m}\right) H(X, \mathbf{1}_0) + \left(\frac{b}{m}\right) H(X, Z) \\ &= \left(1 - \frac{b}{m}\right)(1 - p_0) + \left(\frac{b}{m}\right) H(X, Z), \end{aligned} \tag{28}$$

where $p_0 = \mathbf{P}(X = 0)$. Since $\mathbf{E}(Y) = b$ so, by Theorem 5 and Eq. 28,

$$H(X, Z) \geq \frac{\alpha}{m+1} + p_0\left(\frac{m-b}{b}\right), \tag{29}$$

for any Z with $\mathbf{E}(Z) = m$. Since, by Theorem 5, any optimal X satisfies $\mathbf{P}(X = 0) = 0$ if $b < m$, so (29) can be replaced with

$$H(X, Z) \geq \frac{\alpha}{m+1}, \tag{30}$$

Let $T_{i,j}^m$, with $0 < i \leq m \leq j$ be defined like in proof of Theorem 3. By Eq. 30 for any optimal X we have

$$H(Z, T_{i,j}^m) = \lambda H(X, \mathbf{1}_i) + (1-\lambda) H(X, \mathbf{1}_j) \geq \frac{\alpha}{m+1}.$$

Like in proof of Theorem 3 we take $w_i = H(Z, \mathbf{1}_i)$ to obtain

$$(j-m)w_i + (m-i)w_j \geq \frac{\alpha(j-i)}{m+1}. \tag{31}$$

Since the strategy $(1 - b/m)\,\mathbf{1}_0 + (b/m)U_{\mathrm{E}}^m$ is optimal for player B, so for any optimal X we have equality in (30) for $Z = U_{\mathrm{E}}^m$, as well as for $Z = T_{i,j}^m$, with even $0 \leq i \leq m \leq j \leq 2m$ (c.f. proof of Theorem 3 for similar analysis and arguments used there). Hence for i and j even and such that $0 \leq i \leq m \leq j \leq 2m$, (31) becomes equality.

Suppose that m is odd. Taking $i = m-1$ from (31) we get

$$w_j \geq -(j-m)w_{m-1} + \frac{\alpha(j-m+1)}{m+1} \tag{32}$$

(with equality for even $m \leq j \leq 2m$). Similarly, taking $j = m+1$ we get

$$w_i \geq -(m-i)w_{m+1} + \frac{\alpha(m+1-i)}{m+1} \tag{33}$$

(with equality for even $0 \leq i \leq m$). Since $m-1$ and $m+1$ are even so, from (32) we get

$$w_{m+1} = -w_{m-1} + \frac{2\alpha}{m+1}. \tag{34}$$

From this and from (33) we find out that (32) holds for all $j \geq 0$, with equality for all even $0 \leq j \leq 2m$. For even $0 \leq j \leq 2m$ this implies

$$w_j - w_{j+1} \leq w_{m-1} - \frac{\alpha}{m+1}. \tag{35}$$

On the other hand, for odd $1 \leq j \leq 2m-1$ this implies

$$w_j - w_{j+1} \geq w_{m-1} - \frac{\alpha}{m+1}. \tag{36}$$

Let $p_j = \mathbf{P}(X = j)$. Then $w_j - w_{j+1} = p_j + p_{j+1}$ and, from (35)–(36) we get $p_j + p_{j+1} \leq p_{j+1} + p_{j+2}$ (for all even $0 \leq j \leq 2m-2$). and $p_j + p_{j+1} \geq p_{j+1} + p_{j+2}$ (for all odd $1 \leq j \leq 2m-1$). Thus there exist $d_j \geq 0$ (with $0 \leq j \leq 2m-1$) such that

$$p_j - p_{j+2} + d_j = 0, \text{ for even } 0 \leq j \leq 2m-2 \tag{37}$$

$$-p_j + p_{j+2} + d_j = 0, \text{ for odd } 1 \leq j \leq 2m-1. \tag{38}$$

In the case of even $0 \leq j \leq 2m-2$, (32) becomes equality and it yields

$$w_j - w_{j+2} = 2w_{m-1} - \frac{2\alpha}{m+1},$$

for all even $0 \leq j \leq 2m-2$. Thus $w_j - w_{j+2} = w_{j+2} - w_{j+4}$ (for all even $1 \leq j \leq 2m-4$) and, since $w_i - w_{i+2} = p_i + 2p_{i+1} + p_{i+2}$, so this implies

$$p_j + 2p_{j+1} - 2p_{j+3} - p_{j+4} = 0, \text{ for even } 0 \leq j \leq 2m-4. \tag{39}$$

Moreover, in the case of $j = 2m-2$ we have inequality $w_{2m-2} - w_{2m} \geq w_{2m} - w_{2m+2}$. Thus there exist $d_{2m} \geq 0$ such that

$$p_{2m-2} + 2p_{2m-1} - 2p_{2m+1} - d_{2m} = 0 \tag{40}$$

(recall that, by Theorem 5, $p_{2m+2} = 0$). Equations 37–40 can be obtained for even m as well, taking $i = m-2$ and $j = m+2$.

Observe also that since $\sum_{i=0}^{+\infty} p_i = 1$ and $\sum_{i=0}^{+\infty} i p_i = m$, so $\sum_{i=0}^{+\infty} (i-m)p_i = 0$. Since $Z \leq 2m+1$, so in this case

$$\sum_{i=0}^{2m+1} (i-m)p_i = 0. \tag{41}$$

Lemma 2 *The set of solutions of the system of Eqs. 37–41 with additional constraints:*

$$0 \leq p_i \leq 1, \ \textit{for all } 0 \leq i \leq 2m+1, \tag{42}$$

$$d_i \geq 0, \ \textit{for all } 0 \leq i \leq 2m, \tag{43}$$

$$p_0 + \cdots + p_{2m+1} = 1 \tag{44}$$

is $\sum_{i=0}^{m+1} \lambda_i \begin{bmatrix} \vec{z}_i \\ \vec{d}_i \end{bmatrix}$ *with* $\sum_{i=0}^{m+1} \lambda_i = 1$,

$$\vec{z}_{m+1} = (1-\alpha)\vec{u}_{\mathrm{O}}^{m} + \alpha \vec{u}_{\mathrm{O}}^{m+1},$$

and

– *in the case of* $0 < \alpha \leq \frac{m+1}{2m+1}$: $\lambda_i \geq 0$, *for all* $0 \leq i \leq m$ *and* $i = m+1$, $\lambda_0 + \lambda_m \geq 0$, *and*

$$\vec{z}_0 = (1-\alpha)\vec{u}_{\mathrm{E}}^{m} + \alpha\delta \vec{v}_m^{m} + \alpha(1-\delta)\vec{u}_{\mathrm{O}}^{m}, \quad \vec{z}_i = \alpha\delta \vec{v}_i^{m} + (1-\alpha\delta)\vec{u}_{\mathrm{O}}^{m},$$

where $1 \leq i \leq m$, $\delta = \frac{2m+1}{m+1}$ *and* $\vec{d}_j$, *with* $0 \leq j \leq m+1$, *satisfy Constraints* (43);

– *in the case of* $\frac{m+1}{2m+1} < \alpha < 1$: $\lambda_i \geq 0$, *for all* $0 \leq i \leq m$, *and*

$$\vec{z}_0 = (1-\alpha)\vec{u}_{\mathrm{E}}^{m} + \alpha \vec{u}_{\mathrm{O}}^{m+1}, \quad \vec{z}_i = (1-\alpha)\sigma \vec{v}_i^{m} + (1-(1-\alpha)\sigma)\vec{u}_{\mathrm{O}}^{m+1},$$

where $1 \leq i \leq m$, $\sigma = \frac{2m+1}{m}$ *and* $\vec{d}_j$, *with* $0 \leq j \leq m+1$, *satisfy Constraints* (43).

(Proof of Lemma 2 is moved to the Appendix).

Let $\vec{x}$ be a stochastic vector representing X. If X is optimal, then it must satisfy Eqs. 37–41 with Constraints (42)–(44). Hence, by Lemma 2, it must be that

$$\vec{x} = \sum_{i=0}^{m+1} \lambda_i \vec{z}_i, \tag{45}$$

with $\sum_{i=0}^{m+1} \lambda_i = 1$ and additional properties depending on the value of α.

Suppose first that $0 < \alpha \leq \frac{m+1}{2m+1}$ and $b < m$. Then, by point (iii) of Theorem 5, it must be that $\lambda_0 = 0$ and, consequently, $\lambda_m \geq 0$. Hence any optimal $X \in \mathrm{conv}(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha})$ with $\mathscr{U}^{m,\alpha} = \{(1-\alpha)U_{\mathrm{O}}^{m} + \alpha U_{\mathrm{O}}^{m+1}\}$ and $\mathscr{X}^{m,\alpha} = \alpha\delta\mathscr{V}^{m} + (1-\alpha\delta)\, U_{\mathrm{O}}^{m}$.

Secondly, suppose that $0 < \alpha \leq \frac{m+1}{2m+1}$ and $b = m$. By Lemma 2, it must be that

$$X = \lambda_0 \left((1-\alpha)U_{\mathrm{E}}^{m} + \alpha\delta V_{m}^{m} + \alpha(1-\delta)U_{\mathrm{O}}^{m}\right) + \sum_{j=1}^{m} \lambda_i \left(\alpha\delta V_{j}^{m} + (1-\alpha\delta)U_{\mathrm{O}}^{m}\right)$$
$$+ \lambda_{m+1} \left((1-\alpha)U_{\mathrm{O}}^{m} + \alpha U_{\mathrm{O}}^{m+1}\right)$$

in this case. Consider any distribution $T_{i,2m+1}^{m}$ with even $1 \leq i \leq m$. Let $q_k = \mathbf{P}(T_{i,2m+1}^{m} = k)$. By (4)–(7) and the fact that $\mathbf{E}(T_{i,2m+1}^{m}) = m$ and $q_{2j-1} = 0$, for any $1 \leq j \leq m$, we have

$$\begin{aligned}
H(U_{\mathrm{E}}^{m}, T_{i,2m+1}^{m}) &= 1 - \frac{\mathbf{E}(T_{i,2m+1}^{m}) + 1}{m+1} = 0\\
H(U_{\mathrm{O}}^{m+1}, T_{i,2m+1}^{m}) &= 1 - \frac{\mathbf{E}(T_{i,2m+1}^{m})}{m+1} = \frac{1}{m+1}\\
H(V_{j}^{m}, T_{i,2m+1}^{m}) &= 1 - \frac{2\mathbf{E}(T_{i,2m+1}^{m})}{2m+1} + \frac{q_{2j-1}}{2m+1} = \frac{1}{2m+1}\\
H(U_{\mathrm{E}}^{m}, T_{i,2m+1}^{m}) &= 1 - \left(\frac{1}{m}\right)\sum_{i=1}^{2m} \mathbf{P}(T_{i,2m+1}^{m} \geq i)\\
&= 1 - \left(\frac{1}{m}\right)\left(-q_{2m+1} - \sum_{i=1}^{2m+1} \mathbf{P}(T_{i,2m+1}^{m} \geq i)\right)\\
&= 1 + \frac{q_{2m+1}}{m} - \frac{\mathbf{E}(T_{i,2m+1}^{m})}{m} = \frac{q_{2m+1}}{m}.
\end{aligned}$$

Thus, by (3), we have

$$H(X, T_{i,2m+1}^{m}) = \lambda_0 \left(\frac{\alpha\delta}{2m+1} + \frac{\alpha(1-\delta)}{m} q_{2m+1}\right)$$
$$+ \sum_{j=1}^{m} \lambda_i \left(\frac{\alpha\delta}{2m+1} + \frac{1-\alpha\delta}{m} q_{2m+1}\right)$$

$$
\begin{aligned}
&+\lambda_{m+1}\left(\frac{1-\alpha}{m}q_{2m+1}+\frac{\alpha}{m+1}\right)\\
&=\lambda_0\left(\frac{\alpha}{m+1}-\frac{\alpha}{m+1}q_{2m+1}\right)+\sum_{j=1}^{m}\lambda_i\left(\frac{\alpha}{m+1}+\frac{1-\alpha\delta}{m}q_{2m+1}\right)\\
&+\lambda_{m+1}\left(\frac{1-\alpha}{m}q_{2m+1}+\frac{\alpha}{m+1}\right)\\
&=\frac{\alpha}{m+1}\sum_{i=0}^{m+1}\lambda_i-\lambda_0\frac{\alpha}{m+1}q_{2m+1}\\
&+\frac{1-\alpha\delta}{m}q_{2m+1}\sum_{i=1}^{m}\lambda_i+\lambda_{m+1}\frac{1-\alpha}{m}q_{2m+1}\\
&=\frac{\alpha}{m+1}+q_{2m+1}\left(\frac{1-\alpha\delta}{m}\sum_{i=1}^{m}\lambda_i+\frac{1-\alpha}{m}\lambda_{m+1}-\frac{\alpha}{m+1}\lambda_0\right),
\end{aligned}
$$

On the other hand, by (30), it must be that $H(X, T^m_{i,2m+1}) \geq \frac{\alpha}{m+1}$. Thus it must be that

$$
\frac{\alpha}{m+1}\lambda_0 \leq \frac{1-\alpha\delta}{m}\sum_{i=1}^{m}\lambda_i+\frac{1-\alpha}{m}\lambda_{m+1},
$$

(note that $q_{2m+1} > 0$) which can be reduced to

$$
\lambda_0 \leq \frac{1-\alpha\delta}{1-\alpha}\sum_{i=0}^{m}\lambda_i+\lambda_{m+1},
$$

by adding $\frac{1-\alpha\delta}{1-\alpha}\lambda_0$ to both sides. Thus

$$
\lambda_0 = \beta\left(\frac{1-\alpha\delta}{1-\alpha}\sum_{i=0}^{m}\lambda_i+\lambda_{m+1}\right),
$$

where $0 \leq \beta \leq 1$. From this and from (45) we get $\vec{x} = \lambda_0'\vec{z}_0' + \sum_{i=1}^{m}\left(\lambda_i'\vec{z}_i + \lambda_i''\vec{z}_i'\right) + \lambda_{m+1}' z_{m+1}$ where

$$
\begin{aligned}
\lambda_0' &= \beta\lambda_{m+1}, & \lambda_{m+1}' &= (1-\beta)\lambda_{m+1},\\
\lambda_i' &= \beta\lambda_i, & \lambda_i'' &= (1-\beta)\lambda_i, \quad \text{for } 1 \leq i \leq m-1,\\
\lambda_m' &= \beta\left(\lambda_0+\lambda_m\right), & \lambda_m'' &= (1-\beta)\left(\lambda_0+\lambda_m\right)
\end{aligned}
$$

and

$$\vec{z}_0' = \vec{z}_{m+1} + \vec{z}_0 - \vec{z}_m = \alpha\vec{u}_{\mathrm{O}}^{m+1} + (1-\alpha)\vec{u}_{\mathrm{E}}^{m},$$
$$\vec{z}_i' = \vec{z}_i + \frac{1-\alpha\delta}{1-\alpha}(\vec{z}_0 - \vec{z}_m) = \alpha\delta\vec{v}_i^{m} + (1-\alpha\delta)\vec{u}_{\mathrm{E}}^{m},\ \text{for } 1 \le i \le m.$$

It is easy to see that $\sum_{i=0}^{m+1}\lambda_i' + \sum_{i=1}^{m}\lambda_i'' = 1$, $\lambda_i' \ge 0$, for all $0 \le i \le m+1$, and $\lambda_i'' \ge 0$, for all $1 \le i \le m$. Hence any optimal $X \in \mathrm{conv}(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha})$ with $\mathscr{U}^{m,\alpha} = (1-\alpha)\mathscr{U}^m + \alpha U_{\mathrm{O}}^{m+1}$ and $\mathscr{X}^{m,\alpha} = \alpha\delta\mathscr{V}^m + (1-\alpha\delta)\,\mathscr{U}^m$.

Lastly, suppose that $\frac{m+1}{2m+1} < \alpha < 1$. By Lemma 2, it must be that

$$X = \lambda_0\left((1-\alpha)U_{\mathrm{E}}^m + \alpha U_{\mathrm{O}}^{m+1}\right) + \sum_{j=1}^{m}\lambda_i\left((1-\alpha)\sigma V_j^m + (1-(1-\alpha)\sigma)U_{\mathrm{O}}^{m+1}\right)$$
$$+ \lambda_{m+1}\left((1-\alpha)U_{\mathrm{O}}^m + \alpha U_{\mathrm{O}}^{m+1}\right)$$

in this case.

Like in the previous case, consider any distribution $T_{i,2m+1}^m$ with even $1 \le i \le m$. By (3), (4)–(7) and the fact that $\mathbf{E}(T_{i,2m+1}^m) = m$ and $q_{2j-1} = 0$, for all $1 \le j \le m$, we have

$$H(X, T_{i,2m+1}^m) = \frac{\alpha}{m+1}\sum_{i=0}^{m+1}\lambda_i + \lambda_{m+1}\frac{1-\alpha}{m}q_{2m+1} = \frac{\alpha}{m+1} + \lambda_{m+1}\frac{1-\alpha}{m}q_{2m+1}.$$

On the other hand, by (30), it must be that $H(X, T_{i,2m+1}^m) \ge \frac{\alpha}{m+1}$. Thus it must be that $\lambda_{m+1} \ge 0$. Moreover, by point (iii) of Theorem 5, it must be that $\lambda_0 = 0$ in the case of $b < m$. Hence any optimal $X \in \mathrm{conv}(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha})$ with $\mathscr{X}^{m,\alpha} = (1-\alpha)\sigma\mathscr{V}^m + (1-(1-\alpha)\sigma)\,U_{\mathrm{O}}^{m+1}$ and $\mathscr{U}^{m,\alpha} = (1-\alpha)\mathscr{U}^m + \alpha U_{\mathrm{O}}^{m+1}$ (if $b = m$) and $\mathscr{U}^{m,\alpha} = \{(1-\alpha)U_{\mathrm{O}}^m + \alpha U_{\mathrm{O}}^{m+1}\}$ (if $b < m$).

To see that the strategies found above are optimal, by Theorem 5, it is enough to check that

$$H(X, Y) \ge 1 - \frac{(1-\alpha)b}{m} - \frac{\alpha b}{m+1}, \tag{46}$$

for any $X \in \mathrm{conv}(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha})$ and any Y with $\mathbf{E}(Y) = b$. Using (3) and (4)–(7) it can be easily checked that (46) is satisfied for any $X \in \mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha}$, forany case listed in the theorem. Hence it is also satisfied for any $X \in \mathrm{conv}(\mathscr{U}^{m,\alpha} \cup \mathscr{X}^{m,\alpha})$. □

As an example consider a game $\Gamma(3/2, 1)$. Then the strategy $X = (1/2)\mathbf{1}_1 + (1/2)\mathbf{1}_2$, given as an example in Hart (2008) of one of the strategies not captured by Theorem 5, is $(3/4)V_1^1 + (1/4)U_{\mathrm{O}}^1$. Consider also a game $\Gamma(5/2, 1/2)$. Then the strategy $X = (5/12)\mathbf{1}_1 + (1/4)\mathbf{1}_3 + (1/3)\mathbf{1}_4$, given in Hart (2008) as another example of optimal strategies not captured by Theorem 5, is $(5/6)V_1^2 + (1/6)U_{\mathrm{O}}^2$.

Like in the case of Theorems 3 and 4, it is interesting to see how the uniform distribution, being an optimal strategy of player A in the continuous case, is approximated by discrete distributions. The new results obtained in Theorem 6 show that not all the strategies used as a building blocks for a mixture being an optimal strategy are uniform on some subset of the interval $[0, 2a + 1]$. Secondly, we can see that in the case of $b < \lfloor a \rfloor$ it is possible to have an optimal strategy where non-zero probability is put on even numbers.

Theorem 6 shows also that the case of $b \leq \lfloor a \rfloor$ with non-integer a can be, in fact, divided into two subcases: $a \leq \frac{\lfloor a \rfloor^2 + \lceil a \rceil^2}{\lfloor a \rfloor + \lceil a \rceil}$ and $a > \frac{\lfloor a \rfloor^2 + \lceil a \rceil^2}{\lfloor a \rfloor + \lceil a \rceil}$.[2]

Another interesting thing that the full characterization we have now allows us to see, is how the optimal strategies change with smooth change of constraints a and b. Fix the value of b for example and take an integer $a = m > b$. By Theorem 2, the unique optimal strategy of player A is in this case U_{O}^{m}. When a is increased to $m + \alpha$ ($0 < \alpha < \frac{m+1}{2m+1}$), then new strategies from the set $\mathscr{V}^m \cup U_{\mathrm{O}}^{m+1}$ enter as possible components of an optimal strategy. When α exceeds $\frac{m+1}{2m+1}$, the strategy U_{O}^{m} mixed with the strategies in the set $\mathscr{V}^m$ is replaced with U_{O}^{m+1}, and U_{O}^{m} remains a component of the optimal strategies with a coefficient $\leq 1 - \alpha$, slowly vanishing, as α gets close to 1. The strategies from $\mathscr{V}^m$ remain a component of the optimal strategies with a coefficient $\leq (1 - \alpha)\sigma$ and also vanish slowly as α gets close to 1. Eventually, when $a = m + 1$, strategy U_{O}^{m+1} becomes the unique strategy of player A.

5 Connection to the Colonel Blotto game

The Colonel Blotto game is a classic example of allocation games, where two players compete on different fronts allocating to them their limited resources (see Borel 1921; Tukey 1949; Shubik 1982). The Blotto games were introduced by Borel (1921) and most variations of the classic games remained unsolved (remarkably though, the solution of the continuous variant is known already due to Roberson 2006).

The game $\mathscr{B}(A, B; K)$ is defined as follows. There are two players A and B having $A \geq 1$ and $B \geq 1$ tokens, respectively, to distribute simultaneously over K urns. Thus a pure strategy of player A is a K-partition, $x = \langle x_1, \ldots, x_K \rangle$, of A, so that $x_1 + \cdots + x_K = A$ and each x_i is a natural number. Similarly, a pure strategy of player B is a K-partition, $y = \langle y_1, \ldots, y_K \rangle$, of B, so that $y_1 + \cdots + y_K = B$ and each y_i is a natural number.

After the tokens are distributed, the payoff of each player is computed as follows. For each urn where a player has a strictly larger number of tokens placed he receives the score 1, while for each urn where a player has a strictly smaller number of tokens placed, he receives the score -1. The score on the tied urns is 0 for each player. The overall payoff is the average of payoffs obtained for all urns, that is, given the strategies x and y of A and B, respectively, it is

[2] That is $\alpha \leq \frac{m+1}{2m+1}$ and $\alpha \geq \frac{m+1}{2m+1}$.

$$h_{\mathscr{B}}(x, y) = \frac{1}{K} \sum_{i=1}^{K} \operatorname{sign}(x_i - y_i).$$

The Colonel Blotto is a zero-sum game.

To connect the Colonel Blotto game to the General Lotto game, Hart (2008) proposed first a symmetrized-across-urns variant of this game called the *Colonel Lotto* game. In this game, denoted by $\mathscr{L}(A, B; K)$, the urns are indistinguishable and players simultaneously divide their tokens into K groups, which are then randomly paired. Thus, again, the strategies of the players are K-partitions and the payoff of each player is an average over all possible pairings, that is, given the strategies x and y of A and B, respectively, it is

$$h_{\mathscr{L}}(x, y) = \frac{1}{K^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \operatorname{sign}(x_i - y_j).$$

To see the connection between the Colonel Blotto and Colonel Lotto games, given a pure strategy x of player A, let $\sigma(x)$ denote a mixed strategy that assigns equal probability, $\frac{1}{K!}$, to each permutation of x. Similarly, given a mixed strategy ξ of player A, let $\sigma(\xi)$ denote a mixed strategy obtained by replacing each pure strategy x in the support of ξ by $\sigma(x)$. The strategies $\sigma(x)$ and $\sigma(\xi)$ are called *symmetric across urns*. As was observed in Hart (2008), $h_{\mathscr{B}}(\sigma(\xi), y) = h_{\mathscr{L}}(\xi, y)$, for any pure strategy y of player B. Consequently, $h_{\mathscr{B}}(\sigma(\xi), \eta) = h_{\mathscr{L}}(\xi, \eta)$, for any mixed strategy η of player B. Analogously for the strategies of player B. Hence the following observation can be made

Observation 1 (Hart) *The Colonel Blotto game $\mathscr{B}(A, B; K)$ and the Colonel Lotto game $\mathscr{L}(A, B; K)$ have the same value. Moreover, the mapping σ maps the optimal strategies in the Colonel Lotto game onto the optimal strategies in the Colonel Blotto game that are symmetric across urns.*

Having linked the Colonel Blotto and Colonel Lotto games we are ready to see the link between them and General Lotto games. Notice that any K-partition $\langle z_1, \ldots, z_K \rangle$ of a natural number C can be seen as a discrete random variable Z with values in the set $\{z_1, \ldots, z_K\}$ and the distribution obtained by assigning to each $z_1, \ldots, z_K$ the probability $\frac{1}{K}$. The expected value of Z is then $\mathbf{E}(Z) = \frac{C}{K}$, which is the average number of tokens per urn. This construction links the pure strategies x and y or players A and B in Colonel Lotto game with discrete integer valued random variables X and Y. The strategies of players A and B in Colonel Lotto game could be seen as non-negative, integer valued random variables bounded by A and B and having expectations A/K and B/K, respectively. The payoff $h_{\mathscr{L}}(x, y)$ can be then written as

$$h_{\mathscr{L}}(x, y) = H(X, Y) = \mathbf{P}(X > Y) - \mathbf{P}(X < Y).$$

General Lotto game could be seen as a generalization of Colonel Lotto game which allows for strategies of the players to be unbounded random variables. Notice that every strategy in the Colonel Lotto game $\mathscr{L}(A, B; K)$ is a strategy in the General Lotto game

$\Gamma(A/K, B/K; K)$, although the opposite is not necessarily true. However, every optimal strategy in a General Lotto game which is a strategy in the corresponding Colonel Lotto game is an optimal strategy there. Hence one of the approaches to find optimal strategies for Colonel Lotto games (and, further, for Colonel Blotto games) is to find the optimal strategies in General Lotto games and see which of them are the strategies in the aforementioned games. This was partially done in Hart (2008), where, in particular, the symmetric case of $A = B$ was covered. However, most of non-symmetric cases were only partially solved.

6 Conclusions

In this paper we have found the missing optimal strategies for the players in non-symmetric Discrete General Lotto games. These games are an example of allocation games and have several applications in political competition (Myerson 1993; Sahuguet and Persico 2006; Dekel et al. 2008), all-pay auctions (Sahuguet and Persico 2006) and tournaments (Groh et al. 2010). In particular, they could be used to find full characterization of the optimal strategies for the players in discrete variant of the first price all-pay auctions. This variant was studied by Cohen and Sela (2007), who provide examples of optimal strategies for players in both symmetric and asymmetric cases (with restriction to two players in the latter case). Using the game studied here to obtain full characterization in the multi player case would require, however, studying a natural extension to more than two players.

The full characterization allows us to compare the optimal strategies in the discrete and continuous variants of the game and helps to gain insight into how the discrete restriction affects the equilibrium behaviour. It could be also used for solving the missing cases of Discrete Colonel Blotto games, which we reserve for future research.

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Appendix

In the analysis below we will use standard notation $\mathbf{1}_i$ to denote the i'th unit vector, $\mathbf{I}_n$ to denote the $n \times n$ unit matrix and $\mathbf{0}_{m,n}$ to denote the $m \times n$ zero matrix. We will drop subscripts denoting the dimension of these matrices if it is clear from the context. Given a sequence of elements $a_1 \ \cdots \ a_n$ we will use the notation $(a_1 \ \cdots \ a_n)^m$ to denote a sequence obtained by repeating the sequence m times. Hence, for example, $\left[\, 1 \ (0\ 2)^2 \ 0 \,\right]^T$ denotes the vector $[1\ 0\ 2\ 0\ 2\ 0]$. If $m \leq 0$, then we will use a convention that $(a_1 \ \cdots \ a_n)^m$ denotes the empty sequence. So, for example, $\left[\, 1 \ (0\ 2)^0 \ 0 \,\right]^T$ denotes the vector $[1\ 0]$.

In two of the lemmas we prove below we compute the basis of a null space of matrices of the form $\left[\begin{array}{c|c} f & \mathbf{f} \\ \hline \vec{0} & \mathbf{B}_n \end{array}\right]$ (in the case of Lemma 1) or of the form $\left[\begin{array}{c} \mathbf{f} \\ \hline \mathbf{B}_n \end{array}\right]$ (in the case of Lemma 2), where $\mathbf{f}$ is a row vector and $\mathbf{B}_n$ is a $3(n-1) \times (4n-1)$ matrix of the form

$$\mathbf{B}_n = \left[\begin{array}{c} \mathbf{G}_n \\ \hline \mathbf{H}_n \end{array}\right], \tag{47}$$

where

$$\mathbf{G}_n = \left[\tilde{\mathbf{G}}_n \,\middle|\, \mathbf{I}_{2n-2} \,\middle|\, \vec{0}\right], \quad \tilde{\mathbf{G}}_n = \begin{bmatrix} \mathbf{g}_1 \\ -\mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{2n-3} \\ -\mathbf{g}_{2n-2} \end{bmatrix},$$

$$\mathbf{g}_i = \left[(0)^{i-1}\ 1\ 0\ -1\ (0)^{2n-i-2}\right], \text{ for } 1 \le i \le 2n-2,$$

$$\mathbf{H}_n = \left[\begin{array}{c|c|c} \tilde{\mathbf{H}}_n & \mathbf{0} & \vec{0} \\ \hline \mathbf{h}_{n-1} & (0)^{2n-2} & -1 \end{array}\right], \; \tilde{\mathbf{H}}_n = \begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_{n-2} \end{bmatrix},$$

$$\mathbf{h}_i = \begin{cases} \left[(0)^{2(i-1)}\ 1\ 2\ 0\ -2\ -1\ (0)^{2(n-i)-3}\right], & \text{if } 1 \le i \le n-2, \\ \left[(0)^{2(i-1)}\ 1\ 2\ 0\ -2\right], & \text{if } i = n-1. \end{cases}$$

The computation is by Gaussian elimination and before we give the proofs of the lemmas we show how $\mathbf{B}_n$ can be reduced by Gaussian elimination to a matrix $\mathbf{B}_n^{(2)}$, which will be used in those proofs. The process of elimination is as follows. First we add to each row i of $\mathbf{G}_n$ the sum of rows $j > i$ of $\mathbf{G}_n$ with the same parity as i and multiply even rows of the resulting matrix by -1. By doing this we obtain

$$\mathbf{G}_n^{(1)} = \left[\mathbf{I}_{2n-2} \,\middle|\, -\vec{g}_{-1} \,\middle|\, -\vec{g}_0 \,\middle|\, \tilde{\mathbf{G}}_n^{(1)}\right], \text{ with}$$

$$\vec{g}_{-1} = \left[(1\ 0)^{n-1}\right]^T, \quad \vec{g}_0 = \left[(0\ 1)^{n-1}\right]^T, \quad \tilde{\mathbf{G}}_n^{(1)} = \begin{bmatrix} \mathbf{g}_1^{(1)} \\ -\mathbf{g}_2^{(1)} \\ \vdots \\ \mathbf{g}_{2n-3}^{(1)} \\ -\mathbf{g}_{2n-2}^{(1)} \end{bmatrix}, \text{ where}$$

$$\mathbf{g}_i^{(1)} = \begin{cases} \left[(0)^{2j}\ (1\ 0)^{n-j-1}\ 0\right], & \text{if } i = 2j+1, \\ \left[(0)^{2j}\ (0\ 1)^{n-j-1}\ 0\right], & \text{if } i = 2j+2. \end{cases}$$

Next, we eliminate the first $2n+1$ columns of matrix $\mathbf{H}_n$ using rows of $\mathbf{G}_n^{(1)}$, obtaining:

$$\begin{aligned}
\mathbf{H}_n^{(1)} &= \left[\mathbf{0}\,\middle|\,\hat{e}_{n-1}\,\middle|\,\vec{0}\,\middle|\,\tilde{\mathbf{H}}_n^{(1)}\right], \text{ where}\\
\tilde{\mathbf{H}}_n^{(1)} &= \begin{bmatrix} \mathbf{h}_1^{(1)} \\ \vdots \\ \mathbf{h}_{n-1}^{(1)} \end{bmatrix}, \text{ with}\\
\mathbf{h}_i^{(1)} &= \left[\,(0)^{2(i-1)}\ -1\ 2\ -1\ (0)^{2(n-i-1)}\right], \text{ for } 1 \le i \le n-1.
\end{aligned}$$

We proceed further by adding to each row $2j-1$ of $\mathbf{G}_n^{(1)}$ the sum of rows $k \ge j$ of $\mathbf{H}_n^{(1)}$ with the same parity as j obtaining:

$$\begin{aligned}
\mathbf{G}_n^{(2)} &= \left[\mathbf{I}_{2n-2}\,\middle|\,-\vec{g}'_{-1}\,\middle|\,-\vec{g}_0\,\middle|\,\tilde{\mathbf{G}}_n^{(2)}\right], \textit{where}\\
\vec{g}'_{-1} &= \begin{cases} \left[(1\ 0\ 0\ 0)^{(n-1)/2}\right]^T, & \text{if } n \text{ is odd,} \\ \left[\,(0\ 0\ 1\ 0)^{n/2-1}\ 0\ 0\right]^T, & \text{if } n \text{ is even.} \end{cases} \quad \text{and}\\
\tilde{\mathbf{G}}_n^{(2)} &= \begin{bmatrix} \mathbf{g}_1^{(2)} \\ -\mathbf{g}_2^{(2)} \\ \vdots \\ \mathbf{g}_{2n-3}^{(2)} \\ -\mathbf{g}_{2n-2}^{(2)} \end{bmatrix}, \text{ with}\\
\mathbf{g}_i^{(2)} &= \begin{cases} \left[\,(0)^{2j}\ (0\ 2\ 0\ 0)^{(n-j)/2-1}\ 0\ 2\ -1\right], & \text{if } i = 2j+1 \text{ and } n-j \text{ is even,} \\ \left[\,(0)^{2j}\ (0\ 2\ 0\ 0)^{(n-j+1)/2-1}\ 0\right], & \text{if } i = 2j+1 \text{ and } n-j \text{ is odd,} \\ \mathbf{g}_i^{(1)}, & \text{if } i \text{ is even.} \end{cases}
\end{aligned}$$

Next we add to each row i of $\mathbf{H}_n^{(1)}$ the sum of rows $j > i$ of $\mathbf{H}_n^{(1)}$ with the same parity as i and subtract from it the sum of rows $j > i$ of $\mathbf{H}_n^{(1)}$ with different parity to i. Multiplying the result by -1 we obtain:

$$\begin{aligned}
\mathbf{H}_n^{(2)} &= \left[\mathbf{0}\,\middle|\,\vec{h}_{-1}\,\middle|\,\vec{0}\,\middle|\,\tilde{\mathbf{H}}_n^{(2)}\right], \text{ where}\\
\vec{h}_{-1} &= \begin{cases} \left[(1\ -1)^{(n-1)/2}\right]^T, & \text{if } n \text{ is odd,} \\ \left[\,(-1\ 1)^{n/2-1}\ -1\right]^T, & \text{if } n \text{ is even,} \end{cases} \quad \text{and}\\
\tilde{\mathbf{H}}_n^{(2)} &= \begin{bmatrix} \mathbf{h}_1^{(2)} \\ \vdots \\ \mathbf{h}_{n-1}^{(2)} \end{bmatrix}, \text{ with}\\
\mathbf{h}_i^{(2)} &= \begin{cases} \left[\,(0)^{2(i-1)}\ 1\ -2\ (0\ 2\ 0\ -2)^{(n-i-1)/2}\ 1\right], & \text{if } n-i \text{ is odd,} \\ \left[\,(0)^{2(i-1)}\ 1\ -2\ (0\ 2\ 0\ -2)^{(n-i)/2-1}\ 0\ 2\ -1\right], & \text{if } n-i \text{ is even.} \end{cases}
\end{aligned}$$

Thus we obtain the matrix

$$\mathbf{B}_n^{(2)} = \left[\frac{\mathbf{G}_n^{(2)}}{\mathbf{H}_n^{(2)}}\right]. \tag{48}$$

Now we are ready to give proofs of Lemmas 1–2.

Proof of Lemma 1 Matrix representation of the system of Eqs. 14–18 is

$$\mathbf{A}_a \cdot \begin{bmatrix} \vec{q} \\ \vec{d} \end{bmatrix} = \vec{0}, \qquad \text{where } \mathbf{A}_a = \left[\begin{array}{c|c} f_1 & \mathbf{f}_2 \\ \hline \vec{0} & \mathbf{B}_a \end{array}\right], \tag{49}$$

$\mathbf{B}_a$ is defined in Eq. 47 and $\left[\, f_1 \,\middle|\, \mathbf{f}_2 \,\right] = \mathbf{f} = \left[-a \; -(a-1) \; \cdots \; a-1 \; a \,\middle|\, (0)^{2a-1}\right]$.

Any solution of (49) is an element of the null space of $\mathbf{A}_a$, $\mathbf{Ker}(\mathbf{A}_a)$. To find its basis we proceed by the standard methods, applying Gaussian elimination to $\mathbf{A}_a$ first. Firstly, we reduce $\mathbf{B}_a$ to $\mathbf{B}_a^{(2)}$, as given in Eq. 48. Next, we eliminate first elements in columns $2 \cdots 2a-2$ of $\mathbf{f}$ with rows of $\mathbf{G}_a^{(2)}$. Dividing the result by $-a$ we get:

$$\mathbf{f}^{(1)} = \begin{cases} \left[\,1 \; (0)^{2a-2} \; 0 \; -1 \; (0 \; -1 \; 0 \; 0)^{(a+1)/2-1} \; 0\,\right], & \text{if } a \text{ is odd,} \\ \left[\,1 \; (0)^{2a-2} \; -\frac{1}{2} \; -1 \; (0 \; -1 \; 0 \; 0)^{a/2-1} \; 0 \; -1 \; \frac{1}{2}\,\right], & \text{if } a \text{ is even.} \end{cases}$$

The resulting matrix $\mathbf{A}_a^{(1)}$, written column-wise, is:

$$\mathbf{A}_a^{(1)} = \left[\begin{array}{c|cccccccc} \mathbf{I}_{2a-1} & -\vec{g}_{-1} & -\vec{g}_0 & \vec{0} & -\vec{g}_1 & \cdots & \vec{0} & -\vec{g}_{a-1} & -\vec{g}_a \\ \hline \mathbf{0} & -\vec{h}_{-1} & \vec{0} & \hat{e}_1 & -\vec{h}_1 & \cdots & \hat{e}_{a-2} & -\vec{h}_{a-1} & -\vec{h}_a \end{array}\right],$$

where

$$\begin{aligned}
\vec{g}_{-1} &= \left[0 \; (1\;0\;0\;0)^{(a-1)/2}\right]^T, \qquad \vec{h}_{-1} = \left[(-1\;1)^{(a-1)/2}\right]^T, \text{ if } a \text{ is odd,} \\
\vec{g}_{-1} &= \left[\tfrac{1}{2}\;0\;0\;(1\;0\;0\;0)^{a/2-1}\right]^T, \qquad \vec{h}_{-1} = \left[(1\;-1)^{a/2-1}\;1\right]^T, \text{ if } a \text{ is even,} \\
\vec{g}_0 &= \left[(1\;0)^{a-1}\;1\right]^T, \\
\vec{g}_{2j-1} &= \left[1\;-2\;(1\;0\;1\;-2)^{j-1}\;1\;(0)^{2a-4j}\right]^T, \\
\vec{h}_{2j-1} &= \left[(2\;-2)^{j-1}\;2\;(0)^{a-2j}\right]^T, \; 1 \le j \le \left\lceil \tfrac{a-1}{2} \right\rceil, \\
\vec{g}_{2j} &= \left[0\;0\;(1\;-2\;1\;0)^{j}\;(0)^{2a-4j-3}\right]^T, \\
\vec{h}_{2j} &= \left[(-2\;2)^{j}\;(0)^{a-2j-1}\right]^T, 1 \le j \le \left\lfloor \tfrac{a-1}{2} \right\rfloor, \\
\vec{g}_a &= \left[0\;(0\;0\;1\;0)^{(a-1)/2}\right]^T, \qquad \vec{h}_a = \left[(1\;-1)^{(a-1)/2}\right]^T, \text{ if } a \text{ is odd,} \\
\vec{g}_a &= \left[-\tfrac{1}{2}\;1\;0\;(0\;0\;1\;0)^{a/2-1}\right]^T, \qquad \vec{h}_a = \left[(-1\;1)^{a/2-1}\;-1\right]^T, \text{ if } a \text{ is even,}
\end{aligned}$$

Notice that there are $a+2$ columns of $\mathbf{A}_a^{(1)}$ that are associated with free variables. These are the columns with indexes $2a, 2(a+i)+1$ (with $0 \le i \le a-1$) and $4a-1$, i.e. the columns where in the upper part of the matrix there are vectors $-\vec{g}_i$ with $-1 \le i \le a$.

To obtain the basis for the null space of $\mathbf{A}_a$ we multiply the free variable columns by -1 and then fill in the rows by adding $\hat{e}_i^T$ at positions $i = 2a, 2(a+j)+1$ (with $0 \le j \le a-1$) and $4a-1$. The columns in thus obtained matrix form a basis of the null space, $\mathbf{Ker}(\mathbf{A}_a) = \text{span}\,\{\vec{x}_{-1}, \vec{x}_0, \vec{x}_1, \dots, \vec{x}_a\}$, where

$$\vec{x}_{-1} = \begin{cases} \left[0\ 1\ (0\,0\,0\,1)^{(a-1)/2}\ 0 \,\middle|\, (-1\ 01\ 0)^{(a-1)/2}\ 0\right]^T, & \text{if } a \text{ is odd,} \\ \left[\frac{1}{2}\ (0\,0\,1\,0)^{a/2} \,\middle|\, (1\ 0\ -1\ 0)^{a/2-1}\ 1\ 0\ 0\right]^T, & \text{if } a \text{ is even,} \end{cases}$$

$$\vec{x}_0 = \left[(1\ 0)^a\ 1 \,\middle|\, (0)^{2a-1}\right]^T,$$

$$\vec{x}_{2j-1} = \left[1\ {-2}\ (1\ 0\ 1\ -2)^{j-1}\ 1\ (0)^{2a-4j+2} \,\middle|\, (2\ 0\ -2\ 0)^{j-1}\ 2\ 1\ (0)^{2a-4j+1}\right]^T,$$
$$1 \le j \le \left\lceil \frac{a-1}{2} \right\rceil,$$

$$\vec{x}_{2j} = \left[0\ 0\ (1\ -2\ 1\ 0)^j\ (0)^{2a-4j-1} \,\middle|\, (-2\ 0\ 2\ 0)^{j-1}\ {-2}\ 0\ 2\ 1\ (0)^{2a-4j-1}\right]^T,$$
$$1 \le j \le \left\lfloor \frac{a-1}{2} \right\rfloor,$$

$$\vec{x}_a = \begin{cases} \left[0\ 0\ (0\,1\,0\,0)^{(a-1)/2}\ 0 \,\middle|\, (1\ 0\ -1\ 0)^{(a-1)/2}\ 1\right]^T, & \text{if } a \text{ is odd,} \\ \left[-\frac{1}{2}\ (1\,0\,0\,0)^{a/2} \,\middle|\, (-1\ 0\ 1\ 0)^{a/2-1}\ {-1}\ 0\ 1\right]^T, & \text{if } a \text{ is even,} \end{cases}$$

First we change the basis to $\{\vec{x}'_{-1}, \dots, \vec{x}'_a\}$, where

$$\vec{x}'_{-1} = \frac{1}{a}\,(\vec{x}_{-1} + \vec{x}_a) = \begin{bmatrix} \vec{u}^a_{\mathrm{O}} \\ \vec{d}_{-1} \end{bmatrix},$$
$$\text{where } \vec{d}_{-1} = \frac{1}{m}\left[(0)^{2a-2}\ 1\right]^T,$$
$$\vec{x}'_0 = \frac{1}{a+1}\vec{x}_0 = \begin{bmatrix} \vec{u}^a_{\mathrm{E}} \\ \vec{d}_0 \end{bmatrix}, \text{ where } \vec{d}_0 = \vec{0},$$
$$\vec{x}'_1 = \frac{1}{2a}\left(2\vec{x}'_{-1} + \vec{x}_1\right) = \begin{bmatrix} \vec{w}^a_1 \\ \vec{d}_1 \end{bmatrix}, \text{ where } \vec{d}_1 = \frac{1}{2m}\left[2\ 1\ (0)^{2a-4}\ 2\right]^T,$$
$$\vec{x}'_i = \frac{1}{2a}\left(2\vec{x}'_{-1} + \vec{x}_{i-1} + \vec{x}_i\right) = \begin{bmatrix} \vec{w}^a_i \\ \vec{d}_i \end{bmatrix},$$
$$\text{where } \vec{d}_i = \frac{1}{2m}\left[(0)^{2i-3}\ 1\ 2\ 1\ (0)^{2(a-i-1)}\ 2\right]^T$$
$$\text{and } 2 \le i \le a-2,$$
$$\vec{x}'_{a-1} = \frac{2}{a+1}\vec{x}_{-1} = \begin{bmatrix} \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1} \\ \vec{d}_{a-1} \end{bmatrix},$$
$$\text{where } \vec{d}_{a-1} = \frac{1}{a+1}\left[2\ 0\ 0\right]^T, \text{ if } a = 2,$$
$$\vec{x}'_{a-1} = \frac{1}{a+1}\,(2\vec{x}_{-1} + \vec{x}_{a-2}) = \begin{bmatrix} \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1} \\ \vec{d}_{a-1} \end{bmatrix},$$

$$\text{where } \vec{d}_{a-1} = \frac{1}{a+1}\left[\,(0)^{2a-5}\ 1\ 2\ 0\ 0\,\right]^T, \text{ if } a \geq 3,$$

$$\vec{x}'_a = \frac{1}{a-1}\left(\vec{x}_{a-1} + 2\vec{x}_a\right) = \begin{bmatrix} \vec{u}^a_{Q\uparrow 1} \\ \vec{d}_a \end{bmatrix}, \text{ where } \vec{d}_a = \frac{1}{m-1}\left[\,(0)^{2a-3}\ 1\ 2\,\right]^T.$$

Any solution $\vec{x} = [q_0, \ldots, q_{2a}, d_1, \ldots, d_{2a-1}]$ of (49) is a linear combination of the vectors above, that is

$$\vec{x} = \sum_{i=-1}^{a} \lambda_i \vec{x}'_i.$$

Since $q_1 = \lambda_{-1}\frac{1}{a}, q_{2a} = \lambda_0\frac{1}{a+1}, d_{2a-2} = \lambda_a\frac{1}{a-1}, d_{2a-3} = \lambda_{a-1}\frac{1}{a+1}$ and $d_{2i-1} = \lambda_i\frac{1}{a}$, for $1 \leq i \leq a-2$ so, from Constraints (19)–(20), we get that $\lambda_i \geq 0$, for all $-1 \leq i \leq a$. Additionally, from Constraint (21) and the fact that $\sum_{j=0}^{2a} x'_{ij} = 1$, for all $-1 \leq i \leq a$, we have:

$$\sum_{i=-1}^{a} \lambda_i = \sum_{i=-1}^{a} \lambda_i \sum_{j=0}^{2a} x'_{ij} = \sum_{i=0}^{2a} q_i = 1.$$

Hence the set solutions of the system of Eq. 14–18 with Constraints (19)–(20) is $\mathrm{conv}\left(\{\vec{x}'_{-1}, \ldots, \vec{x}'_a\}\right)$. □

Proof of Lemma 2 Matrix representation of the system of Eqs. 37–41 is

$$\mathbf{A}_m \cdot \begin{bmatrix} \vec{p} \\ \vec{d} \end{bmatrix} = \vec{0}, \qquad \text{where } \mathbf{A}_m = \left[\frac{\mathbf{f}}{\mathbf{B}_{m+1}}\right], \tag{50}$$

$\mathbf{B}_{m+1}$ is defined in Eq. 47 and

$$\mathbf{f} = \left[\,-m-\alpha\ \ -(m-1)-\alpha\ \cdots\ m-\alpha\ \ m+1-\alpha \,\middle|\, (0)^{2m+1}\,\right].$$

Like in proof of Lemma 1 to find solutions of (50) we find a basis of the null space of $\mathbf{A}_m$ using Gaussian elimination. $\mathbf{B}_{m+1}$ can be reduced to $\mathbf{B}^{(2)}_{m+1}$, as given in Eq. 48. Next, we eliminate first $2m$ elements of $\mathbf{f}$ with rows of $\mathbf{G}^{(2)}_{m+1}$. Dividing the result by $(m+1)(1-\alpha)$ we get:

$$\mathbf{f}^{(1)} = \begin{cases} \left[\,(0)^{2m}\ -u\ 1\ (0\ t\ 0\ 0)^{m/2}\ -w\,\right], & \text{if } m \text{ is even,} \\ \left[\,(0)^{2m}\ -u\ 1\ (0\ t\ 0\ 0)^{(m-1)/2}\ 0\ t\ -w\,\right], & \text{if } m \text{ is odd, where} \end{cases}$$

$$u = \begin{cases} \frac{m+2}{2(m+1)}\frac{\alpha}{1-\alpha}, & \text{if } m \text{ is even,} \\ -\frac{1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

$$t = \frac{m+1+\alpha}{(m+1)(1-\alpha)}$$

$$w = \begin{cases} \frac{m}{2(m+1)}\frac{\alpha}{1-\alpha}, & \text{if } m \text{ is even,} \\ \frac{1}{2}\frac{1+\alpha}{1-\alpha}, & \text{if } m \text{ is odd.} \end{cases}$$

Next, we move the first row below block $\mathbf{G}^{(2)}_{m+1}$ obtaining $\mathbf{A}^{(1)}_m = \left[\begin{array}{c}\mathbf{G}^{(2)}_{m+1}\\ \hline \mathbf{f}^{(1)}\\ \hline \mathbf{H}^{(2)}_{m+1}\end{array}\right]$. Adding row $\mathbf{f}^{(1)}$ to rows of $\mathbf{G}^{(2)}_{m+1}$ we get matrix $\mathbf{A}^{(2)}_m$, which, written column-wise, is:

$$\mathbf{A}^{(2)}_m = \left[\begin{array}{c|cccccccccc}\mathbf{I}_{2m} & -\vec{g}_0 & \vec{0} & \vec{0} & -\vec{g}_1 & \vec{0} & -\vec{g}_2 & \cdots & \vec{0} & -\vec{g}_m & -\vec{g}_{m+1}\\ \hline \mathbf{0} & -\vec{h}_0 & \hat{e}_1 & \hat{e}_2 & -\vec{h}_1 & \vec{0} & -\vec{h}_2 & \cdots & \hat{e}_{m+2} & -\vec{h}_m & -\vec{h}_{m+1}\end{array}\right],$$

where

$$\begin{array}{ll}\vec{g}_0 & = \left[0\ u\ (1\ u\ 0\ u)^{(m-1)/2}\right]^T, \quad \vec{h}_{-1} = \left[(1\ -1)^{(m-1)/2}\ 1\right]^T, \text{ if } m \text{ is odd},\\ \vec{g}_0 & = \left[(1\ u\ 0\ u)^{m/2}\right]^T, \quad \vec{h}_{-1} = \left[(-1\ 1)^{m/2}\right]^T, \text{ if } m \text{ is even},\\ \vec{g}_{2j-1} & = \left[(-2\ 1-t\ 0\ 1-t)^{j-1}\ -2\ 1-t\ (0\ -t)^{m-2j+1}\right]^T,\\ \vec{h}_{2j-1} & = \left[(2\ -2)^j\ 2\ (0)^{m-2j-1}\right]^T, 1 \le j \le \lceil \frac{m}{2} \rceil,\\ \vec{g}_{2j} & = \left[(0\ 1\ -2\ 1)^j\ (0)^{2m-4j}\right]^T,\\ \vec{h}_{2j} & = \left[(-2\ 2)^j\ (0)^{m-2j}\right]^T, 1 \le j \le \lfloor \frac{m}{2} \rfloor,\\ \vec{g}_{m+1} & = \left[1\ w\ (0\ w\ 1\ w)^{(m-1)/2}\right]^T, \quad \vec{h}_{m+2} = \left[(-1\ 1)^{(m-1)/2}\ -1\right]^T, \text{ if } m \text{ is odd},\\ \vec{g}_{m+1} & = \left[(0\ w\ 1\ w)^{m/2}\right]^T, \quad \vec{h}_{m+2} = \left[(1\ -1)^{m/2}\right]^T, \text{ if } m \text{ is even}.\end{array}$$

Notice that there are $m+2$ columns of $\mathbf{A}^{(2)}_m$ that are associated with free variables. There are the columns with indexes $2m+1, 2(m+i+1)$ (with $1 \le i \le m$) and $4m+3$, i.e. the columns where in the upper part of the matrix there are vectors $-\vec{g}_i$ with $0 \le i \le m+1$.

To obtain the basis for the null space of $\mathbf{A}_m$ we multiply the free variable columns by -1 and then fill in the rows by adding $\hat{e}_i^T$ at positions $i = 2m+1, 2(m+j+1)$ (with $1 \le j \le m$) and $4m+3$. The columns in thus obtained matrix form a basis of the null space, $\mathbf{Ker}(\mathbf{A}_m) = \text{span}\{\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{m+1}\}$, where

$$\vec{x}_0 = \begin{cases}\left[(0\ u\ 1\ u)^{(m+1)/2} \,\middle|\, (1\ 0\ -1\ 0)^{(m-1)/2}\ 1\ 0\ 0\right]^T, & \text{if } m \text{ is odd},\\ \left[1\ u\ (0\ u\ 1\ u)^{m/2} \,\middle|\, (-1\ 0\ 1\ 0)^{m/2}\ 0\right]^T, & \text{if } m \text{ is even},\end{cases}$$

$$\vec{x}_{2j-1} = \left[(-2\ 1-t\ 0\ 1-t)^{j-1} - 2\ 1-t(0\ -t)^{m-2j+2} | (2\ 0\ -2\ 0)^{j-1} 2\ 1 (0)^{2m-4j+3}\right]^T,$$
$$1 \le j \le \left\lceil \frac{m}{2} \right\rceil,$$

$$\vec{x}_{2j} = \left[(0\ 1\ -2\ 1)^j\ (0)^{2m-4j+2} \,\middle|\, (-2\ 0\ 2\ 0)^{j-1}\ -2\ 0\ 2\ 1\ (0)^{2m-4j+1}\right]^T,$$
$$1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$\vec{x}_{m+1} = \begin{cases}\left[(1\ w\ 0\ w)^{(m+1)/2} \,\middle|\, (-1\ 0\ 1\ 0)^{(m-1)/2}\ -1\ 0\ 1\right]^T, & \text{if } m \text{ is odd},\\ \left[0\ w\ (1\ w\ 0\ w)^{m/2} \,\middle|\, (1\ 0\ -1\ 0)^{m/2}\ 1\right]^T, & \text{if } m \text{ is even},\end{cases}$$

First we change the basis to $\{\vec{x}'_0, \ldots, \vec{x}'_{m+1}\}$, where

$$\vec{x}'_{m+1} = \frac{1-\alpha}{m}\left(\vec{x}_m + 2\vec{x}_{m+1}\right) = \begin{bmatrix} (1-\alpha)\vec{u}^{m}_{\mathrm{O}} + \alpha\vec{u}^{m+1}_{\mathrm{O}} \\ \vec{d}_{m+1} \end{bmatrix},$$
$$\text{with } \vec{d}_{m+1} = \frac{1-\alpha}{m}\left[\,(0)^{2m-1}\ 1\ 2\,\right]^T,$$

and, in the case of $\frac{m+1}{2m+1} < \alpha < 1$,

$$\vec{x}'_0 = \frac{1-\alpha}{m+1}\left(\vec{x}_0 + \vec{x}_{m+1}\right) = \begin{bmatrix} (1-\alpha)\vec{u}^{m}_{\mathrm{E}} + \alpha\vec{u}^{m+1}_{\mathrm{O}} \\ \vec{d}_0 \end{bmatrix}, \text{ with } \vec{d}_0 = \frac{1-\alpha}{m+1}\left[\,(0)^{2m}\ 1\,\right]^T,$$
$$\vec{x}'_1 = \frac{1-\alpha}{m}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}_1\right) = \begin{bmatrix} (1-\alpha)\sigma\vec{v}^{m}_1 + (1-(1-\alpha)\sigma)\vec{u}^{m+1}_{\mathrm{O}} \\ \vec{d}_1 \end{bmatrix},$$
$$\text{with } \sigma = \frac{2m+1}{m} \text{ and } \vec{d}_1 = \frac{1-\alpha}{m}\left[\,2\ 1\ (0)^{2m-2}\ 2\,\right]^T,$$
$$\vec{x}'_i = \frac{1-\alpha}{m}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}_{i-1} + \vec{x}_i\right)$$
$$= \begin{bmatrix} (1-\alpha)\sigma\vec{v}^{m}_i + (1-(1-\alpha)\sigma)\vec{u}^{m+1}_{\mathrm{O}} \\ \vec{d}_i \end{bmatrix}, \text{ for } 2 \le i \le m,$$
$$\text{with } \vec{d}_i = \frac{1-\alpha}{m}\left[\,(0)^{2i-3}\ 1\ 2\ 1\ (0)^{2(m-i)}\ 2\,\right]^T, \text{ for } 2 \le i \le m,$$

while in the case of $0 < \alpha \le \frac{m+1}{2m+1}$,

$$\vec{x}'_1 = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_1\right) = \begin{bmatrix} \alpha\delta\vec{v}^{m}_1 + (1-\alpha\delta)\vec{u}^{m}_{\mathrm{O}} \\ \vec{d}_1 \end{bmatrix},$$
$$\text{with } \delta = \frac{2m+1}{m+1},\ \gamma = \frac{m+1-\alpha(2m+1)}{m\alpha}$$
$$\text{and } \vec{d}_1 = \frac{\alpha}{m+1}\left[\,2\ 1\ (0)^{2m-3}\ \gamma\ 2(1+\gamma)\,\right]^T,$$
$$\vec{x}'_i = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_{i-1} + \vec{x}_i\right) = \begin{bmatrix} \alpha\delta\vec{v}^{m}_i + (1-\alpha\delta)\vec{u}^{m}_{\mathrm{O}} \\ \vec{d}_i \end{bmatrix},$$
$$\text{with } \vec{d}_i = \frac{\alpha}{m+1}\left[\,(0)^{2i-3}\ 1\ 2\ 1\ (0)^{2(m-i)-1}\ \gamma\ 2(1+\gamma)\,\right]^T, \text{ for } 2 \le i \le m-1,$$
$$\vec{x}'_m = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_{m-1} + \vec{x}_m\right) = \begin{bmatrix} \alpha\delta\vec{v}^{m}_i + (1-\alpha\delta)\vec{u}^{m}_{\mathrm{O}} \\ \vec{d}_i \end{bmatrix},$$
$$\text{with } \vec{d}_m = \frac{\alpha}{m+1}\left[\,(0)^{2m-3}\ 1\ 2\ 1+\gamma\ 2(1+\gamma)\,\right]^T,$$
$$\vec{x}'_0 = \frac{1-\alpha}{m+1}\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}'_m - \vec{x}'_{m+1} = \begin{bmatrix} (1-\alpha)\vec{u}^{m}_{\mathrm{E}} + \alpha\delta\vec{v}^{m}_m + \alpha(1-\delta)\vec{u}^{m}_{\mathrm{O}} \\ \vec{d}_0 \end{bmatrix},$$
$$\text{with } \vec{d}_0 = \frac{\alpha}{m+1}\left[\,(0)^{2m-3}\ 1\ 2\ 0\ \tfrac{1-\alpha}{\alpha}\,\right]^T.$$

Any solution $\vec{x} = [p_0, \ldots, p_{2m+1}, d_0, \ldots, d_{2m}]$ of (50) is a linear combination of the vectors above, that is

$$\vec{x} = \sum_{i=0}^{m+1} \lambda_i \vec{x}_i'.$$

From Constraint (44) and the fact that $\sum_{j=0}^{2m+1} x_{ij}' = 1$, for all $0 \le i \le m+1$, we have:

$$\sum_{i=0}^{m+1} \lambda_i = \sum_{i=0}^{m+1} \lambda_i \sum_{j=0}^{2m+1} x_{ij}' = \sum_{i=0}^{2m+1} p_i = 1.$$

Suppose that $\frac{m+1}{2m+1} < \alpha < 1$. Then $p_0 = \lambda_0 \frac{1-\alpha}{m+1}$ and $d_{2i-1} = \lambda_i \frac{2(1-\alpha)}{m}$, for $1 \le i \le m$. Hence, by Constraints (42)–(43), we get that $\lambda_i \ge 0$, for all $0 \le i \le m$.

Now, suppose that $0 < \alpha \le \frac{m+1}{2m+1}$. Then $p_0 = \lambda_0 \frac{1-\alpha}{m+1}$, $d_{2i-1} = \lambda_i \frac{2\alpha}{m+1}$, for $1 \le i \le m-1$, $p_{2m+1} = \lambda_{m+1} \frac{\alpha}{m+1}$ and $d_{2m-1} = (\lambda_m + \lambda_0) \frac{2\alpha}{m+1}$. Thus, by Constraints (43), we get that $\lambda_i \ge 0$, for all $0 \le i \le m-1$ and $i = m+1$, as well as $\lambda_0 + \lambda_m \ge 0$. □

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