

Conchoidal transform of two plane curves

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Abstract

The conchoid of a plane curve C is constructed using a fixed circle B in the affine plane. We generalize the classical definition so that we obtain a conchoid from any pair of curves B and C in the projective plane. We present two definitions, one purely algebraic through resultants and a more geometric one using an incidence correspondence in $\mathbf{P}^2 \times \mathbf{P}^2$. We prove, among other things, that the conchoid of a generic curve of fixed degree is irreducible, we determine its singularities and give a formula for its degree and genus. In the final section we return to the classical case: for any given curve C we give a criterion for its conchoid to be irreducible and we give a procedure to determine when a curve is the conchoid of another.

1 Introduction

The conchoid of a plane curve is a classical construction: given a curve C in the real affine plane, fix a point A and a positive real number r . The conchoid of C is the locus of points Q that are at distance r from a point $P \in C$ on the line AP . Examples of this construction are the conchoid of Nicomede and the limaçon of Pascal (see for example [5], [7]). In [7] one can find an analysis of the basic algebraic properties of the conchoid of an algebraic plane curve over an algebraically closed field of characteristic zero.

When the curve C is algebraic it is easy to obtain the equation of the conchoid from the equation of C . One way to do this is by elimination of variables, using Gröbner bases. However, the conchoid of a curve may have multiple components and this procedure does not always give the correct multiplicities. For example, choosing $A = (0, 0)$ and $r = 1$, for the line $x - 2 = 0$ one finds the irreducible quartic $4y^2 + x^4 + x^2y^2 - 4x^3 - 4xy^2 + 3x^2 = 0$ while for the line $x = 0$ one finds $x(x^2 + y^2 - 1) = 0$. In fact in this last case the component $x = 0$ should be counted twice.

In this paper we give two different ways to define correctly the conchoid. The first is algebraic, and uses resultants instead of Gröbner bases to find the equation of the conchoid. The second is more geometric and uses techniques in algebraic geometry like correspondences and multiple covers of \mathbf{P}^2 .

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Our definitions come from an appropriate generalization of the construction of conchoids. First of all, the notion of distance can be replaced with that of intersection with an assigned circle and hence we can work over any field, not only over \mathbb{R} . Moreover, it is more convenient to work in a projective ambient, so for us curve will mean a divisor in \mathbf{P}^2 . However the conchoid is essentially an affine concept, and so we fix in \mathbf{P}^2 a line L_∞ as line at infinity and a point A in its complement. If B and C are two curves we define the *conchoidal transform of C with respect to B* as the locus of points Q intersection of the line AP , with P a point in C , and the translate of B of a vector \vec{AP} . The translation is well defined in the fixed affine part. When B is a circle with center A and radius r , this definition is the same as the classical one. In this description the two curves play different roles, but we will see that the conchoidal transform is in fact symmetrical in B and C .

Both our definitions are universal on the coefficients of the equations of the curves B and C . This will allow us to reduce many proofs to the case when one of them is a generic line or a union of generic lines, and use deformations.

After some preliminaries, in section 3 we give the definition of conchoid using resultants and we prove some properties, in particular we determine the degree, the singularities and the special components of the conchoid. Then in sections 4 and 5 we give a geometric definition. The construction we make works only under suitable hypotheses, which are made explicit in Assumption 4.1 of Section 4. We then prove that under these hypotheses the two definitions coincide.

We use the word *generic* in the sense of algebraic geometry. For a family of objects parametrized by an algebraic variety, for example the family of all plane curves of given degree, generic means “in the complement of a proper closed algebraic subset”, i.e., outside the locus given by finitely many polynomial equations in the coordinates of the parameter space. Sometimes it is possible to give these equations explicitly, as we do in Assumption 4.1 where every geometric condition can be translated in the vanishing of some polynomials. In other situations it’s enough to know that these equations exist, for example in the proof of Theorem 5.3 where we use the classical Bertini’s theorem.

We show that the conchoid of a generic curve is irreducible and give a formula for its genus that depends on the genera and the degrees of the curves B and C . The case B circle and C rational is studied extensively in [8], where an algorithm is given to determine when the conchoid of C is rational or splits in two rational components and to compute a rational parametrization of each rational component. We also define the concept of *proper conchoid* in analogy of that of proper transform.

In the last section we go back to the classical case: in this situation the multiple cover of \mathbf{P}^2 is a double cover and we use the theory of double planes to give a criterion for the irreducibility of the proper conchoid of any curve. This part requires a bit more algebraic geometry than the rest of the paper, in particular in the proofs we use freely the properties of branched coverings and of normalization. We introduce also the concept of *n -iterated conchoid* and show that all the iterated conchoids of a fixed curve belong to a 1-dimensional flat family. We end with a procedure to determine when an irreducible curve is either the complete or proper conchoid of another.

2 Notation and generalities

We work over a fixed base field k . For a geometrical interpretation it is better to have k algebraically closed, but most definitions make sense on the field of definition of the starting curves. We assume the characteristic of k to be 0 or a prime number p greater than the degrees of the curves we consider, so we can use derivatives to study singularities.

We will denote \mathbf{P}^2 the projective plane over k . As the concept of conchoid is an affine one, we fix a line L_∞ and we denote with \mathbb{A}^2 its complement. It is a fixed affine plane, and inside it we fix a point A . We choose homogeneous coordinates $[x : y : z]$ in \mathbf{P}^2 so that L_∞ has equation $z = 0$, and $A = [0 : 0 : 1]$ is the origin of \mathbb{A}^2 . If $D \subset \mathbf{P}^2$ is the curve given by the homogeneous equation $G(x, y, z) = 0$, we denote by $D^{(a)}$ the *affine part* of D , i.e., $D^{(a)}$ is the curve in \mathbb{A}^2 given by the equation $G(x, y, 1) = 0$.

We fix two projective curves, denoted by B and C , with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ of degrees d and δ and geometric genus g and γ respectively. To avoid trivial cases, we assume that B is the projective closure of $B^{(a)}$, i.e., L_∞ is not a component of B .

The following lemma will allow us to give different but equivalent definitions for the concept of *conchoidal transform of the curves B and C* . Having fixed a point A , the affine plane has a natural vector space structure in which A is the zero element. In the statement of the lemma we are using this structure when we add points or multiply them by a scalar.

Lemma 2.1. *Let B be a projective curve in \mathbf{P}^2 and $B^{(a)}$ its affine part. For every P, Q in \mathbb{A}^2 , such that $P, Q \neq A$, the following are equivalent:*

1. Q is on the line AP and on the translate of $B^{(a)}$ by the vector \vec{AP} ;
2. Q is on the line AP and the point $Q - P$ (i.e., the translate of Q by the vector \vec{PA}) belongs to $B^{(a)}$;
3. $Q = P + S$ with $S \in B^{(a)}$ and A, P and S are collinear;
4. $\exists \lambda \in k$ such that $P = \lambda Q$ and $(1 - \lambda)Q \in B^{(a)}$.

We do not give the proof, which is elementary; we only note that the main reason for the equivalence is the fact that the line AP is invariant under translation by the vector \vec{AP} .

Let $C^{(a)}$ be an affine curve. In the classical construction of a conchoid, to each point $P \in C^{(a)}$ one associates the two points on the line AP at distance 1 from P . These are the points Q satisfying condition 1. of the previous Lemma, when $B^{(a)}$ is the circle of center A and radius 1. In this way one obtains an affine curve. More generally, one may think of the conchoidal transform of the curve C with respect to B as the projective closure of the set of points satisfying one of the equivalent conditions of Lemma 2.1, as P varies on $C^{(a)}$. Using condition 3., we see that the roles of C and B are in fact completely symmetrical. We will use condition 4. to give a general definition of the conchoidal transform of two projective curves: the definition will involve a resultant, and it will be useful both for theoretical purposes and as a computational device, instead of elimination of variables and Groebner basis computations.

3 Conchoidal transforms as resultants

Let B and C be projective plane curves, with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ respectively. Writing down explicitly condition 4 of Lemma 2.1 we see that a point $Q = [a :$

$b : 1]$ in \mathbb{A}^2 different from A is in the conchoid of C with respect to B if the system of two equations in the single unknown λ

$$\begin{cases} F((1-\lambda)a, (1-\lambda)b, 1) = 0 \\ G(\lambda a, \lambda b, 1) = 0 \end{cases}$$

has a solution. Using projective coordinates, we then define:

Definition 3.1. *The conchoidal transform $\mathcal{C}(B, C)$ of B and C (which we will often call simply the conchoid) is the divisor in \mathbf{P}^2 given by the resultant $R(F, G)$ of the two polynomials in the homogeneous variables λ and μ*

$$F((\mu - \lambda)x, (\mu - \lambda)y, \mu z) \quad \text{and} \quad G(\lambda x, \lambda y, \mu z) \quad (1)$$

Write $F(x, y, z) = F_d(x, y) + zF_{d-1} + \dots$ and $G(x, y, z) = G_\delta(x, y) + zG_{\delta-1} + \dots$ as polynomials in z so that F_h e G_h are homogeneous polynomials of degree h in the indeterminates x, y . We have:

$$F((\mu - \lambda)x, (\mu - \lambda)y, \mu z) = \sum_{i=0}^d \lambda^i \mu^{d-i} \Phi_i(x, y, z)$$

where $\Phi_i(x, y, z) = (-1)^i \sum_{j=i}^d \binom{j}{i} F_j(x, y) z^{d-j}$, and

$$G(\lambda x, \lambda y, \mu z) = \sum_{i=0}^{\delta} \lambda^i \mu^{\delta-i} G_i(x, y) z^{\delta-i}$$

hence:

$$R(F, G) = \begin{vmatrix} \Phi_d & \Phi_{d-1} & \dots & \dots & \dots & \Phi_0 & 0 & \dots & \dots & 0 \\ & \dots & & & \dots & & & \dots & & \\ & \dots & & & \dots & & & \dots & & \\ 0 & \dots & \dots & 0 & \Phi_d & \Phi_{d-1} & \dots & \dots & \dots & \Phi_0 \\ G_\delta & zG_{\delta-1} & \dots & \dots & \dots & z^\delta G_0 & 0 & 0 & \dots & 0 \\ & \dots & & & \dots & & & \dots & & \\ & \dots & & & \dots & & & \dots & & \\ 0 & \dots & 0 & G_\delta & zG_{\delta-1} & \dots & \dots & \dots & \dots & z^\delta G_0 \end{vmatrix}.$$

Example 3.2. *Conchoidal transform of two lines. Let $F = ax + by + cz$ and $G = a'x + b'y + c'z$. The conchoidal transform is given by:*

$$\begin{vmatrix} -(ax + by) & ax + by + cz \\ a'x + b'y & c'z \end{vmatrix} = -[(ax + by + cz)(a'x + b'y + c'z) - cc'z^2].$$

This polynomial does not define a curve only if B and C are both the line L_∞ given by $z = 0$. In all the other cases it is the hyperbola passing through the origin A and with asymptotes the lines B and C .

Example 3.3. *Conchoid with respect to a line B . Let $F = ax + by + cz$ as before and G any homogeneous polynomial of degree $\delta \geq 2$. The conchoidal transform is given by:*

$$\begin{vmatrix} -(ax + by) & ax + by + cz & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & \dots & 0 & -(ax + by) & ax + by + cz \\ G_\delta & zG_{\delta-1} & \dots & z^{\delta-1}G_1 & z^\delta G_0 \end{vmatrix}.$$

This polynomial is

$$G(x(ax + by + cz), y(ax + by + cz), (ax + by)z)$$

as we show by induction on the degree δ . For $\delta = 1$ it is true by the computation in the previous example; assume now the statement for $\deg G = \delta - 1$, and expand the determinant along the last row: we obtain $R(F, G) = (ax + by + cz)^\delta G_\delta(x, y) + z(ax + by)R(F, \overline{G})$ where $G = G_\delta + z\overline{G}$, which is the thesis.

We have again an effective divisor, unless B and C have both L_∞ as a component.

We can obtain the following properties of the conchoidal transform from well known properties of the resultant. In particular, 4. and 5. say that if either B or C is in some special position with respect to A or L_∞ , then the conchoidal transform will have certain special components.

Theorem 3.4. *Let B and C be as before. Then:*

1. $\deg \mathcal{C}(B, C) = 2\delta d$;
2. $\mathcal{C}(B, C) = \mathcal{C}(C, B)$;
3. if $C = C_1 + C_2$ then $\mathcal{C}(B, C) = \mathcal{C}(B, C_1) + \mathcal{C}(B, C_2)$;
4. if $P \in L_\infty \cap B \cap C$ and the multiplicities in P of B and C are respectively η and ϵ , then $P \in \mathcal{C}(B, C)$ with multiplicity $\geq \epsilon\delta + \eta d$ and the line AP is a component of $\mathcal{C}(B, C)$ with multiplicity $\geq \epsilon(\eta - \epsilon) + \frac{\epsilon(\epsilon+1)}{2}$ if $\epsilon \leq \eta$;
5. if $A \in C$ with multiplicity ν , then the divisor νB is contained in $\mathcal{C}(B, C)$.

Proof. 1. and 3. follow respectively from the definition and a property of the resultant ([2] Exercise 3 page 79).

To prove 2. we observe that the existence of a non trivial solution (λ, μ) is the same as the existence of a non trivial solution $(\lambda' = \mu - \lambda, \mu' = \mu)$ and with respect to these new variables λ', μ' the roles of C and B are interchanged.

To prove 4. we may assume that P has coordinates $[1 : 0 : 0]$ and so the line AP is given by $y = 0$. Since the η and ϵ are the multiplicities in P of B and C respectively, every monomial in F belongs to $(y, z)^\eta$ and every monomial in G belongs to $(y, z)^\epsilon$. Hence every entry in the first δ rows of the matrix whose determinant is $R(F, G)$ belongs to $(y, z)^\eta$ and every entry in the other d rows belongs to $(y, z)^\epsilon$. This facts clearly imply $R(F, G) \in (y, z)^{\epsilon\delta + \eta d}$.

For the same reason as above, $y^{\eta-i}$ divides Φ_{d-i} for every $i < \eta$ and $y^{\epsilon-i}$ divides $G_{\delta-i}$ for every $i < \epsilon$. Thanks to 2. we may assume $\epsilon \leq \eta$. Moreover, after a change of coordinates $x' = x$, $y' = y$ and $z' = z + uy$ for a general constant u , we may assume that y^ϵ is the maximal power of y dividing G_δ and y^η is the maximal power of y dividing $\Phi_d = F_d$ (note that such a change of coordinates does not modify A nor P). Now the thesis is a consequence of the following Lemma 3.5, where $S = k[x, y, z]_{(y)}$ and M is the matrix of the first η columns of the matrix whose determinant is $R(F, G)$ (without the null rows).

Finally, for 5. we observe that, if $G_0 = \dots = G_{\nu-1} = 0$, then the resultant is multiple of $\Phi_0(x, y, z)^\nu$ where $\Phi_0(x, y, z) = \sum_{j=0}^d F_j(x, y)z^{\delta-j} = F(x, y, z)$. □ □

Lemma 3.5. *Let S be a discrete valuation ring with valuation v , let a, b positive integers, $a \leq b$, and let M be a $2b \times b$ matrix with entries in S of the following type:*

$$M = \begin{pmatrix} g_1 & g_2 & \cdots & \cdots & g_b \\ 0 & g_1 & g_2 & \cdots & g_{b-1} \\ & \cdots & & & \\ 0 & \cdots & 0 & \cdots & g_1 \\ \hline f_1 & f_2 & \cdots & \cdots & f_b \\ 0 & f_1 & f_2 & \cdots & f_{b-1} \\ & \cdots & & & \\ 0 & \cdots & 0 & \cdots & f_1 \end{pmatrix}$$

such that $v(f_1) = b$, $v(g_1) = a$ and $v(f_i) \geq b + 1 - i$ and $v(g_i) \geq a + 1 - i$ for $i = 1, \dots, b$. Then $v(m) \geq a(b - a) + \frac{a(a+1)}{2}$ for every minor m of maximal order of M .

Proof. As g_1 divides f_1, \dots, f_{b-a+1} , we can performe a row reduction on M obtaining a matrix of the following type (where we have deleted a final block of a rows containing only zero entries):

$$M' = \left(\begin{array}{ccc|ccc} g_1 & g_2 & \cdots & & \cdots & \cdots & g_b \\ & \cdots & \cdots & & & & \cdots \\ 0 & 0 & g_1 & & \cdots & & g_{a+1} \\ \hline 0 & \cdots & 0 & g_1 & \cdots & 0 & g_a \\ & \cdots & & & \cdots & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & g_1 \\ 0 & \cdots & 0 & h_{b-a+1} & \cdots & \cdots & h_b \\ & \cdots & \cdots & \cdots & & \cdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & h_{b-a+1} \end{array} \right)$$

Note that the statement holds for M if and only if it holds for M' . Every $b \times b$ minor of M' is the product of the only non-zero $(b - a) \times (b - a)$ minor extracted from the first $b - a$ columns, which is g_1^{b-a} (whose valuation is $a(b - a)$) and an $a \times a$ minor extracted from the lower right block M'' . All the entries of the i -column of M'' have valuation $\geq a + 1 - i$ and then every $a \times a$ minor of M'' has valuation $\geq \frac{a(a+1)}{2}$. \square \square

Remark 3.6. *We computed several examples and we have always found that the multiplicity of the line AP in $\mathcal{C}(B, C)$ satisfies a stronger inequality than the one given in 4. above, namely it is always at least the product $e\eta$ of the multiplicities at P of the curves B and C . We also have some theoretical justification for this fact, but not a proof.*

4 The incidence surface $W_B = \mathcal{C}(B, -)$

The definition just given using resultants is applicable to any pair of curves, gives explicitly the equation of the conchoidal transform and allows to prove some interesting consequences. However, it is hard in general to obtain geometrical properties from the equation alone. Hence we now present a different characterization of the conchoid of two curves, using a more geometrical approach. In this construction the curves B and C will play different roles and the conchoidal transform will appear as obtained from a fixed curve B acting over a general curve C .

The definition will use a surface W_B obtained from the curve B . In this section we define W_B and study its properties. In the next section we will use it to define the conchoid of C . The geometrical construction makes sense only if B is generic enough, so we start by fixing the hypotheses on B .

Assumption 4.1. *B will always be a smooth curve in \mathbf{P}^2 of degree d and genus g (so that $g = 1/2(d-1)(d-2)$), defined by the equation $F(x, y, z) = 0$.*

We also assume that B intersects the fixed line L_∞ in d distinct points P_i , it does not contain the fixed point A and intersects every line through A in at least $(d-1)$ distinct points (i.e., no line through A is a multitangent to B or a flex tangent).

We will denote by L_i the d lines AP_i and by D_j the $d(d-1)$ lines through A that are tangent to B : we do not exclude that $L_i = D_j$ for some i and j may hold.

Finally we will denote by B_- the curve given by $F(-x, -y, z) = 0$, that is the curve whose affine part is symmetric to $B^{(a)}$ with respect to A .

Let us consider the subset of $\mathbf{P}^2 \times \mathbf{P}^2$ containing all the pairs of points (P, Q) that satisfy the equivalent conditions given in Lemma 2.1 and denote by W_B its closure (with respect to the Zariski topology). We can write the equations for the affine part of W_B using condition 2 as follows.

Let $P = [x : y : z]$ and $Q = [X : Y : Z]$ two points not lying on L_∞ . Then $(P, Q) \in W_B$ if and only if $xY - yX = 0$ and $F(zX - xZ, zY - yZ, zZ) = 0$. The first equation corresponds to “ A, P, Q collinear” and the second one to “ $Q - P \in B^{(a)}$ ”. In fact, in the affine open set $z \neq 0, Z \neq 0$ the point $Q - P$ is given by $(\frac{X}{Z} - \frac{x}{z}, \frac{Y}{Z} - \frac{y}{z})$ and so in \mathbf{P}^2 the corresponding point is $[\frac{X}{Z} - \frac{x}{z} : \frac{Y}{Z} - \frac{y}{z} : 1]$ that is $[zX - xZ : zY - yZ : zZ]$.

This computation justifies the following definition in projective coordinates:

Definition 4.2. *In the product of projective planes $\mathbf{P}^2 \times \mathbf{P}^2$ with bihomogeneous coordinates $[x : y : z; X : Y : Z]$, the incidence surface with respect to B is the subvariety W_B defined by the bihomogeneous ideal*

$$I = (F(zX - xZ, zY - yZ, zZ), xY - yX). \quad (2)$$

We will denote by $\pi_1 : W_B \rightarrow \mathbf{P}^2$ e $\pi_2 : W_B \rightarrow \mathbf{P}^2$ the projections on the first and on the second factor.

In a similar way as before, we will use $W_B^{(a)}$ for the affine part of W_B , i.e., $W_B^{(a)}$ is given by the equations $F(X - x, Y - y, 1) = xY - yX = 0$ in the affine space \mathbb{A}^4 given by $z \neq 0, Z \neq 0$.

We note that the subscheme defined by the ideal I could be reducible or non reduced. However we will prove in the next proposition that under Assumption 4.1 W_B is indeed irreducible and reduced, so it is a variety.

Proposition 4.3. *In the above notation:*

1. π_1 e π_2 are surjective;
2. the involution σ of $\mathbf{P}^2 \times \mathbf{P}^2$ given by $(P, Q) \mapsto (Q, P)$ restricts to an isomorphism $W_B \cong W_{B_-}$. Moreover $\pi_1 \circ \sigma = \pi_2$, $\pi_2 \circ \sigma = \pi_1$;

3. the affine part $W_B^{(a)}$ of W_B is a product (though in a non-standard way). More precisely:

$$W_B^{(a)} \cong B^{(a)} \times \mathbb{A}^1$$

(but $W_B^{(a)} \not\cong \pi_1(W_B^{(a)}) \times \pi_2(W_B^{(a)})$);

4. W_B is an irreducible and reduced surface and its affine part $W_B^{(a)}$ is smooth.

Proof. 1. By Assumption 4.1 a general line through A in \mathbb{A}^2 meets the affine curve $B^{(a)}$ in d points. So it is an easy consequence of condition 2. of Lemma 2.1 that for a general point Q on such a line there is a point P such that the condition holds for (P, Q) (and viceversa, for a general P there is at least a Q). Then the image of π_1 (or π_2) is a dense subset of \mathbf{P}^2 . As it is also closed, it must be the whole \mathbf{P}^2 .

2. The isomorphism between W_B and W_{B_-} given by the involution σ directly follows from condition 2. of Lemma 2.1, because $Q - P \in B^{(a)}$ if and only if $P - Q \in B_-^{(a)}$. In the same way we can see that σ exchanges π_1 and π_2 on the affine subsets. Finally the relations obtained on the affine subset can be extended to the projective closure, because σ is also an involution of $(\mathbf{P}^2 \times \mathbf{P}^2) \setminus \mathbb{A}^4$.

3. In the open subset $Z = z = 1$, the affine coordinates are (x, y, X, Y) . The equations defining $W_B^{(a)}$ are $F(X - x, Y - y, 1) = xY - yX = 0$. With the change of coordinates $x' = X - x$, $y' = Y - y$ these equations become $F(x', y', 1) = x'y - y'x = 0$. Thanks to the hypothesis $A \notin B$, all the solutions can be written as $(a, b, \lambda a, \lambda b)$ where $[a : b : 1] \in B^{(a)}$ (and so $(a, b) \neq (0, 0)$). Clearly $W^{(a)} \cong B^{(a)} \times \mathbb{A}^1$.

Finally, 4. is a straightforward consequence of the previous item, because W_B is the closure of $W_B^{(a)}$ in $\mathbf{P}^2 \times \mathbf{P}^2$. □ □

We now investigate the singular locus of W_B , that must be contained in the part at infinity $W_B \setminus W_B^{(a)}$ because of the previous result. Here we will use the hypothesis that either $\text{char}(k) = 0$ or $\text{char}(k) = p$ greater than the degree d of B .

Proposition 4.4. *If $d = 1$, i.e., B is a line, then W_B is smooth. If $d \geq 2$, the singular locus of W_B is the subvariety $W_B \cap (L_\infty \times L_\infty)$ cut by $z = Z = 0$. More precisely, every point in $W_B \cap (L_\infty \times L_\infty)$ has multiplicity at least d .*

Proof. If (P, Q) belongs to the locally closed subset of W_B where $Z = 0$ and $z \neq 0$, then it has coordinates of type $[\lambda a : \lambda b : c; a : b : 0]$ for some λ, a, b such that $\lambda \neq 0$ and either $a \neq 0$ or $b \neq 0$. If for instance $a \neq 0$ we can choose $a = 1$ and consider (P, Q) as a point in the affine 4-space given by $z = X = 1$ and with coordinates (x, y, Y, Z) . In these coordinates, the equations of W_B are

$$F(1 - xZ, Y - yZ, Z) = xY - y = 0.$$

We use F_x , F_y and F_z to denote the derivatives of $F(x, y, z)$ with respect to its original variables. Computing the derivatives with respect to the variables (x, y, Y, Z) using the chain rule and evaluating in the point $(P, Q) = [\lambda : \lambda b : 1; 1 : b : 0]$, the Jacobian matrix of the equations of W_B is:

$$\begin{bmatrix} 0 & 0 & F_y(1, b, 0) & F_z(1, b, 0) \\ b & -1 & \lambda & 0 \end{bmatrix}$$

and has rank 2 because $[1, b, 0] \in B$ and B is smooth. We observe that the last item in the first row should be $-\lambda[F_x(1, b, 0) + bF_y(1, b, 0)] + F_z(1, b, 0)$, but the quantity in square brackets vanishes: in fact by the Euler relation it becomes $F(1, b, 0)$ and $[1 : b : 0] \in B$. So we can conclude that (P, Q) is a smooth point. In the same way we can prove the smoothness of every point in the subset of W_B given by $Z = 0$, $z \neq 0$ and $Y \neq 0$.

The same holds if $z = 0$ and $Z \neq 0$, thanks to the symmetry between W_B and W_{B_-} .

If $Z = z = 0$ then $(P, Q) = [a : b : 0; a : b : 0]$ and either a or b does not vanish. If for instance $x = X = 1$, the entries of the first row of the Jacobian matrix (with coordinates (y, z, Y, Z)) are homogeneous polynomials of degree $d - 1$ with respect to the variables $zX - Zx, zY - Zy, zZ$. If we evaluate the Jacobian matrix in (P, Q) , that is if we set $y = Y = b$ and $z = Z = 0$, then its rank is not maximal if and only if $d \geq 2$. Moreover, the rank is not maximal also if we consider the higher derivatives up to the $(d - 1)$ -th one. Then (P, Q) has multiplicity at least d . \square \square

We study now the properties of the fibers of the projection π_1 . We refer to the beginning of this section for the meaning of D_j , P_i and L_i .

The fibers of π_2 will have the same properties. In fact π_2 can be seen as the first projection from the incidence surface W_{B_-} . Note that B and its symmetric curve B_- share the same tangent lines through A and the same intersection points with the line at infinity L_∞ .

Proposition 4.5. *Let P be any point in \mathbf{P}^2 .*

1. *If P is general (more precisely if it is not one of the points considered in the following items), then $\pi_1^{-1}(P)$ is a set of $d = \deg(B)$ distinct points;*
2. *if $P \in D_j \setminus L_\infty$, then $\pi_1^{-1}(P)$ is given by $d - 1$ distinct points (exactly one of which with multiplicity 2);*
3. *$\pi_1^{-1}(A)$ is the curve Γ in $\mathbf{P}^2 \times \mathbf{P}^2$ of the points (A, Q) such that $F(Q) = 0$, so that in a natural way $\Gamma \cong \pi_2(\Gamma) = B$;*
4. *if $P \in L_\infty \setminus B$, then $\pi_1^{-1}(P)$ is a single point with multiplicity d ;*
5. *if $P = P_i[a_i : b_i : 0] \in L_\infty \cap B$, then $\pi_1^{-1}(P_i)$ is the rational curve Λ_i of the points $[a_i : b_i : 0; \lambda a_i : \lambda b_i : \mu]$, so that $\Lambda_i \cong \pi_2(\Lambda_i) = L_i$.*

Proof. If P is a point in \mathbb{A}^2 , then $\pi_1^{-1}(P)$ can be obtained by first intersecting the line AP with B and then translating by the vector \vec{AP} : this proves 1. and 2. Statement 3. is the case when $P = A$ and easily follows from the equations (2) of W_B .

So it remains to prove the last two items. If $P = [a : b : 0]$, looking at the second equation in (2) we can see that every point Q in $\pi_1^{-1}(P)$ is of the type $Q = [\lambda a : \lambda b : Z]$. If we evaluate the first equation in (P, Q) , we obtain $F(-aZ, -bZ, 0) = 0$ that is $(-Z)^d F(a, b, 0) = 0$. There are two possibilities. If $P \notin B$, then $Z = 0$ and $\pi_1^{-1}(P) = \{[a : b : 0; a : b : 0]\}$ contains a single point $Q = [a : b : 0]$ with multiplicity d . If, on the other hand, $P \in B$, then all values for Z are possible and $\pi_1^{-1}(P)$ is a rational curve with parametric equations $[a : b : 0; \lambda a : \lambda b : \mu]$ in the homogeneous parameters $[\lambda : \mu]$. \square \square

We collect in the following corollary the main results obtained until now.

Corollary 4.6. W_B is a surface in $\mathbf{P}^2 \times \mathbf{P}^2$, that is a reduced and irreducible subvariety of dimension 2. If the degree $d = \deg(B) \geq 2$, its singular locus is the curve given by $Z = z = xY - yX = 0$ and every singular point is d -uple.

The projection $\pi_1 : W_B \rightarrow \mathbf{P}^2$ is a generically finite map of degree d , branched over the $d(d-1)$ lines D_j , i.e., the lines containing A and tangent to B . The exceptional fibers are the one over A , which is the curve Γ , isomorphic to B through π_2 , and those over the d points $P_i \in B \cap L_\infty$, which are the rational curves Λ_i , isomorphic (through π_2) to the lines $L_i = AP_i$.

5 The conchoid of C obtained from W_B

If C is a reduced curve and does not contain any special points (namely A and $P_i \in B \cap L_\infty$), then the curve $\pi_2(\pi_1^{-1}(C))$ is well defined. Thanks to the equivalent conditions of Lemma 2.1, we can easily see that the curve $\pi_2(\pi_1^{-1}(C))$ is precisely the conchoidal transform $\mathcal{C}(B, C)$ defined in Section 3. However, if either C is non reduced or it contains some of the special points or some of the special divisors, the curve $\mathcal{C}(B, C)$ can have some non reduced components and also some components that are in some sense *special components*. This is a very common difficulty in algebraic geometry, when exceptional fibers of morphisms are involved. Similar to the definition of proper transform for a blowing-up morphism, we would like to define a *proper conchoid*, not containing exceptional fibers of the transformation. To this end, we give a new definition of conchoid in a geometric way. We will prove that this definition is equivalent to the previous one, but in it the two starting curves B and C play different roles. More explicitly, for every B and C we will obtain not only a curve $\mathcal{C}_B(C)$, but also a set of exceptional divisors: the curve $\mathcal{C}_B(C)$ is the same as $\mathcal{C}(B, C)$, but the exceptional divisors will depend only on B , so that they are in general a different set from that of $\mathcal{C}_C(B)$. Removing the exceptional divisors, we will finally obtain the definition of the proper conchoid (Definition 5.5).

We recall the definitions from algebraic geometry of pull-back and push-forward of cycles. To keep things simple we state them only for the special case we need. For more information on the general notions, see [3], Chapter 1 or [4], Appendix A.

Let W be a projective variety and $\phi : W \rightarrow \mathbf{P}^2$ a surjective morphism. A k -cycle is a formal linear combination with integer coefficients of reduced and irreducible subvarieties of dimension k . For C a reduced and irreducible subvariety in W , we define the push-forward $\phi_*(C)$ to be 0 if the dimension of the image $\phi(C)$ is strictly less than $\dim C$, otherwise we set

$$\phi_*(C) = m \cdot \phi(C)$$

where m is the *degree* of the map $\phi|_C$, i.e., the number of points in the generic fiber. We extend ϕ_* to all cycles by linearity. We note that the push-forward of a k -cycle is again a k -cycle.

We define the pull-back only for divisors, i.e., cycles of codimension 1. If D is a reduced and irreducible curve in \mathbf{P}^2 , it is given by a single equation $G = 0$. The pull-back $\phi^*(D)$ is given by “pulling back” this equation to W . To make sense of this we cover \mathbf{P}^2 with open affine sets $\{U_i\}$, for example the standard ones obtained by dehomogenizing one variable at the time, and we set g_i the corresponding inhomogeneous equation of C on U_i . Then the $W_i = \phi^{-1}(U_i)$ are an affine open cover of W and we let $\phi_i : W_i \rightarrow U_i$ be the restriction of ϕ

to W_i . Then $\phi^*(D)$ is the divisor with local equation $g_i \circ \phi_i = 0$ in the open set W_i . We note that even if D is reduced, $\phi^*(D)$ may have multiple components since the differential of ϕ may not have maximal rank everywhere. Again we extend ϕ^* to all divisors by linearity.

We can now give the:

Definition 5.1. *If C is a curve of \mathbf{P}^2 , that is a 1-cycle in \mathbf{P}^2 , we will call conchoid of C (with respect to B) the cycle $\mathcal{C}_B(C) = \pi_{2*}(\pi_1^*(C))$.*

If C is reduced and does not contain any special point or divisor for π_1 and π_2 , then $\mathcal{C}_B(C)$ is precisely $\pi_2(\pi_1^{-1}(C))$. We can obtain an equation for its affine part $(\mathcal{C}_B(C))^{(a)}$ after elimination of the variables x, y from the ideal:

$$I = (F(X - x, Y - y, 1), xY - yX, G(x, y, 1)). \quad (3)$$

In all the other cases, we can consider a flat family of curves C_t depending on one or more parameters t such that $C_{t_0} = C$ and, for a general t , C_t is of the previous type. The conchoid of C is the limit of $\mathcal{C}_B(C_t)$ for $t = t_0$.

We can for instance consider the family C_t of all degree δ curves whose equation is a degree δ polynomial with indeterminate coefficients. Then we can formally perform the elimination of the variables x, y and, at the end, specialize t .

It can also be useful to think of the general degree δ polynomial as an element of a vector space generated by all the products of δ linear forms, corresponding to curves split in lines.

Example 5.2. *Let us consider the classical case, when B is the circle $x^2 + y^2 - z^2 = 0$ and $A = [0 : 0 : 1]$. The conchoid of a general line $ax + by + cz = 0$, obtained as just indicated, is given by the equation $(aX + bY + cZ)^2(X^2 + Y^2) - (aX + bY)^2Z^2 = 0$. If we specialize the coefficients a, b, c in order to obtain the conchoid of the line L of equation $x = 0$ (which contains A), we get $X^2(X^2 + Y^2 - Z^2) = 0$, i.e., the divisor $2L + B$ that has degree 4. If instead we eliminate, for example using a Gröbner basis computation, the variables (x, y) from the ideal I in (3) above, the resulting curve has (affine) equation $X(X^2 + Y^2 - 1) = 0$, i.e., it is $L + B$. So we see that specialization does not commute with elimination.*

We can also obtain the conchoid of the infinity line L_∞ : its equation is $Z^2(X^2 + Y^2) = 0$, i.e., the conchoid is $2L_\infty + L_1 + L_2$.

We now state and prove the main result for B and C generic.

Theorem 5.3. *Let B be a curve as in Assumption 4.1, and let C be a generic curve of degree δ and geometric genus γ . Then:*

1. $\mathcal{C}_B(C) = \mathcal{C}(B, C)$.
2. $\mathcal{C}_B(C)$ is irreducible and reduced;
3. $\mathcal{C}_B(C)$ is birational to $\pi_1^{-1}(C)$ (via π_2);
4. $\mathcal{C}_B(C)$ has genus

$$\tilde{g} = d\gamma + \delta g + (d - 1)(\delta - 1);$$

5. $\mathcal{C}_B(C)$ goes through the origin A with multiplicity δd ; the tangent cone in the origin is the union of the lines joining A to the δd points of $B \cap C_-$;

6. $\mathcal{C}_B(C)$ meets the line L_∞ in the points at infinity of B with multiplicity δ and in the points at infinity of C with multiplicity d .

Proof. Let $f_\delta : \mathbf{P}^2 \rightarrow \mathbf{P}^N$ be the δ -uple embedding of \mathbf{P}^2 and $f = f_\delta \circ \pi_1 : W_B \rightarrow \mathbf{P}^N$. Since W_B is an irreducible surface and π_1 is surjective, we can apply Bertini's Theorem (see [6, Theorem 3.3.1, pg. 207]) to obtain that $f^{-1}(H)$ is reduced and irreducible for a general hyperplane $H \subseteq \mathbf{P}^N$. By definition of f_δ , a generic plane curve C of degree δ is the inverse image of a generic hyperplane of \mathbf{P}^N and hence $\pi_1^{-1}(C) = f^{-1}(H)$ is reduced and irreducible for C generic.

As the image of an irreducible subvariety is irreducible we obtain that

$$\mathcal{C}_B(C) = \pi_{2*}(\pi_1^*(C)) = mD$$

where D is a reduced and irreducible plane curve, and m is the degree of $\pi_2|_{\pi_1^{-1}(C)}$. We note that $m \leq d = \text{degree of } \pi_2 = \text{degree of } B$.

We now show that the cycle $\mathcal{C}_B(C)$ has degree $2d\delta$. The degree is the homology class of the cycle $\mathcal{C}_B(C) = \pi_{2*}(\pi_1^*(C))$ in $H^2(\mathbf{P}^2, \mathbb{Z}) \cong \mathbb{Z}$, where π_1 and π_2 are the restrictions to W_B of the projections p_1 and p_2 defined on $\mathbf{P}^2 \times \mathbf{P}^2$. Let H be the class of a line in \mathbf{P}^2 and let p be the class of a point. The homology module of $\mathbf{P}^2 \times \mathbf{P}^2$ is free with generators:

$$\begin{aligned} A_1 &= H \times \mathbf{P}^2 & A_2 &= \mathbf{P}^2 \times H \\ a &= p \times \mathbf{P}^2 & b &= H \times H & c &= \mathbf{P}^2 \times p \\ \alpha &= p \times H & \beta &= H \times p \\ \gamma &= p \times p \end{aligned}$$

As homology classes we have:

$$\pi_{2*}(\pi_1^*(C)) = p_{2*}((p_1^*(C) \cdot W))$$

W is a surface, complete intersection of two hypersurfaces of bidegree $(1, 1)$ and (d, d) and hence its homology class is:

$$[W] = (A_1 + A_2) \cdot (dA_1 + dA_2) = d(A_1 + A_2)^2 = d(a + 2b + c)$$

C is a plane curve of degree δ and hence $[C] = \delta H$. Then $p_1^*(C) = \delta H \times \mathbf{P}^2 = \delta A$. Intersecting with W we get

$$p_1^*(C) \cdot W = d\delta A \cdot (a + 2b + c) = d\delta(2\alpha + \beta)$$

We have $p_{2*}(\beta) = 0$ since the image of β is a point, while $p_{2*}(\alpha) = H$. We conclude

$$[\pi_{2*}(\pi_1^*(C))] = p_{2*}((p_1^*(C) \cdot W)) = 2d\delta H \in H^2(\mathbf{P}^2, \mathbb{Z}).$$

We now let $B = L$ a line not through the point A and let C be reduced and irreducible, not through A and such that $B \cap C \cap L_\infty = \emptyset$. Since $m \leq d = 1$, we have that

$$\mathcal{C}_L(C) = D$$

is reduced and irreducible and has the same degree as $\mathcal{C}(L, C)$. As the affine parts of these two curves have the same support, we conclude that they are equal as cycles. In particular, $\mathcal{C}(L, C)$ is reduced and irreducible for C generic.

Since the definition via resultants is symmetric, taking B as in Assumption 4.1 and L a generic line, we have that $\mathcal{C}(B, L)$ is reduced and irreducible, and by Theorem 3.4 $\mathcal{C}(B, C)$ is reduced when C is the union of δ distinct generic lines. The property of being reduced is an open property, since one can detect multiple roots with the vanishing of system of resultants (the discriminants), and hence we have that $\mathcal{C}(B, C)$ is reduced for C generic.

Consider again $\mathcal{C}_B(C)$ and $\mathcal{C}(B, C)$ for C generic: their affine parts have the same support, $\mathcal{C}(B, C)$ is reduced, $\mathcal{C}_B(C) = mD$ has only one irreducible component, and they have the same degree as cycles. We conclude that they are equal as cycles, and hence they are both reduced and irreducible. Moreover, $m = 1$ and so $\pi_2|_{\pi_1^{-1}(C)}$ is a birational map from $\pi_1^{-1}(C)$ to $\mathcal{C}_B(C)$. This proves 1., 2., and 3.

4. By what we have just proved, it is enough to compute the genus of $\pi_1^{-1}(C) = \tilde{C}$. The map $\pi_1 : \tilde{C} \rightarrow C$ is a covering of degree d , ramified over the points where C meets the ramification of π_1 , i.e., the $d(d-1)$ lines through A tangent to B . By our assumption on B the ramification index is 1 for all these points, and so the Riemann-Hurwitz formula gives:

$$2\tilde{g} - 2 = d(2\gamma - 2) + \delta d(d-1);$$

since B is smooth of degree d , its genus g equals $\frac{(d-1)(d-2)}{2}$, from which the thesis follows. We write the genus formula in this way to point out once again the symmetry between C and B .

5. and 6. now follow from what we have proved, and the fact that they are true when C is a generic line (see Example 3.3). In fact, if C is the line of equation $ax + by + cz = 0$, $\mathcal{C}_B(C)$ has equation

$$F(x(ax + by + cz), y(ax + by + cz), (ax + by)z) = 0$$

Setting $z = 1$, the homogeneous part of minimum degree is $F(cx, cy, ax + by)$ and setting this to zero gives the tangent cone in the origin. Hence we see that the tangent cone is given by the lines through the origin and the points of intersection of B and the line $ax + by - cz = 0$, which is C_- . □ □

Corollary 5.4. *Let B be a curve as in Assumption 4.1. Then for every curve C we have $\mathcal{C}(B, C) = \mathcal{C}_B(C)$. Moreover 5. and 6. of Theorem 5.3 still hold, with the multiplicities greater than or equal to the ones given (instead of just equal).*

Proof. Denote with $G_{\mathbf{t}}(x, y, z)$ the generic form of degree δ in the indeterminates x, y, z , denote with \mathbf{t} its coefficients which we take as indeterminates, and let $C_{\mathbf{t}}$ be the corresponding curve. Using the definition via resultants, we can determine $\mathcal{C}(B, C_{\mathbf{t}})$: as a function of the variables \mathbf{t} it is given by a polynomial $R_{\mathbf{t}}$.

Also $\mathcal{C}_B(C_{\mathbf{t}})$ is given by a polynomial function of the variables \mathbf{t} , and the two polynomials must coincide up to a constant factor since the curves obtained by generic specialization of \mathbf{t} coincide thanks to Theorem 5.3. Hence the two curves $\mathcal{C}(B, C)$ and $\mathcal{C}_B(C)$ coincide for all choices of C .

Since also 5. and 6. are given by properties of the polynomials defining $\mathcal{C}(B, C_{\mathbf{t}})$ and $\mathcal{C}_B(C_{\mathbf{t}})$, the same reasoning shows that they hold in general, as inequalities, by semicontinuity in \mathbf{t} . □ □

We note that if the surface W_B is reducible our proof does not work since we cannot use Bertini's theorem. Our Assumption 4.1 is used not only to ensure the irreducibility of W_B , but also to obtain the genus formula given in the previous theorem.

We want to emphasize that specialization of parameters does not commute with elimination of variables, as we have seen in Example 5.2, while it commutes with the resultants. Our geometric definition coincides with the one given by resultants, and so it commutes with specialization.

Recall the definition and the properties of the divisors Γ and Λ_i in W_B that we will consider as *special*. Γ is the divisor $\pi_1^{-1}(A)$ and $\Lambda_i = \pi_1^{-1}(P_i)$ where $P_i \in B \cap L_\infty$. We have $\pi_2(\pi_1^{-1}(A)) = \pi_2(\Gamma) = B$ and $\pi_2(\Lambda_i) = L_i$.

Definition 5.5. *Let C be a curve. The proper conchoid of C with respect to B is the curve $\tilde{C}_B(C)$ that does not have B and the L_i 's as components and such that $\mathcal{C}_B(C) = aB + \sum_i b_i L_i + \tilde{C}_B(C)$.*

By what has been proved, the integers a, b_i are greater than or equal to the multiplicities of C in the points A, P_i ; they are strictly greater if the tangent cone to C in one of these points contains one of the lines L_i .

Remark 5.6. *This definition has one drawback: it always eliminates the curve B from the conchoid of another curve, even when B should be considered as a non-exceptional component. For example, if B is the circle with center A and radius 1 and C is the circle with the same center A and radius 2, B should be considered a non-exceptional component of the conchoid of C with respect to B , since C does not go through A .*

6 The classical case: W_B and double planes

We want now to apply our results in the classical case, i.e., when B is a circle with center A . In this case we show that W_B is the blow-up in three points of a ramified double cover of \mathbf{P}^2 . The geometry of these surfaces, classically known as *double planes*, is well-known and this will allow us to determine sufficient conditions on the curve C so that its conchoid is irreducible.

The same approach could be followed for curves B of any degree d . W_B is again a blow-up of a ramified cover of \mathbf{P}^2 of degree d , but in this case the geometry of multiple covers is much less known, and little can be said in general.

For a clear exposition in modern language of the classical theory of double planes see for instance the paper by Sernesi [9]. In particular, in that paper one can find necessary and sometimes sufficient conditions on a curve in \mathbf{P}^2 so that its pullback to the double cover is reducible. Stated loosely, the condition is that the curve must be everywhere tangent to the branch locus. We do not use directly this, since it requires that the branch locus is smooth and the curve generic and in our case the branch locus is a pair of lines. However, the statement turns out to be true for the particular double plane we are interested in and valid for all irreducible curves as we will prove.

Let B be a circle with center A or, more generally, a conic with center in A . There are two points P_1 and P_2 in $B \cap L_\infty$ and hence two lines L_1 and L_2 . These lines are also the tangents to B passing through A , previously denoted D_i , since the center of the conic is the pole of the line at infinity.

Let D be the cycle $L_1 + L_2 + 2L_\infty$ in \mathbf{P}^2 , i.e., the curve (reducible and not reduced) with equation $\ell_1 \ell_2 z^2 = 0$, E the double plane branched over D and $p : E \rightarrow \mathbf{P}^2$ the corresponding finite morphism of degree 2.

A point on the surface E is singular if and only if its image under p is a singular point of D , and in this case it is a double point on E . Since D has a multiple component, E is not normal and has a curve of double points that projects onto L_∞ . Moreover, $\tilde{A} = p^{-1}(A)$ is an ordinary double point of E .

Let $n : F \rightarrow E$ be the normalization morphism (see, e.g., [4, I, Exercise 3.17]): the composition $q = p \circ n : F \rightarrow \mathbf{P}^2$ is a double plane branched over the divisor $L_1 + L_2$, and hence F is a quadric cone. For a proof of all of the preceding assertions, see [9], especially Teorema page 19 and Esempio page 8 and 9.

It follows that E is obtained from a quadric cone by identifying two rational curves (that are not lines on the cone, since they project onto the line $z = 0$). In particular, we obtain an isomorphism between the open sets $\pi_1^{-1}(\mathbb{A}^2 \setminus \{0\})$ of W_B and $q^{-1}(\mathbb{A}^2 \setminus \{0\})$ of F .

We summarize the construction in the following diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{n} & E & \xleftarrow{f} & W_B \\
 & \searrow q & \searrow p & \swarrow \pi_1 & \searrow \pi_2 \\
 & & \mathbf{P}^2 & & \mathbf{P}^2
 \end{array} \tag{4}$$

where f is the blow-up of E in \tilde{A} and the two points over P_1 and P_2 , the points at infinity of L_1 and L_2 , as can be seen from the description of the geometry of W_B given in Proposition 4.5.

So the proper conchoid $\tilde{C}_B(C)$ of Definition 5.5 is birational to the corresponding proper transform in F . We already proved that a generic irreducible curve has irreducible conchoid (and hence irreducible proper conchoid). Using this description via double planes we can now characterize completely the curves whose proper conchoid is irreducible.

Theorem 6.1. *Let C be a reduced and irreducible curve in \mathbf{P}^2 of degree δ with equation $G(x, y, z) = 0$. Then $\tilde{C}_B(C)$ is reducible if and only if there exist homogeneous polynomials H_1 and H_2 such that:*

$$1. \quad G = H_1^2 - \ell_1 \ell_2 H_2^2 \quad \text{if } \delta \text{ is even}$$

or

$$2. \quad G = \ell_1 H_1^2 - \ell_2 H_2^2 \quad \text{if } \delta \text{ is odd}$$

where ℓ_1 and ℓ_2 are the equations of the lines L_1 and L_2 respectively.

Proof. By Diagram (4) $\tilde{C}_B(C)$ is irreducible if and only if $q^{-1}(C)$ is. So it is enough to prove the claim for $q^{-1}(C)$, and even restrict ourselves to the affine case. Identifying F with the quadric cone of equation $\ell_1 \ell_2 - t^2$ in \mathbb{A}^3 , the map q is the projection onto the plane \mathbb{A}^2 given by $t = 0$. The quadric cone is normal and the degree map is an isomorphism from its divisor class group to $\mathbb{Z}/2\mathbb{Z}$. In particular a divisor is a complete intersection if and only if it has even degree (see [4], Ch. I, Exercise 3.17 and Ch. II, Example 6.5.2).

If C has equation $G(x, y) = 0$, then $q^{-1}(C)$ is given by the intersection of the cone with the hypersurface in \mathbb{A}^3 of equation $G(x, y) = 0$. If this divisor on the cone is reducible then it has exactly two components, since q is a finite map of degree two. If Y_1 is one of the

components, the other component Y_2 is obtained using the involution $t \mapsto -t$ on the cone. Hence both components have degree δ .

If δ is even, then Y_1 is a complete intersection of the cone with a hypersurface $H(x, y, t)$ of degree $\delta/2$, and since we are taking intersection with the cone $t^2 = \ell_1 \ell_2$, we can assume that $H = H_1 + tH_2$, where $H_1, H_2 \in k[x, y]$. In this case, $H_1 - tH_2$ gives the other component Y_2 and $H_1^2 - t^2 H_2^2$ cuts on the cone the sum $Y_1 + Y_2 = q^{-1}(C)$. Using again the equation $t^2 = \ell_1 \ell_2$ of the cone, we see that $H_1^2 - \ell_1 \ell_2 H_2^2$ cuts the same divisor $Y_1 + Y_2$. Finally, $H_1^2 - \ell_1 \ell_2 H_2^2 = 0$ as a curve in \mathbb{A}^2 contains the curve C and has the same degree, and hence coincide with C . So the equation $G(x, y)$ of C is as claimed.

The proof is similar in the case δ odd. The divisor Y_1 is not principal, and we consider the principal divisor $Y_1 + L$, where L is the line $\ell_1 = t = 0$. As before, $Y_1 + L$ is cut on the cone by a hypersurface that in this case will be of the form $\ell_1 H_1 + tH_2$ with $H_1, H_2 \in k[x, y]$ because it must vanish on L . Then $Y_2 + L$ is cut by $\ell_1 H_1 - tH_2$, $Y_1 + Y_2 + 2L$ is cut by $\ell_1^2 H_1^2 - \ell_1 \ell_2 H_2^2$ and finally $Y_1 + Y_2$ by $\ell_1 H_1^2 - \ell_2 H_2^2$ as ℓ_1 cuts $2L$ on the cone. $\square \quad \square$

Example 6.2. Let B be the circle with center A and any radius. By the previous theorem, a conic C has reducible conchoid if and only if C has equation $\ell^2 - \ell_1 \ell_2 = 0$, where ℓ has degree 1. Notice that ℓ is the equation of the polar line of A with respect to C . Since B is a circle, the points P_1 and P_2 are the cyclic points. Then ℓ_1 and ℓ_2 are two tangents to the conic C from the cyclic points and so their intersection A is a focus of C (by definition of focus, see, e.g., [1, pag. 171]). Note that a complex conic which is not a parabola has 4 foci, and if the conic is real only two of them are real. These are the usual foci of ellipses and hyperbolas. Since a parabola is tangent to the line at infinity, it has always only 1 focus. We conclude that a conic has reducible conchoid if and only if A is one of the foci of C . For instance if B is the circle with center A and radius 1 and C is the parabola $(y+z)^2 - (x^2 + y^2)$, then $\tilde{C}_B(C)$ is the union of the two quartics $x^4 + (y^2 - 2yz)x^2 - 2y^3z + y^2z^2 = 0$ and $x^4 + (y^2 - 2yz - 4z^2)x^2 - 2y^3z - 3y^2z^2 = 0$.

For a different proof and many explicit computations, see [8, Theorem 6].

We assume now that C is reducible. We can again ask if its proper conchoid is reducible or not. To answer this question we introduce the notion of *iterated conchoid*. We begin with an example.

Example 6.3. Let C be a generic line. Let $C_1 = \mathcal{C}_B(C)$ be its conchoid, which is again irreducible, and let us consider $\mathcal{C}_B(C_1)$. This is a divisor of degree 16, whose components are the circle B with multiplicity 2, the two lines L_1 and L_2 each with multiplicity 3, the line C with multiplicity 2 and an irreducible curve C_2 of degree 4 (to check this computation, take C the line of equation $x - hz = 0$ and use resultants). The curve C_2 is in fact the conchoid of C with respect to the circle B_2 with center A and radius twice that of B .

This behaviour is not special to the lines and we prove:

Proposition 6.4. Let C be a generic curve of degree δ , and let $C_1 = \mathcal{C}_B(C)$. Then the conchoid $\overline{C_2} = \mathcal{C}_B(C_1)$ is a divisor of degree 16δ , whose components are:

1. the circle B , with multiplicity 2δ ;
2. the two lines L_1 and L_2 each with multiplicity 3δ ;
3. the curve C with multiplicity 2 ;

4. a curve C_2 of degree 4δ , which is the conchoid of C with respect to the circle B_2 with center A and radius twice that of B .

Proof. As we did earlier, we can consider C as specialization of the curve C_t of degree δ with generic coefficients. The linear system C_t is generated by curves that are product of generic linear forms. Hence every curve $\mathcal{C}_B(\mathcal{C}_B(C_t))$, and so especially $\overline{C_2} := \mathcal{C}_B(\mathcal{C}_B(C))$, must contain B , L_1 and L_2 with at least multiplicity as stated.

Now, let us consider the affine part $\overline{C_2}^{(a)}$ of $\overline{C_2}$. It contains all the points Q of the form $Q = P' + (P + S)$, where $S \in C^{(a)}$ and $P', P \in B^{(a)}$ collinear with A (see Lemma 2.1). The intersection $B \cap AS$ consists of two points P_+ and P_- and hence there are two possibilities for P and two for P' . If either $P = P' = P_+$ or $P = P' = P_-$, the corresponding point Q belongs to B_2 and hence Q belongs to $\mathcal{C}_{B_2}(C)$. Since the total degree is 16δ the components appear with the stated multiplicity, and not higher, and their sum is the whole divisor $\mathcal{C}_B(C_1)$. \square \square

Definition 6.5. *The curve C_2 defined in the previous Proposition is called proper second conchoid of C .*

In this case we discard from $\mathcal{C}_B(C_1)$ not only the exceptional components, but also the curve C .

Remark 6.6. *We can define inductively the proper n -th conchoid C_n of C and see in the same way that it turns out to be the conchoid of C with respect to the circle B_n with center A and radius n times that of B . The infinitely many curves C_n belong to a 1-dimensional flat family. In fact $C_n = \mathcal{C}_{B_n}(C) = \mathcal{C}(B_n, C)$ can be obtained using the resultant $R(F_t, G)$, where $F_t = x^2 + y^2 - t^2 z^2$, and specializing the parameter t to n .*

Proposition 6.7. *The proper conchoid of a reducible curve is reducible.*

Proof. Let C be a reducible curve and $\Delta := \pi_1^{-1}(C) \setminus \{\text{exceptional components}\}$. C is reducible and so Δ has at least two components, since $\pi_1(\Delta) = C$ has a number of components less than or equal to that of Δ . Assume that the proper conchoid $\tilde{C}(C)$ is irreducible. As the map π_2 is generically $2 : 1$, $\pi_2^{-1}(\tilde{C}(C))$ has at most two components and since it contains Δ it must be $\pi_2^{-1}(\tilde{C}(C)) = \Delta$ and hence $\pi_1(\pi_2^{-1}(\tilde{C}(C))) = \pi_1(\Delta) = C$. Since $B = B_-$, the curve $\pi_1(\pi_2^{-1}(\tilde{C}(C)))$ contains the proper second conchoid of C and hence it has at least 3 irreducible non exceptional components, those of C and $\mathcal{C}_{B_2}(C)$. This contradiction proves our claim, since a component of C cannot be equal to $\mathcal{C}_{B_2}(C)$: in fact any curve different from L_∞ or an exceptional curve has only finitely many points in common with its conchoid. \square \square

We conclude giving a computational procedure to establish when an irreducible curve \mathcal{D} is either the conchoid or the proper conchoid of another curve C with respect to some point A (not necessarily the origin) and radius r , i.e., with respect to the circle B with equation $(x - a)^2 + (y - b)^2 - r^2 z^2 = 0$.

In order to decide if \mathcal{D} is a complete conchoid, we start by checking some obvious necessary conditions: first of all the degree must be a multiple of 4. If we set $\deg(\mathcal{D}) = 4\delta$, then \mathcal{D} must meet the line at infinity $z = 0$ in the two cyclic points $([1 : i : 0])$ and $([1 : -i : 0])$ with multiplicity at least δ and all the other points at infinity of \mathcal{D} must be at least double points. Hence, if $H(x, y, z) = 0$ is an equation defining \mathcal{D} , then $H(x, y, 0)$ must split as

$(x^2 + y^2)^\delta H_\delta(x, y)^2$. Moreover, there must be a point on \mathcal{D} (namely the point A) in the affine open set \mathbb{A}^2 with multiplicity at least 2δ .

When all these conditions are fulfilled, the distance r must be twice the distance between a pair of points on \mathcal{D} and collinear with A .

Hence the only possibilities for A and r are finite, and we can check all cases to see if the conchoid of \mathcal{D} with respect to the circle with center A and radius r contains a non-exceptional component with multiplicity 2: for what we proved above this component, if it exists, is a curve whose conchoid is \mathcal{D} .

In order to check if \mathcal{D} is the proper conchoid of a curve C we can use Theorem 6.1 and Proposition 6.4. Excluding the trivial case $\deg(\mathcal{D}) = 1$, a first necessary condition is the existence of the pair of lines ℓ_1, ℓ_2 , each containing a cyclic point, which are everywhere tangent to \mathcal{D} . If they exist and they meet in the affine subset \mathbb{A}^2 , their common point is A and, as above, the distance r must be twice the distance between a pair of points on \mathcal{D} and collinear with A . Hence there are finitely many possibilities for r and we can check all cases to see if the conchoid of \mathcal{D} with respect to the circle with center A and radius r splits as described in Proposition 6.4: if $\mathcal{D} = \mathcal{C}_B(C)$, the curve C is a non-exceptional component with multiplicity 2 of $\mathcal{C}_B(\mathcal{D})$.

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