# A Combinatorial Approach to Involution and $\delta$ -Regularity I: Involutive Bases in Polynomial Algebras of Solvable Type

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Received: date / Revised version: date

**Abstract** Involutive bases are a special form of non-reduced Gröbner bases with additional combinatorial properties. Their origin lies in the Janet-Riquier theory of linear systems of partial differential equations. We study them for a rather general class of polynomial algebras including also non-commutative algebras like those generated by linear differential and difference operators or universal enveloping algebras of (finite-dimensional) Lie algebras. A number of basic properties are derived and we provide concrete algorithms for their construction. Furthermore, we develop a theory for involutive bases with respect to semigroup orders (as they appear in local computations) and over coefficient rings, respectively. In both cases it turns out that generally only weak involutive bases exist.

### **1** Introduction

In the late 19th and early 20th century a number of French mathematicians developed what is nowadays called the Janet-Riquier theory of differential equations [29,30,38,43,47,48]. It is a theory for general systems of differential equations, i. e. also for under- and overdetermined systems, and provides in particular a concrete algorithm for the completion to a so-called passive<sup>1</sup> system. In recent times, the theory has found again considerable interest mainly in the context of Lie symmetry analysis, so that a number of references to modern works and implementations are contained in the review [28].

The defining property of passive systems is that they do not generate any nontrivial integrability conditions. As the precise definition of passivity requires the introduction of a ranking on the set of all derivatives and as every linear system of

<sup>&</sup>lt;sup>1</sup> Sometimes the terminology "involutive" is used which apparently goes back to Lie.

partial differential equations with constant coefficients bijectively corresponds to a polynomial module, it appears natural to relate this theory to the algebraic theory of Gröbner bases [1,5].

Essentially, the Janet-Riquier theory lacks only the concept of reduction to a normal form; otherwise it contains all the ingredients of Gröbner bases. Somewhat surprisingly, a rigorous links has been established only fairly recently by Gerdt and collaborators who introduced a special form of non-reduced Gröbner bases for polynomial ideals [14, 15, 51], the *involutive bases* (a slightly different approach to involutive bases has been proposed by Apel [4]; it will not be used here).

The fundamental idea behind involutive bases (coming from the Janet-Riquier theory) is to assign to each generator in a basis a subset of all variables: its multiplicative variables. This assignment is defined by a so-called involutive division, as it corresponds to a restriction of the usual divisibility relation of terms. We only permit to multiply each generator by polynomials in its multiplicative variables. As we will see later in Part II, for appropriately prepared bases, this approach automatically leads to combinatorial decompositions of polynomial modules.

Like Gröbner bases, involutive bases can be defined in many non-commutative algebras. We introduce a generalisation of the polynomial algebras of solvable type of Kandry-Rodi and Weispfenning [32]. This class also includes the *G*-algebras considered by Apel [2] and Levandovskyy [34,35]. In contrast to these works, we permit that the variables act on the coefficients, so that, say, linear differential operators with *variable* coefficients form a polynomial algebra of solvable type in our sense. Thus our framework automatically includes the work of Gerdt [13] on involutive bases for linear differential equations as well.

This article is the first of two parts. It reviews the basic theory of involutive bases and extends it to polynomial algebras of solvable type. Much of this material may already be found scattered in the literature (though not in the generality presented here and sometimes with incorrect proofs). We give a new definition of involutive bases closer in spirit to the standard definition of Gröbner bases. Furthermore, we introduce the new notion of a weak involutive basis which is still a Gröbner basis but does not lead to a combinatorial decomposition.

The main emphasis in the literature is on optimising the simple completion algorithm of Section 6 and on providing fast implementations; as the experiments reported in [17] demonstrate, the results have been striking. We will, however, ignore this topic and instead study in Part II a number of applications of involutive bases (mainly the special case of Pommaret bases) in the structure analysis of polynomial modules. This will include in particular the relation between involutive bases and the above mentioned combinatorial decompositions. Note that in these applications we will mainly restrict to the ordinary commutative polynomial ring.

This first part is organised as follows. The next section defines involutive divisions and bases within the Abelian monoid  $(\mathbb{N}_0^n, +)$  of multi indices. It also introduces the two most important divisions named after Janet and Pommaret, respectively. Section 3 generalises the definition of polynomial algebras of solvable type and shows the existence of left Gröbner bases in these algebras. The following three sections define (weak) involutive bases for left ideals in solvable algebras and give concrete algorithms for their construction. The next four sections study some extensions of the basic theory. Section 7 analyses the relation between left and right ideals in polynomial algebras of solvable type and the computation of bases for two-sided ideals. The following two sections generalise to semigroup orders and study the use of the Mora normal form. Finally, Sections 10 considers involutive bases over rings. It turns out that in these more general situations usually only weak bases exist.

In a short appendix we fix our conventions for term orders which are inverse to the ones found in most textbooks on Gröbner bases. We also prove an elementary property of the degree reverse lexicographic term order that makes it particularly natural for Pommaret bases.

#### 2 Involutive Divisions

We study the Abelian monoid  $(\mathbb{N}_0^n, +)$  with the addition defined componentwise and call its elements *multi indices*. They may be identified in a natural way with the vertices of an *n*-dimensional integer lattice, so that we can easily visualise subsets of  $\mathbb{N}_0^n$ . For a multi index  $\nu \in \mathbb{N}_0^n$  we introduce its *cone*  $\mathcal{C}(\nu) = \nu + \mathbb{N}_0^n$ , i. e. the set of all multi indices that can be reached from  $\nu$  by adding another multi index. We say that  $\nu$  *divides*  $\mu$ , written  $\nu \mid \mu$ , if  $\mu \in \mathcal{C}(\nu)$ . Given a finite subset  $\mathcal{N} \subset \mathbb{N}_0^n$ , we define its *span* as the monoid ideal generated by  $\mathcal{N}$ :

$$\langle \mathcal{N} \rangle = \bigcup_{\nu \in \mathcal{N}} \mathcal{C}(\nu) . \tag{1}$$

The basic idea of an involutive division is to introduce a restriction of the cone of a multi index, the involutive cone: it is only allowed to add multi indices certain entries of which vanish. This is equivalent to a restriction of the above defined divisibility relation. The final goal will be having a *disjoint* union in (1) by using only these involutive cones on the right hand side. This will naturally lead to the combinatorial decompositions discussed in Part II.

In order to finally give the definition of an involutive division, we need one more notation: let  $N \subseteq \{1, ..., n\}$  be an arbitrary subset of the set of the first n integers; then we write  $\mathbb{N}_N^n = \{\nu \in \mathbb{N}_0^n \mid \nu_j = 0, \forall j \notin N\}$  for the set of all multi indices where the only entries who may be non-zero are those whose indices are contained in N.

**Definition 2.1** An involutive division L is defined on the Abelian monoid  $(\mathbb{N}_0^n, +)$ , if a subset  $N_{L,\mathcal{N}}(\nu) \subseteq \{1, \ldots, n\}$  of multiplicative indices is associated to every multi index  $\nu$  in a finite subset  $\mathcal{N} \subset \mathbb{N}_0^n$  such that the following two conditions on the involutive cones  $\mathcal{C}_{L,\mathcal{N}}(\nu) = \nu + \mathbb{N}_{N_{L,\mathcal{N}}(\nu)}^n \subseteq \mathbb{N}_0^n$  of the multi indices  $\nu \in \mathcal{N}$ are satisfied.

- 1. If there exist two elements  $\mu, \nu \in \mathcal{N}$  with  $\mathcal{C}_{L,\mathcal{N}}(\mu) \cap \mathcal{C}_{L,\mathcal{N}}(\nu) \neq \emptyset$ , either  $\mathcal{C}_{L,\mathcal{N}}(\mu) \subseteq \mathcal{C}_{L,\mathcal{N}}(\nu)$  or  $\mathcal{C}_{L,\mathcal{N}}(\nu) \subseteq \mathcal{C}_{L,\mathcal{N}}(\mu)$  holds.
- 2. If  $\mathcal{N}' \subset \mathcal{N}$ , then  $N_{L,\mathcal{N}}(\nu) \subseteq N_{L,\mathcal{N}'}(\nu)$  for all  $\nu \in \mathcal{N}'$ .

An arbitrary multi index  $\mu \in \mathbb{N}_0^n$  is involutively divisible by  $\nu \in \mathcal{N}$ , written  $\nu \mid_{L,\mathcal{N}} \mu$ , if  $\mu \in \mathcal{C}_{L,\mathcal{N}}(\nu)$ .

Before we discuss the precise meaning of this definition and in particular of the two conditions contained in it, we should stress the following important point: as indicated by the notation, involutive divisibility is always defined with respect to both an involutive division L and a fixed finite set  $\mathcal{N} \subset \mathbb{N}_0^n$ ; only an element of  $\mathcal{N}$  can be an involutive divisor. Obviously, involutive divisibility  $\nu \mid_{L,\mathcal{N}} \mu$  implies ordinary divisibility  $\nu \mid \mu$ .

The involutive cone  $C_{L,\mathcal{N}}(\nu)$  of any multi index  $\nu \in \mathcal{N}$  is a subset of the full cone  $C(\nu)$ . We are not allowed to add arbitrary multi indices to  $\nu$  but may increase only certain entries of  $\nu$  determined by the multiplicative indices. The first condition in the above definition says that involutive cones can intersect only trivially: if two intersect, one must be a subset of the other.

The non-multiplicative indices form the complement of  $N_{L,\mathcal{N}}(\nu)$  in  $\{1,\ldots,n\}$ and are denoted by  $\overline{N}_{L,\mathcal{N}}(\nu)$ . If we remove some elements from the set  $\mathcal{N}$  and determine the multiplicative indices of the remaining elements with respect to the subset  $\mathcal{N}'$ , we obtain in general a different result than before. The second condition for an involutive division says that while it may happen that a non-multiplicative index becomes multiplicative for some  $\nu \in \mathcal{N}'$ , the converse cannot happen.

*Example 2.2* A classical involutive division is the *Janet division J*. In order to define it, we must introduce certain subsets of the given set  $\mathcal{N} \subset \mathbb{N}_0^n$ :

$$(d_k, \dots, d_n) = \left\{ \nu \in \mathcal{N} \mid \nu_i = d_i, \ k \le i \le n \right\}.$$
(2)

The index n is multiplicative for  $\nu \in \mathcal{N}$ , if  $\nu_n = \max_{\mu \in \mathcal{N}} {\{\mu_n\}}$ , and k < n is multiplicative for  $\nu \in (d_{k+1}, \ldots, d_n)$ , if  $\nu_k = \max_{\mu \in (d_{k+1}, \ldots, d_n)} {\{\mu_k\}}$ .

Obviously, this definition depends on the ordering of the variables  $x_1, \ldots, x_n$ and we may obtain variants, if we first apply an arbitrary but fixed permutation  $\pi \in S_n$  to the variables. In fact, Gerdt and Blinkov [14] use an "inverse" definition, i. e. they first apply the permutation  $(n \cdots 21)$ . Our convention is the original one of Janet [30, pp. 16–17].

Gerdt et al. [16] designed a special data structure, the Janet tree, for the fast determination of Janet multiplicative indices and for a number of other operations useful in the construction of Janet bases. As shown in [24], this data structure is based on a special relation between the Janet division and the inverse lexicographic term order (see the appendix). This relation allows us to compute very quickly the multiplicative variables of any set  $\mathcal{N}$  with Algorithm 2.1. The algorithm simply runs two pointers over the inverse lexicographically ordered set  $\mathcal{N}$  and changes accordingly the set  $\mathcal{I}$  of potential multiplicative indices.

**Definition 2.3** The division L is globally defined, if the assignment of the multiplicative indices is independent of the set N; in this case we write simply  $N_L(\nu)$ .

*Example 2.4* An involutive division that will become very important for us in the sequel is the *Pommaret division* P. It assigns the multiplicative indices according to the following simple rule: if  $1 \le k \le n$  is the smallest index such that  $\nu_k > 0$  for some multi index  $\nu \in \mathbb{N}_0^n \setminus \{[0, \ldots, 0]\}$ , then we call k the class of  $\nu$ , written  $\operatorname{cls} \nu$ , and set  $N_P(\nu) = \{1, \ldots, k\}$ . Finally, we define  $N_P([0, \ldots, 0]) = \{1, \ldots, n\}$ .

Algorithm 2.1 Multiplicative variables for the Janet division

**Input:** finite list  $\mathcal{N} = \{\nu^{(1)}, \dots, \nu^{(k)}\}$  of pairwise different multi indices from  $\mathbb{N}_0^n$ **Output:** list  $N = \{N_{J,\mathcal{N}}(\nu^{(1)}), \dots, N_{J,\mathcal{N}}(\nu^{(k)})\}$  of lists with multiplicative variables 1:  $\mathcal{N} \leftarrow \texttt{sort}(\mathcal{N}, \prec_{\texttt{invlex}}); \quad \nu \leftarrow \mathcal{N}[1]$ 2:  $p_1 \leftarrow n$ ;  $\mathcal{I} \leftarrow \{1, \ldots, n\}$ ;  $N[1] \leftarrow \mathcal{I}$ 3: for j from 2 to  $|\mathcal{N}|$  do  $p_2 \leftarrow \max\left\{i \mid (\nu - \mathcal{N}[j])_i \neq 0\right\}; \quad \mathcal{I} \leftarrow \mathcal{I} \setminus \{p_2\}$ 4: 5: if  $p_1 < p_2$  then 6:  $\mathcal{I} \leftarrow \mathcal{I} \cup \{p_1, \dots, p_2 - 1\}$ 7: end\_if  $N[j] \leftarrow \mathcal{I}; \quad \nu \leftarrow \mathcal{N}[j]; \quad p_1 \leftarrow p_2$ 8: 9: end\_for 10: return N

Thus P is globally defined. Note that like the Janet division it depends on the ordering of the variables  $x_1, \ldots, x_n$  and thus one may again introduce simple variants by applying a permutation.

Above we have seen that the Janet division is in a certain sense related to the inverse lexicographic order. The Pommaret division has a special relation to the degree reverse lexicographic order. According to Lemma A.1,  $\prec_{degrevlex}$  is the only term order that respects classes. This will become important for the construction of Pommaret bases, as it implies that for homogeneous polynomials this order always leads to maximal sets of multiplicative indices and thus to smaller bases.

Above we introduced the span of a set  $\mathcal{N} \subset \mathbb{N}_0^n$  as the union of the cones of its elements. Given an involutive division it appears natural to consider also the union of the involutive cones. Obviously, this yields in general only a subset (without any algebraic structure) of the monoid ideal  $\langle \mathcal{N} \rangle$ .

**Definition 2.5** *The* involutive span *of a finite set*  $\mathcal{N} \subset \mathbb{N}_0^n$  *is* 

$$\langle \mathcal{N} \rangle_L = \bigcup_{\nu \in \mathcal{N}} \mathcal{C}_{\mathcal{N},L}(\nu) .$$
 (3)

The set  $\mathcal{N}$  is called weakly involutive for the division L or a weak involutive basis of the monoid ideal  $\langle \mathcal{N} \rangle$ , if  $\langle \mathcal{N} \rangle_L = \langle \mathcal{N} \rangle$ . A weak involutive basis is a strong involutive basis or for short an involutive basis, if the union on the right hand side of (3) is disjoint, i. e. the intersections of the involutive cones are empty. We call any finite set  $\mathcal{N} \subseteq \overline{\mathcal{N}} \subset \mathbb{N}_0^n$  such that  $\langle \overline{\mathcal{N}} \rangle_L = \langle \mathcal{N} \rangle$  a (weak) involutive completion of  $\mathcal{N}$ . An obstruction to involution for the set  $\mathcal{N}$  is a multi index  $\nu \in \langle \mathcal{N} \rangle \setminus \langle \mathcal{N} \rangle_L$ .

*Remark 2.6* An obvious necessary condition for a strong involutive basis is that no multi indices  $\mu, \nu \in \mathcal{N}$  exist such that  $\mu|_{L,\mathcal{N}}\nu$ . Sets with this property are called *involutively autoreduced*. One easily checks that the definition of the Janet division implies that  $\mathcal{C}_{\mathcal{N},J}(\mu) \cap \mathcal{C}_{\mathcal{N},L}(\nu) = \emptyset$  whenever  $\mu \neq \nu$ . Hence for this particular division any set is involutively autoreduced.  $\triangleleft$ 

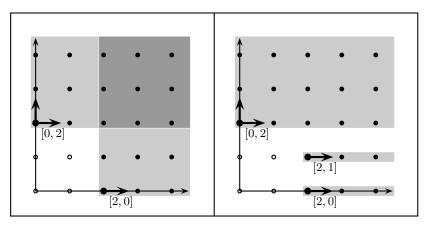


Fig. 1 Left: intersecting cones. Right: involutive cones.

*Example* 2.7 Figure 1 demonstrates the geometric interpretation of involutive divisions for n = 2. In both diagrams one can see the monoid ideal generated by the set  $\mathcal{N} = \{[0, 2], [2, 0]\}$ ; the vertices belonging to it are marked by dark points. The arrows represent the multiplicative indices, i. e. the "allowed directions". The left diagram shows that the full cones of the two elements of  $\mathcal{N}$  intersect in the darkly shaded area and that  $\mathcal{N}$  is not (weakly) involutive, as the multi indices [k, 1] with  $k \geq 2$  are obstructions to involution. The right diagram shows a strong involutive basis of  $\langle \mathcal{N} \rangle$  for both the Janet and the Pommaret division. We must add to  $\mathcal{N}$  the multi index [2, 1] and both for it and for [2, 0] only the index 1 is multiplicative. One clearly sees how  $\langle \mathcal{N} \rangle$  is decomposed into three disjoint involutive cones: one of dimension 2, two of dimension 1.

We are particularly interested in *strong* involutive bases. The following result shows that in the "monomial" case any weak involutive basis can be reduced to a strong one by simply eliminating some elements.

**Proposition 2.8** If  $\mathcal{N}$  is a weakly involutive set, then a subset  $\mathcal{N}' \subseteq \mathcal{N}$  exists such that  $\mathcal{N}'$  is a strong involutive basis of  $\langle \mathcal{N} \rangle$ .

*Proof* This proposition represents a nice motivation for the two conditions in Definition 2.1 of an involutive division. If  $\mathcal{N}$  is not yet a strong involutive basis, the union in (3) is not disjoint and intersecting involutive cones exist. By the first condition, this implies that some cones are contained in other ones; no other form of intersection is possible. If we eliminate the tips of these cones from  $\mathcal{N}$ , we get a subset  $\mathcal{N}' \subset \mathcal{N}$  which has by the second condition the same involutive span, as the remaining elements may only gain additional multiplicative indices. Thus after a finite number of such eliminations we arrive at a strong involutive basis.  $\Box$ 

Recall that for arbitrary monoid ideals a basis  $\mathcal{N}$  is called *minimal*, if it is not possible to remove an element of  $\mathcal{N}$  without losing the property that we have a basis. A similar notion can naturally be introduced for involutive bases.

**Definition 2.9** Let  $\mathcal{I} \subseteq \mathbb{N}_0^n$  be a monoid ideal and L an involutive division. An involutive basis  $\mathcal{N}$  of  $\mathcal{I}$  with respect to L is called minimal, if any other involutive basis  $\mathcal{N}'$  of  $\mathcal{I}$  with respect to L satisfies  $\mathcal{N} \subseteq \mathcal{N}'$ .

Obviously, the minimal involutive basis of a monoid ideal is unique. For globally defined divisions, any involutive basis is unique.

**Proposition 2.10** Let *L* be a globally defined division and  $\mathcal{I} \subseteq \mathbb{N}_0^n$  a monoid ideal. If  $\mathcal{I}$  has an involutive basis for *L*, then it is unique and thus minimal.

*Proof* Let  $\mathcal{N}$  be the minimal basis of  $\mathcal{I}$  and  $\mathcal{N}_1$ ,  $\mathcal{N}_2$  two distinct involutive bases of  $\mathcal{I}$ . Both  $\mathcal{N}_1 \setminus \mathcal{N}_2$  and  $\mathcal{N}_2 \setminus \mathcal{N}_1$  must be non-empty, as otherwise one basis was contained in the other one and thus the larger basis could not be involutively autoreduced with respect to the global division L. Take an arbitrary multi index  $\nu \in \mathcal{N}_1 \setminus \mathcal{N}_2$ . The basis  $\mathcal{N}_2$  contains a unique multi index  $\mu$  such that  $\mu|_L \nu$ . It cannot be an element of  $\mathcal{N}_1$ , as  $\mathcal{N}_1$  is involutively autoreduced. Thus  $\mathcal{N}_1$  must contain a unique multi index  $\lambda$  such that  $\lambda|_L \mu$ . As L is globally defined, this implies that  $\lambda|_L \nu$ , a contradiction.  $\Box$ 

The algorithmic construction of (weak) involutive completions for a given set  $\mathcal{N} \subset \mathbb{N}_0^n$  will be discussed in detail in Section 5. For the moment we only note that we cannot expect that for an arbitrary set  $\mathcal{N}$  and an arbitrary involutive division L an involutive basis  $\mathcal{N}'$  of  $\langle \mathcal{N} \rangle$  exists.

*Example 2.11* We consider the set  $\mathcal{N} = \{[1,1]\}$  for the Pommaret division. As  $\operatorname{cls}[1,1] = 1$ , we get  $N_P([1,1]) = \{1\}$ . So  $\mathcal{C}_P([1,1]) \subset \mathcal{C}([1,1])$ . But any multi index contained in  $\langle \mathcal{N} \rangle$  also has class 1. Hence no *finite* involutive basis of  $\langle \mathcal{N} \rangle$  exists for the Pommaret division. We can generate it involutively only with the infinite set  $\{[1,k] \mid k \in \mathbb{N}\}$ .

**Definition 2.12** An involutive division L is called Noetherian, if any finite subset  $\mathcal{N} \subset \mathbb{N}_0^n$  has a finite involutive completion with respect to L.

#### Lemma 2.13 The Janet division is Noetherian.

*Proof* Let  $\mathcal{N} \subset \mathbb{N}_0^n$  be an arbitrary finite set. We explicitly construct a Janet basis for  $\langle \mathcal{N} \rangle$ . Define the multi index  $\mu$  by  $\mu_i = \max_{\nu \in \mathcal{N}} \nu_i$ . Then the set

$$\bar{\mathcal{N}} = \left\{ \bar{\nu} \in \mathbb{N}_0^n \cap \langle \mathcal{N} \rangle \mid \mu \in \mathcal{C}(\bar{\nu}) \right\}$$
(4)

is an involutive completion of  $\mathcal{N}$  with respect to the Janet division. Indeed,  $\mathcal{N} \subseteq \overline{\mathcal{N}}$ and  $\overline{\mathcal{N}} \subset \langle \mathcal{N} \rangle$ . Let  $\rho \in \langle \mathcal{N} \rangle$  be an arbitrary element. If  $\rho \in \overline{\mathcal{N}}$ , then trivially  $\rho \in \langle \overline{\mathcal{N}} \rangle_J$ . Otherwise set  $I = \{i \mid \rho_i > \mu_i\}$  and define the multi index  $\overline{\rho}$  by  $\overline{\rho}_i = \rho_i$  for  $i \notin I$  and  $\overline{\rho}_i = \mu_i$  for  $i \in I$ , i. e.  $\overline{\rho}_i = \min \{\rho_i, \mu_i\}$ . By construction of the set  $\overline{\mathcal{N}}$  and the definition of  $\mu$ , we have that  $\overline{\rho} \in \overline{\mathcal{N}}$  and  $I \subseteq N_{J,\overline{\mathcal{N}}}(\overline{\rho})$ . But this implies that  $\rho \in \mathcal{C}_{J,\overline{\mathcal{N}}}(\overline{\rho})$  and thus  $\overline{\mathcal{N}}$  is a finite Janet basis for  $\langle \mathcal{N} \rangle$ .  $\Box$ 

### **3** Polynomial Algebras of Solvable Type

We could identify multi indices and monomials and proceed to define involutive bases for polynomial ideals. But as the basic ideas remain unchanged in many different situations, e. g. rings of linear differential or difference operators, we generalise a concept originally introduced by Kandry-Rody and Weispfenning [32] and use *polynomial algebras of solvable type*.

Let  $\mathcal{P} = \mathcal{R}[x_1, \ldots, x_n]$  be a polynomial ring over a ring<sup>2</sup>  $\mathcal{R}$  with unit. If  $\mathcal{R}$  is commutative, then  $\mathcal{P}$  is a commutative ring with unit with respect to the usual multiplication. We equip the  $\mathcal{R}$ -module  $\mathcal{P}$  with alternative multiplications, in particular with non-commutative ones. We allow that both the variables  $x_i$  do not commute any more and that they operate on the coefficients. The usual multiplications is denoted either by a dot  $\cdot$  or by no symbol at all. Alternative multiplications  $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$  are always written as  $f \star g$ .

Like Gröbner bases, involutive bases are defined with respect to a *term order*. It selects in each polynomial  $f \in \mathcal{P}$  a *leading term*  $\operatorname{lt}_{\prec} f = x^{\mu}$  with *leading exponent*  $\operatorname{le}_{\prec} f = \mu$ . The coefficient  $r \in \mathcal{R}$  of  $x^{\mu}$  in f is the *leading coefficient*  $\operatorname{lc}_{\prec} f$  and the product  $rx^{\mu}$  is the *leading monomial*  $\operatorname{lm}_{\prec} f$ . Based on the leading exponents we associate to each finite set  $\mathcal{F} \subset \mathcal{P}$  a set  $\operatorname{le}_{\prec} \mathcal{F} \subset \mathbb{N}_{0}^{n}$  to which we may apply the theory developed in the previous section. But this requires a kind of compatibility between the multiplication  $\star$  and the chosen term order.

**Definition 3.1**  $(\mathcal{P}, \star, \prec)$  *is a* polynomial algebra of solvable type *for the term order*  $\prec$ *, if the multiplication*  $\star : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$  *satisfies three axioms.* 

- (i)  $(\mathcal{P}, \star)$  is a ring with unit 1.
- (ii)  $\forall r \in \mathcal{R}, f \in \mathcal{P} : r \star f = rf.$

(iii)  $\forall \mu, \nu \in \mathbb{N}_0^n, r \in \mathcal{R} : \operatorname{le}_{\prec}(x^{\mu} \star x^{\nu}) = \mu + \nu \land \operatorname{le}_{\prec}(x^{\mu} \star r) = \mu.$ 

Condition (i) ensures that arithmetics in  $(\mathcal{P}, \star, \prec)$  obeys the usual associative and distributive laws. Because of Condition (ii),  $(\mathcal{P}, \star, \prec)$  is a left  $\mathcal{R}$ -module. We do not require that it is a right  $\mathcal{R}$ -module, as this would exclude the possibility that the variables  $x_i$  operate non-linearly on  $\mathcal{R}$ . Condition (iii) ensures the compatibility of the new multiplication  $\star$  and the term order  $\prec$ ; we say that the multiplication  $\star$  respects the term order  $\prec$ . It implies the existence of injective maps  $\rho_{\mu}: \mathcal{R} \to \mathcal{R}$ , maps  $h_{\mu}: \mathcal{R} \to \mathcal{P}$  with  $le_{\prec}(h_{\mu}(r)) \prec \mu$  for all  $r \in \mathcal{R}$ , coefficients  $r_{\mu\nu} \in \mathcal{R} \setminus \{0\}$  and polynomials  $h_{\mu\nu} \in \mathcal{P}$  with  $le_{\prec}h_{\mu\nu} \prec \mu + \nu$  such that

$$x^{\mu} \star r = \rho_{\mu}(r)x^{\mu} + h_{\mu}(r)$$
, (5a)

$$x^{\mu} \star x^{\nu} = r_{\mu\nu} x^{\mu+\nu} + h_{\mu\nu} .$$
 (5b)

**Lemma 3.2** The maps  $\rho_{\mu}$  and the coefficients  $r_{\mu\nu}$  satisfy for arbitrary multi indices  $\mu, \nu, \lambda \in \mathbb{N}_0^n$  and for arbitrary ring elements  $r \in \mathcal{R}$ 

$$\rho_{\mu}(\rho_{\nu}(r))r_{\mu\nu} = r_{\mu\nu}\rho_{\mu+\nu}(r) , \qquad (6a)$$

$$\rho_{\mu}(r_{\nu\lambda})r_{\mu,\nu+\lambda} = r_{\mu\nu}r_{\mu+\nu,\lambda} .$$
(6b)

Furthermore, all maps  $\rho_{\mu}$  are ring endomorphisms.

<sup>&</sup>lt;sup>2</sup> For us a ring is always associative.

*Proof* The first assertion is a trivial consequence of the associativity of the multiplication  $\star$ . The equations correspond to the leading coefficients of the equalities  $x^{\mu} \star (x^{\nu} \star r) = (x^{\mu} \star x^{\nu}) \star r$  and  $x^{\mu} \star (x^{\nu} \star x^{\lambda}) = (x^{\mu} \star x^{\nu}) \star x^{\lambda}$ , respectively. The second assertion follows mainly from Condition (i).  $\Box$ 

If  $\mathcal{R}$  is an integral domain, then for arbitrary polynomials  $f, g \in \mathcal{P}$  a ring element  $r \in \mathcal{R} \setminus \{0\}$  and a polynomial  $h \in \mathcal{P}$  satisfying  $e_{\prec}h \prec e_{\prec}(f \cdot g)$  exist such that

$$f \star g = r \left( f \cdot g \right) + h \ . \tag{7}$$

Under this assumption we may formulate (iii) in the alternative form

(iii)' 
$$\forall f, g \in \mathcal{P} : \operatorname{le}_{\prec}(f \star g) = \operatorname{le}_{\prec}f + \operatorname{le}_{\prec}g.$$

For the case that  $\mathcal{R}$  is even a field, the same class of non-commutative algebras was introduced in [9] under the name *PBW algebras* (see Example 3.5 below for an explanation of this name).

**Proposition 3.3** *The product*  $\star$  *is fixed, as soon as the following data are given:* constants  $r_{ij} \in \mathcal{R} \setminus \{0\}$ , polynomials  $h_{ij} \in \mathcal{P}$  and maps  $\rho_i : \mathcal{R} \to \mathcal{R}$ ,  $h_i : \mathcal{R} \to \mathcal{P}$ such that for  $1 \leq i \leq n$ 

$$x_i \star r = \rho_i(r)x_i + h_i(r) , \quad \forall r \in \mathcal{R} ,$$
 (8a)

$$x_i \star x_j = r_{ij} x_j \star x_i + h_{ij} , \quad \forall 1 \le j < i .$$
(8b)

*Proof* The set of all "monomials"  $x_{i_1} \star x_{i_2} \star \cdots \star x_{i_q}$  with  $i_1 \leq i_2 \leq \cdots \leq i_q$  forms a basis of  $\mathcal{P}$ , as because of (iii) the map  $x_{i_1} \star x_{i_2} \star \cdots \star x_{i_q} \mapsto x_{i_1} \cdot x_{i_2} \cdots x_{i_q}$ is an  $\mathcal{R}$ -module automorphism mapping the new basis into the standard basis. Obviously, it is possible to evaluate any product  $f \star g$  by repeated applications of the rewrite rules (8) provided f and g are expressed in the new basis.  $\Box$ 

Note that this proof is non-constructive in the sense that we are not able to determine the multiplication in terms of the standard basis, as we do not know explicitly the transformation between the new and the standard basis. The advantage of this proof is that it is valid for arbitrary coefficient rings  $\mathcal{R}$ . Making some assumptions on  $\mathcal{R}$  (the simplest possibility is to require that it is a field), one could use Lemma 3.2 to express the coefficients  $r_{\mu\nu}$  and  $\rho_{\mu}$  in (5) by the data in (8). This would yield a constructive proof.

Of course, the data in Proposition 3.3 cannot be chosen arbitrarily. Besides the obvious conditions on the leading exponents of the polynomials  $h_{ij}$  and  $h_i(r)$ imposed by (iii), each map  $\rho_i$  must be an injective  $\mathcal{R}$ -endomorphism and each map  $h_i$  must satisfy  $h_i(r + s) = h_i(r) + h_i(r)$  and a kind of pseudo-Leibniz rule  $h_i(rs) = \rho_i(r)h_i(s) + h_i(r)s$ . The associativity of  $\star$  imposes further rather complicated conditions. For the case of a *G*-algebra with the multiplication defined by rewrite rules they have been explicitly determined by Levandovskyy [34,35] who called them *non-degeneracy conditions*. *Example 3.4* An important class of non-commutative polynomials was originally introduced by Noether and Schmeidler [40] and later systematically studied by Ore [42]; our exposition follows [8]. It includes in particular linear differential and difference operators (with variable coefficients). Such rings are not considered in [32], as the terms operate on the coefficients. Our more general definition of solvable algebras can handle this case.

Let  $\mathbb{F}$  be an arbitrary commutative ring and  $\sigma : \mathbb{F} \to \mathbb{F}$  an injective endomorphism. A *pseudo-derivation* with respect to  $\sigma$  is a map  $\delta : \mathbb{F} \to \mathbb{F}$  such that (i)  $\delta(f+g) = \delta(f) + \delta(g)$  and (ii)  $\delta(f \cdot g) = \sigma(f) \cdot \delta(g) + \delta(f) \cdot g$  for all  $f, g \in \mathbb{F}$ . If  $\sigma = \operatorname{id}_{\mathbb{F}}$ , the identity map, (ii) is the standard Leibniz rule for derivations. If  $\sigma \neq \operatorname{id}_{\mathbb{F}}$ , one can show that there exists an  $h \in \mathbb{F}$  such that  $\delta = h(\sigma - \operatorname{id}_{\mathbb{F}})$ . And conversely, if  $\delta \neq 0$ , there exists an  $h \in \mathbb{F}$  such that  $\sigma = h\delta + \operatorname{id}_{\mathbb{F}}$ . Ore called  $\sigma(f)$  the *conjugate* and  $\delta(f)$  the *derivative* of f.

Given  $\sigma$  and  $\delta$ , the ring  $\mathbb{F}[\partial; \sigma, \delta]$  of univariate *Ore polynomials* consists of all formal polynomials in  $\partial$  with coefficients in  $\mathbb{F}$ , i. e. of expressions of the form  $\theta = \sum_{i=0}^{q} f_i \partial^i$  with  $f_i \in \mathbb{F}$  and  $q \in \mathbb{N}_0$ . The addition is defined as usual. The variable  $\partial$  operates on an element  $f \in \mathbb{F}$  according to the rule

$$\partial \star f = \sigma(f)\partial + \delta(f) \tag{9}$$

which is extended associatively and distributively to define the multiplication in  $\mathbb{F}[\partial; \sigma, \delta]$ : given two elements  $\theta_1, \theta_2 \in \mathbb{F}[\partial; \sigma, \delta]$ , we can transform the product  $\theta_1 \star \theta_2$  to the above normal form by repeatedly applying (9). The injectivity of the endomorphism  $\sigma$  ensures that deg  $(\theta_1 \star \theta_2) = \deg \theta_1 + \deg \theta_2$ . We call  $\mathbb{F}[\partial; \sigma, \delta]$  the *Ore extension* of  $\mathbb{F}$  generated by  $\sigma$  and  $\delta$ .

A simple concrete example is given by choosing for  $\mathbb{F}$  some ring of differentiable functions in the real variable x, say  $\mathbb{F} = \mathbb{Q}[x]$ ,  $\delta = \frac{d}{dx}$  and  $\sigma = \operatorname{id}_{\mathbb{F}}$ yielding *linear ordinary differential operators* with polynomial functions as coefficients (i. e. the Weyl algebra over  $\mathbb{Q}$ ). Similarly, we obtain *linear recurrence* and *difference operators*. We set  $\mathbb{F} = \mathbb{C}(n)$ , the space of sequences with complex elements, and take for  $\sigma$  the shift operator, i. e. the automorphism mapping  $s_n$ to  $s_{n+1}$ . Then  $\Delta = \sigma - \operatorname{id}_{\mathbb{F}}$  is a pseudo-derivation.  $\mathbb{F}[E; \sigma, 0]$  consists of linear ordinary recurrence operators,  $\mathbb{F}[E; \sigma, \Delta]$  of linear ordinary difference operators.

So far it is not clear why we call the elements of these classical examples "operators", but the elements of any Ore algebra  $\mathbb{O} = \mathbb{F}[\partial; \sigma, \delta]$  may be interpreted as operators acting on  $\mathbb{F}$ -modules. Indeed, let  $\mathcal{V}$  be an  $\mathbb{F}$ -module and A a so-called  $\mathbb{F}$ -pseudo-linear map  $A : \mathcal{V} \to \mathcal{V}$ , i.e. A(u + v) = A(u) + A(v) and  $A(fu) = \sigma(f)A(u) + \delta(f)u$  for all  $f \in \mathbb{F}$  and  $u, v \in \mathcal{V}$ . We introduce an action  $\alpha : \mathbb{O} \times \mathcal{V} \to \mathcal{V}$  mapping  $(\theta = \sum f_i \partial^i, u)$  to  $A_{\theta}u = \sum f_i A^i(u)$ . For  $\mathcal{V} = \mathbb{F}^m$ a natural choice for A is  $A(v_1, \ldots, v_m) = (\delta(v_1), \ldots, \delta(v_m))$ . If  $\delta = 0$ , we may also take  $A(v_1, \ldots, v_m) = (\sigma(v_1), \ldots, \sigma(v_m))$ . With these definitions the above examples of Ore algebras contain linear operators in the familiar sense.

For multivariate Ore polynomials we take a set  $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$  of  $\mathbb{F}$ endomorphisms and a set  $\Delta = \{\delta_1, \ldots, \delta_n\}$  where each  $\delta_i$  is a pseudo-derivation with respect to  $\sigma_i$ . For each pair  $(\sigma_i, \delta_i)$  we introduce a variable  $\partial_i$  satisfying a commutation rule (9). If we require that all the maps  $\sigma_i, \delta_j$  commute with each other, i. e.  $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i, \delta_i \circ \delta_j = \delta_j \circ \delta_i$  and  $\sigma_i \circ \delta_j = \delta_j \circ \sigma_i$  for all  $i \neq j$ ,

one easily checks that  $\partial_i \star \partial_j = \partial_j \star \partial_i$ , i. e. the variables  $\partial_i$  commute. Setting  $\mathcal{D} = \{\partial_1, \ldots, \partial_n\}$ , we denote by  $\mathbb{F}[\mathcal{D}; \Sigma, \Delta]$  the ring of multivariate Ore polynomials. Because of the commutativity of the  $\partial_i$  we may write the terms as  $\partial^{\mu}$  with multi indices  $\mu \in \mathbb{N}_0^n$ , so that it indeed makes sense to speak of a polynomial ring. Comparing with Proposition 3.3, we see that we are in the special case where the maps  $h_i$  always yield constant polynomials and the variables  $x_i$  commute.

Finally, we show that  $(\mathbb{F}[\mathcal{D}; \Sigma, \Delta], \star, \prec)$  is an algebra of solvable type for any term order  $\prec$ . The product of two monomial operators  $a\partial^{\mu}$  and  $b\partial^{\nu}$  is given by

$$a\partial^{\mu} \star b\partial^{\nu} = \sum_{\lambda+\kappa=\mu} {\mu \choose \lambda} a\sigma^{\lambda} (\delta^{\kappa}(b)) \partial^{\lambda+\nu}$$
(10)

where  $\binom{\mu}{\lambda}$  is a shorthand for  $\prod_{i=1}^{n} \binom{\mu_i}{\lambda_i}$ ,  $\sigma^{\lambda} = \sigma_1^{\lambda_1} \circ \cdots \circ \sigma_n^{\lambda_n}$  and similarly for  $\delta$ . By the properties of a term order this implies

$$\operatorname{le}_{\prec}\left(a\partial^{\mu}\star b\partial^{\nu}\right) = \mu + \nu = \operatorname{le}_{\prec}\left(a\partial^{\mu}\right) + \operatorname{le}_{\prec}\left(b\partial^{\nu}\right),\tag{11}$$

as any term  $\partial^{\lambda+\nu}$  appearing on the right hand side of (10) divides  $\partial^{\mu+\nu}$  and thus  $\partial^{\lambda+\nu} \leq \partial^{\mu+\nu}$  for any term order  $\prec$ .

*Example 3.5* Bell and Goodearl [6] introduced the *Poincaré-Birkhoff-Witt extension* (for short *PBW extension*) of a ring  $\mathcal{R}$  as a ring  $\mathcal{P} \supseteq \mathcal{R}$  containing a finite number of elements  $x_1, \ldots, x_n \in \mathcal{P}$  such that (i)  $\mathcal{P}$  is freely generated as a left  $\mathcal{R}$ -module by the monomials  $x^{\mu}$  with  $\mu \in \mathbb{N}_0^n$ , (ii)  $x_i \star r - r \star x_i \in \mathcal{R}$  for all  $r \in \mathcal{R}$  and (iii)  $x_i \star x_j - x_j \star x_i \in \mathcal{R} + \mathcal{R}x_1 + \cdots \mathcal{R}x_n$ . Obviously, any such extension is a polynomial algebra of solvable type in the sense of Definition 3.1 for any degree compatible term order. Other term orders generally do not respect the multiplication in  $\mathcal{P}$ .

The classical example of such a PBW extension is the *universal enveloping* algebra  $U(\mathfrak{g})$  of a finite-dimensional Lie algebra  $\mathfrak{g}$  which also explains the name: the Poincaré-Birkhoff-Witt theorem asserts that the monomials form a basis of these algebras [50]. They still fit into the framework developed by Kandry-Rody and Weispfenning [32], as the  $x_i$  do not act on the coefficients. This is no longer the case for the more general *skew enveloping algebras*  $\mathcal{R} \# U(\mathfrak{g})$  where  $\mathcal{R}$  is a k-algebra on which the elements of  $\mathfrak{g}$  act as derivations [37, Sect. 1.7.10].

*Example 3.6* In all these examples, the coefficients  $r_{\mu\nu}$  appearing in (5) are one; thus (8b) are classical commutation relations. This is no longer true in the *quantised enveloping algebras*  $U_h(\mathfrak{g})$  introduced by Drinfeld [12] and Jimbo [31]. For these algebra it is non-trivial that a Poincaré-Birkhoff-Witt theorem holds; it was shown for general Lie algebras  $\mathfrak{g}$  by Lusztig [36]. Berger [7] generalised this result later to a larger class of associative algebras, the so-called *q-algebras*. They are characterised by the fact that the polynomials  $h_{ij}$  in (8b) are at most quadratic with the additional restriction that  $h_{ij}$  may contain only those quadratic terms  $x_k x_\ell$  that satisfy  $i < k \le \ell < j$  and  $k - i = j - \ell$ . Thus any such algebra is a polynomial algebra of solvable type for any degree compatible term order. As simple concrete example is the *q*-Heisenberg algebra for a real q > 0 (and  $q \neq 1$ ). Let f be a function of a real variable x lying in some appropriate function space. Then we introduce the operators

$$\delta_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \tau_q f(x) = f(qx), \quad \hat{x} f(x) = x f(x).$$
(12)

It is straightforward to verify that that these three operators satisfy the following *q*-deformed form of the Heisenberg commutation rules

$$\delta_q \star \hat{x} = \hat{x} \star \delta_q + \tau_q \,, \quad \delta_q \star \tau_q = \tau_q \star \delta_q \,, \quad \tau_q \star \hat{x} = q\hat{x} \star \tau_q \,. \tag{13}$$

Hence the algebra  $\mathbb{k}[\delta_q, \tau_q, \hat{x}]$  is a polynomial algebra of solvable type for any degree compatible term order (but also for any lexicographic order with  $\tau_q \prec \delta_q$  and  $\tau_q \prec \hat{x}$ ).  $\triangleleft$ 

*Example 3.7* Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type with a degree compatible term order  $\prec$ . Then  $\mathcal{P}$  is a filtered ring with respect to the standard filtration  $\mathcal{F}_q = \bigoplus_{i=0}^q \mathcal{P}_i$  and we may introduce the *associated graded algebra* by setting  $(\operatorname{gr} \mathcal{P})_q = \mathcal{F}_q/\mathcal{F}_{q-1}$ . It is easy to see that  $\operatorname{gr} \mathcal{P}$  is again a polynomial algebra of solvable type for  $\prec$ . If in (8) deg  $h_i(r) = 0$ , deg  $h_{ij} \leq 1$ ,  $\rho_i = \operatorname{id}_{\mathcal{R}}$  and  $r_{ij} = 1$  (which is for example the case for all Poincaré-Birkhoff-Witt extensions), then in fact  $\operatorname{gr} \mathcal{P} = (\mathcal{P}, \cdot)$ , the commutative polynomial ring. Such algebras are sometimes called *almost commutative* [37, Sect. 8.4.2].

**Proposition 3.8** *If the ring*  $\mathcal{R}$  *is an integral domain, then any polynomial algebra*  $(\mathcal{P}, \star, \prec)$  *of solvable type over it is an integral domain, too, and a left Ore domain.* 

*Proof* The first assertion is a trivial consequence of (7): if  $\mathcal{R}$  has no zero divisors, then  $f \cdot g \neq 0$  implies  $f \star g \neq 0$ . Hence  $\mathcal{P}$  does not contain any zero divisors.

For the second one we must verify the *left Ore conditions* [10,41]. We must show that one can find for any two polynomials  $f, g \in \mathcal{P}$  with  $f \star g \neq 0$  two further polynomials  $\phi, \psi \in \mathcal{P} \setminus \{0\}$  such that  $\phi \star f = \psi \star g$ . We describe now a concrete algorithm for this task.

We set  $\mathcal{F}_0 = \{f, g\}$  and choose coefficients  $r_0, s_0 \in \mathcal{R}$  such that in the difference  $r_0g \star f - s_0f \star g = \bar{h}_1$  the leading terms cancel. Then we compute a normal form  $h_1$  of  $\bar{h}_1$  with respect to  $\mathcal{F}_0$ . This leads to an equation of the form

$$(r_0 g + \phi_0) \star f - (s_0 f + \psi_0) \star g = h_1 \tag{14}$$

where  $e_{\prec}h_1 \notin \langle e_{\prec}\mathcal{F}_0 \rangle$ . If  $h_1 = 0$ , we are done and the polynomials  $\phi = r_0g - \phi_0$ and  $\psi = s_0f - \psi_0$  form a solution of our problem. By Part (iii) of Definition 3.1 we have  $e_{\prec}\bar{h}_1 \prec e_{\prec}f + e_{\prec}g$ . This implies by the monotonicity of term orders that  $e_{\prec}\phi_0 \prec e_{\prec}g$  and  $e_{\prec}\psi_0 \prec e_{\prec}f$ . Thus we have found a non-trivial solution.

Otherwise we set  $\mathcal{F}_1 = \mathcal{F}_0 \cup \{h_1\}$  and multiply (14) with f from the left. For suitably chosen coefficients  $r_1, s_1 \in \mathcal{R}$  we have  $r_1 f \star h_1 = s_1 h_1 \star f + \bar{h}_2$  with  $le_{\prec}\bar{h}_2 \prec le_{\prec}(f \cdot h_1)$ . We determine a normal form  $h_2$  of  $\bar{h}_2$  with respect to  $\mathcal{F}_1$ . This yields the equation

$$\left[f \star (r_0 g + \phi_0) - s_1 h_1 + \phi_1\right] \star f - \left[f \star (s_0 f + \psi_0) + \psi_1\right] \star g + \rho_1 \star h_1 = h_2$$
(15)

where  $le_{\prec}h_2 \notin \langle le_{\prec}\mathcal{F}_1 \rangle$  If  $h_2 = 0$ , we are done, as we can substitute  $h_1$  from (14) and obtain thus the solution  $\phi = (r_1f + \rho_1) \star (r_0g + \phi_0) - s_1h_1 + \phi_1$  and  $\psi = (r_1f + \rho_1) \star (s_0f + \psi_0) + \psi_1$ . By the same reasoning on the leading exponents as above, it is non-trivial.

Otherwise we iterate: we set  $\mathcal{F}_2 = \mathcal{F}_1 \cup \{h_2\}$ , multiply (15) with f from the left, rewrite  $r_2f \star h_2$  as  $s_2h_2 \star f + \bar{h}_3$  with some coefficient  $c_2 \in \mathbb{k}$ , compute a normal form  $h_3$  of  $\bar{h}_3$  with respect to  $\mathcal{F}_2$  and so on. If the iteration stops, i.e. if the remainder  $h_N$  vanishes for some value  $N \in \mathbb{N}$ , then we can construct non-zero polynomials  $\phi$ ,  $\psi$  with  $\phi \star f = \psi \star g$  by substituting all remainders  $h_i$ by their defining equations. The iteration terminates by a simple Noetherian argument:  $\langle \text{le}_{\prec} \mathcal{F}_0 \rangle \subset \langle \text{le}_{\prec} \mathcal{F}_1 \rangle \subset \langle \text{le}_{\prec} \mathcal{F}_2 \rangle \subset \cdots$  is a strictly ascending chain of monoid ideals in  $\mathbb{N}_0^n$  and thus cannot be infinite.  $\Box$ 

Obviously, we can show by the same argument that  $\mathcal{P}$  is a right Ore domain. We have given here a direct and in particular constructive proof that  $\mathcal{P}$  satisfies the Ore conditions. Instead we could have invoked Theorem 2.1.15 of [37] stating that any right Noetherian integral domain is also a right Ore domain (using Proposition 3.11 below). Note that our construction is not unique, as we could equally well multiply at each step with g or alternate between f and g etc.

In a left Ore domain  $\mathcal{P}$  we can define for all  $a, \bar{a} \in \mathcal{P} \setminus \{0\}$  and  $b, \bar{b} \in \mathcal{P}$  an equivalence relation:  $a^{-1} \star b \sim \bar{a}^{-1} \star \bar{b}$ , if there exist two elements  $\phi, \bar{\phi} \in \mathcal{P} \setminus \{0\}$  such that  $\phi \star a = \bar{\phi} \star \bar{a}$  and  $\phi \star b = \bar{\phi} \star \bar{b}$ . If  $\mathcal{P}$  is an integral domain, this allows us the definition of a skew field of fractions of the form  $a^{-1} \star b$  [10, Section 12.1]. Similarly one may introduce for a right Ore domain a skew field of fractions of the form  $b \star a^{-1}$ .

*Example 3.9* In the commutative polynomial ring one has always the trivial solution  $\phi = g$  and  $\psi = f$ . One might expect that in the non-commutative case one only has to add some lower terms to it. However, this is not the case. Consider the universal enveloping algebra of the Lie algebra  $\mathfrak{so}(3)$ . We may write it as  $\mathcal{P} = \Bbbk[x_1, x_2, x_3]$  with the multiplication  $\star$  defined by the relations:

$$x_1 \star x_2 = x_1 x_2 , \qquad x_2 \star x_1 = x_1 x_2 - x_3 ,$$
  

$$x_1 \star x_3 = x_1 x_3 , \qquad x_3 \star x_1 = x_1 x_3 + x_2 ,$$
  

$$x_2 \star x_3 = x_2 x_3 , \qquad x_3 \star x_2 = x_2 x_3 - x_1 .$$
  
(16)

This multiplication obviously respects any degree compatible term order but not the lexicographic order. Choosing  $f = x_1$  and  $g = x_2$ , possible solutions for  $\phi \star f = \psi \star g$  are  $\phi = x_2^2 - 1$  and  $\psi = x_1x_2 - 2x_3$  or  $\phi = x_1x_2 + x_3$  and  $\psi = x_1^2 - 1$ . They are easily constructed using the algorithm of the proof of Proposition 3.8 once with f and once with g. Here we must use polynomials of degree 2; it is not possible to find a solution of degree 1.  $\triangleleft$ 

Because of Condition (iii) in Definition 3.1 we can define Gröbner bases for ideals in algebras of solvable type. In the case that  $\mathcal{R}$  is a (commutative) field  $\Bbbk$ , this is straightforward and we will from now on restrict to this case; the general case will be discussed only in Section 10. If we endow  $\mathcal{P}$  with a non-commutative

multiplication, we must in principle distinguish left, right and two-sided ideals. However, with the exception of Section 7, we will exclusively work with left ideals and thus do not introduce special notations.

**Definition 3.10** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type over a field  $\Bbbk$  and  $\mathcal{I} \subseteq \mathcal{P}$  a left ideal. A finite set  $\mathcal{G} \subset \mathcal{P}$  is a Gröbner basis of  $\mathcal{I}$  (for the term order  $\prec$ ), if  $\langle \text{le}_{\prec} \mathcal{G} \rangle = \text{le}_{\prec} \mathcal{I}$ .

For the ordinary multiplication this definition reduces to the classical one. The decisive point, explaining the conditions imposed in Definition 3.1, is that normal forms with respect to a finite set  $\mathcal{F} \subset \mathcal{P}$  may be computed in algebras of solvable type in precisely the same way as in the ordinary polynomial ring. Assume we are given a polynomial  $f \in \mathcal{P}$  such that  $\lg_{\prec}g \mid \lg_{\prec}f$  for some  $g \in \mathcal{G}$  and set  $\mu = \lg_{\prec}f - \lg_{\prec}g$ . If we consider  $g_{\mu} = x^{\mu} \star g$ , then by (iii)  $\lg_{\prec}g_{\mu} = \lg_{\prec}f$ . Setting  $d = \lg_{\prec}f/\lg_{\#}$ , we find by (ii) that  $\lg_{\prec}(f - dg_{\mu}) \prec \lg_{\#}f$ . Hence we may use the usual algorithms for computing normal form; in particular, they always terminate by the same argument as in the ordinary case. Note that in general  $d \neq \lg_{\prec}f/\lg_{\prec}g$ , if  $c \neq 1$  in (7), and that normal form computations are typically more expensive due to the appearance of the additional polynomial h in (7).

The classical Gröbner basis theory can be extended straightforwardly to polynomial algebras of solvable type [2,9,32,34,35], as most proofs are based on the computation of normal forms. The remaining arguments mostly take place in  $\mathbb{N}_0^n$  and thus can be applied without changes. In particular, we get the following result crucial for the termination of Buchberger's algorithm.

**Proposition 3.11** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type over a Noetherian ring  $\mathcal{R}$ . Then  $\mathcal{P}$  is a Noetherian ring, too. If  $\mathcal{R}$  is a field, then every left ideal  $\mathcal{I} \subseteq \mathcal{P}$  possesses a Gröbner basis with respect to  $\prec$ .

*Proof* For the ordinary multiplication the first part is Hilbert's Basis Theorem. We recall here a simple proof that remains valid in our more general situation. Let  $\mathcal{I} \subseteq \mathcal{P}$  be a left ideal. By Dickson's Lemma the monoid ideal  $le_{\prec}\mathcal{I}$  has a finite basis  $\mathcal{N}$ . As  $\mathcal{R}$  is assumed to be Noetherian, the  $\mathcal{R}$ -ideal

$$\mathcal{I}_{\nu} = \{ r \in \mathcal{R} \mid \exists f \in \mathcal{I} : \mathrm{Im}_{\prec} f = rx^{\nu} \}$$
(17)

has for each  $\nu \in \mathcal{N}$  a finite generating set. Thus we can choose a finite set  $\mathcal{G} \subset \mathcal{I}$ such that  $e_{\prec}\mathcal{G}$  is a finite basis of the monoid ideal  $e_{\prec}\mathcal{I}$  and for each  $\nu \in e_{\prec}\mathcal{G}$ the set  $\{lc_{\prec}g \mid g \in \mathcal{G} \land le_{\prec}g = \nu\}$  is a finite basis of  $\mathcal{I}_{\nu}$ .

It follows from a simple normal form argument that  $\mathcal{G}$  generates  $\mathcal{I}$  and hence  $\mathcal{G}$  is a finite generating set of  $\mathcal{I}$ . Thus  $\mathcal{P}$  is left Noetherian. By essentially the same argument,  $\mathcal{P}$  is right Noetherian and thus Noetherian. If  $\mathcal{R}$  is a field, the set  $\mathcal{G}$  is obviously a Gröbner basis.  $\Box$ 

We do not give more details, as they can be found in the above cited references. Instead we will present in the next section a completely different approach leading to involutive bases.

### **4** Involutive Bases

We proceed to define involutive bases for left ideals in polynomial algebras of solvable type. In principle, we could at once consider submodules of free modules over such an algebra. But this only complicates the notation. So we treat only the ideal case and the extension to submodules goes as for Gröbner bases.

**Definition 4.1** Let  $(\mathcal{P}, \star, \prec)$  be an algebra of solvable type over a field  $\Bbbk$  and  $\mathcal{I} \subseteq \mathcal{P}$  a left ideal. A finite set  $\mathcal{H} \subset \mathcal{P}$  is a weak involutive basis of  $\mathcal{I}$  for an involutive division L on  $\mathbb{N}_0^n$ , if  $\mathbb{l}_{\prec}\mathcal{H}$  is a weak involutive basis of the monoid ideal  $\mathbb{l}_{\prec}\mathcal{I}$ . The set  $\mathcal{H}$  is a (strong) involutive basis of  $\mathcal{I}$ , if  $\mathbb{l}_{\prec}\mathcal{H}$  is a strong involutive basis of  $\mathbb{l}_{\prec}$  and no two elements of  $\mathcal{H}$  have the same leading exponents.

This definition is a natural extension of our Definition 3.10 of a Gröbner basis in  $\mathcal{P}$ . It implies immediately that any weak involutive basis is a Gröbner basis. In [45] a more general notion of a weak involutive basis was used. But those bases which do not satisfy the definition above are of no interest, as they do not possess any real structure. As in Section 2, we call any finite set  $\mathcal{F} \subset \mathcal{P}$  (weakly) involutive, if it is a (weak) involutive basis of the ideal  $\langle \mathcal{F} \rangle$  generated by it.

Gerdt and Blinkov [14] gave a different definition of an involutive basis (they considered only strong bases) and we will show next that the two definitions are equivalent. This requires the introduction of some further concepts.

**Definition 4.2** Let  $\mathcal{F} \subset \mathcal{P}$  be a finite set and L an involutive division on  $\mathbb{N}_0^n$ . We assign each element  $f \in \mathcal{F}$  a set of multiplicative variables

$$X_{L,\mathcal{F},\prec}(f) = \left\{ x_i \mid i \in N_{L, \mathrm{le}_{\prec}\mathcal{F}}(\mathrm{le}_{\prec}f) \right\}.$$
(18)

The involutive span of  $\mathcal{F}$  is then the set

$$\langle \mathcal{F} \rangle_{L,\prec} = \sum_{f \in \mathcal{F}} \Bbbk[X_{L,\mathcal{F},\prec}(f)] \star f \subseteq \langle \mathcal{F} \rangle .$$
 (19)

An important aspect of Gröbner bases is the existence of standard representations for ideal elements. For (weak) involutive bases a similar characterisation exists and in the case of strong bases we even obtain unique representations.

**Theorem 4.3** Let  $\mathcal{I} \subseteq \mathcal{P}$  be a non-zero ideal,  $\mathcal{H} \subset \mathcal{I}$  a finite set and L an involutive division on  $\mathbb{N}_{0}^{n}$ . Then the following two statements are equivalent.

- (i) The set  $\mathcal{H}$  is a weak involutive basis of  $\mathcal{I}$  with respect to L and  $\prec$ .
- (ii) Every polynomial  $f \in \mathcal{I}$  can be written in the form

$$f = \sum_{h \in \mathcal{H}} P_h \star h \tag{20}$$

where the coefficients  $P_h \in \mathbb{k}[X_{L,\mathcal{H},\prec}(h)]$  satisfy  $\operatorname{lt}_{\prec}(P_h \star h) \preceq \operatorname{lt}_{\prec} f$  for all polynomials  $h \in \mathcal{H}$ .

 $\mathcal{H}$  is a strong involutive basis, if and only if the representation (20) is unique.

*Proof* Let us first assume that  $\mathcal{H}$  is a weak involutive basis. Take an arbitrary polynomial  $f \in \mathcal{I}$ . According to Definition 4.1, its leading exponent  $e_{\prec}f$  lies in the involutive cone  $\mathcal{C}_{L, e_{\prec} \mathcal{H}}(h)$  of at least one element  $h \in \mathcal{H}$ . Let  $\mu = e_{\prec}f - e_{\prec}h$  and set  $f_1 = f - cx^{\mu} \star h$  where the coefficient  $c \in \mathbb{k}$  is chosen such that the leading terms cancel. Obviously,  $f_1 \in \mathcal{I}$  and  $\operatorname{lt}_{\prec} f_1 \prec \operatorname{lt}_{\prec} f$ . Iteration yields a sequence of polynomials  $f_i \in \mathcal{I}$ . After a finite number of steps we must reach  $f_N = 0$ , as the leading terms are always decreasing and by assumption the leading exponent of *any* polynomial in  $\mathcal{I}$  possesses an involutive divisor in  $e_{\prec}\mathcal{H}$ . But this implies the existence of a representation of the form (20).

Now assume that  $\mathcal{H}$  is even a strong involutive basis and take an involutive standard representation (20). By definition of a strong basis, there exists one and only one generator  $h \in \mathcal{H}$  such that  $lt_{\prec}(P_h \star h) = lt_{\prec}f$ . This determines uniquely  $lt_{\prec}P_h$ . Applying the same argument to  $f - (lt_{\prec}P_h) \star h$  shows by recursion that the representation (20) is indeed unique.

For the converse note that (ii) trivially implies that  $le_{\prec}f \in \langle le_{\prec}\mathcal{H} \rangle_{L,\prec}$  for any polynomial  $f \in \mathcal{I}$ . Thus  $le_{\prec}\mathcal{I} \subseteq \langle le_{\prec}\mathcal{H} \rangle_{L,\prec}$ . As it is obvious that we have in fact an equality,  $\mathcal{H}$  is a weak involutive basis.

Now let us assume that the set  $\mathcal{H}$  is only a weak but not a strong involutive basis of  $\mathcal{I}$ . This implies the existence of two generators  $h_1, h_2 \in \mathcal{H}$  such that  $\mathcal{C}_{L, \mathrm{le}_{\prec} \mathcal{H}}(\mathrm{le}_{\prec} h_2) \subset \mathcal{C}_{L, \mathrm{le}_{\prec} \mathcal{H}}(\mathrm{le}_{\prec} h_1)$ . Hence we have  $\mathrm{Im}_{\prec} h_2 = \mathrm{Im}_{\prec}(cx^{\mu} \star h_1)$  for suitably chosen  $c \in \mathbb{k}$  and  $\mu \in \mathbb{N}_0^n$ . Consider the polynomial  $h_2 - cx^{\mu} \star h_1 \in \mathcal{I}$ . If it vanishes, we have found a non-trivial involutive standard representation of 0. Otherwise an involutive standard representation  $h_2 - cx^{\mu} \star h_1 = \sum_{h \in \mathcal{H}} P_h \star h$ with  $P_h \in \mathbb{k}[X_{L,\mathcal{H},\prec}(h)]$  exists. Setting  $P'_h = P_h$  for all generators  $h \neq h_1, h_2$ and  $P'_{h_1} = P_{h_1} + cx^{\mu}, P'_{h_2} = P_{h_2} - 1$  yields again a non-trivial involutive standard representation  $0 = \sum_{h \in \mathcal{H}} P'_h \star h$ . The existence of such a non-trivial representation of 0 immediately implies that (20) cannot be unique. Thus we have given an indirect proof that for a strong involutive basis the involutive standard representation is unique.  $\Box$ 

**Corollary 4.4** *Let the set*  $\mathcal{H}$  *be a weak involutive basis of the left ideal*  $\mathcal{I} \subseteq \mathcal{P}$ *. Then*  $\langle \mathcal{H} \rangle_{L,\prec} = \mathcal{I}$ *.* 

*Proof* It follows immediately from Theorem 4.3 that  $\mathcal{I} \subseteq \langle \mathcal{H} \rangle_{L,\prec}$ . But as  $\mathcal{H}$  is also a Gröbner basis of  $\mathcal{I}$ , we have in fact equality.  $\Box$ 

*Example 4.5* It is *not* true that any set  $\mathcal{F}$  with  $\langle \mathcal{F} \rangle_{L,\prec} = \mathcal{I}$  is a weak involutive basis of the ideal  $\mathcal{I}$ . Consider in the ordinary polynomial ring  $\mathbb{k}[x, y]$  the ideal  $\mathcal{I}$  generated by the two polynomials  $f_1 = y^2$  and  $f_2 = y^2 + x^2$ . If we order the variables as  $x_1 = x$  and  $x_2 = y$ , then the set  $\mathcal{F} = \{f_1, f_2\}$  trivially satisfies  $\langle \mathcal{F} \rangle_{J,\prec} = \mathcal{I}$ , as with respect to the Janet division all variables are multiplicative for each generator. However,  $\mathbb{le}_{\prec}\mathcal{F} = \{[0, 2]\}$  does *not* generate  $\mathbb{le}_{\prec}\mathcal{I}$ , as obviously  $[2, 0] \in \mathbb{le}_{\prec}\mathcal{I} \setminus \langle \{[0, 2]\} \rangle$ . Thus  $\mathcal{F}$  is not a weak Janet basis (neither is the autoreduced set  $\mathcal{F}' = \{y^2, x^2\}$ , as  $x^2y \notin \langle \mathcal{F}' \rangle_{J,\prec}$ ).

Note that Corollary 4.4 implies the equivalence of our definition of an involutive basis and the original one by Gerdt and Blinkov [14]. By a generalisation of Proposition 2.8, any weak involutive basis  $\mathcal{H}$  contains a strong involutive basis.

**Proposition 4.6** Let  $\mathcal{I} \subseteq \mathcal{P}$  be an ideal and  $\mathcal{H} \subset \mathcal{P}$  a weak involutive basis of it for the involutive division L. Then there exists a subset  $\mathcal{H}' \subseteq \mathcal{H}$  which is a strong involutive basis of  $\mathcal{I}$ .

*Proof* If the set  $e_{\prec}\mathcal{H}$  is already a strong involutive basis of  $e_{\prec}\mathcal{I}$ , we are done. Otherwise  $\mathcal{H}$  contains polynomials  $h_1$ ,  $h_2$  such that  $|e_{\prec}h_1|_{L, |e_{\prec}\mathcal{H}} |e_{\prec}h_2$ . Consider the subset  $\mathcal{H}' = \mathcal{H} \setminus \{h_2\}$ . As in the proof of Proposition 2.8 one easily shows that  $|e_{\prec}\mathcal{H}' = |e_{\prec}\mathcal{H} \setminus \{|e_{\prec}h_2\}$  is still a weak involutive basis of  $|e_{\prec}\mathcal{I}|$  and thus  $\mathcal{H}'$ is still a weak involutive basis of  $\mathcal{I}$ . After a finite number of such eliminations we must reach a strong involutive basis.  $\Box$ 

Given this result, one may wonder why we have introduced the notion of a weak basis. The reason is that in more general situations like computations in local rings or polynomial algebras over coefficient rings (treated in later sections) strong bases rarely exist.

**Definition 4.7** Let  $\mathcal{F} \subset \mathcal{P}$  be a finite set and L an involutive division. A polynomial  $g \in \mathcal{P}$  is involutively reducible with respect to  $\mathcal{F}$ , if it contains a term  $x^{\mu}$  such that  $e_{\prec}f|_{L, e_{\prec}\mathcal{F}}\mu$  for some  $f \in \mathcal{F}$ . It is in involutive normal form with respect to  $\mathcal{F}$ , if it is not involutively reducible. The set  $\mathcal{F}$  is involutively autoreduced, if no polynomial  $f \in \mathcal{F}$  contains a term  $x^{\mu}$  such that another polynomial  $f' \in \mathcal{F} \setminus \{f\}$  exists with  $e_{\prec}f'|_{L, e_{\prec}\mathcal{F}}\mu$ .

*Remark 4.8* The definition of an involutively autoreduced set *cannot* be formulated more concisely by saying that each  $f \in \mathcal{F}$  is in involutive normal form with respect to  $\mathcal{F} \setminus \{f\}$ . If we are not dealing with a global division, the removal of f from  $\mathcal{F}$  will generally change the assignment of the multiplicative indices and thus affect the involutive divisibility.

Involutive reducibility is obviously a restriction of ordinary reducibility. An *obstruction to involution* is a polynomial  $g \in \langle \mathcal{F} \rangle \setminus \langle \mathcal{F} \rangle_{L,\prec}$  possessing a (non-involutive) standard representation with respect to  $\mathcal{F}$ . We will later see that these elements make the difference between an involutive and an arbitrary Gröbner basis. In a differential equations context, elements of  $\langle \mathcal{F} \rangle$  which do not possess a standard representation with respect to  $\mathcal{F}$  are called *integrability conditions*. Note that this notion depends on the chosen term order  $\prec$ ; only for degree compatible term orders it coincides with the usual notion of integrability conditions. Obviously,  $\mathcal{F}$  is a Gröbner basis, if and only if no "integrability conditions" exist.

*Example 4.9* Consider the set  $\mathcal{F} = \{f_1, f_2, f_3\} \subset k[x, y, z]$  with the polynomials  $f_1 = z^2 - xy$ ,  $f_2 = yz - x$  and  $f_3 = y^2 - z$ . For any degree compatible term order, the leading terms of  $f_2$  and  $f_3$  are unique. For  $f_1$  we have two possibilities: if we use the degree lexicographic order (i. e. for  $x \prec y \prec z$ ), it is  $z^2$ , for the degree inverse lexicographic order (i. e. for  $x \succ y \succ z$ ) the leading term is xy.

In the first case, neither "integrability conditions" nor obstructions to involution for the Janet division exist, as  $\langle \mathcal{F} \rangle_{J, \prec_{\text{deglex}}} = \langle \mathcal{F} \rangle$ . Thus  $\mathcal{F}$  is a Janet basis, i. e. an involutive basis with respect to the Janet division, for this term order, although we have not yet the necessary tools to prove this fact. In the second case,  $f_4 = z^3 - x^2 = zf_1 + xf_2$  is an "integrability condition". Hence adding it to  $\mathcal{F}$  yields a Gröbner basis  $\mathcal{G}$  of  $\langle \mathcal{F} \rangle$ , as one may easily check. But this makes z non-multiplicative for  $f_2$  and  $f_5 = zf_2$  is now an obstruction to involution of  $\mathcal{G}$ , as it is not involutively reducible with respect to the Janet division. In fact, the set  $\mathcal{F}' = \{f_1, f_2, f_3, f_4, f_5\}$  is the smallest Janet basis of  $\mathcal{I}$  for this term order, as it is not possible to remove an element. Note that this second basis is not only larger but also contains polynomials of higher degree.

It often suffices, if one does not consider all terms in g but only the leading term  $lt_{\prec}g$ : the polynomial g is *involutively head reducible*, if  $le_{\prec}f|_{L,le_{\prec}\mathcal{F}}$   $le_{\prec}g$  for some  $f \in \mathcal{F}$ . Similarly, the set  $\mathcal{F}$  is *involutively head autoreduced*, if no leading exponent of an element  $f \in \mathcal{F}$  is involutively divisible by the leading exponent of another element  $f' \in \mathcal{F} \setminus \{f\}$ . Note that the definition of a strong involutive basis immediately implies that it is involutively head autoreduced.

As involutive reducibility is a restriction of ordinary reducibility, involutive normal forms can be determined with trivial adaptions of the familiar algorithms. The termination follows by the same argument as usual, namely that any term order is a well-order. If g' is an involutive normal form of  $g \in \mathcal{P}$  with respect to the set  $\mathcal{F}$  for the division L, then we write  $g' = \operatorname{NF}_{\mathcal{F},L,\prec}(g)$ , although involutive normal forms are in general not unique (like ordinary normal forms). Depending on the order in which reductions are applied different results are obtained.

The ordinary normal form is unique, if and only if it is computed with respect to a Gröbner basis; this property is often used as an alternative definition of Gröbner bases. The situation is somewhat different for the involutive normal form.

**Lemma 4.10** The sum in (19) is direct, if and only if the finite set  $\mathcal{F} \subset \mathcal{P}$  is involutively head autoreduced with respect to the involutive division L, .

*Proof* One direction is obvious. For the converse, let  $f_1$ ,  $f_2$  be two distinct elements of  $\mathcal{F}$  and  $X_i = X_{L,\mathcal{F},\prec}(f_i)$  their respective sets of multiplicative variables for the division L. Assume that two polynomials  $P_i \in \mathbb{k}[X_i]$  exist with  $P_1 \star f_1 = P_2 \star f_2$  and hence  $e_{\prec}(P_1 \star f_1) = e_{\prec}(P_2 \star f_2)$ . As the multiplication  $\star$  respects the term order  $\prec$ , this implies that  $\mathcal{C}_{L, \mathrm{le}_{\prec}\mathcal{F}}(\mathrm{le}_{\prec}f_1) \cap \mathcal{C}_{L, \mathrm{le}_{\prec}\mathcal{F}}(\mathrm{le}_{\prec}f_2) \neq \emptyset$ . Thus one of the involutive cones is completely contained in the other one and either  $e_{\prec}f_1 \mid_{L, \mathrm{le}_{\prec}\mathcal{F}} e_{\prec}f_2$  or  $e_{\prec}f_2 \mid_{L, \mathrm{le}_{\prec}\mathcal{F}} e_{\prec}f_1$  contradicting that  $\mathcal{F}$  is involutively head autoreduced.  $\Box$ 

**Proposition 4.11** If the finite set  $\mathcal{F} \subset \mathcal{P}$  is involutively head autoreduced, every polynomial  $g \in \mathcal{P}$  has a unique involutive normal form  $\operatorname{NF}_{\mathcal{F},L,\prec}(g)$ .

*Proof* If 0 is an involutive normal form of g, then obviously  $g \in \langle \mathcal{F} \rangle_{L,\prec}$ . Conversely, assume that  $g \in \langle \mathcal{F} \rangle_{L,\prec}$ , i.e. the polynomial g can be written in the form  $g = \sum_{f \in \mathcal{F}} P_f \star f$  with  $P_f \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f)]$ . As  $\mathcal{F}$  is involutively head autoreduced, the leading terms of the summands never cancel (see the proof of Lemma 4.10). Thus  $lt_{\prec}g = lt_{\prec}(P_f \star f)$  for some  $f \in \mathcal{F}$  and any polynomial  $g \in \langle \mathcal{F} \rangle_{L,\prec}$  is involutively head reducible with respect to  $\mathcal{F}$ . Each reduction step in an involutive normal form algorithm leads to a new polynomial  $g' \in \langle \mathcal{F} \rangle_{L,\prec}$  with  $lt_{\prec}g' \preceq lt_{\prec}g$ . If the leading term is reduced, we even get  $lt_{\prec}g' \prec lt_{\prec}g$ .

each terminating normal form algorithm must sooner or later reduce the leading term, we eventually obtain 0 as unique involutive normal form of any  $g \in \langle \mathcal{F} \rangle_{L,\prec}$ .

Let  $g_1$  and  $g_2$  be two involutive normal forms of the polynomial g. Obviously,  $g_1 - g_2 \in \langle \mathcal{F} \rangle_{L,\prec}$ . By definition of a normal form, neither  $g_1$  nor  $g_2$  contain any term involutively reducible with respect to  $\mathcal{F}$  and the same holds for  $g_1 - g_2$ . Hence the difference  $g_1 - g_2$  is also in involutive normal form and by our considerations above we must have  $g_1 - g_2 = 0$ .  $\Box$ 

**Proposition 4.12** *The ordinary and the involutive normal form of any polynomial*  $g \in \mathcal{P}$  *with respect to a finite weakly involutive set*  $\mathcal{F} \subset \mathcal{P}$  *are identical.* 

*Proof* Recalling the proof of the previous proposition, we see that we used the assumption that  $\mathcal{F}$  was involutively head autoreduced only for proving the existence of a generator  $f \in \mathcal{F}$  such that  $\operatorname{lt}_{\prec} f|_{L,\operatorname{le}_{\prec}\mathcal{F}} \operatorname{lt}_{\prec} g$  for every polynomial  $g \in \langle \mathcal{F} \rangle_{L,\prec}$ . But obviously this property is also implied by the definition of a weak involutive basis. Thus by the same argument as above, we conclude that the involutive normal form with respect to a weakly involutive set is unique. For Gröbner bases the uniqueness of the ordinary normal form is a classical property and any weak involutive basis is also a Gröbner basis. As a polynomial in ordinary normal form with respect to  $\mathcal{F}$  is trivially in involutive normal form with respect to  $\mathcal{F}$ , too, the two normal forms must coincide.  $\Box$ 

Finally, we extend the notion of a minimal involutive basis from  $\mathbb{N}_0^n$  to  $\mathcal{P}$ . This is done in the same manner as in the theory of Gröbner bases.

**Definition 4.13** Let  $\mathcal{I} \subseteq \mathcal{P}$  be an ideal and L an involutive division. An involutive basis  $\mathcal{H}$  of  $\mathcal{I}$  with respect to L is minimal, if  $le_{\prec}\mathcal{H}$  is a minimal involutive basis of the monoid ideal  $le_{\prec}\mathcal{I}$  for the division L.

By Proposition 2.10, we find that for a globally defined division like the Pommaret division any involutive basis is minimal. Uniqueness requires two additional assumptions. First of all, our definition of an involutive basis requires only that it is involutively head autoreduced; for uniqueness we obviously need a full involutive autoreduction. Secondly, we must normalise the leading coefficients to one, i. e. we must take a *monic* basis.

**Proposition 4.14** Let  $\mathcal{I} \subseteq \mathcal{P}$  be an ideal and L an involutive division. Then  $\mathcal{I}$  has at most one monic, involutively autoreduced, minimal involutive basis for L.

*Proof* Assume that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are two different monic, involutively autoreduced, minimal involutive bases of  $\mathcal{I}$  with respect to L and  $\prec$ . By definition of a minimal involutive bases, this implies that  $lt_{\prec}\mathcal{H}_1 = lt_{\prec}\mathcal{H}_2$ . As  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are not identical, we must have two polynomials  $h_1 \in \mathcal{H}_1$  and  $h_2 \in \mathcal{H}_2$  such that  $lt_{\prec}h_1 = lt_{\prec}h_2$  but  $h_1 \neq h_2$ . Now consider the polynomial  $h = h_1 - h_2 \in \mathcal{I}$ . Its leading term must lie in the involutive span of  $lt_{\prec}\mathcal{H}_1 = lt_{\prec}\mathcal{H}_2$ . On the other hand, the term  $lt_{\prec}h$  must be contained in either  $h_1$  or  $h_2$ . But this implies that either  $\mathcal{H}_1$ or  $\mathcal{H}_2$  is not involutively autoreduced.  $\Box$ 

### **5** Monomial Completion

We turn to the question of the actual construction of involutive bases. Unfortunately, for arbitrary involutive division no satisfying solution is known so far. In the monomial case, one may follow a brute force approach, namely performing a breadth first search through the tree of all possible completions. Obviously, it terminates only, if a finite basis exists. But for divisions satisfying some additional properties one can design a fairly efficient completion algorithm.

The first problem in constructing an involutive completion of a finite subset  $\mathcal{N} \subset \mathbb{N}_0^n$  for a division L is to check whether  $\mathcal{N}$  is already involutive. The trouble is that we do not know a priori where obstructions to involution might lie. As these multi indices must somehow be related to the non-multiplicative indices of the elements of  $\mathcal{N}$ , the multi indices  $\nu + 1_j$  with  $\nu \in \mathcal{N}$  and  $j \in \overline{N}_{L,\mathcal{N}}(\nu)$  are a natural first guess.

**Definition 5.1** The finite set  $\mathcal{N} \subset \mathbb{N}_0^n$  is locally involutive for the involutive division L, if  $\nu + 1_j \in \langle \mathcal{N} \rangle_L$  for every non-multiplicative index  $j \in \overline{N}_{L,\mathcal{N}}(\nu)$  of every multi index  $\nu \in \mathcal{N}$ .

While (weak) involution obviously implies local involution, the converse does not hold. A concrete counter example has been given by Gerdt and Blinkov [14]. But for many important divisions the converse is in fact true.

**Definition 5.2** Let L be an involutive division and  $\mathcal{N} \subset \mathbb{N}_0^n$  a finite set. Let furthermore  $(\nu^{(1)}, \ldots, \nu^{(t)})$  be a finite sequence of elements of  $\mathcal{N}$  where every multiindex  $\nu^{(k)}$  with k < t has a non-multiplicative index  $j_k \in \overline{N}_{L,\mathcal{N}}(\nu^{(k)})$  such that  $\nu^{(k+1)}|_{L,\mathcal{N}} \nu^{(k)} + 1_{j_k}$ . The division L is continuous, if any such sequence consists only of distinct elements, i. e. if  $\nu^{(k)} \neq \nu^{(\ell)}$  for all  $k \neq \ell$ .

**Proposition 5.3** For a continuous division *L*, any locally involutive set  $\mathcal{N} \subset \mathbb{N}_0^n$  is weakly involutive.

*Proof* Let the set  $\Sigma$  contain those obstructions to involution that are of minimal length. We claim that for a continuous division L all multi-indices  $\sigma \in \Sigma$  are of the form  $\nu + 1_j$  with  $\nu \in \mathcal{N}$  and  $j \in \overline{N}_{L,\mathcal{N}}(\nu)$ . This immediately implies our proposition: since for a locally involutive set all such multi-indices are contained in  $\langle \mathcal{N} \rangle_L$ , we must have  $\Sigma = \emptyset$  and thus  $\langle \mathcal{N} \rangle = \langle \mathcal{N} \rangle_L$ .

In order to prove our claim, we choose a  $\sigma \in \Sigma$  for which no  $\nu \in \mathcal{N}$  exists with  $\sigma = \nu + 1_j$ . We collect in  $\mathcal{N}_{\sigma}$  all divisors  $\nu \in \mathcal{N}$  of  $\sigma$  of maximal length. Let  $\nu^{(1)}$  be an element of  $\mathcal{N}_{\sigma}$ ; by assumption the multi index  $\mu^{(1)} = \sigma - \nu^{(1)}$ satisfies  $|\mu^{(1)}| > 1$  and at least one non-multiplicative index  $j_i \in \overline{N}_{L,\mathcal{N}}(\nu^{(1)})$ exists with  $\mu_{j_1}^{(1)} > 0$ . By the definition of  $\Sigma$  we have  $\nu^{(1)} + 1_{j_1} \in \langle \mathcal{N} \rangle_L$ . Thus a multi index  $\nu^{(2)} \in \mathcal{N}$  exists with  $\nu^{(2)}|_{L,\mathcal{N}}\nu^{(1)} + 1_{j_1}$ . This implies  $\nu^{(2)} | \sigma$  and we set  $\mu^{(2)} = \sigma - \nu^{(2)}$ . By the definition of the set  $\mathcal{N}_{\sigma}$  we have  $|\nu^{(2)}| \leq |\nu^{(1)}|$ . Hence  $\nu^{(2)} + 1_j \in \langle \mathcal{N} \rangle_L$  for all j.

Choose a non-multiplicative index  $j_2 \in \overline{N}_{L,\mathcal{N}}(\nu^{(2)})$  with  $\mu_{j_2}^{(2)} > 0$ . Such an index exists as otherwise  $\sigma \in \langle \mathcal{N} \rangle_L$ . By the same arguments as above, a multi

index  $\nu^{(3)} \in \mathcal{N}$  exists with  $\nu^{(3)} |_{L,\mathcal{N}} \nu^{(2)} + 1_{j_2}$  and  $|\nu^{(3)}| \leq |\nu^{(2)}|$ . We can iterate this process and produce an infinite sequence  $(\nu^{(1)}, \nu^{(2)}, \dots)$  where each multi index  $\nu^{(i)} \in \mathcal{N}$  and  $\nu^{(i+1)} |_{L,\mathcal{N}} \nu^{(i)} + 1_{j_i}$  with  $j_i \in \overline{N}_{L,\mathcal{N}}(\nu^{(i)})$ . As  $\mathcal{N}$  is a finite set, the elements of the sequence cannot be all different. This contradicts our assumption that L is a continuous division: by taking a sufficiently large part of this sequence we obtain a finite sequence with all properties mentioned in Definition 5.2 but containing some identical elements. Hence a multi index  $\nu \in \mathcal{N}$  must exist such that  $\sigma = \nu + 1_j$ .  $\Box$ 

### Lemma 5.4 The Janet and the Pommaret division are continuous.

*Proof* Let  $\mathcal{N} \subseteq \mathbb{N}_0^n$  be a finite set and  $(\nu^{(i)}, \ldots, \nu^{(t)})$  a finite sequence where  $\nu^{(i+1)}|_{L,\mathcal{N}} \nu^{(i)} + 1_j$  with  $j \in \bar{N}_{L,\mathcal{N}}(\nu^{(i)})$  for  $1 \leq i < t$ .

We claim that for L = J, the Janet division,  $\nu^{(i+1)} \succ_{\text{lex}} \nu^{(i)}$  implying that the sequence cannot contain any identical entries. Set  $k = \max\{i \mid \mu_i \neq \nu_i\}$ . Then  $j \leq k$ , as otherwise  $j \in N_{J,\mathcal{N}}(\nu^{(i+1)})$  entails  $j \in N_{J,\mathcal{N}}(\nu^{(i)})$  contradicting our assumption that j is non-multiplicative for the multi index  $\nu^{(i)}$ . But j < k is also not possible, as then  $\nu_k^{(i+1)} < \nu_k^{(i)}$  and so k cannot be multiplicative for  $\nu^{(i+1)}$ . There remains as only possibility j = k. In this case  $\nu_j^{(i+1)} = \nu_j^{(i)} + 1$ , as otherwise j could not be multiplicative for  $\nu^{(i+1)}$ . Thus we conclude that  $\nu^{(i+1)} \succ_{\text{lex}} \nu^{(i)}$  and the Janet division is continuous.

The proof for the case L = P, the Pommaret division, is slightly more subtle.<sup>3</sup> The condition  $j \in \bar{N}_P(\nu^{(i)})$  implies that  $\operatorname{cls}(\nu^{(i)} + 1_j) = \operatorname{cls}\nu^{(i)}$  and if  $\nu^{(i+1)} |_P \nu^{(i)} + 1_j$ , then  $\operatorname{cls}\nu^{(i+1)} \ge \operatorname{cls}\nu^{(i)}$ , i. e. the class of the elements of the sequence is monotonously increasing. If  $\operatorname{cls}\nu^{(i+1)} = \operatorname{cls}\nu^{(i)} = k$ , then the involutive divisibility requires that  $\nu_k^{(i+1)} \le \nu_k^{(i)}$ , i. e. among the elements of the sequence of the same class the corresponding entry is monotonously decreasing. And if finally  $\nu_k^{(i+1)} = \nu_k^{(i)}$ , then we must have  $\nu^{(i+1)} = \nu^{(i)} + 1_j$ , i. e. the length of the elements is strictly increasing. Hence all elements of the sequence are different and the Pommaret division is continuous.  $\Box$ 

**Definition 5.5** Let *L* be an involutive division and  $\mathcal{N} \subset \mathbb{N}_0^n$  a finite set of multi indices. Choose a multi index  $\nu \in \mathcal{N}$  and a non-multiplicative index  $j \in \overline{N}_{L,\mathcal{N}}(\nu)$  such that:

(i)  $\nu + 1_j \notin \langle \mathcal{N} \rangle_L$ ;

(ii) if there exists  $\mu \in \mathcal{N}$  and  $k \in \overline{N}_{L,\mathcal{N}}(\nu)$  such that  $\mu + 1_k \mid \nu + 1_j$  but  $\mu + 1_k \neq \nu + 1_j$ , then  $\mu + 1_k \in \langle \mathcal{N} \rangle_L$ .

The division L is constructive,<sup>4</sup> if for any such set  $\mathcal{N}$  and any such multi index  $\nu + 1_j$  no multi index  $\rho \in \langle \mathcal{N} \rangle_L$  with  $\nu + 1_j \in \mathcal{C}_{L,\mathcal{N} \cup \{\rho\}}(\rho)$  exists.

<sup>4</sup> In [14] constructivity was introduced only for continuous divisions. Although we will see later that both concepts are indeed mostly used together (in fact, because of the re-

<sup>&</sup>lt;sup>3</sup> It is tempting to tackle the Pommaret division in the same manner as the Janet division using  $\prec_{\text{revlex}}$  instead of  $\prec_{\text{lex}}$ ; in fact, such a "proof" is contained in [14]. Unfortunately, it is not correct, as  $\prec_{\text{revlex}}$  is not a term order: if  $\nu^{(i+1)} = \nu^{(i)} + 1_j$ , then  $\nu^{(i+1)} \prec_{\text{revlex}} \nu^{(i)}$  although the latter multi index is a divisor of the former one. Thus the sequences considered in Definition 5.2 are in general not strictly ascending with respect to  $\prec_{\text{revlex}}$ .

In words, constructivity may roughly be explained as follows. The conditions imposed on  $\nu$  and j ensure a kind of minimality: no proper divisor of  $\nu + 1_j$  is of the form  $\mu + 1_k$  for a  $\mu \in \mathcal{N}$  and not contained in the involutive span  $\langle \mathcal{N} \rangle_L$ . The conclusion implies that it is useless to add multi indices to  $\mathcal{N}$  that lie in some involutive cone, as none of them can be an involutive divisor of  $\nu + 1_j$ . An efficient completion algorithm for a constructive division should consider only non-multiplicative indices.

# **Lemma 5.6** Any globally defined division (and thus the Pommaret division) is constructive. The Janet division is constructive, too.

*Proof* For a globally defined division the proof is very simple. For any multi index  $\rho \in \langle \mathcal{N} \rangle_L$  there exists a multi index  $\mu \in \mathcal{N}$  such that  $\rho \in \mathcal{C}_L(\mu)$ . As for a globally defined division the multiplicative indices are independent of the reference set, we must have by the definition of an involutive division that  $\mathcal{C}_L(\rho) \subseteq \mathcal{C}_L(\mu)$ . Hence adding such a multi index to  $\mathcal{N}$  cannot change the involutive span and if  $\nu + 1_i \notin \langle \mathcal{N} \rangle_L$ , then also  $\nu + 1_i \notin \langle \mathcal{N} \cup \{\rho\} \rangle_L$ . This implies constructivity.

The proof of the constructivity of the Janet division is more involved. The basic idea is to show that if it was not constructive, it could not be continuous either. Let  $\mathcal{N}, \nu, j$  be as described in Definition 5.5. Assume for a contradiction that a multi index  $\rho \in \langle \mathcal{N} \rangle_J$  exists with  $\nu + 1_j \in \mathcal{C}_{J,\mathcal{N} \cup \{\rho\}}(\rho)$ . We write  $\rho = \nu^{(1)} + \mu$  for a multi index  $\nu^{(1)} \in \mathcal{N}$  with  $\rho \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(1)})$ . As  $\nu + 1_j \notin \langle \mathcal{N} \rangle_J$ , we must have  $|\mu| > 0$ . Set  $\lambda = \nu + 1_j - \rho$  and let m, l be the maximal indices such that  $\mu_m > 0$  and  $\lambda_l > 0$ , respectively.

We claim that  $j > \max\{m, l\}$ . Indeed, if  $j \le m$ , then  $\nu_m^{(1)} < \nu_m$  and, by definition of the Janet division, this implies that  $m \notin N_{J,\mathcal{N}}(\nu^{(1)})$ , a contradiction. Similarly, we cannot have j < l, as then  $l \notin N_{J,\mathcal{N}\cup\{\rho\}}(\rho)$ . Finally, j = l is not possible. As we know already that j > m, we have in this case that  $\rho_i = \nu_i^{(1)} = \nu_i$  for all i > j and  $\rho_j \le \nu_j$ . Hence  $j \in \overline{N}_{J,\mathcal{N}\cup\{\rho\}}(\nu)$  and this implies furthermore  $j \in \overline{N}_{J,\mathcal{N}\cup\{\rho\}}(\rho)$ , a contradiction.

We construct a sequence as in Definition 5.2 of a continuous division. Choose an index  $j_1$  with  $\lambda_{j_1} > 0$  and  $j_1 \in \overline{N}_{J,\mathcal{N}}(\nu^{(1)})$ . Such an index exists, as otherwise  $\nu + 1_j \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(1)}) \subset \langle \mathcal{N} \rangle_J$ . We write  $\nu + 1_j = (\nu^{(1)} + 1_{j_1}) + \mu + \lambda - 1_{j_1}$ . Because of  $|\mu| > 0$ , the multi index  $\nu^{(1)} + 1_{j_1}$  is a proper divisor of  $\nu + 1_j$  and according to our assumptions a  $\nu^{(2)} \in \mathcal{N}$  exists with  $\nu^{(1)} + 1_{j_1} \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(2)})$ . By the same arguments as above an index  $j_2 \in \overline{N}_{J,\mathcal{N}}(\nu^{(2)})$  must exist with  $(\mu + \lambda - 1_{j_1})_{j_2} > 0$  and a multi index  $\nu^{(3)} \in \mathcal{N}$  with  $\nu^{(2)} + 1_{j_2} \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(3)})$ . Thus we can iterate and produce an infinite sequence  $(\nu^{(1)}, \nu^{(2)}, \dots)$  such that everywhere  $\nu^{(i+1)} = \nu + \nu^{(i)} + 1$ , with  $i_1 \in \overline{N}_{J,\mathcal{N}}(\nu^{(i)})$ . By the continuity of the

By the same arguments as above an index  $j_2 \in \bar{N}_{J,\mathcal{N}}(\nu^{(2)})$  must exist with  $(\mu + \lambda - 1_{j_1})_{j_2} > 0$  and a multi index  $\nu^{(3)} \in \mathcal{N}$  with  $\nu^{(2)} + 1_{j_2} \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(3)})$ . Thus we can iterate and produce an infinite sequence  $(\nu^{(1)}, \nu^{(2)}, \dots)$  such that everywhere  $\nu^{(i+1)}|_{J,\mathcal{N}}\nu^{(i)} + 1_{j_i}$  with  $j_i \in \bar{N}_{J,\mathcal{N}}(\nu^{(i)})$ . By the continuity of the Janet division all members of the sequence must be different. However, every multi index  $\nu^{(i)}$  is a divisor of  $\nu + 1_j$ , so only finitely many of them can be different. Thus the sequence must terminate which only happens, if  $\nu + 1_j \in \mathcal{C}_{J,\mathcal{N}}(\nu^{(i)})$  for some *i* contradicting our assumptions.  $\Box$ 

striction to multi indices of the form  $\nu + 1_j$ , constructivity is really important only for continuous divisions), they are a priori independent and hence we define constructivity for arbitrary divisions.

We present now an algorithm for determining weak involutive completions of finite set  $\mathcal{N} \subset \mathbb{N}_0^n$ . As mentioned above, for arbitrary involutive divisions, nobody has so far been able to find a reasonable approach. But if we assume that the division is continuous and constructive, then a very simple algorithm exists.

Algorithm 5.1 Completion in  $(\mathbb{N}_0^n, +)$ 

**Input:** a finite set  $\mathcal{N} \subset \mathbb{N}_{0}^{n}$ , an involutive division L, a term order  $\prec$  **Output:** a weak involutive completion  $\overline{\mathcal{N}}$  of  $\mathcal{N}$ 1:  $\overline{\mathcal{N}} \leftarrow \mathcal{N}$ 2: **repeat** 3:  $\mathcal{S} \leftarrow \{\nu + 1_{j} \mid \nu \in \overline{\mathcal{N}}, j \in \overline{N}_{L,\overline{\mathcal{N}}}(\nu), \nu + 1_{j} \notin \langle \overline{\mathcal{N}} \rangle_{L} \}$ 4:  $\overline{\mathcal{N}} \leftarrow \overline{\mathcal{N}} \cup \{\min_{\prec} \mathcal{S}\}$ 5: **until**  $\mathcal{S} = \emptyset$ 6: **return**  $\overline{\mathcal{N}}$ 

The strategy behind Algorithm 5.1 is fairly natural given the results above. In Line /3/ it collects in a set S all "minimal" obstructions to involution. For a continuous division L, the set N is weakly involutive, if and only if  $S = \emptyset$ . Furthermore, for a constructive division L it does not make sense to add elements of  $\langle N \rangle_L$  to N in order to complete. Thus we add in Line /4/ an element of S. This element is chosen with the help of a term order.

**Proposition 5.7** Let the finite set  $\mathcal{N} \subset \mathbb{N}_0^n$  possess a finite (weak) involutive completion with respect to the continuous and constructive division L. Then Algorithm 5.1 terminates with a weak involutive completion  $\overline{\mathcal{N}}$  of  $\mathcal{N}$ .

*Proof* If Algorithm 5.1 terminates, its correctness is obvious. The criterion for its termination,  $S = \emptyset$ , is equivalent to local involution of  $\overline{N}$ . By Proposition 5.3, local involution implies for a continuous division weak involution. Thus the result  $\overline{N}$  is a weak involutive completion of N, as by construction  $N \subseteq \overline{N} \subset \langle N \rangle$ .

If the input set  $\mathcal{N}$  is already weakly involutive, Algorithm 5.1 leaves it unchanged and thus obviously terminates. Let us assume that  $\overline{\mathcal{N}}$  is not yet weakly involutive. In the first iteration of the repeat loop a multi index of the form  $\nu + 1_j$ is added to  $\mathcal{N}$ . It is not contained in  $\langle \mathcal{N} \rangle_L$  and minimal with respect to  $\prec$  among all such non-multiplicative "multiples" of multi indices in  $\mathcal{N}$ . If  $\mathcal{N}_L$  is an arbitrary weak involutive completion of  $\mathcal{N}$ , it must contain a multi index  $\mu \notin \mathcal{N}$  such that  $\mu \mid_{L,\mathcal{N}_L} \nu + 1_j$ . We claim that  $\mu = \nu + 1_j$ .

Assume that  $\mu \neq \nu + 1_j$ . Since  $\mathcal{N}_L \subset \langle \mathcal{N} \rangle$ ,  $\mu$  must lie in the cone of some multi index  $\nu^{(1)} \in \mathcal{N}$ . We will show that, because of the continuity of L,  $\mu \in \langle \mathcal{N} \rangle_L$ contradicting the constructivity of L. If  $\nu^{(1)}|_{L,\mathcal{N}}\mu$ , we are done. Otherwise we write  $\mu = \nu^{(1)} + \rho^{(1)}$  for some multi index  $\rho^{(1)} \in \mathbb{N}_0^n$ . A non-multiplicative index  $j_1 \in \overline{N}_{L,\mathcal{N}}(\nu^{(1)})$  with  $\rho_{j_1}^{(1)} > 0$  must exist. Consider the multi index  $\nu^{(1)} + 1_{j_1}$ . Because of  $\nu^{(1)} + 1_{j_1} \mid \mu$  and  $\mu \mid \nu + 1_j$  the inequality  $\nu^{(1)} + 1_{j_1} \prec (\nu + 1_j)$ holds. As  $\nu + 1_j$  is minimal with respect to  $\prec$ , we must have  $\nu^{(1)} + 1_{j_1} \in \langle \mathcal{N} \rangle_L$ . Thus a multi index  $\nu^{(2)} \in \mathcal{N}$  exists such that  $\nu^{(2)} \mid_{L,\mathcal{N}} \nu^{(1)} + 1_{j_1}$ . By iteration, we construct a sequence  $(\nu^{(1)}, \nu^{(2)}, \dots)$  where each element  $\nu^{(i)}$  is a divisor of  $\mu$  and where  $\nu^{(i+1)}|_{L,\mathcal{N}}\nu^{(i)} + 1_{j_i}$  with a non-multiplicative index  $j_i \in \overline{N}_{L,\mathcal{N}}(\nu^{(i)})$ . This sequence cannot become infinite for a continuous division, as  $\mu$  has only finitely many different divisors and all the multi indices  $\nu^{(i)}$  in arbitrary finite pieces of the sequence must be different. The sequence only stops, if some  $\nu^{(i)} \in \mathcal{N}$  exists such that  $\nu^{(i)}|_{L,\mathcal{N}}\mu$  and hence  $\mu \in \langle \mathcal{N} \rangle_L$ .

Thus any weak involutive completion  $\mathcal{N}_L$  of the given set  $\mathcal{N}$  must contain the multi index  $\nu + 1_j$ . In the next iteration of the repeat loop, Algorithm 5.1 treats the enlarged set  $\mathcal{N}_1 = \mathcal{N} \cup \{\nu + 1_j\}$ . As any weak involutive completion  $\mathcal{N}_L$  of  $\mathcal{N}$  is also a weak involutive completion of  $\mathcal{N}_1$ , we may apply the same argument again. As a completion  $\mathcal{N}_L$  is by definition a finite set, we must reach after a finite number k of iterations a weak involutive basis  $\mathcal{N}_k$  of  $\langle \mathcal{N} \rangle$ .  $\Box$ 

Note the crucial difference between this result and the termination proof of Buchberger's algorithm for the construction of Gröbner bases. In the latter case, we can show the termination for arbitrary input, i. e. the theorem provides a constructive proof for the existence of such a basis. Here we are only able to prove the termination under the assumption that a finite (weak) involutive basis exists; the existence has to be shown separately. For example, Lemma 2.13 guarantees us that any monoid ideal possesses a finite weak Janet basis.

Recall that by Proposition 2.8 any weak involutive basis can be made strongly involutive by simply eliminating some redundant elements. Thus we obtain an algorithm for the construction of a strong involutive basis of  $\langle N \rangle$  by adding an involutive autoreduction as last step to Algorithm 5.1. Alternatively, we could perform the involutive autoreduction as first step. Indeed, if the input set N is involutively autoreduced, then all intermediate sets  $\overline{N}$  constructed by Algorithm 5.1 are also involutively autoreduced. This is a simple consequence of the second condition in Definition 2.1 of an involutive division that involutive cones may only shrink, if we add elements to the set N.

An interesting aspect of the proof above is that it shows that our completion algorithm is largely independent of the term order  $\prec$ . It affects only the order in which multi indices are added but not which or how many multi indices are added during the completion. We may even use different term orders at each step, as for the proof it is only important that we always add a multi index that is minimal in S with respect to *some* term order.

**Corollary 5.8** If Algorithm 5.1 terminates, its output  $\overline{N}$  is independent of the chosen term order  $\prec$ . This holds even, if the term order is changed in every iteration of the repeat loop. Furthermore, if  $\mathcal{N}_L$  is any weak involutive completion of  $\mathcal{N}$ with respect to the division L, then  $\overline{\mathcal{N}} \subseteq \mathcal{N}_L$ .

*Proof* Consider the set  $\mathcal{L}(\mathcal{N})$  of all weak involutive completions of  $\mathcal{N}$  with respect to the division L and define

$$\tilde{\mathcal{N}} = \bigcap_{\mathcal{N}_L \in \mathcal{L}(\mathcal{N})} \mathcal{N}_L .$$
(21)

We claim that Algorithm 5.1 determines this set  $\tilde{\mathcal{N}}$  independent of the used term order. Obviously, this implies our corollary.

Above we showed that the multi indices added in Algorithm 5.1 are contained in *any* weak involutive completion of  $\mathcal{N}$ . Thus all these multi indices are elements of  $\tilde{\mathcal{N}}$ . As our algorithm terminates with a weak involutive completion, its output is also an element of  $\mathcal{L}(\mathcal{N})$  and hence must be  $\tilde{\mathcal{N}}$ .  $\Box$ 

Any monoid ideal in  $\mathbb{N}_0^n$  has a *unique* minimal basis: take an arbitrary basis and eliminate all multi indices having a divisor in the basis. Obviously, these eliminations do not change the span and the result is a minimal basis. Similarly we have seen in Section 2 that if a monoid ideal  $\mathcal{I} \subseteq \mathbb{N}_0^n$  has a finite involutive basis for a given division L, then a unique minimal involutive basis exists. By the same argument as in the proof of Corollary 5.8, it can easily be constructed by taking the unique minimal basis of  $\mathcal{I}$  as input for Algorithm 5.1.

#### **6** Polynomial Completion

A trivial method to compute an involutive basis for an ideal  $\mathcal{I}$  in a polynomial algebra  $(\mathcal{P}, \star, \prec)$  of solvable type goes as follows: we determine first a Gröbner basis  $\mathcal{G}$  of  $\mathcal{I}$  and then with Algorithm 5.1 the involutive completion of  $le_{\prec}\mathcal{G}$ . In fact, a similar approach is proposed by Sturmfels and White [46] for the construction of Stanley decompositions (cf. Part II). However, we prefer to extend the ideas behind Algorithm 5.1 to a direct completion algorithm for polynomial ideals, as we believe that this is more efficient.

First, we need two subalgorithms: *involutive normal forms* and *involutive head autoreductions*. The design of an algorithm NormalForm<sub>L, \prec</sub>(g,  $\mathcal{H}$ ) determining an involutive normal form of the polynomial g with respect to the finite set  $\mathcal{H} \subset \mathcal{P}$ is trivial. We may use the standard algorithm for normal forms in the Gröbner theory, if we replace the ordinary divisibility by its involutive version. Obviously, this does not affect the termination. Actually, for our purposes it is not even necessary to compute a full normal form; we may stop as soon as we have obtained a polynomial that is not involutively head reducible.

The design of an algorithm InvHeadAutoReduce<sub>L, $\prec$ </sub>( $\mathcal{F}$ ) for an involutive head autoreduction of a finite set  $\mathcal{F}$  is also obvious. Again one may use the standard algorithm with the ordinary divisibility replaced by involutive divisibility.

Based on these two subalgorithms, we propose Algorithm 6.1 for the computation of involutive bases in  $\mathcal{P}$ . It follows the same strategy as the monomial algorithm. We multiply each generator by its non-multiplicative variables. Then we look whether the result is already contained in the involutive span of the basis; if not, it is added. The check whether a polynomial lies in the involutive span is performed via an involutive normal form computation; for each element of the span the involutive normal form is zero. In order to obtain reasonable multiplicative variables, we take care that our set is always involutively head autoreduced.

The way in which we select in Line /7/ the next polynomial  $\bar{g}$  to be treated is more or less identical with the normal selection strategy in the theory of Gröbner bases. There this strategy is known to work very well for degree compatible term

Algorithm 6.1 Completion in  $(\mathcal{P}, \star, \prec)$ 

**Input:** a finite set  $\mathcal{F} \subset \mathcal{P}$ , an involutive division L **Output:** an involutive basis  $\mathcal{H}$  of  $\mathcal{I} = \langle \mathcal{F} \rangle$  with respect to L and  $\prec$ 1:  $\mathcal{H} \leftarrow \texttt{InvHeadAutoReduce}_{L,\prec}(\mathcal{F})$ 2: loop  $\mathcal{S} \leftarrow \left\{ x_j \star h \mid h \in \mathcal{H}, \, x_j \in \bar{X}_{L,\mathcal{H},\prec}(h), \, x_j \star h \notin \langle \mathcal{H} \rangle_{L,\prec} \right\}$ 3: 4: if  $S = \emptyset$  then return  $\mathcal{H}$ 5: 6: else 7:  $\bar{q} \leftarrow \min_{\prec} \mathcal{S}$ 8:  $q \leftarrow \text{NormalForm}_{L,\prec}(\bar{q},\mathcal{H})$ 9:  $\mathcal{H} \leftarrow \texttt{InvHeadAutoReduce}_{L,\prec}(\mathcal{H} \cup \{g\})$ 10: end\_if 11: end\_loop

orders but not so well for other orders like the purely lexicographic one. Here we will see below that the normal selection strategy is important for proving the termination of the algorithm. It is currently unclear to what extent other strategies may be used for the selection of  $\bar{g}$ .

**Definition 6.1** A finite set  $\mathcal{F} \subset \mathcal{P}$  is locally involutive for the division L, if for every polynomial  $f \in \mathcal{F}$  and for every non-multiplicative variable  $x_j \in \overline{X}_{L,\mathcal{F},\prec}(f)$  the product  $x_i \star f$  has an involutive standard representation with respect to  $\mathcal{F}$ .

Note that for an involutively head autoreduced set  $\mathcal{F}$ , we may equivalently demand that  $x_j \star f \in \langle \mathcal{F} \rangle_{L,\prec}$ ; because of Lemma 4.10 this automatically implies the existence of an involutive standard representation. In fact, the criterion appears in this form in Line /3/ of Algorithm 6.1. In any case, local involution may be effectively verified by computing an involutive normal form of  $x_j \star f$  in the usual manner, i. e. always performing head reductions.

**Proposition 6.2** If the finite set  $\mathcal{F} \subset \mathcal{P}$  is locally involutive for the continuous division L, then  $\langle \mathcal{F} \rangle_{L,\prec} = \langle \mathcal{F} \rangle$ .

*Proof* We claim that if  $\mathcal{F}$  is locally involutive (with respect to a continuous division), then every product  $x^{\mu} \star f_1$  of an arbitrary term  $x^{\mu}$  with a polynomial  $f_1 \in \mathcal{F}$  possesses an involutive standard representation. This entails trivially our proposition, as any polynomial in  $\langle \mathcal{F} \rangle$  consists of a finite linear combination of such products. Adding the corresponding involutive standard representations proves that the polynomial is contained in  $\langle \mathcal{F} \rangle_{L,\prec}$ .

In order to prove our claim, it suffices to show the existence of a representation

$$x^{\mu} \star f_1 = \sum_{f \in \mathcal{F}} \left( P_f \star f + \sum_{\nu \in \mathbb{N}_0^n} c_{\nu,f} x^{\nu} \star f \right)$$
(22)

where  $P_f \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f)]$  and  $e_{\prec}(P_f \star f) = e_{\prec}(x^{\mu} \star f_1)$  (or  $P_f = 0$ ) and where the coefficients  $c_{\nu,f} \in \mathbb{k}$  vanish for all multi-indices  $\nu \in \mathbb{N}_0^n$  such that  $e_{\prec}(x^{\nu} \star f) \succeq e_{\prec}(x^{\mu} \star f_1)$ . Our claim follows then by an obvious induction.

If  $x^{\mu} \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f_1)]$ , i.e. it contains only variables that are multiplicative for  $e_{\prec}f_1$ , nothing has to be shown. Otherwise we choose a non-multiplicative index  $j_1 \in \overline{N}_{L, e_{\prec}\mathcal{F}}(e_{\prec}f_1)$  such that  $\mu_{j_1} > 0$ . As  $\mathcal{F}$  is locally involutive, an involutive standard representation  $x_{j_1} \star f_1 = \sum_{f \in \mathcal{F}} P_f^{(1)} \star f$  exists. Let  $\mathcal{F}_2 \subseteq \mathcal{F}$ contain all polynomials  $f_2$  such that  $e_{\prec}(P_{f_2}^{(1)} \star f_2) = e_{\prec}(x_{j_1} \star f_1)$ . If we have  $x^{\mu-1_{j_1}} \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f_2)]$  for all polynomials  $f_2 \in \mathcal{F}_2$ , then we are done, as at least  $\lim_{\prec} (x^{\mu-1_{j_1}} \star P_{f_2}^{(1)}) \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f_2)]$ .

least  $\lim_{\prec} (x^{\mu-1_{j_1}} * P_{f_2}^{(1)})$  for an polynomials  $f_2 \in \mathfrak{c}_2$ , and we are done, as a least  $\lim_{\prec} (x^{\mu-1_{j_1}} * P_{f_2}^{(1)}) \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f_2)]$ . Otherwise we consider the subset  $\mathcal{F}'_2 \subset \mathcal{F}_2$  of polynomials  $f_2$  for which  $x^{\mu-1_{j_1}} \notin \mathbb{k}[X_{L,\mathcal{F},\prec}(f_2)]$  and iterate over it. For each polynomial  $f_2 \in \mathcal{F}'_2$  we choose a non-multiplicative index  $j_2 \in \overline{N}_{L, \ker_{\prec}}(\ker_{d_2} f_2)$  such that  $(\mu-1_{j_1})_{j_2} > 0$ . Again the local involution of the set  $\mathcal{F}$  implies the existence of an involutive standard representation  $x_{j_2} \star f_2 = \sum_{f \in \mathcal{F}} P_f^{(2)} \star f$ . We collect in  $\mathcal{F}_3 \subseteq \mathcal{F}$  all polynomials  $f_3$  such that  $\mathbb{I}_{\prec}(P_{f_3}^{(2)} \star f_3) = \mathbb{I}_{\prec}(x_{j_2} \star f_2)$ . If we introduce the multi index  $\nu = \mathbb{I}_{\prec}(x_{j_1} \star f_1) - \mathbb{I}_{\prec} f_2$ , then  $\mathbb{I}_{\prec}(x^{\mu} \star f_1) = \mathbb{I}_{\prec}(x^{\mu+\nu-1_{j_1}-1_{j_2}} \star f_3)$  for all  $f_3 \in \mathcal{F}_3$ . If  $x^{\mu+\nu-1_{j_1}-1_{j_2}} \in \mathbb{K}[X_{L,\mathcal{F},\prec}(f_3)]$  for all  $f_3 \in \mathcal{F}_3$ , we are done.

Otherwise we continue in the same manner: we collect in a subset  $\mathcal{F}'_3 \subseteq \mathcal{F}_3$ all polynomials  $f_3$  which are multiplied by non-multiplicative variables, for each of them we choose a non-multiplicative index  $j_3 \in \mathbb{K}[X_{L,\mathcal{F},\prec}(f_3)]$  such that  $(\mu - 1_{j_1} - 1_{j_2})_{j_3} > 0$ , determine an involutive standard representation of  $x_{j_3} \star f_3$ and analyse the leading terms. If they are still multiplied with non-multiplicative variables, this leads to sets  $\mathcal{F}'_4 \subseteq \mathcal{F}_4$  and so on. This process yields a whole tree of cases and each branch leads to a sequence  $(\nu^{(1)} = \mathrm{le}_{\prec}f_1, \nu^{(2)} = \mathrm{le}_{\prec}f_2, \ldots)$ where all contained multi indices  $\nu^{(k)}$  are elements of the finite set  $\mathrm{le}_{\prec}\mathcal{F}$  and where to each  $\nu^{(k)}$  a non-multiplicative index  $j_k \in \bar{N}_{L,\mathrm{le}_{\prec}\mathcal{F}}(\nu^{(k)})$  exists such that  $\nu^{(k+1)}|_{L,\mathrm{le}_{\prec}\mathcal{F}}\nu^{(k)}+1_{j_k}$ . By the definition of a continuous division, this sequence cannot become infinite and thus each branch must terminate. But this implies that we may construct for each polynomial  $f_1 \in \mathcal{F}$  and each non-multiplicative variables  $x_j \in \bar{X}_{L,\mathcal{F},\prec}(f_1)$  a representation of the claimed form (22).  $\Box$ 

Note that the proposition only asserts that the involutive span equals the normal span. It does *not* say that  $\mathcal{F}$  is weakly involutive (indeed, the set  $\mathcal{F}$  studied in Example 4.5 would be a simple counterexample). If  $g = \sum_{\mu \in \mathbb{N}_0^n} \sum_{f \in \mathcal{F}} c_{\mu,f} x^{\mu} \star f$  is an arbitrary polynomial in  $\langle \mathcal{F} \rangle$ , then adding the involutive standard representations of all the products  $x^{\mu} \star f$  for which  $c_{\mu,f} \neq 0$  yields a representation  $g = \sum_{f \in \mathcal{F}} P_f \star f$  where  $P_f \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f)]$ . But in general it will not satisfy the condition  $e_{\prec}(P_f \star f) \leq e_{\prec}g$  for all  $f \in \mathcal{F}$ . This is guaranteed only for involutively head autoreduced sets, as there it is impossible that the leading terms cancel (Lemma 4.10). For such sets the above proof simplifies, as all the sets  $F_i$  consist of precisely one element and thus no branching is necessary.

**Corollary 6.3** For a continuous division L an involutively head autoreduced set  $\mathcal{F} \subset \mathcal{P}$  is involutive, if and only if it is locally involutive.

As in the proof of Proposition 5.7, local involution of  $\mathcal{H}$  is obviously equivalent to the termination condition  $\mathcal{S} = \emptyset$  of the repeat loop in Algorithm 6.1. Thus we are now in the position to prove the following result.

**Theorem 6.4** Let *L* be a continuous and constructive division and  $(\mathcal{P}, \star, \prec)$  a polynomial algebra of solvable type. Assume that the ideal  $\mathcal{I} \subseteq \mathcal{P}$  is such that  $\operatorname{le}_{\prec}\mathcal{I}$  possesses a finite involutive completion. Then Algorithm 6.1 terminates for any finite generating set  $\mathcal{F}$  of  $\mathcal{I}$  with an involutive basis.

*Proof* We begin by proving the *correctness* of the algorithm under the assumption that it terminates. The relation  $\mathcal{I} = \langle \mathcal{H} \rangle$  remains valid throughout, although  $\mathcal{H}$  changes. But the only changes are the addition of further elements of  $\mathcal{I}$  and involutive head autoreductions; both operations do not affect the ideal generated by  $\mathcal{H}$ . When the algorithm terminates, we have  $\mathcal{S} = \emptyset$  and thus the output  $\mathcal{H}$  is locally involutive and by Corollary 6.3 involutive.

There remains the problem of *termination*. Algorithm 6.1 produces a sequence  $(\mathcal{H}_1, \mathcal{H}_2, ...)$  with  $\langle \mathcal{H}_i \rangle = \mathcal{I}$ . The set  $\mathcal{H}_{i+1}$  is determined from  $\mathcal{H}_i$  in Line /9/. We distinguish two cases, namely whether or not during the computation of the involutive normal form in Line /8/ the leading exponent changes. If  $le_{\prec}\bar{g} = le_{\prec}g$ , then  $\langle le_{\prec}\mathcal{H}_i \rangle = \langle le_{\prec}\mathcal{H}_{i+1} \rangle$ , as  $le_{\prec}g = le_{\prec}h + 1_j$  for some  $h \in \mathcal{H}_i$ . Otherwise we claim that  $\langle le_{\prec}\mathcal{H}_i \rangle \subsetneq \langle le_{\prec}\mathcal{H}_{i+1} \rangle$ .

By construction, g is in involutive normal form with respect to the set  $\mathcal{H}_i$  implying that  $|e_{\prec}g| \in \langle |e_{\prec}\mathcal{H}_i\rangle \setminus \langle |e_{\prec}\mathcal{H}_i\rangle_L$ . If we had  $\langle |e_{\prec}\mathcal{H}_i\rangle = \langle |e_{\prec}\mathcal{H}_{i+1}\rangle$ , a polynomial  $h \in \mathcal{H}_i$  would exist such that  $|e_{\prec}g| = |e_{\prec}h + \mu$  where the multiindex  $\mu$  has a non-vanishing entry  $\mu_j$  for at least one non-multiplicative index  $j \in \overline{N}_{L,|e_{\prec}\mathcal{H}_i}(h)$ . This implies that  $|e_{\prec}h + 1_j \leq |e_{\prec}g| \prec |e_{\prec}\overline{g}$ . But we choose the polynomial  $\overline{g}$  in Line /7/ such that its leading exponent is minimal among all non-multiplicative products  $x_k \star h$  with  $h \in \mathcal{H}_i$ ; hence  $|e_{\prec}\overline{g} \leq |e_{\prec}h + 1_j$ . As this is a contradiction, we must have  $\langle |e_{\prec}\mathcal{H}_i\rangle \subsetneq \langle |e_{\prec}\mathcal{H}_{i+1}\rangle$ .

So the loop of Algorithm 6.1 generates an ascending chain of monoid ideals  $\langle le_{\prec} \mathcal{H}_1 \rangle \subseteq \langle le_{\prec} \mathcal{H}_2 \rangle \subseteq \cdots \subseteq le_{\prec} \mathcal{I}$ . As  $\mathbb{N}_0^n$  is Noetherian, the chain must become stationary at some index N. It follows from the considerations above that in all iterations of the loop after the Nth one  $le_{\prec} \bar{g} = le_{\prec} g$  in Line /8/. At this stage Algorithm 6.1 reduces to an involutive completion of the set  $le_{\prec} \mathcal{H}_N$  using Algorithm 5.1. Indeed, in Line /7/ we choose the polynomial  $\bar{g}$  such that  $le_{\prec} \bar{g}$  is the same multi index as Algorithm 5.1 adds in Line /4/. By Proposition 5.7, the latter algorithm terminates under the made assumptions and hence Algorithm 6.1 terminates, too.  $\Box$ 

## **Corollary 6.5** Let *L* be a Noetherian division. Then every ideal $\mathcal{I} \subseteq \mathcal{P}$ possesses a finite involutive basis with respect to the division *L*.

*Example 6.6* Now we are finally in the position to prove the claims made in Example 4.9. With respect to the degree reverse lexicographic term order the Janet (and the Pommaret) division assigns the polynomial  $f_1 = z^2 - xy$  the multiplicative variables  $\{x, y, z\}$  and the polynomials  $f_2 = yz - x$  and  $f_3 = y^2 - z$  the multiplicative variables  $\{x, y\}$ . Thus we must check the two non-multiplicative products:  $zf_2 = yf_1 + xf_3$  and  $zf_3 = yf_2 - f_1$ . As both possess an involutive standard representation, the set S in Line /3/ of Algorithm 6.1 is empty in the first iteration and thus  $\mathcal{F}$  is a Janet (and a Pommaret) basis of the ideal it generates.

The situation changes, if we use the degree inverse lexicographic term order, as then  $lt_{\prec}f_1 = xy$ . Now  $X_{J,\mathcal{F},\prec}(f_1) = \{x\}, X_{J,\mathcal{F},\prec}(f_2) = \{x, y, z\}$  and  $X_{J,\mathcal{F},\prec}(f_3) = \{x, y\}$ . In the first iteration we find  $\mathcal{S} = \{zf_1\}$ . Its involutive normal form is  $f_4 = z^3 - x^2$  and we add this polynomial to  $\mathcal{F}$  to obtain  $\mathcal{H}_1 = \{f_1, f_2, f_3, f_4\}$  (the involutive head autoreduction does not change the set). For  $f_4$  all variables are multiplicative; for the other generators there is one change: z is no longer multiplicative for  $f_2$ . Thus in the second iteration  $\mathcal{S} = \{zf_2\}$ . It is easy to check that this polynomial is already in involutive normal form with respect to  $\mathcal{H}_1$  and we obtain  $\mathcal{H}_2$  by adding  $f_5 = yz^2 - xz$  to  $\mathcal{H}_1$ . In the next iteration  $\mathcal{S}$  is empty, so that  $\mathcal{H}_2$  is indeed the Janet basis of  $\langle \mathcal{F} \rangle$  for the degree inverse lexicographic term order.  $\triangleleft$ 

Our proof of Theorem 6.4 has an interesting consequence. Assume that the term order  $\prec$  is of type  $\omega$ , i. e. for any two multi indices  $\mu$ ,  $\nu$  with  $\mu \prec \nu$  only finitely many multi indices  $\rho^{(i)}$  exist with  $\mu \prec \rho^{(1)} \prec \rho^{(2)} \prec \cdots \prec \nu$ . Then even if our algorithm does *not* terminate, it determines in a finite number of steps a Gröbner basis of the ideal  $\mathcal{I}$ .

**Proposition 6.7** Let the term order  $\prec$  be of type  $\omega$ . Then Algorithm 6.1 determines for any finite input set  $\mathcal{F} \subset \mathcal{P}$  in a finite number of steps a Gröbner basis of the ideal  $\mathcal{I} = \langle \mathcal{F} \rangle$ .

*Proof* Above we introduced the set  $\mathcal{H}_N$  such that  $\langle le_{\prec} \mathcal{H}_{N+\ell} \rangle = \langle le_{\prec} \mathcal{H}_N \rangle$  for all  $\ell > 0$ . We claim that  $\mathcal{H}_N$  is a Gröbner basis of  $\mathcal{I}$ .

Let  $f \in \mathcal{I}$  be an arbitrary element of the ideal. As  $\mathcal{H}_N$  is a basis of  $\mathcal{I}$ , we find for each  $h \in \mathcal{H}_N$  a polynomial  $g_h \in \mathcal{P}$  such that

$$f = \sum_{h \in \mathcal{H}_N} g_h \star h .$$
<sup>(23)</sup>

 $\mathcal{H}_N$  is a Gröbner basis, if and only if we can choose the coefficients  $g_h$  such that  $\operatorname{lt}_{\prec}(g_h \star h) \preceq \operatorname{lt}_{\prec} f$ . Assume that for f no such standard representation exists and let  $\mu = \max_{h \in \mathcal{H}_N} \{\operatorname{le}_{\prec} g_h + \operatorname{le}_{\prec} h\}$ . If we denote by  $\overline{\mathcal{H}}_N$  the set of all polynomials  $\overline{h} \in \mathcal{H}_N$  for which  $\operatorname{le}_{\prec} g_{\overline{h}} + \operatorname{le}_{\prec} \overline{h} = \mu$ , then we must have a non-trivial syzygy  $\sum_{\overline{h} \in \overline{\mathcal{H}}_N} (\operatorname{le}_{\prec} g_{\overline{h}} + \operatorname{le}_{\prec} \overline{h}) = 0$ . It is easy to see (cf. Lemma 4.1 in Part II) that at least one generator  $\overline{h} \in \overline{\mathcal{H}}_N$  exists such that some non-multiplicative variable  $x_j \in \overline{X}_{L,\mathcal{H}_N}(\overline{h})$  divides  $\operatorname{lt}_{\prec} g_{\overline{h}}$ .

As  $\prec$  is of type  $\omega$ , after a finite number of steps the non-multiplicative product  $x_j \star \bar{h}$  is analysed in Algorithm 6.1. Thus for some  $n_1 \geq 0$  the set  $\mathcal{H}_{N+n_1}$  contains an element  $\bar{h}'$  with  $le_{\prec}\bar{h}' = le_{\prec}(x_j \star \bar{h})$ . Let  $\mu = le_{\prec}g_{\bar{h}}, x^{\mu-1_j} \star x_j = cx^{\mu} + r_1$  and  $\bar{h}' = dx_j \star \bar{h} + r_2$ . Then we may rewrite

$$g_{\bar{h}} \star \bar{h} = \frac{\mathrm{lc}_{\prec} g_{\bar{h}}}{cd} \left[ x^{\mu - 1_j} \star (\bar{h}' - r_2) - dr_1 \star \bar{h} \right] + \left( g_{\bar{h}} - \mathrm{lm}_{\prec} g_{\bar{h}} \right) \star \bar{h} .$$
(24)

As  $\bar{h}'$  was determined via an involutive normal form computation applied to the product  $x_j \star \bar{h}$  and as we know that at this stage of the algorithm the leading term does not change during the computation, the leading term on the right hand side of (24) is  $lt_{\prec}(x^{\mu-1_j} \star \bar{h}')$ . If the term  $x^{\mu-1_j}$  contains a non-multiplicative variable  $x_k \in \bar{X}_{L,\mathcal{H}_{N+n_1}}(\bar{h}')$ , we repeat the argument obtaining a polynomial  $\bar{h}'' \in \mathcal{H}_{N+n_1+n_2}$  such that  $le_{\prec}\bar{h}'' = le_{\prec}(x_k \star \bar{h}')$ .

Obviously, this process terminates after a finite number of steps, even if we do it for each  $\bar{h} \in \bar{\mathcal{H}}_N$ . Thus after  $N + \ell$  iterations we obtain a set  $\mathcal{H}_{N+\ell}$  such that, after applying all the found relations (24), f can be expressed in the form  $f = \sum_{h \in \mathcal{H}_{N+\ell}} \tilde{g}_h \star h$  where still  $\mu = \max_{h \in \mathcal{H}_{N+\ell}} \{ \text{le}_{\prec} \tilde{g}_h + \text{le}_{\prec} h \}$ . Denote again by  $\bar{\mathcal{H}}_{N+\ell} \subseteq \mathcal{H}_{N+\ell}$  the set of all polynomials  $\bar{h}$  achieving this maximum.

By construction, no term  $\operatorname{lt}_{\prec} \bar{g}_{\bar{h}}$  with  $\bar{h} \in \bar{\mathcal{H}}_{N+\ell}$  contains a variable that is non-multiplicative for  $\bar{h}$ . Hence, it is not possible (cf. again Lemma 4.1 in Part II) that  $\sum_{\bar{h} \in \bar{\mathcal{H}}_{N+\ell}} (\operatorname{le}_{\prec} \tilde{g}_{\bar{h}} + \operatorname{le}_{\prec} \bar{h}) = 0$ . But this contradicts  $\mu \succ \operatorname{lt}_{\prec} f$ . Thus each polynomial  $f \in \mathcal{P}$  possesses a standard representation already with respect to  $\mathcal{H}_N$ and this set is a Gröbner basis.  $\Box$ 

Note that in the given form this result is only of theoretical interest, as in general no efficient method exists for checking whether the current basis is already a Gröbner basis. Using standard criteria would destroy all potential advantages of the involutive algorithm. For the special case of Pommaret bases, Apel [3] found a simple criterion that allows us to use a variant of Algorithm 6.1 for the construction of Gröbner bases independent of the existence of a finite involutive basis.

In contrast to the monomial case, one does not automatically obtain a minimal involutive basis by making some minor modifications of Algorithm 6.1. In particular, it does not suffice to apply it to a minimal basis in the ordinary sense. Gerdt and Blinkov [15] presented an algorithm that always returns a minimal involutive basis provided a finite involutive basis exists. While it still follows the same basic strategy of study all products with non-multiplicative variables, it requires a more complicated organisation of the algorithm. We omit here the details.

### 7 Right and Two-Sided Bases

In this section we briefly discuss the relation between left and right involutive bases and the computation of bases for two-sided ideals. We use in this section the following notations: the left ideal generated by  $\mathcal{F} \subset \mathcal{P}$  is denoted by  $\langle \mathcal{F} \rangle^{(l)}$ , the right ideal by  $\langle \mathcal{F} \rangle^{(r)}$  and the two-sided ideal by  $\langle \mathcal{F} \rangle^{\rangle}$  and corresponding notations for the left, right and two-sided involutive span. It turns out that the results of Kandry-Rodi and Weispfenning [32, Sects. 4/5] remain valid for our larger class of non-commutative algebras and can be straightforwardly extended to involutive bases. For this reason, we will discuss only the case of involutive bases and do not treat separately Gröbner bases.

**Lemma 7.1** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type and  $\mathcal{F} \subset \mathcal{P}$  a finite set. A polynomial  $f \in \mathcal{P}$  is left (involutively) reducible modulo  $\mathcal{F}$ , if and only if it is right (involutively) reducible.

*Proof* Obvious, as reducibility depends only on multi indices.

**Proposition 7.2** Let  $\mathcal{H}_l$  be a monic, involutively left autoreduced, minimal left involutive set and  $\mathcal{H}_r$  a monic, involutively right autoreduced, minimal right involutive set for an involutive division L. If  $\langle \mathcal{H}_l \rangle^{(l)} = \langle \mathcal{H}_r \rangle^{(r)} = \mathcal{I}$ , then  $\mathcal{H}_l = \mathcal{H}_r$ .

*Proof* By definition of a minimal basis, the sets  $le_{\prec} \mathcal{H}_l$  and  $le_{\prec} \mathcal{H}_r$  must both be minimal involutive bases of the monoid ideal  $le_{\prec}\mathcal{I}$ . Obviously, this implies that the two sets are identical. Assume now that  $(\mathcal{H}_l \setminus \mathcal{H}_r) \cup (\mathcal{H}_r \setminus \mathcal{H}_l) \neq \emptyset$  and let f be the element of this set with the minimal leading exponent with respect to  $\prec$ . Without loss of generality, we assume that  $f \in \mathcal{H}_l \setminus \mathcal{H}_r$ . Because of the condition  $\langle \mathcal{H}_l \rangle^{(l)} = \langle \mathcal{H}_r \rangle^{(r)}$ , we have  $f \in \langle \mathcal{H}_r \rangle_{L,\prec}^{(r)}$ . Thus the (by Proposition 4.11 unique) right involutive normal form of f with respect to  $\mathcal{H}_r$  is 0. This implies in particular that f is right involutively reducible with respect to some  $h \in \mathcal{H}_r$ with  $le_{\prec}h \prec le_{\prec}f$ . As f was chosen as the minimal element of the symmetric difference of  $\mathcal{H}_l$  and  $\mathcal{H}_r$ , we find that  $h \in \mathcal{H}_l$ , too. Hence, by Lemma 7.1, f is also left involutively reducible with respect to h (because of  $le_{\prec} \mathcal{H}_l = le_{\prec} \mathcal{H}_r$  the multiplicative variables of h are the same in both cases). But this contradicts the assumption that  $\mathcal{H}_l$  is involutively left autoreduced.  $\Box$ 

A direct derivation of a theory of two-sided involutive bases along the lines of Section 4 fails, as two-sided linear combinations are rather unwieldy objects. A general polynomial  $f \in \langle \langle \mathcal{H} \rangle \rangle$  for some finite set  $\mathcal{H} \subset \mathcal{P}$  is of the form

$$f = \sum_{h \in \mathcal{H}} \sum_{i=1}^{n_h} \ell_i \star h \star r_i$$
(25)

with polynomials  $\ell_i, r_i \in \mathcal{P}$ . The definition of a unique involutive standard representation would require control over the numbers  $n_h$  which seems rather difficult. Therefore we will take another approach and construct left involutive bases even for two-sided ideals. The following proposition is an involutive version of Theorem 5.4 in [32].

**Proposition 7.3** Let  $\mathcal{H} \subset (\mathcal{P}, \star, \prec)$  be a finite set. Then the following five statements are equivalent.

- (i)  $\langle \mathcal{H} \rangle_{L,\prec}^{(l)} = \langle\!\langle \mathcal{H} \rangle\!\rangle.$
- (i)  $\langle \mathcal{H} \rangle_{L,\prec}^{(r)} = \langle\!\langle \mathcal{H} \rangle\!\rangle$ . (ii)  $\langle \mathcal{H} \rangle_{L,\prec}^{(l)} = \langle\!\langle \mathcal{H} \rangle\!\rangle^{(l)}$  and we have  $h \star x_i \in \langle \mathcal{H} \rangle^{(l)}$  for all generators  $h \in \mathcal{H}$  and all variables  $x_i$ .
- (iv)  $\langle \mathcal{H} \rangle_{L,\prec}^{(r)} = \langle \mathcal{H} \rangle^{(l)}$  and we have  $x_i \star h \in \langle \mathcal{H} \rangle^{(r)}$  for all generators  $h \in \mathcal{H}$  and all variables  $x_i$ .
- (v) To every polynomial  $f \in \langle \langle \mathcal{H} \rangle \rangle$  a unique generator  $h \in \mathcal{H}$  exists such that  $\operatorname{le}_{\prec} h \mid_{L, \operatorname{le}_{\prec} \mathcal{H}} \operatorname{le}_{\prec} f.$

*Proof* We begin with the equivalence of the first two statements. (i) implies that  $\langle \mathcal{H} \rangle_{L,\prec}^{(l)} = \langle \mathcal{H} \rangle^{(l)} = \langle \mathcal{H} \rangle^{(l)} = \langle \mathcal{H} \rangle^{(r)}$ . Obviously, the same equality follows from (ii). The equivalence is now a corollary to Lemma 7.1.

Next we consider the equivalence of (i) and (iii); the equivalence of (ii) and (iv) follows by the same argument. One direction is trivial. For the converse, we note that (iii) implies obviously that  $f \star t \in \langle \mathcal{H} \rangle^{(l)}$  for all terms  $t \in \mathbb{T}$ . Now (i) follows from (25).

The equivalence of (i) or (ii), respectively, with (v) is a trivial consequence of the definition of an involutive basis.  $\Box$ 

This proposition leads to the simple Algorithm 7.1 for the construction of a left involutive basis of the two-sided ideal  $\langle\!\langle \mathcal{F} \rangle\!\rangle$ . It first constructs in Line /1/ a left involutive basis  $\mathcal{H}$  of the left ideal  $\langle\!\langle \mathcal{F} \rangle\!\rangle^{(l)}$  (using Algorithm 6.1). The while loop in Lines /2–13/ extends the set  $\mathcal{H}$  to a left generating set of the two-sided ideal  $\langle\!\langle \mathcal{F} \rangle\!\rangle$  according to (iii) in Proposition 7.3. Finally, we complete in Line /14/ this set to an involutive basis. Note that in Line /1/ it is not really necessary to compute a left involutive basis; any left Gröbner basis would suffice as well.

Algorithm 7.1 Left Involutive basis for two-sided ideal in  $(\mathcal{P}, \star, \prec)$ Input: finite set  $\mathcal{F} \subset \mathcal{P}$ , involutive division L Output: left involutive basis  $\mathcal{H}$  of  $\langle\langle \mathcal{F} \rangle\rangle$ 

Supple left involutive busis / of ((5 //
$1: \ \mathcal{H} \leftarrow \texttt{LeftInvBasis}_{L,\prec}(\mathcal{F}); \ \ \mathcal{S} \leftarrow \mathcal{H}$
2: while $S \neq \emptyset$ do
3: $\mathcal{T} \leftarrow \emptyset$
4: for all $f \in S$ do
5: <b>for</b> <i>i</i> <b>from</b> 1 <b>to</b> <i>n</i> <b>do</b>
6: $h \leftarrow \texttt{InvLeftNormalForm}_{L,\prec}(f \star x_i, \mathcal{H})$
7: <b>if</b> $h \neq 0$ <b>then</b>
8: $\mathcal{H} \leftarrow \mathcal{H} \cup \{h\};  \mathcal{T} \leftarrow \mathcal{T} \cup \{h\}$
9: end_if
10: <b>end_for</b>
11: end_for
12: $\mathcal{S} \leftarrow \mathcal{T}$
13: end_while
14: return LeftInvBasis $_{L,\prec}(\mathcal{H})$

### 8 Involutive Bases for Semigroup Orders I: Lazard's Approach

For a number of applications it is of interest to compute involutive or Gröbner bases with respect to more general term orders, namely *semigroup orders* (see Appendix A). If 1 is no longer the smallest term, then normal form algorithms do not terminate for some input. So we can no longer apply Algorithm 6.1 directly for the computation of involutive bases.

*Example 8.1* The Weyl algebra  $\mathbb{W}_n$  is the polynomial algebra in the 2n variables  $x_1, \ldots, x_n$  and  $\partial_1, \ldots, \partial_n$  with the following non-commutative product  $\star$ : for all  $1 \leq i \leq n$  we have  $\partial_i \star x_i = x_i \partial_i + 1$  and  $\star$  is the normal commutative product in all other cases. It is easy to see that  $\mathbb{W}_n$  is a polynomial algebra of solvable type for any monoid order. For semigroup orders compatibility requires that  $1 \prec x_i \partial_i$  for all *i*. In [44] such orders are called *multiplicative monomial orders*.

An important class of semigroup orders is defined via real weight vectors. Let  $(\xi, \zeta) \in \mathbb{R}^{2n}$  be such that  $\xi + \zeta \in \mathbb{R}^n$  is non-negative and let  $\prec$  be an arbitrary monoid order. Then we define  $x^{\mu}\partial^{\nu} \prec_{(\xi,\zeta)} x^{\sigma}\partial^{\tau}$ , if either  $\mu \cdot \xi + \nu \cdot \zeta < \sigma \cdot \xi + \tau \cdot \zeta$  or  $\mu \cdot \xi + \nu \cdot \zeta = \sigma \cdot \xi + \tau \cdot \zeta$  and  $x^{\mu}\partial^{\nu} \prec x^{\sigma}\partial^{\tau}$ . This yields a monoid order, if and

only if  $(\xi, \zeta)$  is non-negative. A special case are the orders with weight vectors of the form  $(\xi, -\xi)$  arising from the action of the algebraic torus  $(\mathbb{k}^*)^n$  on the Weyl algebra. They have numerous applications in the theory of  $\mathcal{D}$ -modules [44].

As normal form computations do not necessarily terminate for semigroup orders, we must slightly modify our definitions of (weak) involutive or Gröbner bases. The proof of Theorem 4.3 (and thus also the one of Corollary 4.4 showing that a weak involutive basis of an ideal  $\mathcal{I}$  is indeed a basis of  $\mathcal{I}$ ) requires normal form computations and thus this theorem is no longer valid. The same problem occurs for Gröbner bases. Therefore we must explicitly include this condition in our definition.

**Definition 8.2** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type where  $\prec$  is an arbitrary semigroup order. Let furthermore  $\mathcal{I} \subseteq \mathcal{P}$  be a left ideal. A Gröbner basis of  $\mathcal{I}$  is a finite set  $\mathcal{G}$  such that  $\langle \mathcal{G} \rangle = \mathcal{I}$  and  $\langle \text{le}_{\prec} \mathcal{G} \rangle = \text{le}_{\prec} \mathcal{I}$ . The set  $\mathcal{G}$  is a weak involutive basis of  $\mathcal{I}$  for the involutive division L, if in addition the set  $\text{le}_{\prec} \mathcal{G}$  is weakly involutive for L. It is a (strong) involutive basis, if it is furthermore involutively head autoreduced.

In the case of Gröbner bases, a classical trick due to Lazard [33] consists of homogenising the input and lifting the semigroup order to a monoid order on the homogenised terms. One can show that computing first a Gröbner basis for the ideal spanned by the homogenised input and then dehomogenising yields a Gröbner basis with respect to the semigroup order. Note, however, that in general we cannot expect that *reduced* Gröbner bases exist.

We extend now this approach to involutive bases. Here we encounter the additional difficulty that we must lift not only the order but also the used involutive division. In particular, we must show that properties like Noetherity or continuity are preserved by the lift which is non-trivial. For the special case of involutive bases in the Weyl algebra, this problem was first solved in [27].

Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type where  $\prec$  is any semigroup order that respects the multiplication  $\star$ . We set  $\mathcal{P}_h = \Bbbk[x_0, x_1, \ldots, x_n]$  and extend the multiplication  $\star$  to  $\mathcal{P}_h$  by defining that  $x_0$  commutes with all other variables and the elements of the field  $\Bbbk$ . For a polynomial  $f = \sum c_\mu x^\mu \in \mathcal{P}$  of degree q, we introduce as usual its *homogenisation*  $f^{(h)} = \sum c_\mu x_0^{q-|\mu|} x^\mu \in \mathcal{P}_h$ . Conversely, for a polynomial  $\tilde{f} \in \mathcal{P}_h$  we denote its projection to  $\mathcal{P}$  as  $f = \tilde{f}|_{x_0=1}$ .

We denote by  $\mathbb{T}_h$  the set of terms in  $\mathcal{P}_h$ ; obviously, it is as monoid isomorphic to  $\mathbb{N}_0^{n+1}$ . We use in the sequel the following convention. Multi indices in  $\mathbb{N}_0^{n+1}$ always carry a tilde:  $\tilde{\mu} = [\mu_0, \dots, \mu_n]$ . The projection to  $\mathbb{N}_0^n$  defined by dropping the first entry (i. e. the exponent of the homogenisation variable  $x_0$ ) is signalled by omitting the tilde; thus  $\mu = [\mu_1, \dots, \mu_n]$ . For subsets  $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{n+1}$  we also simply write  $\mathcal{N} = \{\nu \mid \tilde{\nu} \in \tilde{\mathcal{N}}\} \subset \mathbb{N}_0^n$ .

We first lift the semigroup order  $\prec$  on  $\mathbb{T}$  to a monoid order  $\prec_h$  on  $\mathbb{T}_h$  by defining  $x^{\tilde{\mu}} \prec_h x^{\tilde{\nu}}$ , if either  $|\tilde{\mu}| < |\tilde{\nu}|$  or both  $|\tilde{\mu}| = |\tilde{\nu}|$  and  $x^{\mu} \prec x^{\nu}$ . It is trivial to check that this yields indeed a monoid order and that  $(\mathcal{P}_h, \star, \prec_h)$  is again a polynomial algebra of solvable type. For lifting the involutive division, we proceed somewhat similarly to the definition of the Janet division: the homogenisation variable  $x_0$  is multiplicative only for terms which have maximal degree in  $x_0$ .

**Proposition 8.3** Let *L* be an arbitrary involutive division on  $\mathbb{N}_0^n$ . For any finite set  $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{n+1}$  and every multi index  $\tilde{\mu} \in \tilde{\mathcal{N}}$ , we define  $N_{L_{k},\tilde{\mathcal{N}}}(\tilde{\mu})$  by:

 $\begin{array}{l} - \ 0 \in N_{L_h, \tilde{\mathcal{N}}}(\tilde{\mu}), \ \text{if and only if } \mu_0 = \max_{\tilde{\nu} \in \tilde{\mathcal{N}}} \{\nu_0\}, \\ - \ 0 < i \in N_{L_h, \tilde{\mathcal{N}}}(\tilde{\mu}), \ \text{if and only if } i \in N_{L_h, \mathcal{N}}(\mu). \end{array}$ 

*This determines an involutive division*  $L_h$  *on*  $\mathbb{N}_0^{n+1}$ *.* 

*Proof* Both conditions for an involutive division are easily verified. For the first one, let  $\tilde{\rho} \in C_{L_h,\tilde{\mathcal{N}}}(\tilde{\mu}) \cap C_{L_h,\tilde{\mathcal{N}}}(\tilde{\nu})$  with  $\tilde{\mu}, \tilde{\nu} \in \tilde{\mathcal{N}}$ . If  $\tilde{\rho}_0 = \tilde{\mu}_0 = \tilde{\nu}_0$ , the first entry can be ignored, and the properties of the involutive division L implies the desired result. If  $\tilde{\rho}_0 = \tilde{\mu}_0 > \tilde{\nu}_0$ , the index 0 must be multiplicative for  $\tilde{\nu}$  contradicting  $\tilde{\mu}_0 > \tilde{\nu}_0$ . If  $\tilde{\rho}_0$  is greater than both  $\tilde{\mu}_0$  and  $\tilde{\nu}_0$ , the index 0 must be multiplicative for both implying  $\tilde{\mu}_0 = \tilde{\nu}_0$ . In this case we may again ignore the first entry and invoke the properties of L.

For the second condition we note that whether a multiplicative index i > 0 becomes non-multiplicative for some element  $\tilde{\nu} \in \tilde{\mathcal{N}}$  after adding a new multi index to  $\tilde{\mathcal{N}}$  is independent of the first entry and thus only determined by the involutive division L. If the new multi index has a higher first entry than all elements of  $\tilde{\mathcal{N}}$ , then 0 becomes non-multiplicative for all elements in  $\tilde{\mathcal{N}}$  but this is permitted.  $\Box$ 

Now we check to what extent the properties of L are inherited by the lifted division  $L_h$ . Given the similarity of the definition of  $L_h$  and the Janet division, it is not surprising that we may reuse many ideas of proofs for the latter.

### **Proposition 8.4** If L is a Noetherian division, then so is $L_h$ .

*Proof* Let  $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{n+1}$  be an arbitrary finite subset. In order to prove the existence of an  $L_h$ -completion of  $\tilde{\mathcal{N}}$ , we first take a finite *L*-completion  $\hat{\mathcal{N}} \subset \mathbb{N}_0^n$  of  $\mathcal{N}$  which always exists, as by assumption *L* is Noetherian. Next, we define a finite subset  $\tilde{\mathcal{N}}' \subset \langle \tilde{\mathcal{N}} \rangle$  by setting

$$\tilde{\mathcal{N}}' = \left\{ \tilde{\mu} \in \mathbb{N}_0^{2n+1} \mid \mu \in \hat{\mathcal{N}} \land \mu_0 \le \max_{\tilde{\nu} \in \tilde{\mathcal{N}}} \nu_0 \right\} \cap \langle \tilde{\mathcal{N}} \rangle .$$

We claim that this set  $\tilde{\mathcal{N}}'$  is an  $L_h$ -completion of  $\tilde{\mathcal{N}}$ . By construction, we have both  $\tilde{\mathcal{N}}' \subset \langle \tilde{\mathcal{N}} \rangle$  and  $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{N}}'$ , so that we must only show that  $\tilde{\mathcal{N}}'$  is involutive.

Let  $\tilde{\mu} \in \langle \tilde{\mathcal{N}}' \rangle$  be arbitrary. By construction of  $\tilde{\mathcal{N}}'$ , we can find  $\tilde{\nu} \in \tilde{\mathcal{N}}'$  with  $\nu \mid_{L,\hat{\mathcal{N}}} \mu$ . Moreover, the definition of  $\tilde{\mathcal{N}}'$  guarantees that we can choose  $\tilde{\nu}$  in such a way that either  $\nu_0 = \mu_0$  or  $\nu_0 = \max_{\tilde{\rho} \in \tilde{\mathcal{N}}'} \rho_0 < \mu_0$  holds. In the former case, we trivially have  $\tilde{\nu} \mid_{L_h, \tilde{\mathcal{N}}'} \tilde{\mu}$ ; in the latter case we have  $0 \in N_{L_h, \tilde{\mathcal{N}}}(\tilde{\nu})$  (see Proposition 8.3). Thus in either case  $\tilde{\mu} \in \langle \tilde{\mathcal{N}}' \rangle_{L_h}$ .  $\Box$ 

**Proposition 8.5** If L is a continuous division, then so is  $L_h$ .

*Proof* Let  $(\tilde{\nu}^{(1)}, \ldots, \tilde{\nu}^{(r)})$  with  $\tilde{\nu}^{(i)} \in \tilde{\mathcal{N}}$  be a finite sequence as described in the definition of continuity. First we note that the integer sequence  $(\nu_0^{(1)}, \ldots, \nu_0^{(r)})$  is monotonically increasing. If  $\nu_0^{(i)}$  is not maximal among the entries  $\mu_0$  for  $\tilde{\mu} \in \tilde{\mathcal{N}}$ , no multiplicative divisor of  $\tilde{\nu}^{(i)} + 1_j$  in  $\tilde{\mathcal{N}}$  can have a smaller first entry: if  $\nu_0^{(i)}$ 

is maximal, the index 0 is multiplicative for  $\tilde{\nu}^{(i)}$  and any involutive divisor in  $\tilde{\mathcal{N}}$  must also be maximal in the first entry. Thus it suffices to look at those parts of the sequence where equality in the zero entries holds. But there the inequality of the multi indices  $\tilde{\nu}^{(i)}$  follows from the continuity of the underlying division L.  $\Box$ 

Unfortunately, it is much harder to show that constructivity is preserved. We could prove this only for globally defined divisions and the Janet division.

# **Proposition 8.6** Let L be an involutive division on $\mathbb{N}_0^n$ . If L is either globally defined or the Janet division, then the lifted division $L_h$ is constructive.

*Proof* We give a proof only for the case of a globally defined division. For the Janet division J one must only make a few modifications of the proof that J itself is constructive. We omit the details; they can be found in [27].

We select a finite set  $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{n+1}$ , a multi index  $\tilde{\mu} \in \tilde{\mathcal{N}}$  and a non-multiplicative index i of  $\tilde{\mu}$  such that the conditions in the definition of constructivity are fulfilled. Assume that there exists a  $\tilde{\rho} \in \tilde{\mathcal{N}}$  such that  $\tilde{\mu} + 1_i = \tilde{\rho} + \tilde{\sigma} + \tilde{\tau}$  with  $\tilde{\rho} + \tilde{\sigma} \in \mathcal{C}_{L_h, \tilde{\mathcal{N}}}(\tilde{\rho})$  and  $\tilde{\rho} + \tilde{\sigma} + \tilde{\tau} \in \mathcal{C}_{L_h, \tilde{\mathcal{N}} \cup \{\tilde{\rho} + \tilde{\sigma}\}}(\tilde{\rho} + \tilde{\sigma})$ . Let L be a globally defined division. If i = 0, then  $\mu_0 + 1 = \rho_0 + \sigma_0 + \tau_0$  implies that  $\sigma_0 = \tau_0 = 0$ : for  $\sigma_0 > 0$ , we would have (0 is multiplicative for  $\tilde{\rho}$ )  $\rho_0 > \mu_0 \ge \rho_0 + \sigma_0 > \rho_0$ . For  $\sigma_0 = 0$  and  $\tau_0 > 0$  a similar contradiction appears. If i > 0, the argumentation is simple. A global division is always constructive, as adding further elements to  $\mathcal{N}$  does not change the multiplicative indices. But the same holds for the indices k > 0 in the lifted division  $L_h$ . Thus under the above conditions  $\tilde{\mu} + 1_i \in \langle \tilde{\mathcal{N}} \rangle_{L_h}$ contradicting the assumptions.  $\Box$ 

Based on these results, Algorithm 6.1 may be extended to semigroup orders. Given a finite set  $\mathcal{F} \in \mathcal{P}$ , we first determine its homogenisation  $\mathcal{F}^{(h)} \in \mathcal{P}_h$  and then compute an involutive basis of  $\langle F^{(h)} \rangle$  with respect to  $L_h$  and  $\prec_h$ . What remains to be done is first to show that the existence of a finite involutive basis is preserved under the lifting to  $\mathcal{P}_h$  and then to study the properties of the dehomogenisation of this basis.

**Proposition 8.7** If the left ideal  $\mathcal{I} = \langle \mathcal{F} \rangle \subseteq \mathcal{P}$  possesses an involutive basis with respect to the Noetherian division L and the semigroup order  $\prec$ , then the left ideal  $\tilde{\mathcal{I}} = \langle \mathcal{F}^{(h)} \rangle \subseteq \mathcal{P}_h$  generated by the homogenisations of the elements in the finite set  $\mathcal{F}$  possesses an involutive basis with respect to the lifted division  $L_h$  and the monoid order  $\prec_h$ .

*Proof* By Proposition 3.11, the ideal  $\tilde{\mathcal{I}} \subseteq \mathcal{P}_h$  possesses a Gröbner basis  $\tilde{\mathcal{G}}$  with respect to the monoid order  $\prec_h$ . It follows from Proposition 8.4 that a finite  $L_h$ -completion  $\hat{\mathcal{N}}$  of the set  $e_{\prec_h} \tilde{\mathcal{G}}$  exists. Moreover, as  $\tilde{\mathcal{G}}$  is a Gröbner basis of  $\tilde{\mathcal{I}}$ , the monoid ideals  $\langle le_{\prec_h} \tilde{\mathcal{G}} \rangle$  and  $le_{\prec_h} \tilde{\mathcal{I}}$  coincide. Thus  $\hat{\mathcal{N}}$  is an involutive basis of  $le_{\prec_h} \tilde{\mathcal{I}}$  with respect to the division  $L_h$ . Thus an involutive basis  $\tilde{\mathcal{H}}$  of  $\tilde{\mathcal{I}}$  with respect to the division  $L_h$ .

$$\tilde{\mathcal{H}} = \left\{ x^{\tilde{\mu}} \star \tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}} \land \operatorname{le}_{\prec_h} (x^{\tilde{\mu}} \star \tilde{g}) \in \hat{\mathcal{N}} \right\}.$$
(26)

This set is obviously finite.  $\Box$ 

Hence our lifting leads to a situation where we can apply Theorem 6.4. Unfortunately, the dehomogenisation of the strong involutive basis computed in  $\mathcal{P}_h$  does not necessarily lead to a *strong* involutive basis in  $\mathcal{P}$ , but we obtain always at least a weak involutive basis and thus in particular a Gröbner basis.

**Theorem 8.8** Let  $\tilde{\mathcal{H}}$  be a strong involutive basis of the left ideal  $\tilde{\mathcal{I}} \subseteq \mathcal{P}_h$  with respect to  $L_h$  and  $\prec_h$ . Then the dehomogenisation  $\mathcal{H}$  is a weak involutive basis of the left ideal  $\mathcal{I} \subseteq \mathcal{P}$  with respect to L and  $\prec$ .

*Proof* An integer  $k \ge 0$  exists for any  $f \in \mathcal{I}$  such that  $\tilde{f} = x_0^k f^{(h)} \in \tilde{\mathcal{I}}$ . Hence  $\tilde{f}$  possesses a unique involutive standard representation

$$\tilde{f} = \sum_{\tilde{h} \in \tilde{\mathcal{H}}} \tilde{P}_{\tilde{h}} \tilde{h}$$
(27)

with  $\tilde{P}_{\tilde{h}} \in \mathbb{k}[(X_{L_{h}, \mathrm{le}_{\prec_{h}}\tilde{\mathcal{H}}}(\tilde{h})]$  and  $\mathrm{le}_{\prec_{h}}(\tilde{P}_{\tilde{h}}\tilde{h}) \preceq_{h} \mathrm{le}_{\prec_{h}}\tilde{f}$ . Setting  $x_{0} = 1$  in (27) yields a representation of f with respect to the dehomogenised basis<sup>5</sup>  $\mathcal{H}$  of the form  $f = \sum_{h \in \mathcal{H}} P_{h}h$  with  $P_{h} \in \mathbb{k}[(X_{L, \mathrm{le}_{\prec}\mathcal{H}}(h)]$ . This obviously implies that  $\langle \mathcal{H} \rangle = \mathcal{I}$ . By the definition of the lifted order  $\prec_{h}$  and the homogeneity of all involved polynomials, we have furthermore that  $\mathrm{le}_{\prec}(P_{h}h) \preceq \mathrm{le}_{\prec}f$  and hence that  $\mathrm{le}_{\prec}\mathcal{H}$  is a weak involutive basis of  $\mathrm{le}_{\prec}\mathcal{I}$ . Hence all conditions of Definition 8.2 are satisfied and  $\mathcal{H}$  is a weak involutive basis of the ideal  $\mathcal{I}$  for the division L.  $\Box$ 

It is not a shortcoming of our proof that in general we do not get a strong involutive basis, but actually some ideals do not possess strong involutive bases. In particular, there is no point in invoking Proposition 4.6 for obtaining a strong basis. While we may surely obtain by elimination a subset  $\mathcal{H}' \subseteq \mathcal{H}$  such that  $le_{\prec}\mathcal{H}$  is a strong involutive basis of  $le_{\prec}\mathcal{H}$ , in general  $\langle \mathcal{H}' \rangle \subsetneq \mathcal{I}$ .

*Example 8.9* Consider in the Weyl algebra  $\mathbb{W}_2 = \mathbb{k}[x, y, \partial_x, \partial_y]$  the left ideal generated by the set  $\mathcal{F} = \{\underline{1} + x_1 + x_2, \underline{\partial_2} - \partial_1\}$ . We take the semigroup order induced by the weight vector (-1, -1, 1, 1) and refined by a term order for which  $\partial_2 \succ \partial_1 \succ x_2 \succ x_1$ . Then the underlined terms are the leading ones. One easily checks that  $\mathcal{F}$  is a Gröbner basis for this order. Furthermore, all variables are multiplicative for each generator with respect to the Pommaret division and thus  $\mathcal{F}$  is a weak Pommaret basis, too.

Obviously,  $\mathcal{F}$  is neither a reduced Gröbner basis nor a strong Pommaret basis, as 1 is a (multiplicative) divisor of  $\partial_2$ . However, it is easy to see that the left ideal  $\mathcal{I} = \langle \mathcal{F} \rangle$  does not possess a reduced Gröbner basis or a strong Pommaret basis. Indeed, we have  $le_{\prec}\mathcal{I} = \mathbb{N}_0^4$  and thus such a basis had to consist of only a single generator; but  $\mathcal{I}$  is not a principal ideal.

A special situation arises for the Janet division. Recall from Remark 2.6 that any finite set  $\mathcal{N} \subset \mathbb{N}_0^n$  is automatically involutively autoreduced with respect to the Janet division. Thus any weak Janet basis is a strong basis, if all generators

<sup>&</sup>lt;sup>5</sup> Note that  $\mathcal{H}$  is in general smaller than  $\tilde{\mathcal{H}}$ , as some elements of  $\tilde{\mathcal{H}}$  may differ only in powers of  $x_0$ .

have different leading exponents. If we follow the above outlined strategy of applying Algorithm 6.1 to a homogenised basis and then dehomogenising the result, we cannot generally expect this condition to be satisfied. However, with a minor modification of the algorithm we can achieve this goal.

**Theorem 8.10** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type where  $\prec$  is an arbitrary semigroup order. Then every left ideal  $\mathcal{I} \subseteq \mathcal{P}$  possesses a strong Janet basis for  $\prec$ .

**Proof** Assume that at some intermediate stage of Algorithm 6.1 the basis  $\hat{\mathcal{H}}$  contains two polynomials  $\tilde{f}$  and  $\tilde{g}$  such that  $e_{\prec_h}(\tilde{g}) = e_{\prec_h}(\tilde{f}) + 1_0$ , i. e. the leading exponents differ only in the first entry. If  $\tilde{g} = h\tilde{f}$ , we will find f = g after the dehomogenisation and no obstruction to a strong basis appears. Otherwise we note that, by definition of the lifted Janet division  $J_h$ , the homogenisation variable  $x_0$  is non-multiplicative for  $\tilde{f}$ . Thus at some later stage the algorithm must consider the non-multiplicative product  $x_0\tilde{f}$  (if it was already treated,  $\tilde{\mathcal{H}}$  would not be involutively head autoreduced).

In the usual algorithm, we then determine the involutive normal form of the polynomial  $x_0 \tilde{f}$ ; the first step of this computation is to replace  $x_0 \tilde{f}$  by  $x_0 \tilde{f} - \tilde{g}$ . Alternatively, we may proceed instead as follows. The polynomial  $\tilde{g}$  is removed from the basis  $\tilde{\mathcal{H}}$  and replaced by  $x_0 \tilde{f}$ . Then we continue by determining the involutive normal form of  $\tilde{g}$  with respect to the new basis. Note that this modification concerns only the situation that a multiplication by  $x_0$  has been performed and that the basis  $\tilde{\mathcal{H}}$  contains already an element with the same leading exponent as the obtained polynomial.

If the final output  $\mathcal{H}$  of the thus modified completion algorithm contains two polynomials  $\tilde{f}$  and  $\tilde{g}$  such that  $e_{\prec_h}(\tilde{g})$  and  $e_{\prec_h}(\tilde{f})$  differ only in the first entry, then either  $\tilde{g} = x_0^k \tilde{f}$  or  $\tilde{f} = x_0^k \tilde{g}$  for some  $k \in \mathbb{N}$ . Thus the dehomogenisation yields a basis  $\mathcal{H}$  where all elements possess different leading terms and  $\mathcal{H}$  is a strong Janet basis. Looking at the proof of Theorem 6.4, it is easy to see that this modification does not affect the correctness and the termination of the algorithm. As the Janet division is Noetherian, these considerations prove together with Proposition 8.4 the assertion.  $\Box$ 

Note that our modification only achieves its goal, if we really restrict in Algorithm 6.1 to head reductions. Otherwise some other terms than the leading term in  $x_0 \tilde{f}$  might be reducible but not the corresponding terms in  $\tilde{f}$ . Then we could still find after the dehomogenisation two generators with the same leading term.

*Example 8.11* Let us consider in the Weyl algebra  $\mathbb{W}_3$  the left ideal generated by the set  $\mathcal{F} = \{\partial_z - y\partial_x, \partial_y\}$ . If we apply the usual involutive completion Algorithm 6.1 (to the homogenisation  $\mathcal{F}^{(h)}$ ), we obtain for the weight vector (-1, 0, 0, 1, 0, 0) refined by the degree reverse lexicographic order and the Janet division the following weak basis with nine generators:

$$\mathcal{H}_1 = \left\{ \partial_x, \ \partial_y, \ \partial_z, \ \partial_x \partial_z, \ \partial_y \partial_z, \ y \partial_x, \ y \partial_x + \partial_z, \ y \partial_x \partial_z, \ y \partial_x \partial_z + \partial_z^2 \right\}.$$
(28)

As one easily sees from the last four generators, it is not a strong basis.

Applying the modified algorithm for the Janet division yields the following basis with only seven generators:

$$\mathcal{H}_2 = \left\{ \partial_x + \partial_y \partial_z, \ \partial_y, \ \partial_z, \ \partial_x \partial_z, \ \partial_y \partial_z, \ y \partial_x + \partial_z, \ y \partial_x \partial_z + \partial_z^2 \right\}.$$
(29)

Obviously, we now have a strong basis, as all leading terms are different.

This example also demonstrates the profound effect of the homogenisation. A strong Janet or Pommaret basis of  $\langle \mathcal{F} \rangle$  is simply given by  $\mathcal{H} = \{\partial_x, \partial_y, \partial_z\}$  which is simultaneously a reduced Gröbner basis. In  $\langle \mathcal{F}^{(h)} \rangle$  many reductions are not possible because the terms contain different powers of  $x_0$ . However, this is a general problem of all approaches to Gröbner bases for semigroup orders and not specific for the involutive approach.

In this particular case, one could have applied the involutive completion algorithm directly to the original set  $\mathcal{F}$  and it would have terminated with the minimal basis  $\mathcal{H}$ , although we are using a order which is not a monoid order. Unfortunately, it is not clear how to predict when infinite reduction chains appear in normal form computations with respect to such orders, so that one does not know in advance whether one may dispense with the homogenisation.

# 9 Involutive Bases for Semigroup Orders II: Mora's Normal Form

One computational disadvantage of the approach outlined in the previous section is that the basis  $\tilde{\mathcal{H}}$  in the homogenised algebra  $\mathcal{P}_h$  is often much larger than the final basis  $\mathcal{H}$  in the original algebra  $\mathcal{P}$ , as upon dehomogenisation generators may become identical. An alternative approach was proposed by Greuel and Pfister [22] and independently by Gräbe [20,21] extending ideas of Mora [39] on computing tangent cones; an extensive discussion is contained in [11, Chapt. 4]. The basic idea is to dispense with computing an involutive basis of the ideal  $\langle \mathcal{F}^{(h)} \rangle \subseteq \mathcal{P}_h$ . Instead we determine a generally smaller set  $\tilde{\mathcal{H}}$  whose dehomogenisation still leads to a set  $\mathcal{H}$  which is an involutive basis of the ideal  $\langle \mathcal{F} \rangle$  – not over the ring  $\mathcal{P}$  but over a larger ring of fractions.

**Proposition 9.1** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type where  $\prec$  is an arbitrary semigroup order. Then the subset

$$\mathcal{S}_{\prec} = \{ f \in \mathcal{P} \mid |\mathsf{lt}_{\prec} f = 1 \} . \tag{30}$$

is multiplicatively closed and the left localisation  $\text{Loc}_{\prec}\mathcal{P} = \mathcal{S}_{\prec}^{-1} \star \mathcal{P}$  is a well defined ring of left fractions.

*Proof* Obviously,  $1 \in S_{\prec}$ . If 1 + f and 1 + g are two elements in  $S_{\prec}$ , then the compatibility of the order  $\prec$  with the multiplication  $\star$  ensures that their product is of the form  $(1 + f) \star (1 + g) = 1 + h$  with  $\operatorname{lt}_{\prec} h \prec 1$ . Hence the set  $S_{\prec}$  is multiplicatively closed.

As polynomial algebras of solvable type do not possess zero divisors, a sufficient condition for the existence of the ring of left fractions  $S_{\prec}^{-1} \star P$  is that for all

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 $f \in S_{\prec}$  and  $g \in \mathcal{P}$  the intersection  $(\mathcal{P} \star f) \cap (S_{\prec} \star g)$  is not empty [10, Sect. 12.1]. But this follows from our proof of Proposition 3.8. If we retrace the steps of the algorithm implied by that proof, we first note that  $\operatorname{lt}_{\prec}\bar{h}_1 \prec \operatorname{lt}_{\prec}(g \star f) = \operatorname{lt}_{\prec}g$ . Taking the usual normal form algorithm, this implies that  $\psi_0 = 0$ . Similarly, we find in the next step that  $\operatorname{lt}_{\prec}\bar{h}_2 \prec \operatorname{lt}_{\prec}(f \star h_1) = \operatorname{lt}_{\prec}h_1 \preceq \operatorname{lt}_{\prec}\bar{h}_1$ . Thus we have also  $\psi_1 = 0$  and so on. This proves that  $\psi \in S_{\prec}$  and hence our lemma.  $\Box$ 

Mora [39] introduced the notion of the *écart* of a polynomial f as the difference between the lowest and the highest degree of a term in f (or, alternatively, we may write  $ecart(f) = deg_{x_0} f^{(h)}$ ), and based a new normal form algorithm on it. The main difference between it and the usual algorithm lies in the possibility to reduce also with respect to intermediate results (see Line /5/ in Algorithm 9.1). We do not use the écart but instead follow the description given by Greuel and Pfister [22] and work in the homogenised algebra  $\mathcal{P}_h$ . We also give immediately the involutive version of the Mora normal form algorithm.

Algorithm 9.1 Involutive Mora normal form for a semigroup order  $\prec$  on  $\mathcal{P}$ 

**Input:** homogeneous  $\tilde{f} \in \mathcal{P}_h$ , finite homogeneous set  $\tilde{\mathcal{F}} \subset \mathcal{P}_h$ , involutive division L **Output:** involutive Mora normal form  $\tilde{h}$  of  $\tilde{f}$  with respect to  $\tilde{\mathcal{F}}$ 1:  $\tilde{h} \leftarrow \tilde{f}$ ;  $\tilde{\mathcal{G}} = \tilde{\mathcal{F}}$ 2: while  $\exists \tilde{g} \in \tilde{\mathcal{G}}, k \in \mathbb{N}_0 : \operatorname{le}_{\prec_h} \tilde{g} \mid_{L_h, \operatorname{le}_{\prec_h} \tilde{\mathcal{G}}} \operatorname{le}_{\prec_h} (x_0^k \tilde{h})$  do choose such  $\tilde{q}$  with minimal k 3: if k > 0 then 4: 5:  $\tilde{\mathcal{G}} \leftarrow \tilde{\mathcal{G}} \cup \{\tilde{h}\}$ end\_if 6:  $\mu \leftarrow \operatorname{le}_{\prec_h}(x_0^k \tilde{h}) - \operatorname{le}_{\prec_h} \tilde{g}; \quad \tilde{h} \leftarrow x_0^k \tilde{h} - \frac{\operatorname{lc}_{\prec_h}(x_0^k \tilde{h})}{\operatorname{lc}_{\prec_h}(x^\mu \tilde{g})} x^\mu \star \tilde{g}$ 7: choose maximal  $\ell$  such that  $x_0^{\ell} \mid \tilde{h}$ 8:  $\tilde{h} \leftarrow \tilde{h} / x_0^\ell$ 9: 10: end\_while 11: return  $\tilde{h}$ 

**Proposition 9.2** Algorithm 9.1 always terminates. If  $\tilde{h}$  is the output for the input  $\tilde{f}$  and  $\tilde{\mathcal{G}}$ , then there exists a representation

$$\tilde{u} \star \tilde{f} = \sum_{\tilde{g} \in \tilde{\mathcal{G}}} \tilde{P}_{\tilde{g}} \star \tilde{g} + \tilde{h}$$
(31)

where the coefficients  $\tilde{u} \in \mathcal{P}_h$  and  $\tilde{P}_{\tilde{g}} \in \mathbb{k}[X_{L_h,\tilde{\mathcal{G}}}(\tilde{g})]$  satisfy  $\operatorname{lt}_{\prec_h} \tilde{u} = x_0^k$  and the equality  $k + \operatorname{deg} \tilde{f} = \operatorname{deg} (\tilde{P}_{\tilde{g}} \star \tilde{g}) = \operatorname{deg} \tilde{h}$  (whenever  $\tilde{P}_{\tilde{g}} \neq 0$ ). Furthermore, none of the leading terms  $\operatorname{lt}_{\prec_h} \tilde{g}$  involutively divides  $\operatorname{lt}_{\prec_h} (x_0^\ell \tilde{h})$  with respect to the set  $\operatorname{lt}_{\prec_h} \tilde{\mathcal{G}}$  for any  $\ell \geq 0$ . If  $\prec$  is a monoid order on  $\mathbb{T}$ , then  $\tilde{u} = x_0^k$ .

*Proof* The proof given by Greuel and Pfister [22] is not affected by the noncommutativity of  $\star$ , as it operates mainly on the leading terms. Thus we may reuse it with obvious modifications due to the restriction to involutive divisors. It is trivial that the coefficients  $\tilde{P}_{\tilde{q}}$  depend only on multiplicative variables.  $\Box$  As all elements of  $S_{\prec}$  are units in  $\operatorname{Loc}_{\prec} \mathcal{P}$ , we may extend the notions of leading term, monomial or exponent: if  $h = f/(1+g) \in \operatorname{Loc}_{\prec} \mathcal{P}$ , then we set  $\operatorname{lt}_{\prec} h = \operatorname{lt}_{\prec} f$  etc. It follows immediately from the compatibility of  $\star$  and  $\prec$  that this yields a well-defined result. Indeed, if f/(1+g) = f'/(1+g'), then there must exist a polynomial  $s \in S_{\prec}$  such that either  $s \star f = f'$  or  $s \star f' = f$ . In either case we find  $\operatorname{lt}_{\prec} f = \operatorname{lt}_{\prec} f'$ .

Proposition 9.2 implies in particular that after a dehomogenisation  $u \in S_{\prec}$ . Thus we obtain as a simple corollary that with the help of Algorithm 9.1 we may compute for any  $f \in \operatorname{Loc}_{\prec} \mathcal{P}$  a representation of the form  $f = \sum_{g \in \mathcal{G}} P_g \star g + h$ with coefficients  $P_g \in \mathbb{k}[X_{L,\mathcal{G}}(g)]$  satisfying  $\operatorname{lt}_{\prec}(P_g \star g) \preceq \operatorname{lt}_{\prec} f$  and where  $\operatorname{lt}_{\prec} h \preceq \operatorname{lt}_{\prec} f$ . It follows immediately that h = 0 implies that  $f \in \langle \mathcal{G} \rangle \subseteq \operatorname{Loc}_{\prec} \mathcal{P}$ . The converse, however, holds of course only for a Gröbner basis.

As a consequence, we are able to construct a complete theory of involutive bases over  $\text{Loc}_{\prec}\mathcal{P}$ . Definition 8.2 of Gröbner and involutive bases may be extended without changes from the ring  $\mathcal{P}$  to  $\text{Loc}_{\prec}\mathcal{P}$ . Proposition 3.11 on the existence of Gröbner bases generalises to  $\text{Loc}_{\prec}\mathcal{P}$ , as its proof is only based on the leading exponents and a simple normal form argument remaining valid due to our considerations above. This in turn implies immediately the existence of weak involutive bases for any ideal  $\mathcal{I} \subseteq \text{Loc}_{\prec}\mathcal{P}$  such that  $\text{le}_{\prec}\mathcal{I}$  possesses a (weak) involutive basis, as we may complete any Gröbner basis of it to a weak involutive basis with the help of Algorithm 5.1.

Note that even if the set  $\hat{\mathcal{G}}$  is involutively head autoreduced, we cannot conclude in analogy to Proposition 4.11 that the involutive Mora normal form is unique, as we only consider the leading term in Algorithm 9.1 and hence the lower terms in  $\tilde{h}$  may still be involutive divisible by the leading term of some generator  $\tilde{g} \in \tilde{\mathcal{G}}$ . However, Theorem 4.3 remains valid.

For concrete computations we restrict to ideals  $\mathcal{I} \subseteq \text{Loc}_{\prec} \mathcal{P}$  generated by finite sets  $\mathcal{F} \subset \mathcal{P}$  of *polynomials*, as then we may completely avoid the explicit use of fractions. As in the previous section, the main idea is to move to the homogenised algebra  $(\mathcal{P}_h, \star, \prec_h)$ .

**Theorem 9.3** Let  $(\mathcal{P}, \star, \prec)$  be a polynomial algebra of solvable type where  $\prec$  is an arbitrary semigroup order. For a finite set  $\mathcal{F} \subset \mathcal{P}$  of polynomials let  $\mathcal{I} = \langle \mathcal{F} \rangle$ be the left ideal generated by it in the localisation  $\text{Loc}_{\prec}\mathcal{P}$ . Let furthermore L be an involutive division on  $\mathbb{N}_0^n$  such that the lifted division  $L_h$  is continuous and constructive and assume that the monoid ideal  $\text{le}_{\prec}\mathcal{I}$  possesses an involutive basis for L. If we apply Algorithm 6.1 with the involutive Mora normal form instead of the usual one to the homogenised set  $\mathcal{F}^{(h)}$  and the lifted division  $L_h$ , then it terminates with a set  $\tilde{\mathcal{H}} \subset \mathcal{P}_h$  whose dehomogenisation  $\mathcal{H}$  is a weak involutive basis of the ideal  $\mathcal{I}$  for  $\prec$  and L over  $\text{Loc}_{\prec}\mathcal{P}$ .

*Proof* The termination of Algorithm 6.1 under the made assumptions was shown in Proposition 6.2 and Theorem 6.4. One easily verifies that their proofs are not affected by the substitution of the normal form algorithm, as they rely mainly on Theorem 4.3 and on the fact that the leading term of the normal form is not involutively divisible by the leading term of any generator. Both properties remain valid for the Mora normal form.

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Note, however, that generally the output  $\tilde{\mathcal{H}}$  is not an involutive basis of  $\langle \mathcal{F}^{(h)} \rangle$ ! But we claim that the following property holds: if f is an arbitrary element of  $\mathcal{I}$ , then the involutive Mora normal form of its homogenisation  $f^{(h)}$  with respect to  $\tilde{\mathcal{H}}$  vanishes. Indeed, one can show analogously to the proof of Proposition 6.2 that an integer  $k \geq 0$  exists such that  $x_0^k f^{(h)} \in \langle \mathcal{H} \rangle_{L_h, \prec_h}$ . Thus we find for each  $f \in \mathcal{I}$  a representation  $\tilde{u} \star f^{(h)} = \sum_{\tilde{h} \in \tilde{\mathcal{H}}} \tilde{P}_{\tilde{h}} \star \tilde{h}$  where the

Thus we find for each  $f \in \mathcal{I}$  a representation  $\tilde{u} \star f^{(h)} = \sum_{\tilde{h} \in \tilde{\mathcal{H}}} P_{\tilde{h}} \star h$  where the coefficients are as in Proposition 9.2. This allows us to show in a similar manner as in the proof of Theorem 8.8 that the dehomogenisation  $\mathcal{H}$  is a weak involutive basis of  $\mathcal{I}$  over the localisation  $\text{Loc}_{\prec}\mathcal{P}$ .  $\Box$ 

*Example 9.4* We continue with Example 8.11. Applying the algorithm outlined in Theorem 9.3 we obtain the following basis of  $\langle \mathcal{F} \rangle$ 

$$\mathcal{H}_3 = \{ \partial_x + \partial_y \partial_z, \ \partial_y, \ \partial_z, \ y \partial_x + \partial_z \} . \tag{32}$$

Obviously, it is considerably smaller than the bases obtained with Lazard's approach; in fact, we obtain almost the minimal basis. But the true power of Mora's normal form becomes evident only, if we compare the sizes of the corresponding bases in the homogenised Weyl algebra. Both  $\tilde{\mathcal{H}}_1$  and  $\tilde{\mathcal{H}}_2$  consist of 21 generators, whereas  $\tilde{\mathcal{H}}_3$  has only 7 elements.

As in the previous section we may always construct *strong* Janet bases by a simple modification of Algorithm 6.1.

# 10 Involutive Bases over Rings

Finally, we consider the general case that  $\mathcal{P} = \mathcal{R}[x_1, \ldots, x_n]$  is a polynomial algebra of solvable type over a Noetherian ring  $\mathcal{R}$ . In the commutative case, Gröbner bases for such algebras have been studied in [18,49] (see [1, Chapt. 4] for a more extensive discussion); for PBW extensions (recall Example 3.5) a theory of Gröbner bases was recently developed in [19]. We will follow the approach developed in these references and assume in the sequel that the following two operations may be effectively performed in  $\mathcal{R}$ .

- (i) Given elements  $s, r_1, \ldots, r_k \in \mathcal{R}$ , we can decide whether  $s \in \langle r_1, \ldots, r_k \rangle$ (the left ideal in  $\mathcal{R}$  generated by  $r_1, \ldots, r_k$ ).
- (ii) Given elements  $r_1, \ldots, r_k \in \mathcal{R}$ , we can construct a finite basis of the module of left syzygies  $s_1r_1 + \cdots + s_kr_k = 0$ .

In this case one often says that linear equations are solvable in  $\mathcal{R}$ .

The first operation is obviously necessary for the algorithmic reduction of polynomials with respect to a set  $\mathcal{F} \subset \mathcal{P}$ . The necessity of the second operation will become evident later. Compared with the commutative case, reduction is a more complicated process, in particular due to the possibility that in (5) the maps  $\rho_{\mu}$  may be different from the identity and the coefficients  $r_{\mu\nu}$  unequal one.

Let  $\mathcal{G} \subset \mathcal{P}$  be a finite set. We introduce for any polynomial  $f \in \mathcal{P}$  the sets  $\mathcal{G}_f = \{g \in \mathcal{G} \mid \exists e_{\prec}g \mid e_{\prec}f\}$  and

$$\bar{\mathcal{G}}_f = \left\{ x^{\mu} \star g \mid g \in \mathcal{G}_f \land \mu = \operatorname{le}_{\prec} f - \operatorname{le}_{\prec} g \land \operatorname{le}_{\prec} (x^{\mu} \star g) = \operatorname{le}_{\prec} f \right\}$$
(33)

Note that the last condition is redundant only, if  $\mathcal{R}$  is an integral domain. Otherwise it may happen that  $|\bar{\mathcal{G}}_f| < |\mathcal{G}_f|$ , namely if  $\rho_\mu(r)r_{\mu\nu} = 0$  for  $\lim_{\prec} g = rx^{\nu}$ . We say that f is *head reducible* with respect to  $\mathcal{G}$ , if  $lc_{\prec}g \in \langle lc_{\prec}\bar{\mathcal{G}}_f \rangle$ . *Involutive head reducibility* is defined analogously via sets  $\mathcal{G}_{f,L}$  and  $\bar{\mathcal{G}}_{f,L}$  where only involutive divisors with respect to the division L on  $\mathbb{N}_0^n$  are taken into account, i. e. we set  $\mathcal{G}_{f,L} = \{g \in \mathcal{G} \mid le_{\prec}f \in \mathcal{C}_{L,le_{\prec}\mathcal{G}}(le_{\prec}g)\}$ . Thus the set  $\mathcal{G}$  is *involutively head autoreduced*, if  $lc_{\prec}g \notin \langle lc_{\prec}(\bar{\mathcal{G}}_{g,L} \setminus \{g\}) \rangle$  for all polynomials  $g \in \mathcal{G}$ . This is now a much weaker notion than before; in particular, Lemma 4.10 is no longer valid.

**Definition 10.1** Let  $\mathcal{I} \subseteq \mathcal{P}$  be a left ideal in the polynomial algebra  $(\mathcal{P}, \star, \prec)$ of solvable type over a ring  $\mathcal{R}$  in which linear equations can be solved. A finite set  $\mathcal{G} \subset \mathcal{P}$  is a Gröbner basis of  $\mathcal{I}$ , if for every polynomial  $f \in \mathcal{I}$  the condition  $lc_{\prec}f \in \langle lc_{\prec}\bar{\mathcal{G}}_{f} \rangle$  is satisfied. The set  $\mathcal{G}$  is a weak involutive basis for the involutive division L, if for every polynomial  $f \in \mathcal{I}$  the condition  $lc_{\prec}f \in \langle lc_{\prec}\bar{\mathcal{G}}_{f,L} \rangle$  is satisfied. A weak involutive basis is a strong involutive basis, if every set  $\bar{\mathcal{G}}_{f,L}$ contains precisely one element.

It is easy to see that our proof of Proposition 3.11 shows the existence of a Gröbner basis for any left ideal  $\mathcal{I}$  and that the characterisation of (weak) involutive bases via the existence of involutive standard representations (Theorem 4.3) remains valid. Indeed, only the first part of the proof requires a minor change: the polynomial  $f_1$  is now of the form  $f_1 = f - \sum_{h \in \mathcal{H}_{f,L}} r_h h$  where the coefficients  $r_h \in \mathcal{R}$  are chosen such that  $lt \prec f_1 \prec lt \prec f$ .

*Example 10.2* As in the previous two sections, we cannot generally expect strong involutive bases to exist. As a simple concrete example, also demonstrating the need of the second assumption on  $\mathcal{R}$ , we consider in  $\mathbb{k}[x,y][z]$  (with the ordinary multiplication) the ideal  $\mathcal{I}$  generated by the set  $\mathcal{F} = \{x^2z - 1, y^2z + 1\}$ . Obviously, both generators have the same leading exponent [1]; nevertheless none is reducible by the other one due to the relative primeness of the coefficients. Furthermore, the syzygy  $\mathbf{S} = x^2\mathbf{e}_2 - y^2\mathbf{e}_1 \in \mathbb{k}[x,y]^2$  connecting the leading coefficients leads to the polynomial  $x^2 + y^2 \in \mathcal{I}$ . It is easy to see that a Gröbner and weak Janet basis of  $\mathcal{I}$  is obtained by adding it to  $\mathcal{F}$ . A strong Janet basis does not exist, as none of these generators may be removed from the basis.

This example shows that simply applying our completion algorithm 6.1 will generally not suffice. Obviously, with respect to the Janet division z is multiplicative for both elements of  $\mathcal{F}$  so that no non-multiplicative variables exist and thus it is not possible to generate the missing generator by multiplication with a non-multiplicative variable. We must substitute in Algorithm 6.1 the involutive head autoreduction by a more comprehensive operation.

**Definition 10.3** Let  $\mathcal{F} \subset \mathcal{P}$  be a finite set and L an involutive division. We consider for each  $f \in \mathcal{F}$  the syzygies  $\sum_{\bar{f} \in \bar{\mathcal{F}}_{f,L}} s_{\bar{f}} \operatorname{lc}_{\prec} \bar{f} = 0$  connecting the leading coefficients of the elements of the set  $\bar{\mathcal{F}}_{f,L}$ . The set  $\mathcal{F}$  is involutively  $\mathcal{R}$ -saturated for the division L, if for any such syzygy  $\mathbf{S}$  the polynomial  $\sum_{\bar{f} \in \bar{\mathcal{F}}_{f,L}} s_{\bar{f}} \bar{f}$  possesses an involutive standard representation with respect to  $\mathcal{F}$ .

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Note that an element  $f \in \mathcal{F}$  is involutively head reducible by the other elements of  $\mathcal{F}$ , if and only if  $\operatorname{Syz}(\operatorname{lc}_{\prec}\bar{\mathcal{F}}_{f,L})$  contains a syzygy with  $s_f = 1$ . For this reason it is easy to combine an involutive  $\mathcal{R}$ -saturation with an involutive head autoreduction, as one can see in Algorithm 10.1. The loop in Lines /5-13/ takes care of the involutive head autoreduction; the loop in Lines /17-22/ checks the  $\mathcal{R}$ -saturation. In each iteration of the outer while loop we analyse from the remaining polynomials (collected in  $\mathcal{S}$ ) those with the highest leading exponent. The set  $\mathcal{S}$  is reset to the full basis, whenever a new element has been put into  $\mathcal{H}$ ; this ensures that all new reduction possibilities are taken into account. In Line /15/ it does not matter which element  $f \in \mathcal{S}_{\nu}$  we choose, as the set  $\mathcal{H}'_{f,L}$  depends only on le $_{\prec} f$  and all elements of  $\mathcal{S}_{\nu}$  possess by construction the same leading term  $\nu$ .

#### Algorithm 10.1 Involutive $\mathcal{R}$ -saturation (and head autoreduction)

**Input:** finite set  $\mathcal{F} \subset \mathcal{P}$ , involutive division L on  $\mathbb{N}_0^n$ **Output:** involutively  $\mathcal{R}$ -saturated and head autoreduced set  $\mathcal{H}$  with  $\langle \mathcal{H} \rangle = \langle \mathcal{F} \rangle$ 1:  $\mathcal{H} \leftarrow \mathcal{F}$ ;  $\mathcal{S} \leftarrow \mathcal{F}$ 2: while  $S \neq \emptyset$  do 3:  $\nu \leftarrow \max_{\prec} \operatorname{le}_{\prec} \mathcal{S}; \quad \mathcal{S}_{\nu} \leftarrow \{ f \in \mathcal{H} \mid \operatorname{le}_{\prec} f = \nu \}$  $\mathcal{S} \leftarrow \mathcal{S} \setminus \mathcal{S}_{\nu}; \quad \mathcal{H}' \leftarrow \mathcal{H}$ 4: 5: for all  $f \in S_{\nu}$  do  $h \leftarrow \texttt{HeadReduce}_{L,\prec}(f,\mathcal{H})$ 6: if  $f \neq h$  then 7:  $\mathcal{S}_{\nu} \leftarrow \mathcal{S}_{\nu} \setminus \{f\}; \quad \mathcal{H}' \leftarrow \mathcal{H}' \setminus \{f\}$ 8: 9: if  $h \neq 0$  then  $\mathcal{H}' \leftarrow \mathcal{H}' \cup \{h\}$ 10: 11: end\_if 12: end\_if end\_for 13: 14: if  $S_{\nu} \neq \emptyset$  then 15: choose  $f \in S_{\nu}$  and determine the set  $\overline{\mathcal{H}}'_{f,L}$ 16: compute basis  $\mathcal{B}$  of  $\operatorname{Syz}(\operatorname{lc}_{\prec}\bar{\mathcal{H}}'_{f,L})$ for all  $\mathbf{S} = \sum_{\bar{f} \in \bar{\mathcal{H}}'_{f,L}} s_{\bar{f}} \mathbf{e}_{\bar{f}} \in \mathcal{B}$  do 17:  $h \leftarrow \texttt{NormalForm}_{L,\prec}(\sum_{\bar{f} \in \bar{\mathcal{H}}'_{f,L}} s_{\bar{f}} \bar{f}, \mathcal{H}')$ 18: 19: if  $h \neq 0$  then 20:  $\mathcal{H}' \leftarrow \mathcal{H}' \cup \{h\}$ 21: end\_if 22: end\_for 23: end\_if if  $\mathcal{H}' \neq \mathcal{H}$  then 24: 25:  $\mathcal{H} \leftarrow \mathcal{H}'; \quad \mathcal{S} \leftarrow \mathcal{H}$ 26: end\_if 27: end\_while 28: return  $\mathcal{H}$ 

**Proposition 10.4** Algorithm 10.1 terminates for any input  $\mathcal{F}$  with an involutively  $\mathcal{R}$ -saturated and head autoreduced set  $\mathcal{H}$  such that  $\langle \mathcal{H} \rangle = \langle \mathcal{F} \rangle$ .

Proof The correctness of the algorithm is trivial. The termination is the consequence of the fact that both  $\mathcal{R}$  and  $\mathcal{P}$  are Noetherian. Whenever we add a polynomial h to the set  $\mathcal{H}'$ , we have either that  $le_{\prec}h \notin \langle le_{\prec}\mathcal{H}' \rangle$  or  $lc_{\prec}h \notin \langle lc_{\prec}\mathcal{H}'_{h}\rangle$ . As neither in  $\mathcal{R}$  nor in  $\mathcal{P}$  infinite ascending chains of ideals are possible, the algorithm must terminate after a finite number of steps.  $\Box$ 

An obvious idea is now to substitute in the completion Algorithm 6.1 the involutive head autoreduction by an involutive  $\mathcal{R}$ -saturation. Recall that Proposition 6.2 (and Corollary 6.3) was the crucial step for proving the correctness of Algorithm 6.1. Our next goal is thus to show that for involutively  $\mathcal{R}$ -saturated sets local involution implies weak involution. Unfortunately, it seems that this does not hold for arbitrary polynomial algebras of solvable type.

**Proposition 10.5** Let  $\mathcal{P}$  be a polynomial algebra of solvable type such that in (5) for all  $\mu, \nu \in \mathbb{N}_0^n$  and all  $r \in \mathcal{R} \setminus \{0\}$  the coefficients  $r_{\mu\nu}$  and the values  $\rho_{\mu}(r)$  are units in  $\mathcal{R}$ . Then a finite, involutively  $\mathcal{R}$ -saturated set  $\mathcal{F} \subset \mathcal{P}$  is weakly involutive, if and only if it is locally involutive.

*Proof* We first note that Proposition 6.2 remains true. Its proof only requires a few trivial modifications, as all appearing coefficients (for example, when we rewrite  $x^{\mu} \rightarrow x^{\mu-1_j} \star x_j$  are in fact units under our assumption and thus we may proceed as for a field. Hence if  $\mathcal{F}$  is locally involutive, then  $\mathcal{I} = \langle \mathcal{F} \rangle = \langle \mathcal{F} \rangle_{L,\prec}$  implying that any polynomial  $g \in \mathcal{I}$  may be written in the form  $g = \sum_{f \in \mathcal{F}} P_f \star f$  with  $P_f \in \mathcal{R}[X_{L,\mathcal{F},\prec}(f)]$ . We are done, if we can show that the coefficients  $P_f$  may be chosen such that  $le_{\prec}(P_f \star f) \preceq le_{\prec}g$ .

If the representation that comes out of the proof of Proposition 6.2 already satisfies this condition on the leading exponents, nothing has to be done. Otherwise we set  $\nu = \max_{\prec} \{ \operatorname{le}_{\prec}(P_f \star f) \mid f \in \mathcal{F} \}$  and  $\mathcal{F}_{\nu} = \{ f \in \mathcal{F} \mid \operatorname{le}_{\prec}(P_f \star f) = \nu \}.$ As  $\nu \in \bigcap_{f \in \mathcal{F}_{\nu}} \mathcal{C}_{L, \mathrm{le}_{\prec} \mathcal{F}}(\mathrm{le}_{\prec} f)$ , the properties of an involutive division imply that we can label the elements of  $\mathcal{F}_{\nu}$  in such a way that  $le_{\prec}f_1 \mid le_{\prec}f_2 \mid \cdots \mid le_{\prec}f_k$ 

with  $k = |\mathcal{F}_{\nu}|$ . Obviously,  $\mathcal{F}_{\nu} \subseteq \mathcal{F}_{f_k,L}$ . By construction,  $\sum_{f \in \mathcal{F}_{\nu}} \operatorname{lc}_{\prec}(P_f \star f) = 0$ . We want to relate this equality with the syzygies considered in Definition 10.3 of an involutively  $\mathcal{R}$ -saturated set. We use the following abbreviations:  $\lim_{\prec} f = r_f x^{\nu_f}$  and  $\lim_{\prec} P_f = s_f x^{\mu_f}$ . If we introduce furthermore the multi indices  $\bar{\nu} = \nu - \nu_{f_k}$  and  $\bar{\mu}_f = \mu_f - \bar{\nu}$ , then each element  $f \in \mathcal{F}_{\nu}$  contributes to the set  $\overline{\mathcal{F}}_{f_k,L}$  the polynomial  $\overline{f} = x^{\overline{\mu}_f} \star f$ . Using (5) and our assumptions, we see that the polynomials  $P_f \star f$  and  $x^{\bar{\nu}} \star \left[ \rho_{\bar{\nu}}(s_f r_{\bar{\nu}\bar{\mu}_f}^{-1}) \right]^{-1} \bar{f}$ possess the same leading monomials. This implies the existence of the R-syzygy  $\sum_{i=1}^{k} \left[ \rho_{\bar{\nu}}(s_{f_i} r_{\bar{\nu}\bar{\mu}_{f_i}}^{-1}) \right]^{-1} \mathrm{lc}_{\prec} \bar{f}_i = 0.$ As the set  $\mathcal{F}$  is involutively  $\mathcal{R}$ -saturated, there exists an involutive standard

representation

$$\sum_{i=1}^{k} \left[ \rho_{\bar{\nu}}(s_{f_i} r_{\bar{\nu}\bar{\mu}_{f_i}}^{-1}) \right]^{-1} \bar{f}_i = \sum_{f \in \mathcal{F}} Q_f \star f , \qquad (34)$$

i.e.  $Q_f \in \mathbb{k}[X_{L,\mathcal{F},\prec}(f)]$  and  $\mathbb{le}_{\prec}(Q_f \star f) \prec \nu_{f_k}$ . Introducing the polynomials  $Q'_f = Q_f - \left[\rho_{\bar{\nu}}(s_f r_{\bar{\nu}\bar{\mu}_f}^{-1})\right]^{-1} x^{\bar{\mu}_f}$  for  $f \in \mathcal{F}_{\nu}$  and  $Q'_f = Q_f$  otherwise, we obtain the syzygy  $\sum_{f \in \mathcal{F}} Q'_f \star f = 0$ . If we set  $P'_f = P_f - x^{\bar{\nu}} \star Q'_f$ , then, by construction,  $g = \sum_{f \in \mathcal{F}} P'_f \star f$  is another involutive representation of the polynomial g with  $\nu' = \max_{\prec} \{ \operatorname{le}_{\prec}(P'_f \star f) \mid f \in \mathcal{F} \} \prec \nu.$ 

Repeating this procedure for a finite number of times hence obviously yields an involutive standard representation of the polynomial g. As g was an arbitrary element of the ideal  $\mathcal{I} = \langle \mathcal{F} \rangle$ , this implies that  $\mathcal{F}$  is indeed weakly involutive.  $\Box$ 

Note that it suffices, if the ring elements  $r_{\mu\nu}$  and  $\rho_{\mu}(r)$  are left invertible.

**Theorem 10.6** Let  $\mathcal{P}$  be a polynomial algebra of solvable type satisfying the assumptions of Proposition 10.5. If the subalgorithm InvHeadAutoReduce<sub>L, \prec</sub> is substituted in Algorithm 6.1 by Algorithm 10.1, then the completion will terminate with a weak involutive basis of  $\mathcal{I} = \langle \mathcal{F} \rangle$  for any finite input set  $\mathcal{F} \subset \mathcal{P}$  such that the monoid ideal  $le_{\prec}\mathcal{I}$  possesses a weak involutive basis.

*Proof* The correctness of the modified algorithm follows immediately from Proposition 10.5. For the termination we may use the same argument as in the proof of Theorem 6.4, as it depends only on the leading exponents.  $\Box$ 

#### **11 Conclusions**

We introduced a rather general class of non-commutative polynomial algebras. It contains all the algebras studied by Kandry-Rody and Weispfenning [32] and by Apel [2] and Levandovskyy [34,35], respectively. In contrast to these works, we permitted that the variables act on the coefficients, so that it is possible to treat operator algebras. Thus our approach automatically includes linear differential operators with coefficients from some function field and it is no longer necessary to treat this case separately as in [13]. For the case that the coefficients form a field, the same class of algebras was already studied in [9] where a theory of Gröbner bases was developed for them.

Our approach is very closely modelled on that of Kandry-Rody and Weispfenning [32]. However, we believe that the third condition in Definition 3.1 (the compatibility between term order  $\prec$  and non-commutative product  $\star$ ) is more natural than the stricter axioms in [32]. In particular, we could not see where Kandry-Rody and Weispfenning needed their stricter conditions, as all their main results hold in our more general situation, too.

Comparing with [2,34,35], one must say that there used approach is more constructive than ours. More precisely, these authors specify the non-commutative product via commutation relations and thus have automatically a concrete algorithm for evaluating any product. As we have seen in the proof of Proposition 3.3, the same data suffices to fix our axiomatically described product, but it does not provide us with an algorithm. However, we showed that we can always map to their approach via a basis transformation.

We showed that the polynomial algebra of solvable type from a natural framework for involutive bases. This is not surprising, if one takes into account that the main part of the involutive theory happens in the monoid  $\mathbb{N}_0^n$  and the decisive third condition in Definition 3.1 of a polynomial algebra of solvable type ensures that its product  $\star$  does not interfere with the leading exponents.

We extended the theory of involutive bases to semigroup orders and to polynomials over rings. It turned out that the novel concept of a *weak* involutive basis is crucial for such generalisations, as in both cases strong bases rarely exist. These weak bases are still Gröbner bases and involutive standard representations still exist (though they are no longer unique). It seems that in local computations the Janet division has a distinguished position, as strong Janet bases always exist.

Concerning involutive bases over rings, we will study in Part II the special case that the coefficient ring is again a polynomial algebra of solvable type. Using the syzygy theory that will be developed there, we will be able to obtain stronger results and a "purely involutive" completion algorithm. The current approach contains hidden in the concept of  $\mathcal{R}$ -saturation parts of the Buchberger algorithm for the construction of Gröbner bases over rings.

It appears doubtful that for rings  $\mathcal{R}$  not satisfying the assumptions made in Proposition 10.5 an effective Gröbner bases theory exists. These assumptions seems to be decisive to reduce syzygies of arbitrarily high degrees to a finite number of syzygies between the leading monomials of the basis. Without the possibility of such a reduction no finite criterion for a Gröbner basis can exist.

Definition 2.1 represents the currently mainly used definition of an involutive division. While it appears quite natural, one problem is that in some sense too many involutive divisions exist, in particular rather weird ones with unpleasant properties. This effect has lead us to the introduction of such technical concepts like continuity and constructivity. One could imagine that there should exist a stricter definition of involutive divisions that automatically ensures that Algorithm 5.1 terminates without having to resort to these technicalities.

Most of these weird divisions are globally defined and multiplicative indices are assigned only to finitely many multi indices. Such divisions are obviously of no interest, as more or less no monoid ideal possesses an involutive basis for them. One way to eliminate these divisions would be to require that for every  $q \in \mathbb{N}_0$ the monoid ideal  $(\mathbb{N}_0^n)_{\geq q} = \{\nu \in \mathbb{N}_0^n \mid q \leq |\nu|\}$  has an involutive basis. All the involutive divisions used in practice satisfy this condition, but it is still a long way from this simple condition to the termination of Algorithm 5.1.

Our Definition 4.1 of an involutive basis is not the original one proposed by Gerdt and Blinkov [14]. We believe that it is more natural and closer to the classical definition of Gröbner bases. In particular, the fact that any (weak) involutive basis is a Gröbner basis becomes trivial in this approach. The notion of a weak involutive basis is new, but as we have seen in the last sections, in more general situations like in local computations or with coefficient rings such a weaker notion is necessary. If one is only interested in using Algorithm 6.1 as an alternative to Buchberger's algorithm, weak bases are sufficient. However, most of the more advanced applications of involutive bases in Part II will require strong bases.

We did not discuss the efficiency of the here presented algorithms. Much of the literature on involutive bases is concerned with their use as an alternative approach to the construction of Gröbner bases. In particular, recent experiments by Gerdt et al. [17] comparing a specialised C/C++ program for the construction of

Janet bases with the Gröbner bases package of SINGULAR [23] indicate that the involutive approach is highly competitive. This is quite remarkable, if one takes into account that SINGULAR is based on the results of several decades of intensive research on Gröbner bases by many groups, whereas involutive bases are still very young and only a few researchers have actively worked on them. The results in Part II will offer some heuristic explanations for this observation.

Finally, we mention that most of the algorithms discussed in this article have been implemented (for general polynomial algebras of solvable type) by M. Hausdorf [25,26] in the computer algebra system *MuPAD*.<sup>6</sup> The implementation does not use the simple completion Algorithm 6.1 but a more optimised version yielding minimal bases developed by Gerdt and Blinkov [15]. It also includes the modified algorithm for determining strong Janet bases in local rings.

#### A Term Orders

We use in this article non-standard definitions of some basic term orders. More precisely, we revert the order of the variables: our definitions become the standard ones, if one transforms  $(x_1, \ldots, x_n) \rightarrow (x_n, \ldots, x_1)$ . The reason for this reversal is that this way the definitions fit better to the conventions in the theory of involutive systems of differential equations. Furthermore, they appear more natural in some applications like the determination of the depth in Part II.

A term order  $\prec$  is for us a total order on the set  $\mathbb{T}$  of all terms  $x^{\mu}$  satisfying the following two conditions: (i)  $1 \leq t$  for all terms  $t \in \mathbb{T}$  and (ii)  $s \leq t$  implies  $r \cdot s \leq r \cdot t$  for all terms  $r, s, t \in \mathbb{T}$ . If a term order fulfils in addition the condition that  $s \prec t$  whenever deg  $s < \deg t$ , it is called *degree compatible*. As  $\mathbb{T}$  and  $\mathbb{N}_0^n$ are isomorphic as monoids, we may also speak of term orders on  $\mathbb{N}_0^n$ . In fact, most term orders are defined via multi indices.

The *lexicographic* order is defined by  $x^{\mu} \prec_{\text{lex}} x^{\nu}$ , if the last non-vanishing entry of  $\mu - \nu$  is negative. Thus  $x_2^2 x_3 \prec_{\text{lex}} x_1 x_3^2$ . With respect to the *reverse lexicographic* order  $x^{\mu} \prec_{\text{revlex}} x^{\nu}$ , if the first non-vanishing entry of  $\mu - \nu$  is positive. Now we have  $x_1 x_3^2 \prec_{\text{revlex}} x_2^2 x_3$ . However,  $\prec_{\text{revlex}}$  is *not* a term order, as it violates the first condition:  $x_1 \prec_{\text{revlex}} 1$ . The reverse lexicographic order should not be confused with the *inverse lexicographic* order  $\prec_{\text{invlex}}$  which arises from  $\prec_{\text{lex}}$ by inverting the order of the variables, i. e.  $x^{\mu} \prec_{\text{invlex}} x^{\nu}$ , if the first non-vanishing entry of  $\mu - \nu$  is negative.

Degree compatible versions exist of all these orders.  $x^{\mu} \prec_{\text{deglex}} x^{\nu}$ , if  $|\mu| < |\nu|$ or if  $|\mu| = |\nu|$  and  $x^{\mu} \prec_{\text{lex}} x^{\nu}$ . Similarly,  $x^{\mu} \prec_{\text{degrevlex}} x^{\nu}$ , if  $|\mu| < |\nu|$  or if  $|\mu| = |\nu|$  and  $x^{\mu} \prec_{\text{revlex}} x^{\nu}$ . Note that  $\prec_{\text{degrevlex}}$  is a term order, in fact a very important one! It possesses the following useful characterisation.

**Lemma A.1** Let  $\prec$  be a term order such that the condition  $\operatorname{lt}_{\prec} f \in \langle x_1, \ldots, x_k \rangle$ is equivalent to  $f \in \langle x_1, \ldots, x_k \rangle$  for every homogeneous polynomial  $f \in \mathcal{P}$ . Then  $\prec$  is the degree reverse lexicographic order  $\prec_{\operatorname{degrevlex}}$ .

<sup>&</sup>lt;sup>6</sup> For more information see www.mupad.de.

*Proof* Let  $x^{\mu}$  and  $x^{\nu}$  be two monomials of the same degree such that the first non-vanishing entry of  $\mu - \nu$  is  $\mu_k - \nu_k$ . Without loss of generality, let  $\mu_k > \nu_k$ . Set  $\rho = [\nu_1, \ldots, \nu_k, 0, \ldots, 0]$  and consider the multi indices  $\bar{\mu} = \mu - \rho$  and  $\bar{\nu} = \nu - \rho$ . Obviously,  $x^{\bar{\mu}} \in \langle x_1, \ldots, x_k \rangle$  whereas  $x^{\bar{\nu}} \notin \langle x_1, \ldots, x_k \rangle$ . Considering the homogeneous polynomial  $f = x^{\bar{\mu}} + x^{\bar{\nu}}$ , the assumption of the lemma implies that the leading term of f must be  $x^{\bar{\nu}}$ . By the monotonicity of term orders, we conclude that  $x^{\mu} \prec x^{\nu}$ . But by definition, we also have  $x^{\mu} \prec_{\text{degrevlex}} x^{\nu}$ .  $\Box$ 

We say that a term order *respects classes*, if for multi indices  $\mu$ ,  $\nu$  of the same length  $\operatorname{cls} \mu < \operatorname{cls} \nu$  implies  $x^{\mu} \prec x^{\nu}$ . It is now easy to see that by Lemma A.1 only one class respecting term order on  $\mathbb{T}$  exists: the degree reverse lexicographic order. If we consider free polynomial modules, only TOP lifts [1] of  $\prec_{\operatorname{degrevlex}}$  are class respecting.

A more appropriate name for term orders might be *monoid orders*, as the two conditions above say nothing but that these orders respect the monoid structure of  $\mathbb{T}$ . A more general class of (total) orders are *semigroup orders* where we skip the first condition, i. e. we only take the semigroup structure of  $\mathbb{T}$  into account. It is a well-known property of such orders that they are no longer well-orders. This implies in particular the existence of infinite descending sequences so that normal form algorithms do not necessarily terminate.

Acknowledgements The author would like to thank V.P. Gerdt for a number of interesting discussions on involutive bases. M. Hausdorf and R. Steinwandt participated in an informal seminar at Karlsruhe University where most ideas of this article were presented and gave many valuable comments. In April 2002 the author taught a one-week intensive course on involutive bases at Universität Kaiserslautern; the remarks of the very active participants lead to further improvements. Some constructive remarks of the anonymous referees were also helpful. This work was financially supported by Deutsche Forschungsgemeinschaft and INTAS grant 99-1222.

### References

- 1. W.W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics 3. AMS, Providence, 1994.
- 2. J. Apel. Gröbnerbasen in Nichtkommutativen Algebren und ihre Anwendung. PhD thesis, Universität Leipzig, 1988.
- J. Apel. The computation of Gröbner bases using an alternative algorithm. In M. Bronstein, J. Grabmeier, and V. Weispfenning, editors, *Symbolic Rewriting Techniques*, Progress in Computer Science and Applied Logic 15, pages 35–45. Birkhäuser, Basel, 1998.
- 4. J. Apel. Theory of involutive divisions and an application to Hilbert function computations. J. Symb. Comp., 25:683–704, 1998.
- Th. Becker and V. Weispfenning. *Gröbner Bases*. Graduate Texts in Mathematics 141. Springer-Verlag, New York, 1993.
- A.D. Bell and K.R. Goodearl. Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions. *Pacific J. Math.*, 131:13–37, 1988.
- 7. R. Berger. The quantum Poincaré-Birkhoff-Witt theorem. *Comm. Math. Phys.*, 143:215–234, 1992.

- 8. M. Bronstein and M. Petkovšek. An introduction to pseudo-linear algebra. *Theor. Comp. Sci.*, 157:3–33, 1996.
- J.L. Buesco, J. Gómez Torrecillas, F.J. Lobillo, and F.J. Castro. An introduction to effective calculus in quantum groups. In S. Caenepeel and A. Verschoren, editors, *Rings, Hopf Algebras, and Brauer Groups*, Lecture Notes in Pure and Applied Mathematics 197, pages 55–83. Marcel Dekker, New York, 1998.
- 10. P.M. Cohn. Algebra II. John Wiley, London, 1977.
- 11. D. Cox, J. Little, and D. O'Shea. Using Algebraic Geometry. Graduate Texts in Mathematics 185. Springer-Verlag, New York, 1998.
- 12. V.G. Drinfeld. Hopf algebras and the quantum Yang-Baxter equations. *Sov. Math. Dokl.*, 32:254–258, 1985.
- V.P. Gerdt. Completion of linear differential systems to involution. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing* — *CASC 1999*, pages 115–137. Springer-Verlag, Berlin, 1999.
- V.P. Gerdt and Yu.A. Blinkov. Involutive bases of polynomial ideals. *Math. Comp. Simul.*, 45:519–542, 1998.
- V.P. Gerdt and Yu.A. Blinkov. Minimal involutive bases. *Math. Comp. Simul.*, 45:543– 560, 1998.
- V.P. Gerdt, Yu.A. Blinkov, and D.A. Yanovich. Construction of Janet bases I: Monomial bases. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing — CASC 2001*, pages 233–247. Springer-Verlag, Berlin, 2001.
- V.P. Gerdt, Yu.A. Blinkov, and D.A. Yanovich. Construction of Janet bases II: Polynomial bases. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing CASC 2001*, pages 249–263. Springer-Verlag, Berlin, 2001.
- P. Gianni, B. Trager, and G. Zacharias. Gröbner bases and primary decomposition of polynomial ideals. J. Symb. Comp., 6:149–167, 1988.
- M. Giesbrecht, G. Reid, and Y. Zhang. Non-commutative Gröbner bases in Poincaré-Birkhoff-Witt extensions. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing — CASC 2002.* Springer-Verlag, Berlin, 2002 (to appear).
- H.-G. Gräbe. The tangent cone algorithm and homogenization. J. Pure Appl. Alg., 97:303–312, 1994.
- 21. H.-G. Gräbe. Algorithms in local algebra. J. Symb. Comp., 19:545-557, 1995.
- 22. G.-M. Greuel and G. Pfister. Advances and improvements in the theory of standard bases and syzygies. *Arch. Math.*, 66:163–176, 1996.
- 23. G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 2.0 A computer algebra system for polynomial computations. Technical report, Centre for Computer Algebra, University of Kaiserslautern, 2001. www.singular.uni-kl.de.
- 24. M. Hausdorf and W.M. Seiler. An efficient algebraic algorithm for the geometric completion to involution. *Appl. Alg. Eng. Comm. Comp.*, 13:163–207, 2002.
- M. Hausdorf and W.M. Seiler. Involutive bases in *MuPAD* I: Involutive divisions. *mathPAD*, 11:51–56, 2002.
- M. Hausdorf and W.M. Seiler. Involutive bases in *MuPAD* II: Polynomial algebras of solvable type. In preparation, 2002.
- 27. M. Hausdorf, W.M. Seiler, and R. Steinwandt. Involutive bases in the Weyl algebra. *J. Symb. Comp.*, to appear.
- W. Hereman. Review of symbolic software for the computation of Lie symmetries of differential equations. *Euromath Bull.*, 2:45–82, 1994.
- M. Janet. Sur les Systèmes d'Équations aux Dérivées Partielles. J. Math. Pure Appl., 3:65–151, 1920.

- M. Janet. Leçons sur les Systèmes d'Équations aux Dérivées Partielles. Cahiers Scientifiques, Fascicule IV. Gauthier-Villars, Paris, 1929.
- 31. M. Jimbo. A q-difference analogue of  $U(\mathfrak{g})$  and the Yang-Baxter equations. Lett. Math. Phys., 10:63–69, 1985.
- A. Kandry-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. J. Symb. Comp., 9:1–26, 1990.
- D. Lazard. Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations. In J.A. Hulzen, editor, *Proc. EUROCAL*, Lecture Notes in Computer Science 162, pages 146–156. Springer-Verlag, 1983.
- 34. V. Levandovskyy. Gröbner bases of a class of non-commutative algebras. Master's thesis, Universität Kaiserslautern, 2000.
- 35. V. Levandovskyy. On Gröbner bases for non-commutative G-algebras. In J. Calmet, M. Hausdorf, and W.M. Seiler, editors, Proc. Under- and Overdetermined Systems of Algebraic or Differential Equations, pages 99–118. Fakultät für Informatik, Universität Karlsruhe, 2002.
- 36. G. Lusztig. Quantum groups at roots of 1. Geom. Dedi., 35:89-113, 1990.
- 37. J.C. McConnell and J.C. Robson. Non-commutative Noetherian Rings. Wiley, 1987.
- C. Méray and C. Riquier. Sur la convergence des développements des intégrales ordinaires d'un système d'équations différentielles partielles. *Ann. Sci. Ec. Norm. Sup.*, 7:23–88, 1890.
- T. Mora. An algorithm to compute the equations of tangent cones. In J. Calmet, editor, *Proc. EUROCAM* '82, Lecture Notes in Computer Science 144, pages 158–165. Springer-Verlag, Berlin, 1982.
- E. Noether and W. Schmeidler. Moduln in nichtkommutativen Bereichen, insbesondere aus Differential- und Differenzausdrücken. *Math. Zeit.*, 8:1–35, 1920.
- 41. O. Ore. Linear equations in non-commutative fields. Ann. Math., 32:463-477, 1931.
- 42. O. Ore. Theory of non-commutative polynomials. Ann. Math., 34:480-508, 1933.
- 43. C. Riquier. *Les Systèmes d'Équations aux Derivées Partielles*. Gauthier-Villars, Paris, 1910.
- M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeomet*ric Differential Equations. Algorithms and Computation in Mathematics 6. Springer-Verlag, Berlin, 2000.
- W.M. Seiler. Involution the formal theory of differential equations and its applications in computer algebra and numerical analysis. Habilitation thesis, Dept. of Mathematics, Universität Mannheim, 2001.
- B. Sturmfels and N. White. Computing combinatorial decompositions of rings. *Combinatorica*, 11:275–293, 1991.
- 47. J.M. Thomas. Differential Systems. AMS, New York, 1937.
- A. Tresse. Sur les invariants différentiels des groupes continus de transformations. Acta Math., 18:1–88, 1894.
- W. Trinks. Über B. Buchbergers Verfahren, Systeme algebraischer Gleichungen zu lösen. J. Num. Th., 10:475–488, 1978.
- V.S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representations*. Graduate Texts in Mathematics 102. Springer-Verlag, New York, 1984.
- A.Yu. Zharkov and Yu.A. Blinkov. Involution approach to solving systems of algebraic equations. In G. Jacob, N.E. Oussous, and S. Steinberg, editors, *Proc. Int. IMACS Symp. Symbolic Computation*, pages 11–17, Lille, 1993.