Math Meth Oper Res (2002) 55:383–400

Mathematical Methods
of Operations Research

Linear bilevel problems: Genericity results and an efficient method for computing local minima

Georg Still

University of Twente, Faculty of Mathematical Science, P.O. Box 217, 7500 AE Enschede, The Netherlands (e-mail: g.still@math.utwente.nl)

Manuscript received: September 2000/Final version received: December 2001

Abstract. The paper is concerned with linear bilevel problems. These nonconvex problems are known to be NP-complete. So, no theoretically efficient method for solving the global bilevel problem can be expected. In this paper we give a genericity analysis of linear bilevel problems and present a new algorithm for efficiently computing local minimizers. The method is based on the given structural analysis and combines ideas of the Simplex method with projected gradient steps.

Key words: linear bilevel programming, genericity results, numerical methods

Mathematical Subject Classification 1991: 90C26

1 Introduction

This paper deals with linear bilevel problems of the form

$$\text{(LBL)}: \min_{x,y} a^1x + b^1y \quad \text{s.t.} \quad A^1x + B^1y \leq c^1$$

$$\text{and } y \text{ is a solution of}$$

$$Q(x): \min_{y} a^2x + b^2y \quad \text{s.t.} \quad A^2x + B^2y \leq c^2.$$

with given matrices $A^1 \in \mathbb{R}^{k_1 \cdot n}$, $B^1 \in \mathbb{R}^{k_1 \cdot m}$, $A^2 \in \mathbb{R}^{k_2 \cdot n}$, $B^2 \in \mathbb{R}^{k_2 \cdot m}$, vectors a^1, $a^2 \in \mathbb{R}^n$, b^1, $b^2 \in \mathbb{R}^m$, $c^1 \in \mathbb{R}^{k_1}$, $c^2 \in \mathbb{R}^{k_2}$ and variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

Throughout the paper we omit the transposed sign in some expressions. For example ax denotes the inner product $a^T x$ in $\mathbb{R}^n$ and uB^2 the left multiplication of the matrix B^2 by the vector $u \in \mathbb{R}^{k_2}$.

The linear bilevel problem can be considered as a game between an upper level player and a lower level player which for given $x \in \mathbb{R}^n$ has to solve the lower level problem $Q(x)$. The constraints $A^1x + B^1y \leq c^1$, resp. $A^2x + B^2y \leq c^2$ are called upper-resp. lower level constraints. For a theoretical and practical introduction into bilevel programming the reader is referred to [12] and [2]. LBL-problems are non-convex problems which are NP-complete (cf. [3]).

The aim of the present paper is twofold. Firstly we develop genericity results for linear bilevel problems. By genericity results we roughly mean statements which assert that for *almost all* LBL-problems certain nice properties are fulfilled. Secondly, since the LBL-problem is NP-complete, it could be of interest to develop an algorithm which is able to compute at least a local minimizer efficiently. We present such an algorithm for the LBL-problems without upper level constraints. The algorithm is based on the genericity analysis and combines ideas of the Simplex method in linear programming with projected gradient steps.

The paper is organized as follows. In the second section we give an overview on the structure of the LBL-problems. Section 3 is concerned with genericity results. In Section 4 we introduce our new algorithm for computing local minimizer of LBL and discuss complexity questions. In the last section we report on numerical experiments by comparing the performance of our local minimization algorithm with a Kuhn-Tucker method of Bard/Moore for solving the global LBL-problem.

2 Preliminary results

We firstly introduce some notation. With $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ we define,

$M_2(x) = \{y \mid A^2x + B^2y \leq c^2\}$	feasible set of $Q(x)$
$M_2 = \{(x, y) \mid A^2x + B^2y \leq c^2\}$	lower level feasible set
$X_2 = \{x \mid M_2(x) \neq \emptyset\}$	projection of M_2 onto $\mathbb{R}^n$
$S(x) = \{y \mid y \text{ solves } Q(x)\}$	set of solutions of $Q(x)$
$S = \{(x, y) \mid y \in S(x)\}$	the graph of $S(x)$
$v(x) = a^2x + b^2y \quad \text{with } y \in S(x)$	value function of $Q(x)$
$M_1 = \{(x, y) \mid A^1x + B^1y \leq c^1\}$	upper level feasible set
$M_{\text{sem}} = M_1 \cap M_2$	the semi-feasible set
$X = \{x \mid (x, y) \in M_{\text{sem}}, \text{ for some } y\}$	projection of M_{sem} onto $\mathbb{R}^n$
$M = \{(x, y) \mid (x, y) \in M_{\text{sem}}, y \in S(x)\}$	feasible set of LBL

Remark 1. The polyhedra M_2, M_{sem} and their projections X_2, X are closed sets.

Throughout the paper the following abbreviations are used: We put $N = n + m$, $z = (x, y) \in \mathbb{R}^N$ and

$$A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}, \quad c = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix},$$

$$d^1 = \begin{pmatrix} a^1 \\ b^1 \end{pmatrix}, \quad d^2 = \begin{pmatrix} a^2 \\ b^2 \end{pmatrix}, \quad C = [A\ B].$$

We define $J^1 = \{1, \dots, k_1\}$, $J^2 = \{k_1 + 1, \dots, k_1 + k_2\}$, $J = J^1 \cup J^2$ and denote by C_j, A_j, B_j the j^{th} rows of C, A, B, $j \in J$. Then, the semi-feasible polyhedron can be written as

$$M_{\text{sem}} = \{z \in \mathbb{R}^N \mid C_j z \leq c_j, j \in J\}.$$

For a given index set $J_0 \subset J$ let C_{J_0} be the sub-matrix of C containing only the rows C_j with indices $j \in J_0$. A_{J_0}, B_{J_0}, c_{J_0} are defined accordingly.

A subset $f_0 \subset M_{\text{sem}}$ is called a face of the polyhedron M_{sem} if there exists an index set $J_0 \subset J$ such that

$$f_0 = f(J_0) := \{z \in M_{\text{sem}} \mid C_j z = c_j, j \in J_0\}. \tag{1}$$

Given $J_0 \subset J$ and the related face $f_0 = f(J_0)$ of M_{sem}, we say that f_0 has dimension d, $0 \leq d \leq N$, if there exists an element $z_0 \in f_0$ such that

$$C_j z_0 < c_j, \quad j \in J \backslash J_0 \quad \text{and} \quad \dim span\{C_j, j \in J_0\} = N - d. \tag{2}$$

The d-dimensional face $f_0 = f(J_0)$ is said to be non-degenerate if $|J_0| = N - d$. A vertex of M_{sem} is a face of dimension 0. If $\operatorname{int} M_{\text{sem}} \neq \emptyset$ then M_{sem} is a face of dimension N.

The following assumptions will play an important role. (In practice these assumptions do not imply a restriction. If necessary, we can add appropriate (large) box constraints to bound the feasible sets.)

A1: For all $x \in X_2$ the solution set $S(x)$ is bounded (and thus compact).
A2: The polyhedron M_{sem} is bounded (thus compact).

The following theorem contains the main results on the structure of linear bilevel problems.

Theorem 1. *For an LBL-problem the following holds.*

(a) *The set $X_2 \subset \mathbb{R}^n$ is a polyhedron (thus closed and convex).*
(b) *The feasible set M consist of a union $f_1 \cup f_2 \cup \cdots \cup f_l$ of faces of the polyhedron M_{sem}. In particular, M is a closed set in R^{n+m}.*

If moreover the assumptions $A1$ and $A2$ hold we have:

(c) *The value function $v(x)$ of $Q(x)$ is convex and Lipschitz continuous on X_2.*
(d) *A global solution of LBL occurs at a vertex of M_{sem}.*
(e) *For any local minimizer (x_k, y_k) of LBL on a face f_k there exist a local minimizer $(\bar{x}_k, \bar{y}_k) \in f_k$ which is a vertex of M_{sem} with the same value $a^1 x_k + b^1 y_k = a^1 \bar{x}_k + b^1 \bar{y}_k$.*

Proof. For a proof of (a)–(d) we refer to [12] (or to [13] for a slightly more general bilevel problem).

(e) This is obvious by noticing that a local minimizer (x_k, y_k) of LBL on a face f_k is a global minimizer on the (bounded) polyhedron f_k. □

Th feasible set of an LBL-problem with upper level constraints need not be connected. However the following holds.

Theorem 2. *For all $x \in X_2$ let the set $S(x)$ be non-empty (fulfilled if A2 holds). Then if no upper level constraints are present, the feasible set M of the LBL-problem is path-wise connected.*

Proof. We have to show that for any two points (x^1, y^1), (x^2, y^2) in M there is a path in M from (x^1, y^1) to (x^2, y^2). Suppose that this is not the case. Consider the maximal path-connected component M^1 in M containing (x^1, y^1) and suppose that M^1 does not coincide with M. Since M consists of the union of say K faces and any face is convex (thus path-wise connected), M^1 consists of a number of these faces say $f_1, \dots, f_{K_1}$, $K_1 < K$, i.e. $M^1 = \bigcup_{k=1}^{K_1} f_k$. It now follows that

$$\left(\bigcup_{k=1}^{K_1} f_k\right) \cap f_j = \emptyset, \quad j = K_1 + 1, \dots, K.$$

In fact, $\bar{x} \in f_j \cap M^1$ implies that $f_j \subset M^1$. Consequently, with the closed set $M^2 := \bigcup_{j=K_1+1}^{K} f_j$, we must have

$$M^1 \cup M^2 = M, \quad M^1 \cap M^2 = \emptyset.$$

Let X^i, $i \in \{1, 2\}$, denote the projections of M^i onto $\mathbb{R}^n$. We have $X = X^1 \cup X^2$ and X^1, X^2 are closed (projections of polyhedra are polyhedra). Since there are no upper level constraints it follows that $M_{\text{sem}} = M_2$ and $X = X_2$. Moreover for any $x \in X$ we have $S(x) \neq \emptyset$. Thus the projection onto $\mathbb{R}^n$ of M coincides with X and X is a polyhedron (in particular convex). Consider the line segment L between the points $x^1 \in X^1$ and $x^2 \in X^2$. Since $L \subset X = X^1 \cup X^2$ and X^1, X^2 are closed, there must exist a point $\bar{x} \in L$ belonging to both sets X^1 and X^2. Consequently, there are points $(\bar{x}, \bar{y}^1) \in M^1$ and $(\bar{x}, \bar{y}^2) \in M^2$. Since $S(\bar{x})$ is convex, the whole line segment between $(\bar{x}, \bar{y}^1)$ and $(\bar{x}, \bar{y}^2)$ lies in M. This contradicts the assumption that M^1 is not path-wise connected with M^2. □

In Section 4 we develop a new algorithm for computing local solutions of linear bilevel problems without upper level constraints. This raises the question whether it is possible to avoid upper level constraints in the LBL-model. If there are upper level constraints then the the players could change the model by passing the upper level constraints to the lower level. Such a strategy may change the model but it can only be an advantage for the upper level player. For the lower level player it can lead to a better but also to an inferior objective value depending on whether his objective is 'similar' or 'opposite' to the upper level objective.

Let LBL_0 be the bilevel problem obtained from LBL by passing the constraints $A^1x + B^1y \le c^1$ to the lower level and let M_0 be the feasible set of LBL_0. Then the following result is obvious.

Lemma 1. *For the the feasible sets M_0 and M of LBL_0 and LBL we have*

$$M \subset M_0.$$

In particular, for the corresponding minimal values the inequality $v_0 \le v$ holds.

3 Genericity results for LBL

In this section we study the structure of the feasible set and the set of local minimizers of LBL from a generic viewpoint.

Throughout the paper, by a generic subset G of $\mathbb{R}^K$ we mean a set which is open and has a complement $G^c = \mathbb{R}^K \setminus G$ of measure zero (notation $\mu(G^c) = 0$). Note that $\mu(G^c) = 0$ implies that the set G is dense in $\mathbb{R}^K$. For details on genericity and stratification theory we refer to [8].

Our genericity analysis will be based on the following 'non-trivial' result (see [8]).

Lemma 2. *Let $p : \mathbb{R}^K \to \mathbb{R}$ be a polynomial function, $p \not\equiv 0$. Then, the solution set $p^{-1}(0) = \{w \in \mathbb{R}^K \mid p(w) = 0\}$ is a closed set of measure zero. Equivalently the complement $G = \mathbb{R}^K \setminus p^{-1}(0)$ is a generic set in $\mathbb{R}^K$.*

By noticing that $\det A = \sum_{\pi \in \Pi_l} \operatorname{sign} \pi a_{1\pi(1)} \dots a_{l\pi(l)}$ defines a polynomial mapping $p : \mathbb{R}^{l \cdot l} \to \mathbb{R}$ we directly are led to the following result which will be used repeatedly.

Lemma 3. *Let V_l denote the set of real $(l \times l)$-matrices, $V_l = \{A = (a_{ij})_{i,j=1,\dots,l} \mid a_{ij} \in \mathbb{R}\} \equiv \mathbb{R}^{l \cdot l}$. Then, the set $V_l^0 = \{A \in V_l \mid \det A = 0\}$ is a closed set of measure zero in $\mathbb{R}^{l \cdot l}$. Equivalently the set $V_l^r = V_l \setminus V_l^0$ of regular matrices is generic in $\mathbb{R}^{l \cdot l}$.*

Remark 2. In the proof of Theorem 3 later on we implicitly make use of the following elementary facts:

Let V be a generic subset in $\mathbb{R}^q$. Then $\mathbb{R}^s \times V$ is generic in $\mathbb{R}^s \times \mathbb{R}^q$.
Let $V_1, \dots, V_r$ be generic subsets of $\mathbb{R}^q$. Then the intersection $V = \bigcap_{i=1}^r V_i$ is generic in $\mathbb{R}^q$.

It is well-known that for common linear programs generically all vertices of the feasible set are non-degenerate. In the following we generalize such a genericity result to linear bilevel problems.

Firstly we introduce a set which formally describes a linear bilevel problem as a point in $\mathbb{R}^K$. Let $p = (n, m, k_1, k_2)$ be fixed (p gives the 'size' of the LBL). A problem LBL can be seen as an element from the set

$$\mathscr{P}_p = \{P = (A^1, A^2, B^1, B^2, c^1, c^2, a^1, a^2, b^1, b^2)\}$$

with arrays $A^1, \dots, b^2$ as defined in Section 1. The set $\mathscr{P}_p$ can be identified with $\mathbb{R}^K$,

$$\mathscr{P}_p \equiv \mathbb{R}^K, \quad K := (k_1 + k_2 + 2)(n + m) + k_1 + k_2.$$

The following theorem contains the main genericity results.

Theorem 3. *The problem set $\mathscr{P}_p \equiv \mathbb{R}^K$ contains a generic subset $\mathscr{V}$ such that for any LBL-problem P in $\mathscr{V}$ the following holds.*

(a) *All faces of the semi-feasible polyhedron M_{sem} of problem P are non-degenerate faces. In particular all vertices of M_{sem} are non-degenerate.*
(b) *For any $x \in X$, if $Q(x)$ has a solution, then there is a unique solution $y(x)$ of $Q(x)$ attained at a vertex of $M_2(x)$.*
(c) *All local minimizers $z_\nu = (x_\nu, y_\nu)$, $\nu = 1, \dots, q$, of P are locally unique minimizers and (non-degenerate) vertices of M_{sem}. All values $v_\nu = d^1 z_\nu$, $\nu = 1, \dots, q$, are different. In particular, P has a unique global minimizer.*
(d) *The feasible set M of P consist of a union $f_1 \cup f_2 \cup \cdots \cup f_l$ of non-degenerate faces f_i of M_{sem} of dimension n.*

Proof. (a) Suppose $f_0 = f(J_0)$ is a face of M_{sem} of dimension d, $0 \le d \le N$, i.e. with an index $J_0 \subset J$ and a point $z_0 \in f_0$ we have

$$C_j z_0 < c_j, \quad j \in J \backslash J_0 \quad \text{and} \quad \dim span\{C_j, j \in J_0\} = N - d. \tag{3}$$

We now show that generically the face is nondegenerate, i.e. $|J_0| = N - d$.

Case $d = 0$: Then $f_0 = \{z_0\}$ is a vertex of M_{sem}. Suppose that $|J_0| > N$ holds. Then there is some subset $J_1 \subset J_0$ such that $|J_1| = N$,

$$C_{J_1} z_0 = c_{J_1} \quad \text{and} \quad C_{J_1} \text{ is a regular } (N \times N)\text{-matrix.}$$

Choose $j_0 \in J_0 \backslash J_1$ arbitrarily. Then for the vertex z_0 we have

$$C_{j_0} z_0 \neq c_{j_0} \Leftrightarrow C_{J_1, j_0} := \begin{pmatrix} C_{J_1} & c_{J_1} \\ C_{j_0} & c_{j_0} \end{pmatrix} \text{ is a regular matrix.} \tag{4}$$

By Lemma 2 the set $\mathscr{V}_{J_1, j_0} = \{P \in \mathscr{P}_p \mid \det C_{J_1, j_0} \neq 0\}$ is generic in $\mathscr{P}_p$. By Remark 2 also the intersection $\mathscr{V}_1 := \bigcap_{j_0 \in J_0 \backslash J_1} \mathscr{V}_{J_1, j_0}$ is generic. By construction, in this set $\mathscr{V}_1$ the vertex z_0 is non-degenerate.

Case $d > 0$: Suppose that $|J_0| > N - d$. Assume for brevity $J_0 = \{1, \dots, k\}$, $(k = |J_0| > n - d)$. Let C be the $(N \times k)$-matrix $C = [C_1, \dots, C_k]$. Relation (3) implies rank $C = N - d$ and then

$$\det(C_{ij})_{i,j=1,\dots,N-d+1} = 0.$$

By Lemma 2 this can generically be excluded.

For the proof of (c) we moreover now show that for a generic subset of problems in $\mathscr{P}_p$ all vertices $\bar{z} = (\bar{x}, \bar{y})$ of M_{sem} have different function values

$\bar{v} = d^1\bar{z}$. To do so let us assume that z_0 and $z_1, z_0 \neq z_1$, are (non-degenerate) vertices of M_{sem}. Then with corresponding index sets $J_0, J_1 \subset J,\ J_0 \neq J_1$, $|J_0| = |J_1| = n + m$ we have

$$z_0 = C_{J_0}^{-1} c_{J_0}, \quad z_1 = C_{J_1}^{-1} c_{J_1}.$$

With the adjoint $C_{J_0}^{\text{ad}}$ of C_{J_0} we can write $C_{J_0}^{-1} = \dfrac{1}{\det C_{J_0}} C_{J_0}^{\text{ad}}$ and accordingly $C_{J_1}^{-1} = \dfrac{1}{\det C_{J_1}} C_{J_1}^{\text{ad}}$. Now, the values v_0 and v_1 are the same, i.e. $d^1z_0 - d^1z_1 = 0$, if and only if

$$p(C_{J_0}, c_{J_0}, C_{J_1}, c_{J_1}, d^1) := \det C_{J_1} \cdot d^1 C_{J_0}^{\text{ad}} c_{J_0} - \det C_{J_0} \cdot d^1 C_{J_1}^{\text{ad}} c_{J_1} = 0.$$

This relation represents a polynomial equation $p = 0$ with a non-vanishing polynomial p. In View of Lemma 2 the set $S_{J_0,J_1} := p^{-1}(0)$ is closed and of measure zero in $\mathscr{P}_p$. Thus, the complement $\mathscr{V}_{J_0,J_1} = \mathscr{P}_p \backslash S_{J_0,J_1}$ is generic. By construction, for P in $\mathscr{V}_{J_0,J_1}$ the vertices z_0, z_1 have different values. Since there are only finitely many such sets $J_0, J_1 \subset J$ the intersection of all corresponding sets $\mathscr{V}_{J_0,J_1}$ is generic in $\mathscr{P}_p$.

(b) Choose $x_0 \in X$ arbitrarily and consider the lower level problem

$$Q(x_0)\colon \min_y d^2y \quad \text{st. } B_jy \leq b_j - A_jx_0, \quad j \in J^2.$$

Suppose y_0 is a solution of $Q(x_0)$. Then there exist J_0, $J_0 \subset J^2$, $|J_0| \leq m$ (by Caratheodory's Theorem), $0 < u_0 \in \mathbb{R}^{|J_0|}$ such that

$$u_0 B_{J_0} = -b^2, \quad B_jy_0 = b_j - A_jx_0, \quad j \in J_0.$$

Generically, $|J_0| \geq m$, i.e. we can assume $|J_0| = m$. In fact, if $|J_0| < m$ then in view of $u_0B_{J_0} = -b^2$ the $(|J_0| + 1) \times (|J_0| + 1)$-matrix (assume for brevity $J_0 = \{1, \ldots, |J_0|\}$)

$$\hat{B} := [(B_{ij})_{\substack{i=1,\ldots,|J_0|+1 \\ j=1,\ldots,|J_0|}} \hat{b}] \quad \text{with } \hat{b} := (b_1^2, \ldots, b_{|J_0|+1}^2)^T$$

would satisfy $\det(\hat{B}) = 0$ which can generically be avoided. Since generically (with $|J_0| = m$) the matrix B_{J_0}, is regular, a solution y_0 of $Q(x_0)$ is generically a vertex of the polyhedron $M_2(x_0)$. Moreover since the multiplier vector u_0 is positive it is not difficult to show that y_0 is the unique solution.

(c) Let $z_0 = (x_0, y_0)$ be a local minimizer of the bilevel problem P. The feasible point z_0 belongs to a face $f_0 = f(J_0)$ of M_{sem} and by Theorem 1(b) we can assume $f_0 \in M$. Since f_0 is a polyhedron, z_0 is a global minimizer of the linear program

$$\min d^1z \quad \text{st.} \quad z \in f_0 = f(J_0) := \{z \in \mathbb{R}^{n+m} \mid C_z \leq c, C_{J_0}z = c_{J_0}\}. \tag{5}$$

With the same arguments as in part (b) we can show that generically the solution of this program occurs at a vertex z_1 of the polyhedron f_0 and that the solution is unique. The vertex z_0 of the face f_0 is also a (non-degenerate) vertex of M_{sem}. By the arguments in the proof of (a) all vertices have different values.

(d) Choose $z_1 = (x_1, y_1) \in M$. Since y_1 solves $Q(x_1)$ there exist $J_1 \subset J$, $J_1^2 \subset J_1 \cap J^2$, $|J_1^2| = m$, $u_1 \geq 0$, $u_1 \in \mathbb{R}^m$ such that

$$C_j z_1 < c_j, \quad j \in J \backslash J_1, \quad C_j z_1 = c_j, \quad j \in J_1 \quad u_1 B_{J_1^2} = -b^2. \tag{6}$$

We now show that generically z_1 is contained in a face f_0 of dimension n given by (1) with $J_0 = J_1^2$. (Since $|J_1^2| = m$ by definition (see Section 2) this face is non-degenerate.)

We firstly notice that for a generic subset of $\mathscr{P}_p$ we have $|J_1| \leq n+m$. Otherwise $|J_1| > n+m$ and with some $J_1^0 \subset J_1$, $|J_1^0| = n+m$, $j_0 \in J_1 \backslash J_1^0$ the quadratic matrix $C_{J_1^0 j_0}$ in (4) would be singular which generically can be avoided.

Moreover, for a generic subset in $\mathscr{P}_p$ we have

$$\operatorname{rank} C_{J_1} = |J_1| \quad \text{and} \quad \operatorname{rank} C_{J_1^2} = |J_1^2| = m. \tag{7}$$

This holds since the condition $\operatorname{rank} C_{J_1} < |J_1|$ or $\operatorname{rank} C_{J_1^2} < |J_1^2|$ would imply that

$$\det(C_{ij})_{i,j \in J_1} = 0 \quad \text{or} \quad \det(C_{ij})_{i,j \in J_1^2} = 0$$

which by Lemma 2 can generically be excluded.

We now show that z_1 is contained in an n-dimensional (non-degenerate) face. Using $\operatorname{rank} C_{J_1} = |J_1| \leq n+m$ (see (7)) there exist a vector $\xi \in \mathbb{R}^{n+m}$ satisfying

$$C_{J_1^2} \xi = 0, \quad C_j \xi = -1, \quad j \in J_1 \backslash J_1^2.$$

Then, for $z_0 := z_1 + t\xi$, $t > 0$ small enough, we have (cf. (6))

$$C_{J_1^2} z_0 = c_{J_1^2}, \quad C_j z_0 < c_j, \quad j \in J_1 \backslash J_1^2, \quad u_1 B_{J_1^2} = -b^2. \tag{8}$$

Thus z_1 and z_0 are contained in the feasible face

$$f_0 = \{z \in M_{\mathrm{sem}} \mid C_{J_1^2} z = c_{J_1^2}\}$$

of dimension $d = n + m - |J_1^2| = n$.

Suppose now that z_1 is contained in a feasible face f_2 of dimension greater than n. Then by definition there exist a feasible point $z_2 = (x_2, y_2) \in f_2$ and index sets $J_2 \subset J$, $J_2^2 := J_2 \cap J^2$ and $u_2 \geq 0$, $u_2 \in \mathbb{R}^{J_2^2}$ such that

$$C_{J_2} z_2 = c_{J_2}, \quad C_j z < c_j, \quad j \in J \backslash J_2, \quad u_2 B_{J_2^2} = -b^2$$

and $\dim \operatorname{span}\{C_j, j \in J_2\} = n + m - d < m$ (i.e. $d > n$). Generically we can assume that C_{J_2} has full rank$|J_2|$ (see (7)). This implies $|J_2^2| \leq |J_2| < m$ and y_2 is not a vertex solution of $Q(x_2)$. However this can generically be excluded as shown in the proof of part (b). $\square$

We say that the semi-feasible set M_{sem} satisfies the Slater condition if there is a point z_0 such that

$$Cz_0 < c.$$

Such a point z_0 is an inner point of M_{sem}. For the numerical computations we want to restrict the problem set to the following set of linear bilevel problems.

$$\begin{aligned}\mathscr{P}_p^r = \{P \in \mathscr{P}_p \mid\ & M_{\text{sem}} \text{ fulfills the Slater condition, } M_{\text{sem}} \text{ compact,} \\ & S(x) \text{ is compact } \forall x \in X\}.\end{aligned}$$

In this set, for any $x \in X$ a solution of $Q(x)$ exists. The following stability statement holds.

Lemma 4. *The problem set* $\mathscr{P}_p^r$ *is open in* $\mathbb{R}^K$.

Proof. For $\bar{P} \in \mathscr{P}_p$ let the Slater condition be satisfied with $\bar{z}$, i.e. $\bar{C}\bar{z} < \bar{c}$ ($\bar{C}, \bar{c}$ defining the constraints of $\bar{P}$). Then, obviously for a whole neighbourhood of problems $P \in \mathscr{P}_p$ the condition $C\bar{z} < c$ holds.

To show that M_{sem} is compact it suffice to prove boundedness. We show: Given $\bar{P} \in \mathscr{P}_p^r$ with bounded $M_{\text{sem}}(\bar{P})$ there exists some ε such that

$$\bigcup_{\|P-\bar{P}\|<\varepsilon} M_{\text{sem}}(P) \text{ is bounded.} \tag{9}$$

Suppose (9) does not hold. Then there exists a sequence of problems $P_k \in \mathscr{P}_p$ and vectors $z_k \in M_{\text{sem}}(P_k)$ such that (with C_k, c_k corresponding to P_k)

$$P_k \to \bar{P}, \quad C_k z_k \leq c_k, \quad \text{and} \quad \|z_k\| \to \infty \quad \text{for } k \to \infty. \tag{10}$$

By dividing the constraints by $\|z_k\|$ and assuming (take a subsequence) $\dfrac{z_k}{\|z_k\|} \to \hat{z}$ we find for $k \to \infty$,

$$\bar{C}\hat{z} \leq 0.$$

Choosing a point $\bar{z} \in M_{\text{sem}}(\bar{P})$ also $z(t) := \bar{z} + t\hat{z} \in M_{\text{sem}}(\bar{P})$ for all $t > 0$ contradicting the boundedness of $M_{\text{sem}}(\bar{P})$.

We finally prove the statement for $S(x)$. Let us assume that we have given a problem $\bar{P} \in \mathscr{P}_p^r$ such that the corresponding sets $\bar{S}(x)$ are compact for all $x \in \bar{X}$. We have to show that there exists some ε such that for all P, $\|P - \bar{P}\| < \varepsilon$ with corresponding sets S and X the property

$$S(x) \text{ is compact for all } x \in X \tag{11}$$

holds. We only have to prove boundedness since the solution sets $S(x)$ are always (closed) faces of $M_2(x)$. Suppose now that (11) is not true in a neighborhood of $\bar{P}$. Then there exists a sequence of problems $P_k \in \mathscr{P}_p$, $P_k \to \bar{P}$ and points $x_k \in X_k$, $y_k \in S_k(x_k)$ such that

$$\|y_k\| \to \infty \quad \text{for } k \to \infty.$$

In view of $x_k \in X_k$ we can choose elements $(x_k, \tilde{y}_k) \in M_{\text{sem}}(P_k)$. Using (9) we can assume (taking a subsequence)

$$(x_k, \tilde{y}_k) \to (\bar{x}, \tilde{y}) \in M_{\text{sem}}(\bar{P}). \tag{12}$$

By assumption, $S(\bar{x})$ is bounded. Since the sequence $\|(x_k, \tilde{y}_k)\|$ is bounded the following inequalities hold with some $\rho > 0$,

$$a_k^2 x_k + b_k^2 y_k \le a_k^2 x_k + b_k^2 \tilde{y}_k \le \rho.$$

Consequently

$$A_k^2 x_k + B_k^2 y_k \le c_k^2, \quad a_k^2 x_k + b_k^2 y_k \le \rho.$$

Dividing these relations by $\|y_k\|$ and assuming $\dfrac{y_k}{\|y_k\|} \to \hat{y}$, $\|\hat{y}\| = 1$ we find using $\|y_k\| \to \infty$ that

$$\bar{B}^2 \hat{y} \le 0 \quad \text{and} \quad \bar{b}^2 \hat{y} \le 0. \tag{13}$$

We choose some $\bar{y} \in \bar{S}(\bar{x})$ and define $y(t) := \bar{y} + t\hat{y}$. In view of (13), for all $t > 0$,

$$\bar{A}^2 \bar{x} + \bar{B}^2 y(t) \le \bar{c}^2 \quad \text{and} \quad \bar{a}^2 \bar{x} + \bar{b}^2 y(t) \le \bar{a}^2 \bar{x} + \bar{b}^2 \bar{y},$$

i.e. $y(t) \in \bar{S}(\bar{x})$. This contradicts the fact that $\bar{S}(\bar{x})$ is bounded. $\square$

These genericity results in particular mean that given a LBL-problem P which does not have the nice properties in Theorem 3 (i.e. $P \notin \mathscr{V}$) by almost all arbitrarily small perturbations we obtain a problem in $\mathscr{V}$. However, in contrast to the situation for linear programs, where a 'small' perturbation of the problem data leads to a 'small' perturbation of the minimal value, here for LBL-problems an arbitrarily small perturbation of the problem may lead to a large perturbation in the minimal value.

4 A new algorithm for computing local minima of LBL

Different methods for solving linear bilevel problems have been designed. For example the algorithm of Bard/Moore in [1] which combines a Kuhn-Tucker approach with a branch and bound method, the penalty method (see e.g. White/Anandalingam [14]) and the subgradient method (see e.g. Falk/Liu [4]). An overview of numerical methods is to be found in [12] and [2].

It is well-known that the LBL-problem (also the problem without upper level constraints) is NP-complete (see [3]). So, (unless P = NP) no efficient (polynomial) algorithm can be expected to solve the global minimization problem for LBL. Therefore it could be of interest to have a method which is able to compute at least a local minimizer of LBL efficiently.

In this section we describe such an algorithm for the bilevel problem without upper level constraints: With $z = (x, y) \in \mathbb{R}^{n+m}$, $C = [A \;\; B]$

$$(\mathrm{LBL}_0):\ \min_{x,y}\ a^1x + b^1y \quad (\text{or } d^1z) \quad \text{s.t. } y \text{ is a solution of}$$

$$Q(x):\ \min_{y}\ b^2y \quad \text{s.t.} \quad Ax + By \le c \quad (\text{or } Cz \le c).$$

Again, C_j denotes the j-th row of C, $j \in J := \{1, \dots, k\}$. As usual, for z satisfying $Cz \le c$ the active index set is defined by $J(z) = \{j \in J \mid C_jz = c_j\}$. For $J_k \subset J$ we introduce the linear subspace

$$S(J_k) = \{z \in \mathbb{R}^{n+m} \mid C_{J_k}z = 0\}.$$

In every step of the algorithm below we have to compute the projection of the objective vector $-d^1$ onto a space $S(J_k)$ corresponding to a face $f(J_k)$ of M_{sem}.

Our method is based on the analysis of the structure of the feasible set in Section 2 and combines projected gradient steps with ideas of the Simplex method. The conceptual method is as follows:

Phase I: Compute a starting feasible point $z_0 = (x_0, y_0)$ of LBL_0.
Phase II: Compute a local minimizer $\bar{z} = (\bar{x}, \bar{y})$ by proceeding with projected gradient steps along feasible faces of dimension $(n - \kappa)$, $\kappa = \kappa_0, \dots, n$.

We now describe our algorithm in detail.

Phase I: (Computation of a feasible point z_0 and a descent direction d_0 in z_0)

1. Compute a solution $\hat{z} = (\hat{x}, \hat{y})$ of the LP-relaxation of LBL_0:

$$\min_{z}\ d^1z \quad \text{s.t.} \quad Cz \le c.$$

   (If $\hat{z}$ is feasible for LBL_0, i.e. if $\hat{y}$ solves $Q(\hat{x})$ then stop: $\hat{z}$ is a solution of LBL_0.)
2. Compute a solution y_0 of $Q(\hat{x})$; $z_0 := (\hat{x}, y_0)$ is a feasible point.
3. Put $J_0 = J(z_0)$ and compute the projection s_0 of $-d^1$ onto $S(J_0)$.

Phase II: (Computation of a local minimizer)

We start with the feasible point z_0 and the direction s_0 computed in Phase I and end up with a local minimizer $\bar{z}$ of LBL_0.

Step $k \to k+1$: We have given

a feasible point	$z_k = (x_k, y_k) \in \mathbb{R}^{n+m}$
a feasible descent direction	$s_k \in \mathbb{R}^{n+m}$
a multiplier	$u_k \ge 0$
an index set	$J_k \subset J(z_k)$, $m \le \lvert J_k \rvert \le n + m - 1$

such that

1. z_k and $z_k(t) := z_k + ts_k$ ($t \ge 0$ small) are contained in $f(J_k)$ and

$$C_jz_k(t) < c_j, \quad j \in J \backslash J_k, \quad \text{for all } t > 0 \text{ small}.$$

2. $d^1 s_k < 0$
3. $u_k B_{J_k} = -b^2$

(i): Move along the (feasible) ray $z_k(t) := z_k + ts_k$, $t \geq 0$ to the 'boundary' of M_{sem}. The maximum step-length is

$$t_k := \min_{j \notin J_k,\, C_j s_k > 0} \frac{c_j - C_j z_k}{C_j s_k} \quad \text{with } j_k^+ \in \operatorname{argmin}\{t_k\}.$$

Put $z_{k+1} := z_k + t_k s_k$.
(Since M_{sem} is bounded we must have $t_k < \infty$.)

(ii): Change to a new feasible face depending on the number $|J_k|$, $m \leq |J_k| \leq n + m - 1$. We distinguish between three cases

(A)	$\lvert J_k\rvert = m$	(face of 'maximum' dimension n)
(B)	$m < \lvert J_k\rvert \leq n + m - 2$	
(C)	$\lvert J_k\rvert = n + m - 1$	(feasible edge)

(A) $|J_k| = m$ (*try to move to a new feasible face of dimension* n)
Compute the solution $\bar{u}$ of

$$\bar{u} B_{J_k} = B_{j_k^+}.$$

case $\bar{u} \not\leq 0$:

(a) (*feasibility test w.r.t. the multipliers of the lower level*) Compute

$$\rho_k = \min_{j \in J_k, \overline{u_j} > 0} \left\{ \frac{(u_k)_j}{\overline{u_j}} \right\}, \quad j_k^- \in \operatorname{argmin}\{\rho_k\}.$$

and put $J_* = J_k \cup \{j_k^+\} \backslash \{j_k^-\}$.
(Note that by construction $B_{J_k}^T (u_k - \rho_k \bar{u}) + \rho_k B_{j_k^+} = -b^2$.)

(b) (*feasibility test w.r.t. constraints*) Compute the projection s_{k+1} of $-d^1$ onto $S(J_*)$.
If $s_{k+1} C_{j_k^-} < 0$ put $J_{k+1} = J_k \cup \{j_k^+\} \backslash \{j_k^-\}$.
(s_{k+1} a is feasible descent direction in z_{k+1} on the face $f(J_{k+1})$ of dimension n)
If $s_{k+1} C_{j_k^-} \geq 0$ put $J_{k+1} = J_k \cup \{j_k^+\}$ and compute the projection s_{k+1} of $-d^1$ onto $f(J_{k+1})$.
(s_{k+1} is feasible direction of descent in z_{k+1} on the face $f(J_{k+1})$ of dimension $n - 1$)

case $\bar{u} \leq 0$: (face $f(J_*)$ is not feasible)
Put $J_{k+1} = J_k \cup \{j_k^+\}$ and compute the projection s_{k+1} of $-d^1$ onto $S(J_{k+1})$.

(B) $m < |J_k| \leq n + m - 2$ (move to a face of dimension $n + m - |J_k| - 1$.)
Put $J_{k+1} = J_k \cup \{j_k^+\}$. and compute the projection s_{k+1} of $-d^1$ onto $S(J_{k+1})$.

(C) $|J_k| = n + m - 1$ (*find a feasible descent edge emanating from the vertex* z_{k+1})
Put $J_* = J_k \cup \{j_k^+\}$ (and assume for brevity $J_* = \{1, \ldots, n+m\}$).
For $i \in J_k$:

(1) Compute the solution s^i of $C_{J_*} s = -e_i$
(s^i is the direction of the edge emanating from z_{k+1})
if $d^1 s^i \geq 0$ goto next i, if $d^1 s^i < 0$ goto (2).
(2) Put $J^i = J_* \backslash \{i\}$ and solve the feasibility condition

$$u^i B_{J^i} = -b^2, \quad u^i \geq 0.$$

If a solution exists then put $s_{k+1} = s^i$, $J_{k+1} = J_k \cup \{j_k^+\} \backslash \{i\}$, $k \to k+1$. otherwise goto next i.

If no feasible edge of descent is found in z_{k+1} then the vertex z_{k+1} is a local minimizer of LBL_0 (see Lemma 5).

Lemma 5. *Given $z_{k+1} \in M_{sem}$ and let $J_* := J(z_{k+1})$ where z_{k+1} is a non-degenerate vertex. Suppose for all directions s^i (of the edges emanating from z_{k+1}), $i \in J_*$, at least one of the following holds:*

1. *The vector s^i is not a descent direction ($s^i d^1 \geq 0$)*
2. *The points $z_{k+1}(t) := z_{k+1} + ts^i$, with $t > 0$ small, are not feasible.*

Then the vertex z_{k+1} is a local minimum of LBL_0.

Proof. Assume z_{k+1} is not a local minimizer. Then a descent direction d must exist, such that

$$\text{For } t > 0 \text{ small, } z_{k+1}(t) = z_{k+1} + td \text{ is feasible and } dd^1 < 0. \tag{14}$$

Direction d can be written as positive combination of the directions s^i. So,

$$d = \sum_{i \in \bar{J}} \alpha_i s^i$$

with some $\bar{J} \subset J_*$ and $\alpha_i > 0$, $i \in \bar{J}$. (Again, assume for brevity $J_* = \{1, \ldots, n+m\}$.) Consequently, in view of $C_i s^i = -1$, $i \in \bar{J}$, the indices in $\bar{J}$ are no longer active for $z_{k+1}(t)$ for $t > 0$ small, i.e. $J(z_{k+1}(t)) = J_* \backslash \bar{J} =: J_0$. Moreover, since $z_{k+1}(t)$ is feasible, with some $\bar{u}$ we have

$$B_{J_0}^T \bar{u} = -b^2, \quad \bar{u} \geq 0. \tag{15}$$

Thus, for all $i \in \bar{J} = J_* \backslash J_0$ the multiplier $\bar{u}$ in (15) gives a solution of

$$B_{J_* \backslash \{i\}}^T u = -b^2, \quad u \geq 0.$$

In other words, s^i is a feasible direction in z_{k+1}. By assumption $s^i d^1 \geq 0$. Therefore,

$$dd^1 = \sum_{i\in \bar{J}} \alpha_i s^i d^1 \geq 0.$$

in contradiction to the second condition in (14). □

Our algorithm is a special instance of a method for solving general mathematical programs with equilibrium constraints (MPEC) in [6]. To write the problem LBL_0 in (MPEC) form we only have to express the minimization in the lower level equivalently by the Kuhn-Tucker condition. This leads to the following problem (with the notation above):

$$\min d^1 z \quad \text{s.t.} \quad c - Cz \geq 0, \quad \mu \geq 0, \quad \mu^T(c - Cz) = 0, \quad B^T u = -b^2.$$

The method in [6] is based on the computation of feasible descent directions in each step. Under certain regularity assumptions the algorithm is proven to converge to a so-called B-stationary point, i.e. a point $\bar{z}$ such that with the objective function $f(z)$ and the tangent cone $T(\bar{z}, F)$ at $\bar{z}$ w.r.t. the feasible set F:

$$\nabla f(\bar{z})d \geq 0 \quad \text{for all } d \in T(\bar{z}, F).$$

Note that in the algoritm above we move along a feasible descent direction s_k in each step. Moreover the algorithm converges to a local minimizer z_{k+1} satisfying the assumptions of Lemma 5 which implies that z_{k+1} is B-stationary.

Remark 3. (*Finiteness of the algorithm*)

If all feasible faces attained during the algorithm are non-degenerate and $s_k \neq 0$ for all projections (which is generically satisfied) the algorithm above computes a local minimizer after finitely many steps. This can be seen as follows. Suppose that in step (ii) A of the algorithm we arrive at a point z_{k+1} with active indices $J(z_{k+1}) = J_k \cup \{j_k^+\}$. Then if $s_{k+1}C_{j_k^-} < 0$ holds we pass to a new face $f(J_{k+1})$ with $J_{k+1} = J_k \cup \{j_k^+\} \setminus\cup \{j_k^-\}$ of dimension n and we never can come back to a point in the relative interior of the face $f(J(z_{k+1}))$. In the other case where $s_{k+1}C_{j_k^-} \geq 0$ we pass successively to faces of dimension $n-1, n-2, \ldots, 0$. Finally we end up with steps proceeding from vertex to vertex of the polyhedron M_{sem} with strictly decreasing objective value. So, during the algorithm we never can reach two points z_k, z_l with the same active index sets $J(z_k) = J(z_l)$. The result follows since there are only finitely many possible active sets.

Remark 4. The different steps of our algorithm can be implemented efficiently by using appropriate update formulas for the LR-decomposition (or QR-decomposition) of the 'basis matrices' B_{J_k} in each step (cf. e.g. [9]).

Remark 5. We restricted our algorithm to problems LBL_0 without upper level constraints. The reason is that for these problems a feasible starting point can be found efficiently (by solving two LP's in Phase I).

Unfortunately, if upper level constraints are present, then the point z_0 computed in Phase I need not satisfy the upper level constraints. In this case z_0 is not feasible for LBL. We did not succeed in finding an 'efficient' Phase I procedure for problems with upper level constraints. Note that this problem is

NP-complete. In fact finding a feasible point of a LBL with a single upper level constraint $a_1 x + b_1 y \le c_1$ is equivalent with solving the LBL-program

$$\min a_1 x + b_1 y \quad \text{s.t.} \quad y \text{ solves: } \min_y a^1 x + b^1 y \quad \text{s.t.} \quad A^2 x + B^2 y \le c^2$$

(with value not exeeding c_1) which is known to be (strongly) NP-complete. The modification of Phase II to general linear bilevel problems with upper level constraints does not make any problems.

We emphasize that our algorithm could be used to 'accelerate' branch and bound methods (for example, the Bard/Moore algorithm).

The 'pivot-strategy' for selecting a new feasible face can by modified in various directions. By Theorem 2, since the feasible set is path-wise connected, in principle we can reach the global minimizer of LBL_0 from our starting point z_0.

5 Computational experiments

In this section we report on some numerical experiments with our algorithm. We compare the computing time for our local search with the time needed for the global minimization by an implementation of the Bard/Moore method in Hamming [10] on the same machine and on the same randomly generated problems. (Note however, that there are more efficient methods for solving LBL-problems, e.g. the HJS-method (cf. [11], [2]) and the method in [7]).

Remark 6. The problems are randomly generated in the following way: The components of the vectors (a^1, b^1), (a^2, b^2) in the objectives and C_j in the constraints $Cz \le c$ are generated randomly in $[-100, 100]$. The right-hand side components c_j corresponding to C_j are given by $c_j = q_j \|C_j\|$ with q_j random in [1, 100].

Some results are presented in table (6.1), in which we use the abbreviations:

n:	Number of leader variables.
m:	Number of follower variables.
k:	Number of constraints (in addition we added the constraints $x, y \ge 0$).
$\lvert J_0 \rvert$:	Number of active indices at the feasible starting point z_0 (see Phase I).
N_{it}:	Number of iterations k in Phase II.
$v_0 = d^1 z_0$:	Objective value of the leader in the feasible starting point.
v_{loc}:	Objective value of the leader in the local minimum.
v_{glob}:	Objective function value of the leader in the global minimum.
t_{loc}:	time (in sec.) needed for computing the local minimizer (our algorithm).
t_{glob}:	time (in sec.) needed for computing the global minimizer (implementation of the Bard/Moore method).

In 12 of the 25 test problems our local method ended up with the global solution.

Table 6.1. Results of the computation of local minimum versus global minimum

n	m	k	$\|J_0\|$	N_{it}	v_0	v_{loc}	v_{glob}	t_{loc}	t_{glob}
2	6	8	8	1	7.68		7.68	0.3	8
2	8	10	10	1	−3.90		−3.90	0.4	19
2	10	12	11	4	−44.88	−61.20	−61.20	1.3	25
4	2	6	6	1	−21.67		−21.67	0.3	2
4	4	8	7	2	−1.69	−8.53	−9.08	0.5	7
4	6	10	9	2	−0.13	−5.59	−7.16	0.7	15
4	10	14	13	3	80.72	65.36	−15.22	1.9	220
6	2	8	8	1	−186.08		−186.08	0.3	3
6	4	10	8	4	−100.99	−121.58	−148.51	0.8	21
6	6	12	11	2	−155.01	−159.39	−159.39	0.9	62
6	8	14	14	1	−328.92		−328.92	0.9	154
6	10	16	15	3	−30.11	−32.30	−34.77	1.9	133
8	2	10	10	1	−129.85		−129.85	0.5	3
8	4	12	11	2	−8.89	−30.10	−38.22	0.8	28
8	6	14	14	1	−98.12		−98.12	0.8	15
8	8	16	14	3	−128.92	−131.08	−142.52	1.7	150
8	10	18	16	3	−69.48	−96.81	−102.55	2.5	691
10	2	12	12	1	−23.95		−23.95	0.6	9
10	4	14	14	1	−131.04		−131.04	0.8	24
10	6	16	11	6	−76.81	−131.69	−149.34	1.8	213
10	8	18	15	4	−101.81	−112.57	−122.42	2.3	429
10	10	20	19	2	−88.42	−88.76	−88.76	3.6	763
12	12	24	23	2	−1.26	−30.82	−113.65	5.6	3318
16	16	32	29	4	−6.73	−18.38	−47.28	13	11413
20	20	40	36	6	−11.09	−16.94	−17.57	21	13938

The next two tables contain results with problems for constant $n+m$ and different n, m. In the first table, for 8 of the 15, and in the second, for 6 of the 9 problems, the local method computed the global solution. In table 6.3, N_{ver} gives the number of vertex to vertex steps in Phase II iiC.

Table 6.2. Results for computing local minima versus global minima for constant $n+m=12$

(n,m,k)	$\|J_0\|$	v_0	v_{loc}	v_{glob}	t_{loc}	t_{glob}
$(4,8,16)$	11	−58.07	−61.52	−61.52	1	78
	10	−107.53	−127.18	−141.50	1	133
	11	−33.01	−33.74	−33.74	1	123
	11	−18.44	−40.29	−40.29	1	169
	11	−39.40	−73.24	−78.41	2	243
$(8,4,16)$	12	−107.82		−107.82	1	66
	11	−44.19	−56.84	−95.90	1	62
	11	12.79	1.63	−70.82	2	43
	12	−1.08		−1.08	1	12
	12	−1.02		−1.02	1	10
$(6,6,16)$	12	−8.89		−8.89	1	51
	11	−14.98	−19.62	−43.29	1	92
	11	−115.81	−116.36	−126.66	1	78
	11	−116.12	−123.14	−123.14	1	187
	10	−102.42	−129.60	−129.60	1	209

Table 6.3. Results of computation of local minimum versus global minimum for for constant $n+m=16$

(n,m,k)	$\|J_0\|$	N_{ver}	v_0	v_{loc}	v_{glob}	t_{loc}	t_{glob}
$(8,8,20)$	16	0	−135.96		−135.96	2	594
	13	6	−139.56	−173.84	−173.84	4	1005
	14	0	−60.40	−60.80	−60.80	2	923
$(4,12,20)$	15	0	151.87	150.50	−0.66	4	3366
	15	3	−85.15	−130.98	−140.70	4	1531
	14	0	−36.12	−42.10	−59.03	3	1781
$(12,4,20)$	16	0	−84.65		−84.65	2	139
	14	5	−60.98	−89.25	−89.25	3	209
	12	0	−121.34	−162.97	−162.97	2	75

The next table gives the computation time for the local search for increasing problem size (average of 3 randomly generated problems). The results suggest a polynomial behavior; doubling the problem size leads to a factor of about 10 in the computing time.

Table 6.4. Mean computing times for local minimum

(n,m,k)	mean t_{loc}	(n,m,k)	mean t_{loc}
$(2,2,6)$	0.3	$(14,14,42)$	22
$(4,4,12)$	0.7	$(16,16,48)$	38
$(6,6,18)$	1.6	$(18,18,54)$	34
$(8,8,24)$	3.0	$(20,20,60)$	97
$(10,10,30)$	7.8	$(24,24,72)$	140
$(12,12,36)$	11	$(28,28,84)$	302

Surprisingly, in all our computations we never had to start after Phase I on a feasible face of maximal dimension n. In many cases the starting feasible point z_0 in Phase I, coincides with the global minimizer. In most of the other cases the point z_0 was situated 'near' the local (or even global) solution, such that only few steps in Phase II had to be performed. This explains why in our experiments the computing time of our local search seems to behaves polynomial in contrast to the drastic increase in the computing time for the global search (compare for example the results in Tables 6.2 and 6.3 for $(n,m,k) = (4,8,16)$ and $(n,m,k) = (4,12,20)$; and also the experiments in [10])).

Acknowledgments. The author wish to thank Theo Frederiks for his numerical experiments in [5]. He is also indebted to the referees for their valuable comments in particular one of the referees for clarifying the NP-completeness of the feasibility problem for LBL with upper level constraints in Remark 5.

References

[1] Bard JF, Moore JT (1990) A branch and bound algorithm for the bilevel programming problem. SIAM J. Sci. Stat. Comput. 11:281–292
[2] Bard JF (1998) Practical bilevel optimization. Kluwer, Dordrecht
[3] Ben-Ayed O, Blair CE (1990) Computational difficulties of bilevel linear programming. Oper. Res. 38(3):556–560
[4] Falk JF, Liu J (1995) On bilevel programming, Part I. Mathematical Programming 70(1):47–72
[5] Fredericks T (2000) Linear bi-level problems: An efficient method for computing local minima. Master's Thesis, Department Applied Mathematics at University of Twente
[6] Fukushima M, Tseng P, An implementable active-set algorithm for computing a B-stationary point of a mathematical program with linear complementarity constraints
[7] Gendreau M, Marcotte P, Savard G (1996) A hybrid tabu-ascent algorithm for the linear bilevel programming problem. J. of Global Optimization 8:217–232
[8] Gibson CG, Wirthműller K, Du Plessis AA, Looijenga EJN (1976) Topological stability of smooth mappings. Lecture Notes in Math., vol. 552, Springer-Verlag, Berlin
[9] Golub G, Loan F van (1986) Matrix computions. North Oxford, London
[10] Hamming R (1998) Linear bi-level optimization problems. Master's Thesis, Department Applied Mathematics at University of Twente
[11] Hansen P, Jaumard B, Savard G (1992) New branch-and-bound rules for linear bilevel programming. SIAM J. of Scientific and Statistical Computing 13(5):1194–1217
[12] Shimizu K, Ishizuka Y, Bard J (1997) Nondifferentiable and two-level mathematical programming. Kluwer Academic Publishers, Boston
[13] Stein O, Still G, Generalized semi-infinite optimization and bilevel optimization. To appear in European Journal of Operational Research
[14] White DJ (1993) Anandalingam, a penalty function approach for solving bilevel linear problems. J. Global Optim. 3(4):397–419