

# CONVOLUTIONAL CODES OF GOPPA TYPE

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ABSTRACT. A new kind of Convolutional Codes generalizing Goppa Codes is proposed. This provides a systematic method for constructing convolutional codes with prefixed properties. In particular, examples of Maximum-Distance Separable (MDS) convolutional codes are obtained.

## 1. INTRODUCTION

The aim of this paper is to propose a definition of Convolutional Goppa Codes (CGC). This definition will provide an algebraic method for constructing Convolutional Codes with prescribed invariants.

We propose a definition of CGC in terms of families of curves  $X \rightarrow \mathbb{A}^1$  parametrized by the affine line  $\mathbb{A}^1 = \text{Spec } \mathbb{F}_q[z]$  over a finite field  $\mathbb{F}_q$ . In this setting, the usual definition of a Goppa Code as the code obtained by evaluation of sections at several rational points, is translated as a code obtained by evaluation (of sections of some invertible sheaf over  $X$ ) along several sections of the fibration  $X \rightarrow \mathbb{A}^1$ .

The paper is organized as follows.

In §2 we offer a summary on Goppa Codes following [3], [6], and using the standard notations of Algebraic Geometry [2].

§3 is devoted to giving the general definition of CGC and gives some general results.

In §4 we study the case of a trivial fibration of projective lines over  $\mathbb{A}^1$  and we conclude giving some explicit examples of MDS convolutional codes.

We freely use the standard notations of abstract Algebraic Geometry as can be found in [2]. After the works of V. Lomadze [4], J. Rosenthal and R. Smarandache [7], [8], there is evidence that the use of methods of Algebraic Geometry can be relevant to the study of Convolutional Codes. This paper is a step in favor of that evidence.

## 2. BACKGROUND ON ALGEBRAIC GEOMETRY AND GOPPA CODES

In this Section we summarize the basic definitions about Goppa Codes, constructed using methods of Algebraic Geometry (see [3], [6]).

Let  $X$  be a geometrically irreducible, smooth and projective curve over the finite field  $\mathbb{F}_q$ . Let  $p_1, \dots, p_n$  be  $n$  different  $\mathbb{F}_q$ -rational points of  $X$ , and  $D$  the divisor  $D = p_1 + \dots + p_n$ . Let  $G$  be another effective divisor with support disjoint from  $D$ . The Goppa code  $C(G, D)$  defined by  $(G, D)$  is the linear code of length  $n$  over  $\mathbb{F}_q$  defined as the image of the linear map

$$\begin{aligned} \alpha: L(G) &\rightarrow \mathbb{F}_q^n \\ f &\mapsto (f(p_1), \dots, f(p_n)), \end{aligned}$$

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where  $L(G)$  is the complete linear series defined by  $G$ . That is, let  $\mathbb{F}_q(X)$  be the field of rational functions over the curve  $X$ ,

$$L(G) = \{f \in \mathbb{F}_q(X) \text{ such that } \text{Div}(f) + G \geq 0\}.$$

The Goppa code has dimension

$$k = \dim C(G, D) = \dim L(G) - \dim L(G - D).$$

Let  $g$  be the genus of  $X$ ; if we assume the inequality  $2g - 2 < \deg(G) < n$ , then one has

$$k = \deg(G) - g + 1,$$

and the minimum distance  $d$  of  $C(G, D)$  satisfies the inequality

$$d \geq n - \deg(G).$$

Let  $\mathcal{O}_X(D)$  be the invertible sheaf on  $X$  defined by the divisor  $D$ . One has the following exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0,$$

where  $\mathcal{O}_D \simeq \mathcal{O}_{p_1}/\mathfrak{m}_{p_1} \times \cdots \times \mathcal{O}_{p_n}/\mathfrak{m}_{p_n} \simeq \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ . Tensoring the above exact sequence by  $\mathcal{O}_X(G)$ , one obtains

$$0 \rightarrow \mathcal{O}_X(G - D) \rightarrow \mathcal{O}_X(G) \rightarrow \mathcal{O}_D \rightarrow 0.$$

By taking global sections, we obtain an exact sequence of cohomology

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_X(G - D)) \rightarrow H^0(X, \mathcal{O}_X(G)) \xrightarrow{\alpha} \mathcal{O}_D \rightarrow H^1(X, \mathcal{O}_X(G - D)) \rightarrow \\ \rightarrow H^1(X, \mathcal{O}_X(G)) \rightarrow 0, \end{aligned}$$

where  $L(G) = H^0(X, \mathcal{O}_X(G))$  and  $\alpha$  is the evaluation map defined above.

In the case  $2g - 2 < \deg(G) < n$ , one has the exact sequence

$$(2.1) \quad 0 \rightarrow H^0(X, \mathcal{O}_X(G)) \xrightarrow{\alpha} \mathcal{O}_D \rightarrow H^1(X, \mathcal{O}_X(G - D)) \rightarrow 0.$$

Let  $\omega_X$  be the dualizing sheaf of  $X$ , which is isomorphic to the sheaf of regular 1-forms over  $X$ ;  $H^0(X, \omega_X)$  is the  $\mathbb{F}_q$ -vector space of global regular 1-forms over  $X$ , which is of dimension  $g = \text{genus of } X$ .

By Serre's duality ([2]), there exist canonical isomorphisms of  $\mathbb{F}_q$ -vector spaces

$$H^1(X, \mathcal{L})^* \simeq H^0(X, \omega_X \otimes \mathcal{L}^{-1})$$

for every invertible sheaf  $\mathcal{L}$  on  $X$ . Given a divisor  $D$  over  $X$ , we shall denote by  $\Omega(D)$  the vector space  $H^0(X, \omega_X \otimes \mathcal{O}_X(-D))$ .

The dual Goppa code,  $C^*(G, D)$ , associated with the Goppa code  $C(G, D)$  is defined as the linear code of length  $n$  over  $\mathbb{F}_q$  given by the image of the linear map

$$\begin{aligned} \alpha^*: \Omega(G - D) \rightarrow \mathbb{F}_q^n \\ \eta \mapsto (\text{Res}_{p_1}(\eta), \dots, \text{Res}_{p_n}(\eta)), \end{aligned}$$

Let us take duals in the exact sequence (2.1):

$$0 \rightarrow H^1(X, \mathcal{O}_X(G - D))^* \xrightarrow{\beta} \mathcal{O}_D^* \xrightarrow{\alpha^t} H^0(X, \mathcal{O}_X(G))^* \rightarrow 0.$$

By Serre's duality, one has isomorphisms

$$\begin{aligned} H^1(X, \mathcal{O}_X(G - D))^* &\simeq \Omega(G - D), \\ H^0(X, \mathcal{O}_X(G))^* &\simeq H^1(X, \omega_X \otimes \mathcal{O}_X(-G)), \end{aligned}$$

and the above sequence is the cohomology sequence induced by the exact sequence of sheaves

$$0 \rightarrow \omega_X(-G) \rightarrow \omega_X(D - G) \rightarrow \omega_X(D - G) \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow 0,$$

where we denote  $\omega_X(-G) = \omega_X \otimes \mathcal{O}_X(-G)$ , and  $\beta$  is precisely the map  $\alpha^*$  defining  $C^*(G, D)$ .

Given a linear series  $\Gamma \subseteq H^0(X, \mathcal{O}_X(G))$ , that is, a vector subspace defining a family of divisors linearly equivalent to  $G$ , we define the Goppa code  $C(\Gamma, D)$  associated with  $\Gamma$  and  $D$  as the image of the homomorphism  $\alpha|_{\Gamma}$ :

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(G)) & \xrightarrow{\alpha} & \mathcal{O}_D \\ \cup & \nearrow \alpha|_{\Gamma} & \\ \Gamma & & \end{array}$$

When  $\Gamma \subsetneq H^0(X, \mathcal{O}_X(G))$ , we shall say that  $C(\Gamma, D)$  is a non-complete Goppa code.

### 3. CONVOLUTIONAL GOPPA CODES

We shall construct a kind of convolutional code that generalizes the notion of Goppa codes. These codes will be associated with families of algebraic curves.

Given an algebraic variety  $S$  over  $\mathbb{F}_q$ , a family of projective algebraic curves parametrized by  $S$  is a morphism of algebraic varieties  $\pi: X \rightarrow S$ , such that  $\pi$  is a projective and flat morphism whose fibres  $X_s = \pi^{-1}(s)$  are smooth and geometrically irreducible curves over  $\mathbb{F}_q(s)$  (the residue field of  $s \in S$ ).

Let us consider a family of curves  $X \xrightarrow{\pi} U$  parametrized by  $U = \text{Spec } \mathbb{F}_q[z] = \mathbb{A}^1$ . Given a closed point  $u \in U$  with residue field  $\mathbb{F}_q(u)$ , the fibre  $X_u = \pi^{-1}(u)$  is a curve over the finite field  $\mathbb{F}_q(u)$ .

Let  $p_i$ ,  $1 \leq i \leq n$ , be  $n$  different sections,  $p_i: U \rightarrow X$ , of the projection  $\pi$ . These sections define a Cartier divisor on  $X$ :

$$D = p_1(U) + \cdots + p_n(U),$$

which is flat of degree  $n$  over the base  $U$  ([2]).

Note that given a coherent sheaf  $\mathcal{F}$  on  $X$ , the cohomology groups  $H^i(X, \mathcal{F})$  are finite  $\mathbb{F}_q[z]$ -modules and  $H^i(X, \mathcal{F}) = 0$  for  $i \geq 0$  (see [2] III).

Let  $\mathcal{L}$  be an invertible sheaf over  $X$ . One has an exact sequence of sheaves on  $X$

$$(3.1) \quad 0 \rightarrow \mathcal{L}(-D) \rightarrow \mathcal{L} \rightarrow \mathcal{O}_D \rightarrow 0,$$

which induces a long exact cohomology sequence

$$(3.2) \quad 0 \rightarrow H^0(X, \mathcal{L}(-D)) \rightarrow H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \rightarrow H^1(X, \mathcal{L}(-D)) \rightarrow H^1(X, \mathcal{L}) \rightarrow 0.$$

Let  $r$  be the degree of  $\mathcal{L}$  in each fibre of  $\pi$  (which is independent of the fibre) and let  $g$  be the genus of any fibre of  $\pi$  (also independent of the fibres).

**Proposition 3.1.** *Let us assume that  $2g - 2 < r$ . Then, one has that  $H^1(X, \mathcal{L}) = 0$  and  $H^0(X, \mathcal{L})$  is a free  $\mathbb{F}_q[z]$ -module of rank  $r - g + 1$*

*Proof.* Under the condition  $2g - 2 < r$ , one has that  $H^1(X_u, \mathcal{L}|_{X_u}) = 0$  for every point  $u \in U$ . Note that  $H^i(X, \mathcal{F}) \cong R^i \pi_* \mathcal{F}$  for every coherent sheaf  $\mathcal{F}$  on  $X$  ([2] III), and applying ([2] III Corollary 12.9) one concludes the proof.  $\square$

Under the hypothesis of Proposition 3.1, there exists an exact sequence of  $\mathbb{F}_q[z]$ -modules

$$(3.3) \quad 0 \rightarrow H^0(X, \mathcal{L}(-D)) \rightarrow H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \rightarrow H^1(X, \mathcal{L}(-D)) \rightarrow 0.$$

where  $H^0(X, \mathcal{O}_D)$  is a free  $\mathbb{F}_q[z]$ -module of rank  $n$ .

*Remark 3.2.* Let  $\eta \in U$  be the generic point of  $U$ , whose residue field is  $\mathbb{F}_q(z)$ ; the fibre  $X_\eta = \pi^{-1}(\eta)$  is a smooth, irreducible curve over  $\mathbb{F}_q(z)$ . Note that  $p_1(\eta), \dots, p_n(\eta)$  are  $n$  different  $\mathbb{F}_q(z)$ -rational points of the curve  $X_\eta$ . One then has a canonical decomposition of  $H^0(X, \mathcal{O}_D)_\eta$  as a  $\mathbb{F}_q(z)$ -algebra

$$H^0(X, \mathcal{O}_D)_\eta = \mathbb{F}_q(z) \times \dots \times \mathbb{F}_q(z).$$

Given a  $\mathbb{F}_q[z]$ -module  $M$ , let us denote by  $M_\eta$  the  $\mathbb{F}_q(z)$ -vector space

$$M_\eta = M \otimes_{\mathbb{F}_q[z]} \mathbb{F}_q(z).$$

The sequence (3.3) induces an exact sequence of  $\mathbb{F}_q(z)$ -vector spaces

$$(3.4) \quad 0 \rightarrow H^0(X, \mathcal{L}(-D))_\eta \rightarrow H^0(X, \mathcal{L})_\eta \xrightarrow{\alpha_\eta} H^0(X, \mathcal{O}_D)_\eta \rightarrow H^1(X, \mathcal{L}(-D))_\eta \rightarrow 0.$$

**Definition 3.3.** The complete convolutional Goppa code associated with  $\mathcal{L}$  and  $D$  is the image of the homomorphism  $\alpha_\eta$

$$\mathcal{C}(\mathcal{L}, D) = \text{Im} \left( H^0(X, \mathcal{L})_\eta \xrightarrow{\alpha_\eta} H^0(X, \mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n \right).$$

Given a free submodule  $\Gamma \subseteq H^0(X, \mathcal{L})$ , the convolutional Goppa code associated with  $\Gamma$  and  $D$  is the image of  $\alpha_\eta|_{\Gamma_\eta}$

$$\mathcal{C}(\Gamma, D) = \text{Im} \left( \Gamma_\eta \xrightarrow{\alpha_\eta} \mathbb{F}_q(z)^n \right).$$

*Remark 3.4.* We use definition 2.4 of [5] as definition of convolutional codes. Any matrix defining  $\alpha_\eta$  (respectively  $\alpha_\eta|_{\Gamma_\eta}$ ) is a generator matrix of rational functions for the code  $\mathcal{C}(\mathcal{L}, D)$  (resp.  $\mathcal{C}(\Gamma, D)$ ).

The canonical decomposition  $H^0(X, \mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n$  as  $\mathbb{F}_q(z)$ -algebras does not extend (in general) to a decomposition  $H^0(X, \mathcal{O}_D) \simeq \mathbb{F}_q[z]^n$  as rings. In fact, one has a canonical isomorphism of rings  $H^0(X, \mathcal{O}_D) \xrightarrow{\phi} \mathbb{F}_q[z]^n$  only when  $p_1(U), \dots, p_n(U)$  are disjoint sections. However,  $H^0(X, \mathcal{O}_D)$  is a free  $\mathbb{F}_q[z]$ -module; then, there exist (non-canonical) isomorphisms of  $\mathbb{F}_q[z]$ -modules:

$$H^0(X, \mathcal{O}_D) \xrightarrow{\phi} \mathbb{F}_q[z] \oplus \dots \oplus \mathbb{F}_q[z],$$

which are not (in general) isomorphism of rings.

This allows us to give another definition of convolutional Goppa codes.

**Definition 3.5.** Given a trivialization  $\phi: H^0(X, \mathcal{O}_D) \simeq \mathbb{F}_q[z]^n$  as  $\mathbb{F}_q[z]$ -modules, one defines the convolutional Goppa code  $\mathcal{C}(\mathcal{L}, D, \phi)$  as the image of  $\phi \circ \alpha$

$$H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \xrightarrow{\phi} \mathbb{F}_q[z]^n.$$

Analogously, one defines the convolutional Goppa code  $\mathcal{C}(\Gamma, D, \phi)$ .

Let us assume (for the rest of the paper) that the invariants  $(r, n, g)$  satisfy the inequality

$$2g - 2 < r < n.$$

**Proposition 3.6.** *Under the above conditions on  $(r, n, g)$ ,  $H^0(X, \mathcal{L}(-D)) = 0$  and  $H^1(X, \mathcal{L}(-D))$  is a free  $\mathbb{F}_q[z]$ -module. The following exact sequence is exact*

$$(3.5) \quad 0 \rightarrow H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \rightarrow H^1(X, \mathcal{L}(-D)) \rightarrow 0.$$

and remains exact when we take fibres over every point  $u \in U$ .

*Proof.* If  $2g - 2 < r < n$ ,  $H^0(X_u, \mathcal{L}(-D)|_{X_u}) = 0$  for every point  $u \in U$ ; and applying ([2] III Corollary 12.9) one concludes.  $\square$

**Corollary 3.7.** *The convolutional code  $\mathcal{C}(\mathcal{L}, D, \phi)$  has dimension  $k = r - g + 1$  and length  $n$ . Every matrix defining  $\phi \circ \alpha$  is a basic generator matrix [5] for  $\mathcal{C}(\mathcal{L}, D, \phi)$ .*

*Proof.* This is a direct consequence of the last statement of Proposition 3.6 and the characterization of basic generator matrices of [5].  $\square$

Let us consider the convolutional Goppa code  $\mathcal{C}(\Gamma, D, \phi)$  defined by a submodule  $\Gamma \subseteq H^0(X, \mathcal{L})$  and a trivialization  $\phi$ . With the above restrictions, one has:

**Proposition 3.8.** *Every matrix defining  $\phi \circ \alpha|_{\Gamma}$  is a basic generator matrix for the code  $\mathcal{C}(\Gamma, D, \phi)$  if and only if  $H^0(X, \mathcal{L})/\Gamma$  is a torsion-free  $\mathbb{F}_q[z]$ -module.*

*Proof.* The sequence (3.5) induces a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Gamma & \xrightarrow{\alpha|_{\Gamma}} & H^0(X, \mathcal{O}_D) & \longrightarrow & H^1(X, \Gamma) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{L}) & \longrightarrow & H^0(X, \mathcal{O}_D) & \longrightarrow & H^1(X, \mathcal{L}(-D)) & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & H^0(X, \mathcal{L})/\Gamma & & & & 0 & & \end{array}$$

Then, the kernel of  $H^1(X, \Gamma) \rightarrow H^1(X, \mathcal{L}(-D))$  is isomorphic to  $H^0(X, \mathcal{L})/\Gamma$  and  $H^1(X, \mathcal{L}(-D))$  is free. This implies that the torsion elements of  $H^1(X, \Gamma)$  are contained in  $H^0(X, \mathcal{L})/\Gamma$ , from which one concludes the proof.  $\square$

The above results allow us to construct basic generator matrices for the codes  $\mathcal{C}(\Gamma, D, \phi)$ . If  $p_1(U), \dots, p_n(U)$  are disjoint sections and  $\phi$  the canonical trivialization, this gives us a basic generator matrix for  $\mathcal{C}(\Gamma, D)$ . However, in general the codes  $\mathcal{C}(\Gamma, D)$  and  $\mathcal{C}(\Gamma, D, \phi)$  are different.

Let us describe a geometric way to obtain a basic generator matrix for  $\mathcal{C}(\mathcal{L}, D)$  and  $\mathcal{C}(\Gamma, D)$ .

Assume that the curves  $p_1(U), \dots, p_n(U)$  meet transversally at some points, and let  $\bar{X}$  be the blowing-up [2] of  $X$  at these points. One has morphisms

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\beta} & X \\ & \searrow & \downarrow \pi \\ & & U \end{array} \quad \bar{\pi} = \pi \circ \beta$$

such that the proper transform of  $D$  under  $\pi$  is a divisor  $\bar{D} \subset \bar{X}$  satisfying

$$\bar{D} = p_1(U) \amalg \cdots \amalg p_n(U) \xrightarrow{\beta} D,$$

and one has a canonical homomorphism of rings

$$0 \rightarrow \mathcal{O}_D \rightarrow \beta_* \mathcal{O}_{\bar{D}}$$

which induces

$$0 \rightarrow \pi_* \mathcal{O}_D \xrightarrow{\beta} \bar{\pi}_* \mathcal{O}_{\bar{D}} \simeq \mathbb{F}_q[z]^n,$$

where  $\bar{\pi}_* \mathcal{O}_{\bar{D}} \simeq \mathbb{F}_q[z]^n$  is the canonical isomorphism of sheaves of rings.

$\beta^* \mathcal{L}$  is an invertible sheaf on  $\bar{X}$  and there exists a canonical homomorphism

$$\beta^* \mathcal{L} \rightarrow \mathcal{O}_{\bar{D}} \rightarrow 0,$$

whose kernel is  $(\beta^* \mathcal{L})(-\bar{D})$ . This induces

$$0 \rightarrow \mathcal{L} \rightarrow \beta_* \beta^* \mathcal{L} \rightarrow \beta_* \mathcal{O}_{\bar{D}},$$

and taking global sections one obtains

$$0 \rightarrow H^0(X, \mathcal{L}) \xrightarrow{\gamma} H^0(X, \beta_* \beta^* \mathcal{L}) \xrightarrow{\mu} \mathbb{F}_q[z]^n.$$

The image of  $\mu$  is precisely a free submodule of  $\mathbb{F}_q[z]^n$  that defines a basic generator matrix for  $\mathcal{C}(\mathcal{L}, D)$ .

Let us consider the sequence of homomorphisms

$$0 \rightarrow H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \xrightarrow{\beta} H^0(X, \mathcal{O}_{\bar{D}}) = \mathbb{F}_q[z]^n.$$

$\beta \circ \alpha$  is not in general a basic matrix, since  $H^0(X, \mathcal{O}_{\bar{D}})/H^0(X, \mathcal{O}_D)$  has torsion. Let us define

$$\bar{H}^0(X, \mathcal{L}) = \{p \in \mathbb{F}_q[z]^n \text{ such that } \lambda p \in H^0(X, \mathcal{L}) \text{ for some } \lambda \in \mathbb{F}_q[z]\}.$$

$\bar{H}^0(X, \mathcal{L})/H^0(X, \mathcal{L})$  is a torsion module and  $\mathbb{F}_q[z]^n/\bar{H}^0(X, \mathcal{L})$  is torsion-free. Then, every matrix defining the homomorphism  $\bar{H}^0(X, \mathcal{L}) \hookrightarrow \mathbb{F}_q[z]^n$  is a basic generator matrix for  $\mathcal{C}(\mathcal{L}, D)$ .

This is an algebraic-geometric interpretation of Forney's construction of the basic matrices of a convolutional code [1].

#### 4. CONVOLUTIONAL GOPPA CODES ASSOCIATED WITH THE PROJECTIVE LINE

Let  $\mathbb{P}^1 = \text{Proj } \mathbb{F}_q[x_0, x_1]$  be the projective line over  $\mathbb{F}_q$ , and

$$X = \mathbb{P}^1 \times U \xrightarrow{\pi} U = \text{Spec } \mathbb{F}_q[z]$$

the trivial fibration. Let us denote by  $t = x_1/x_0$  the affine coordinate in  $\mathbb{P}^1$ , and by  $p_\infty$  its infinity point. Let us consider the following  $n$  different sections of  $\pi$

$$p_i: U \rightarrow \mathbb{P}^1 \times U$$

defined in the coordinates  $(t, z)$  by

$$p_i(z) = (\alpha_i z + \beta_i, z), \quad \alpha_i, \beta_i \in \mathbb{F}_q.$$

Let  $D = p_1(U) + \cdots + p_n(U)$  and let  $\mathcal{L}$  be the invertible sheaf on  $X$

$$\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(rp_\infty) \otimes_{\mathbb{F}_q} \mathcal{O}_U, \quad r < n,$$

The exact sequence (3.5) is in this case:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(X, \mathcal{L}) & \xrightarrow{\alpha} & H^0(X, \mathcal{O}_D) & \longrightarrow & H^1(X, \mathcal{L}(-D)) \longrightarrow 0. \\ & & \parallel & & \parallel & & \\ & & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty)) \otimes \mathbb{F}_q[z] & \xrightarrow{\alpha} & \mathbb{F}_q[z]^n & & \end{array}$$

Taking the fibres over the generic point  $\eta$ , and the canonical trivialization  $(\pi_* \mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n$ , the homomorphism  $\alpha_\eta$  is the evaluation map at the points  $p_1(\eta), \dots, p_n(\eta)$

$$\begin{aligned} \alpha_\eta: H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty)) \otimes_{\mathbb{F}_q} \mathbb{F}_q(z) &\rightarrow \mathbb{F}_q(z)^n \\ \alpha_\eta(t^j) &= (t^j(p_1(\eta)), \dots, t^j(p_n(\eta))) = ((\alpha_1 z + \beta_1)^j, \dots, (\alpha_n z + \beta_n)^j), \end{aligned}$$

where  $\{1, t, \dots, t^r\}$  is the ‘‘canonical’’ basis of  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))$  in the affine coordinate  $t$ . The convolutional code  $\mathcal{C}(\mathcal{L}, D)$  is a kind of *generalized Reed-Solomon (RS) code* (for  $z = 0$  we obtain a classical RS-code).

Let  $\Gamma \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))$  be the linear subspace generated by  $\{t^s, \dots, t^r\}$ . The convolutional Goppa code  $\mathcal{C}(\Gamma, D)$  is the image of the homomorphism

$$\begin{aligned} \alpha_\eta: \Gamma \otimes_{\mathbb{F}_q} \mathbb{F}_q(z) &\rightarrow \mathbb{F}_q(z)^n \\ t^j &\longmapsto \alpha_\eta(t^j), \quad \text{for } s \leq j \leq r. \end{aligned}$$

In this case  $H^0(X, \mathcal{L})/\Gamma \simeq (H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))/\Gamma) \otimes_{\mathbb{F}_q} \mathbb{F}_q[z]$  is torsion-free. Then, by Proposition 3.8 every matrix defining

$$\alpha: \Gamma \otimes_{\mathbb{F}_q} \mathbb{F}_q[z] \rightarrow H^0(X, \mathcal{O}_D)$$

is a basic generator matrix. To compute a matrix for  $\alpha$  explicitly, we need to fix an isomorphism of  $\mathbb{F}_q[z]$ -modules

$$H^0(X, \mathcal{O}_D) \xrightarrow{\phi} \mathbb{F}_q[z]^n,$$

and this gives a generator matrix for  $\mathcal{C}(\Gamma, D, \phi)$ . However, it would be desirable to compute basic matrices for the codes  $\mathcal{C}(\Gamma, D)$ . We shall do this in general in a forthcoming paper. Here we shall offer some explicit examples.

*Example 4.1.* Let  $a, b \in \mathbb{F}_q$  be two different non-zero elements, and

$$p_i(z) = (a^{i-1}z + b^{i-1}, z), \quad i = 1, \dots, n, \quad \text{with } n < q.$$

The evaluation map  $\alpha_\eta$  over  $\Gamma$  is defined by the matrix

$$(4.1) \quad \begin{pmatrix} (z+1)^s & (az+b)^s & (a^2z+b^2)^s & \dots & (a^{n-1}z+b^{n-1})^s \\ (z+1)^{s+1} & (az+b)^{s+1} & (a^2z+b^2)^{s+1} & \dots & (a^{n-1}z+b^{n-1})^{s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (z+1)^r & (az+b)^r & (a^2z+b^2)^r & \dots & (a^{n-1}z+b^{n-1})^r \end{pmatrix}.$$

This matrix is a generator matrix for the code  $\mathcal{C}(\Gamma, D)$ . Using this construction we can give concrete examples of CGC of dimension  $k = r - s + 1$  that are Maximum-Distance Separable (MDS) convolutional codes, i.e., whose *free distance* attains the generalized Singleton bound [7].

- If  $s = r$ , the convolutional Goppa code  $\mathcal{C}(\Gamma, D)$  has dimension 1, degree  $r$ , and (4.1) is a *canonical* (reduced and basic [5]) generator matrix. We can list a few examples, where  $k/n$ ,  $\delta$  and  $d$  are respectively the rate, the degree and the free distance of the code.

<i>field</i>	<i>canonical generator matrix</i>	$k/n$	$\delta$	$d$
$\mathbb{F}_3 = \{0, 1, 2\}$	$(z + 1 \quad z + 2)$	1/2	1	4
$\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ where $\alpha^2 + \alpha + 1 = 0$	$(z + 1 \quad z + \alpha \quad z + \alpha^2)$	1/3	1	6
$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$	$((z + 1)^2 \quad (z + 2)^2 \quad (z + 4)^2)$	1/3	2	9

In these examples the sections  $p_1, \dots, p_n$  are disjoint, such that one has  $\mathcal{C}(\Gamma, D) = \mathcal{C}(\Gamma, D, \phi)$ , where  $\phi: H^0(X, \mathcal{O}_D) \simeq \mathbb{F}_q[z]^n$  is the corresponding canonical trivialization.

- If  $s < r$ , let us take  $a \in \mathbb{F}_q$  as a primitive element. Now, the matrix (4.1) is reduced, since the matrix of highest-degree terms in each row is a Vandermonde matrix of rank  $k$ . The sections  $p_1, \dots, p_n$  are not disjoint, but in some cases the matrix (4.1) is actually basic and we do not have to find an isomorphism of  $\mathbb{F}_q[z]$ -modules,  $\phi: H^0(X, \mathcal{O}_D) \simeq \mathbb{F}_q[z]^n$ , in order to compute a basic generator matrix for the code  $\mathcal{C}(\Gamma, D)$ .

We present two examples of this situation.

<i>field</i>	<i>canonical generator matrix</i>	$k/n$	$\delta$	$d$
$\mathbb{F}_4$	$\begin{pmatrix} 1 & 1 & 1 \\ z + 1 & \alpha z + \alpha^2 & \alpha^2 z + \alpha \end{pmatrix}$	2/3	1	3
$\mathbb{F}_5$	$\begin{pmatrix} z + 1 & 2z + 3 & 4z + 4 & 3z + 2 \\ (z + 1)^2 & (2z + 3)^2 & (4z + 4)^2 & (3z + 2)^2 \end{pmatrix}$	1/2	3	8

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