

Approximating Persistent Homology in Euclidean Space Through Collapses

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Abstract

The Čech complex is one of the most widely used tools in applied algebraic topology. Unfortunately, due to the inclusive nature of the Čech filtration, the number of simplices grows exponentially in the number of input points. A practical consequence is that computations may have to terminate at smaller scales than what the application calls for.

In this paper we propose two methods to approximate the Čech persistence module. Both are constructed on the level of spaces, i.e. as sequences of simplicial complexes induced by nerves. We also show how the bottleneck distance between such persistence modules can be understood by how tightly they are sandwiched on the level of spaces. In turn, this implies the correctness of our approximation methods.

Finally, we implement our methods and apply them to some example point clouds in Euclidean space.

1 Introduction

Topological data analysis in general, and persistent homology in particular, have shown great promise as tools for analyzing real-world data arising in the sciences. Examples of successful applications range from image analysis [6, 26], to cancer research [1], virology [7] and sensor networks [13].

Central to persistent homology are standard constructions for recovering the homology of an underlying topological space from a finite sample set, chiefly the Čech and Vietoris–Rips complexes. Unfortunately, due to the inclusive nature of their filtrations, the number of simplices grows exponentially in the number of sample points. This may be unfortunate as simplices added at small scales may contribute little to homology at larger, possibly more interesting, scales.

An extreme example may be a constant region in a measurement signal (perhaps from faulty equipment or downtime) under time-delay embedding [27]. In such a case, a large proportion of the point cloud may lie in, say, a dense lump of N points that contributes nothing to the cloud’s overall homology, yet introduces $\binom{N}{k+1}$ k -simplices in the complex from an early scale.

Preprocessing of the point cloud may sometimes rectify the situation, but such schemes are often decidedly “off-line” in the sense that they require a one-off decision about which sparsifications to effectuate ahead of persistence computations. We propose more “on-line” methods wherein a decision to attempt a simplification of the simplicial complex may be made at any time during computations when it is deemed necessary. The simplification operation itself requires only that the point cloud comes supplied with

its complete linkage hierarchical clustering, which may be computed ahead of time once and for all, or the computation of nets.

1.1 Contributions

The well-known Nerve lemma [23] allows one to capture the topology of a continuous space using discrete structures. However, the lemma works under the assumption of a good cover, i.e. a cover wherein every finite intersection of covering sets is contractible. This means that whenever we have a parametrized sequence of good covers, connected by maps of covers, the persistence diagram captured by the nerves equals the persistence diagram computed by singular homology on the level of spaces.

A central result in this paper is a way to bound the bottleneck distance between these two persistence diagrams when the covers are not necessarily good. Using this result we provide an approximation to the Čech persistence module built on a finite sample from Euclidean space. The method enjoys several favorable properties: it approximates the Čech persistence module with provable error bounds and allows for size reduction on a heuristic basis, i.e. only when the complex becomes too large to store. Unfortunately, computing the weights of the simplices turns out to be expensive, making it inapplicable in most settings. To mend this we propose an easy to compute approximation which performs surprisingly well on real data sets. Using our aforementioned result we also show that the net-tree construction as introduced by Sheehy [28] and Dey *et al.* [15] works well for the Čech complex in Euclidean space. This approach enjoys very powerful theoretical bounds, e.g. a linear growth in the number of simplices as a function of sampled points. In practice, however, it is difficult to prevent the complex from growing too large.

Having implemented an algorithm to compute persistence diagrams of simplicial complexes connected by simplicial maps we conclude the paper by applying our approximations to a variety of point samples in Euclidean space.

To the best of our knowledge, this is the first paper where persistence computations are performed on simplicial complexes connected by more general simplicial maps than inclusions.

1.2 Outline

In Section 2 we review background material and Dey *et al.*'s algorithm [15] for computing persistent homology of simplicial complexes connected by simplicial maps. In particular, we introduce the concept of sequences of covers, and in Section 3 we give a homotopy colimit argument which relates the persistence module associated to a sequence of covers to that formed by the covering sets on the level of spaces. This relation is used in Section 3.1 to prove a sandwich type theorem for sequences of covers. We give two approaches to approximating the Čech persistence module in Section 4. The paper concludes with Section 5 where we compute the persistence diagrams of example point clouds in Euclidean space using the aforementioned approximations.

1.3 Related work

In low-dimensional Euclidean space the alpha complex [18] offers a memory efficient way to compute the persistence diagrams of a point cloud. Unfortunately, the number of simplices grows exponentially in the ambient dimension, making it inefficient in high-dimensional space. The witness complex [12] is a simplicial complex built on a subset of the sample, called landmarks. Unfortunately, the persistence diagrams of the associated filtration may depend heavily on the choice of landmarks. Sheehy [28] and later Dey *et al.* [15] approximate the Vietoris–Rips complex using net-trees, and Kerber

and Sharathkumar [24] arrive at similar results for the Čech complex in Euclidean space using quadrees. Our constructions in Section 4.1 is an adaption on the work of Dey *et al.* [15] to the Čech complex in Euclidean space. The construction in Section 4.2 can be viewed as a particular type of a graph induced complex [16]. Chazal and Oudot [10] prove the results in Section 3 for the case where all the simplicial maps are inclusions.

Recent research [17, 30, 22] provides methods to reduce the size of simplicial complexes after being stored, e.g. to provide faster persistence computations. Such reductions are not discussed in this paper as we seek to compute persistence diagrams of point clouds whose filtered complexes are too large to be stored to begin with.

2 Background material

In this section we survey prerequisite background material and fix notation. We assume familiarity with basic concepts from algebraic topology, and basic knowledge of persistent homology. For introductions see [23] and [19], respectively.

Throughout the paper, all simplicial complexes are assumed to be finite and unoriented. A simplex is considered a set of vertices, and we write a k -simplex $\{i_0, \dots, i_k\}$ as $[i_0, \dots, i_k]$. For a simplicial complex K , we will denote its geometric realization by $|K|$. Moreover, if $f : K \rightarrow L$ is a simplicial map between simplicial complexes, then $|f| : |K| \rightarrow |L|$ denotes the continuous map between their geometric realizations defined by f on the vertices and extended linearly using barycentric coordinates. The p -th singular homology vector space of a topological space X with coefficients in the field \mathbb{Z}_2 will be denoted by $H_p(X)$, and for a continuous map $f : X \rightarrow Y$ we denote its induced map on homology by $f_* : H_p(X) \rightarrow H_p(Y)$. When $X = |K|$ is the geometric realization of a simplicial complex, we will make no distinction between the p -th simplicial homology vector space of K and the p -th singular homology vector space of $|K|$. Cohomology vector spaces over \mathbb{Z}_2 are similarly denoted by $H^p(X)$.

A collection of open sets $\mathcal{U} = \{U_i \mid i \in I\}$ indexed by a finite set I is said to be a **(finite) cover** of $\cup_{i \in I} U_i$. The **nerve** $N\mathcal{U}$ of the cover \mathcal{U} is the simplicial complex with vertex set I and a k -simplex $[i_0, \dots, i_k] \in N\mathcal{U}$ if $U_{i_0} \cap \dots \cap U_{i_k} \neq \emptyset$. Let $\mathcal{U} = \{U_i \mid i \in I\}$ and $\mathcal{V} = \{V_j \mid j \in J\}$ be covers of topological spaces $U \subseteq V$. A map of sets $F : I \rightarrow J$ is said to be a **map of covers** if $U_i \subseteq V_{F(i)}$ for all $i \in I$. It is easy to check that F extends to a simplicial map $F : N\mathcal{U} \rightarrow N\mathcal{V}$ between the nerves of the covers. By a **sequence of covers** we will mean a collection of covers $\{\mathcal{U}(\alpha) \mid \alpha \in A \subset [0, \infty)\}$, each indexed respectively by $I(\alpha)$, together with maps of covers $F^{\alpha, \alpha'} : I(\alpha) \rightarrow I(\alpha')$ such that $F^{\alpha, \alpha} = \text{id}$ and $F^{\alpha, \alpha''} = F^{\alpha', \alpha''} \circ F^{\alpha, \alpha'}$ for all $\alpha'' \geq \alpha' \geq \alpha$. Such a sequence will be denoted by a pair (\mathcal{U}, F) . Similarly, for any sequence of covers we have an induced **sequence of nerves** which will be denoted by $(N\mathcal{U}, F)$.

2.1 Persistence modules

A **persistence module** \mathbb{V} over $A \subseteq \mathbb{R}$ is a collection of \mathbf{k} -vector spaces $\{V(\alpha) \mid \alpha \in A\}$ and linear maps $v^{\alpha, \alpha'} : V(\alpha) \rightarrow V(\alpha')$ for all $\alpha \leq \alpha'$ such that $v^{\alpha, \alpha} = \text{id}$ and $v^{\alpha, \alpha''} = v^{\alpha', \alpha''} \circ v^{\alpha, \alpha'}$. The **direct sum** of two persistence modules \mathbb{U} and \mathbb{W} , both indexed over the same set, is the persistence module $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ where $V(\alpha) = U(\alpha) \oplus W(\alpha)$ and $v^{\alpha, \alpha'} = u^{\alpha, \alpha'} \oplus w^{\alpha, \alpha'}$. We say that \mathbb{V} is **indecomposable** if the only decompositions of \mathbb{V} are the trivial decompositions $0 \oplus \mathbb{V}$ and $\mathbb{V} \oplus 0$.

Definition 1. Let $J \subseteq A$ be an interval, i.e. if $s, t \in J$ and $s < r < t$ then $r \in J$. The

interval module over J is the persistence module \mathbb{I}^J defined by

$$I^J(\alpha) = \begin{cases} \mathbf{k} & \text{if } \alpha \in J \\ 0 & \text{otherwise} \end{cases}$$

and $i^{\alpha, \alpha'} = \text{id} : I^J(\alpha) \rightarrow I^J(\alpha')$ whenever $\alpha \leq \alpha' \in J$ and 0 otherwise.

It is not difficult to show that \mathbb{I}^J is indecomposable, and the Krull–Remak–Schmidt–Azumaya theorem [3] tells us that if

$$\mathbb{V} \cong \bigoplus_{l \in L} \mathbb{I}^{J_l} \qquad \mathbb{V} \cong \bigoplus_{m \in M} \mathbb{I}^{K_m},$$

then there is a bijection $\sigma : L \rightarrow M$ such that $J_l = K_{\sigma(l)}$ for all $l \in L$. So whenever \mathbb{V} admits such a decomposition we can characterize it by the multiset $\{J_l \mid l \in L\}$ of intervals called the **persistence diagram** $\mathbf{D}(\mathbb{V})$ of \mathbb{V} . An interval $(b, d) \in \mathbf{D}(\mathbb{V})$ represents a **feature** of \mathbb{V} with **birth** and **death** time b and d , respectively. A persistence diagram is usually depicted as a collection of points in $(\mathbb{R} \cup \{\pm\infty\})^2$. A recent theorem by Crawley-Boevey [11] asserts that \mathbb{V} admits a decomposition into interval modules if V_α is finite-dimensional for all $\alpha \in \mathbb{R}$. For an example of a persistence module which does not admit an interval decomposition, see [9].

To every sequence of covers (\mathcal{U}, F) we have an associated persistence module $(H_p(N\mathcal{U}), F_*)$ with vector spaces $\{H_p(N\mathcal{U}(\alpha)) \mid \alpha \in A \subseteq [0, \infty)\}$ and maps $(F^{\alpha, \alpha'})_*$. As the covers are finite, all the homology vector spaces will have finite dimension, and thus the persistence diagrams are well-defined. In particular, if $P \subseteq M$ is a finite set of points in a metric space M , and $B(p; \alpha)$ is the open ball of radius α centered at p , we get a sequence of covers by defining $B(p; 0) = \{p\}$, $\mathcal{U}(P; \alpha) = \{B(p; \alpha) \mid p \in P\}$ and $F = \text{id}$. The induced sequence of nerves is known as the **Čech filtration** and the associated persistence module is the **Čech persistence module**. In the remainder of this paper $\mathcal{C}(P; \alpha)$ denotes the nerve of the Čech filtration of P at scale α .

Another popular construction is the Vietoris–Rips complex $\mathcal{R}(P; \alpha)$ which is defined as the largest simplicial complex with the same 1-skeleton as $\mathcal{C}(P; \alpha)$. By definition, it follows that $\mathcal{C}(P; \alpha) \subseteq \mathcal{R}(P; \alpha)$, and for $P \subseteq \mathbb{R}^n$, it is also true that $\mathcal{R}(P; \alpha) \subseteq \mathcal{C}(P; \sqrt{2}\alpha)$ [13].

2.2 Metrics and approximations

Let Δ denote the multiset of all pairs $(x, x) \in (\mathbb{R} \cup \{\pm\infty\})^2$, each with countably infinite multiplicity. A **partial matching** between two persistence diagrams D and D' is a bijection $\gamma : B \cup \Delta \rightarrow B' \cup \Delta$, and we denote all such by $\Gamma(D, D')$.

The following defines a metric on persistence diagrams:

Definition 2. The **bottleneck distance** between two persistence diagrams B and B' is

$$d_B(B, B') = \inf_{\gamma \in \Gamma(D, D')} \sup_{(b, d) \in B} \|(b, d) - \gamma((b, d))\|_\infty$$

where

$$\|(b_1, d_1) - (b_2, d_2)\|_\infty = \max(|b_1 - b_2|, |d_1 - d_2|).$$

The theory of *interleavings* [8] offers a generalization of the bottleneck distance to persistence modules that do not admit a decomposition into indecomposables. Importantly, if there exists an ϵ -interleaving between two persistence modules, then their bottleneck distance is at most ϵ . In this paper we adopt the conventions of [28, 24] and use a slight reformulation of the ordinary theory of interleavings.

Definition 3. Two persistence modules \mathbb{U} and \mathbb{V} indexed over $[0, \infty)$ are said to be c -approximate if there exist a constant $c \geq 1$ and two families of homomorphisms $\{\phi_\alpha : U(\alpha) \rightarrow V(c\alpha)\}_{\alpha \geq 0}$ and $\{\psi_\alpha : V(\alpha) \rightarrow U(c\alpha)\}_{\alpha \geq 0}$ such that the following four diagrams commute for all $\alpha \leq \alpha'$:

$$\begin{array}{ccccc}
U(\frac{\alpha}{c}) & \xrightarrow{\quad\quad\quad} & U(c\alpha') & & U(c\alpha) \xrightarrow{\quad\quad\quad} U(c\alpha') \\
& \searrow & \nearrow & & \nearrow \\
& & V(\alpha) \xrightarrow{\quad\quad\quad} V(\alpha') & & V(\alpha) \xrightarrow{\quad\quad\quad} V(\alpha') \\
& & & & \nearrow \\
& & & & U(c\alpha) \xrightarrow{\quad\quad\quad} U(c\alpha') \\
\\
V(\frac{\alpha}{c}) & \xrightarrow{\quad\quad\quad} & V(c\alpha') & & V(c\alpha) \xrightarrow{\quad\quad\quad} V(c\alpha') \\
& \nearrow & \searrow & & \searrow \\
& & U(\alpha) \xrightarrow{\quad\quad\quad} U(\alpha') & & U(\alpha) \xrightarrow{\quad\quad\quad} U(\alpha') \\
& & \nearrow & & \nearrow \\
& & V(\alpha) \xrightarrow{\quad\quad\quad} V(\alpha') & & V(\alpha) \xrightarrow{\quad\quad\quad} V(\alpha')
\end{array}$$

The following theorem is immediate from the theory of interleavings [8].

Theorem 4. *If \mathbb{U} and \mathbb{V} are c -approximate, then their bottleneck distance is bounded by $\log c$ on the log-scale.*

The above result can be seen as a general version of the relationship between the Čech and Vietoris–Rips filtrations. Indeed, while the bottleneck distance between their persistence diagrams may be arbitrarily large, the inclusions

$$\mathcal{C}(P; \alpha) \subseteq \mathcal{R}(P; \alpha) \subseteq \mathcal{C}(P; \sqrt{2}\alpha)$$

ensure that a feature (b, d) in the Vietoris–Rips persistence module is also a feature in the Čech persistence module if $d - b \geq \sqrt{2}b$, and vice versa.

2.3 Computing persistent homology using annotations

Many widely implemented and used algorithms for computing persistent homology assume that the maps in the persistence module are induced by inclusions of simplicial complexes, i.e. that the underlying sequence is a filtration. As shall become clear, we will need to compute in the setting of general simplicial maps.

Definition 5. A surjective simplicial map $f : K \rightarrow K'$ with the property that there exist distinct $[a], [b] \in K$ such that

$$f(\sigma) = \begin{cases} \sigma \setminus \{b\} & \text{if } a, b \in \sigma \\ \{a\} \cup \sigma \setminus \{b\} & \text{if } a \notin \sigma, b \in \sigma \\ \sigma & \text{otherwise} \end{cases}$$

is called an **edge contraction of $[a, b]$ to $[a]$** . Simplices $\sigma, \sigma' \in K$ are called **mirror simplices** (for f) if $f(\sigma) = f(\sigma')$.

We will often refer to an edge contraction like that above by $[a, b] \mapsto [a]$. Since up to isomorphism any simplicial map $K \rightarrow K'$ decomposes into a finite sequence of inclusions and edge contractions, we only need to deal with those two types and adjust the persistence module indices accordingly to reflect the addition of extra maps. Likewise, as is normal, we decompose inclusions into ones of the form $K \rightarrow K \cup \{\sigma\}$ and refer to these as “adding a simplex σ ”.

We will use Dey *et al.*'s method of *persistence annotations* [15] to compute (the persistence diagrams of) persistence modules with simplicial maps, and now quickly review their algorithm and our implementation details.

The method of annotation tracks homology with \mathbb{Z}_2 coefficients across a persistence module by storing the value of all cohomology generators at each simplex and updating these “annotations” to reflect the inclusion of a simplex or the contraction of an edge. Care should be taken to notice a slight difference in terminology: our definition of annotations reflects Dey's *valid annotations*.

Definition 6. An **annotation** for a simplicial complex K is a linear map $\Phi_p : C_p(K) \rightarrow \mathbb{Z}_2^n$ with the property that

$$\varphi_1 = [c \mapsto \Phi_p(c)_1], \dots, \varphi_n = [c \mapsto \Phi_p(c)_n]$$

is a basis for $H^p(K)$. Here $\Phi_p(c)_i$ denotes the i 'th component of $\Phi_p(c) \in \mathbb{Z}_2^n$.

A key observation is the following: the persistent homology of a sequence of simplicial complexes can be obtained by dualizing on the level of chains and taking cohomology. This is true since when working over \mathbb{Z}_2 (or any field), the map $\alpha : H^p(K) \rightarrow \text{Hom}(H_p(K), \mathbb{Z}_2)$ defined by $\alpha([f])([c]) = f(c)$ is an isomorphism. Thus, intervals in persistent cohomology are dual to intervals in persistent homology. Therefore, we shall interchangeably speak of a homology class born at persistence index i as a cohomology class in the opposite direction dying at persistence index i .

By storing the value of Φ_p at each p -simplex, that simplex' contribution to the (co)homology vector space is known and so allows us to only make changes to homology near the site of a contraction. This “locality” of the changes introduced by an edge contraction is summarized in the following definition [14], proposition [2] and lemmas.

Definition 7. The **link** of a simplex σ in a simplicial complex K is the set

$$\text{lk}_K \sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in K\}.$$

An edge $[a, b] \in K$ satisfies the **link condition** if $\text{lk}_K[a] \cap \text{lk}_K[b] = \text{lk}_K[a, b]$.

When the simplicial complex in question is clear, we shall simply write lk for lk_K .

Proposition 8. The contraction $f : K \rightarrow K'$ of an edge that satisfies the link condition induces a homotopy equivalence $|f| : |K| \rightarrow |K'|$, and hence an isomorphism $f_* : H_*(K) \rightarrow H_*(K')$.

Lemma 9. If $[a, b] \in K$, then $\text{lk}_K[a, b] \subseteq \text{lk}_K[a] \cap \text{lk}_K[b]$.

Proof. Suppose $\eta \in \text{lk}[a, b]$. Then there exists a $\tau \in K$ with $[a, b] \subseteq \tau$ and $\eta = \tau \setminus [a, b]$. Since K is a simplicial complex, it also contains $\tau' = \tau \setminus [a]$ and $\tau'' = \tau \setminus [b]$. We have $[b] \subseteq \tau'$ and $\eta = \tau' \setminus [b]$, so $\eta \in \text{lk}[b]$. The same argument using τ'' gives that $\eta \in \text{lk}[a]$. \square

For the following lemma we shall write $L_K(a, b) = (\text{lk}_K[a] \cap \text{lk}_K[b]) \setminus \text{lk}_K[a, b]$.

Lemma 10. If $\eta \in L_K(a, b)$, then $K' = K \cup \{\eta \cup [a, b]\}$ is also a simplicial complex, and moreover $L_{K'}(a, b) = L_K(a, b) \setminus \{\eta\}$.

Proof. Observe that $[a, b] \not\subseteq \eta$. K' is still a simplicial complex, as all faces of $\eta \cup [a, b]$ are present in K by the assumption that $\eta \in L_K(a, b)$. Note that by definition

$$\text{lk}_{K'}[a] = \text{lk}_K[a] \cup \{\eta \cup [b]\} \quad \text{lk}_{K'}[b] = \text{lk}_K[b] \cup \{\eta \cup [a]\},$$

so $\text{lk}_{K'}[a] \cap \text{lk}_{K'}[b] = \text{lk}_K[a] \cap \text{lk}_K[b]$. It also follows from the definition that

$$\text{lk}_{K'}[a, b] = \text{lk}_K[a, b] \cup \{\eta\},$$

so $L_{K'}(a, b) = L_K(a, b) \setminus \{\eta\}$. \square

In summary, we see that to contract an edge we only need to change the simplicial complex in the vicinity of that edge.

Suppose

$$K = (K_0 \xrightarrow{f_0} K_1 \xrightarrow{f_1} \dots \xrightarrow{f_{m-1}} K_m)$$

is a sequence of simplicial complexes (with the f_i 's simplicial maps) whose persistence module

$$H_*(K) = (H_*(K_0) \xrightarrow{(f_0)_*} H_*(K_1) \xrightarrow{(f_1)_*} \dots \xrightarrow{(f_{m-1})_*} H_*(K_m))$$

has been computed, and write Φ_p^i for the annotation of $H^p(K_i)$ and n for its dimension. To compute the persistence module of

$$K' = (K_0 \xrightarrow{f_0} \dots \xrightarrow{f_{m-1}} K_m \xrightarrow{f_m} K_{m+1}),$$

there are four cases to handle:

1. f_m adds a single p -simplex σ , and...

- (a) $\Phi_{p-1}^m(\partial\sigma) = 0$. This corresponds to a generator of $H_p(K')$ being born at persistence index $m+1$, or equivalently to a generator of $H^p(K')$ dying at m going left (see Proposition 5.2 in [15]). Define $\Phi_p^{m+1} : C_p(K_{m+1}) \rightarrow \mathbb{Z}_2^{n+1}$ by

$$\Phi_p^{m+1}(\tau) = \begin{cases} (\Phi_p^m(\tau)_1, \dots, \Phi_p^m(\tau)_n, 0) & \text{if } \tau \neq \sigma \\ (0, \dots, 0, 1) & \text{if } \tau = \sigma \end{cases}$$

and extending linearly. In other dimensions $q \neq p$, we set $\Phi_q^{m+1} = \Phi_q^m$.

- (b) $\Phi_{p-1}^m(\partial\sigma)_{i_1} = \dots = \Phi_{p-1}^m(\partial\sigma)_{i_l} = 1$ for some $l \geq 1$. In this case σ kills a class in $H_{p-1}(K')$ at $m+1$, or equivalently gives birth to one of the generators $\varphi_{i_1}, \dots, \varphi_{i_l}$ of $H^{p-1}(K')$ in the reverse direction (see Proposition 5.2 in [15]). We kill the youngest homology class, say the one numbered u (so φ_u is born in the reverse direction). Note that $\gamma : K_{m+1} \rightarrow \mathbb{Z}_2^n$ defined by

$$\gamma(\tau) = \begin{cases} \Phi_{p-1}^m(\tau) + \Phi_{p-1}^m(\partial\sigma) & \text{if } \Phi_{p-1}^m(\tau)_u = 1 \\ \Phi_{p-1}^m(\tau) & \text{otherwise} \end{cases}$$

has 0 in component u of all its values. Define $\Phi_{p-1}^{m+1} : C_{p-1}(K_{m+1}) \rightarrow \mathbb{Z}_2^{n-1}$ as γ with the u -th component removed, and extend linearly. In other dimensions $q \neq p-1$, we set

$$\Phi_q^{m+1}(\tau) = \begin{cases} (0, \dots, 0) & \text{if } \tau = \sigma, q = p \\ \Phi_q^m(\tau) & \text{otherwise.} \end{cases}$$

2. f_m contracts $[a, b]$ to $[a]$, and...

- (a) $[a, b]$ satisfies the link condition. Let

$$M_{p-1} = \{\sigma \in K_m \mid \dim \sigma = p-1, a \in \sigma \text{ and } \sigma \text{ has a mirror under } f_m\},$$

and note that to any $\tau \in M_{p-1}$, there is a unique $g_\tau \in K_m$ with $\tau \subseteq g_\tau$, $\dim g_\tau = p$ and $[a, b] \subseteq g_\tau$. Define Φ_p^{m+1} on the p -simplices of K_{m+1} by

$$\Phi_p^{m+1}(\sigma) = \Phi_p^m(\sigma) + \sum_{\sigma \supseteq \tau \in M_{p-1}} \Phi_p^m(g_\tau),$$

noting that the sum may be empty. This corresponds to Dey's "annotation transfers" — see Proposition 4.4 and 4.5 of [15] for a more detailed explanation.

- (b) $[a, b]$ does not satisfy the link condition. Lemma 10 tells us which simplices to add, repeatedly hitting the cases 1a and 1b, until the link condition becomes fulfilled¹. Afterwards contracting $[a, b]$ is handled by case 2a. Some bookkeeping is of course required if one wants to consider the potentially many homology changes from the inclusions as occurring at persistence index $m + 1$.

Dey *et al.* show in [15] (Proposition 5.1) that Φ_*^{m+1} as constructed above is an annotation for $H^*(K_{m+1})$. With $K_0 = \emptyset$ and the associated empty annotation Φ_*^0 , then, the above is a correct algorithm for computing persistent homology.

2.3.1 Some implementation details

As suggested in [4, 5], the *simplex tree* is a data structure that is well-suited for storing the simplicial complex in the above algorithm.

A simplex tree is a *trie* (also called a prefix tree), which is a tree T that stores a simplicial complex K whose vertices V have a total ordering \leq by the following rules:

- T contains a distinguished root.
- Every non-root node $n \in T$ carries the data of a label $L(n) \in V$. The root is labelled by a distinguished symbol, say $*$, and we extend the ordering to $* < v$ for all $v \in V$ to ease notation.
- Nodes have zero or more children.
- If n is a child of p , then $L(n) > L(p)$.
- If n and m both are children of p , then $L(n) \neq L(m)$.

The simplicial complex K to be encoded corresponds to all paths to the root of T , and we write $S(n) \in K$ for the simplex corresponding to the path from $n \in T$. We will also refer to the root having **depth** 0, and in general a node as having depth $k + 1$ if its parent has depth k . Thus $\text{depth}(n) = \dim S(n) + 1$.

In terms of implementation, every node holds a pointer to its parent and a dictionary² of pointers to its children, keyed on their labels. Furthermore, we augment the tree by adding to each node a “cousin pointer”: We call m a **cousin** of n if $\text{depth}(m) = \text{depth}(n)$ and $L(m) = L(n)$. Every node holds a pointer to one of its cousins in such a way that they form a cyclic linked list that visits every cousin at the same depth precisely once (per cycle). In addition, an arbitrary representative of each such cyclic linked list is maintained in a dictionary keyed on labels and depths.

Figure 1 shows an example of the basic part of a simplex tree, along with an example of annotations (intermediate data structures are dropped from the figure, and annotations are attached directly to the simplices for ease of visualization).

Boissonnat and Maria show that this data structure allows us to efficiently insert and remove simplices, and compute their faces and cofaces. For details, see [5].

To tie the simplex tree to the annotations discussed earlier, we want to associate to each node (i.e. each simplex) its annotation value. Since multiple simplices are likely to share the same annotation value, we go by way of a union find structure. Each node thus contains a pointer to a node in a forest, wherein each tree represents an annotation

¹This must happen after a finite number of steps since Lemma 10 shows that the size of $(\text{lk}[a] \cap \text{lk}[b]) \setminus \text{lk}[a, b]$ is reduced by one every time one of the new simplices is added. Moreover, one can in practice expect the number of simplices added to be small compared to the size of K_m since only cofaces of $[a, b]$ are added.

²A dictionary is here any data structure with logarithmic lookup time complexity for keys.

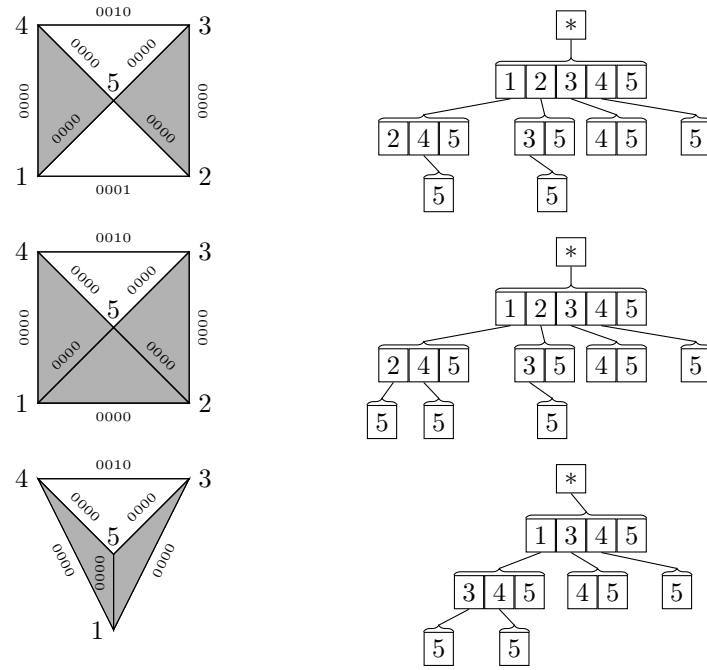


Figure 1: A somewhat simplified simplex tree representation of a simplicial complex. Annotation values on the 1-simplices are included for a persistence module in which the simplices are added in the order $\dots, [1, 5], [4, 5], [2, 5], [3, 5], [1, 4], [2, 3], [1, 4, 5], [2, 3, 5], [3, 4], [1, 2]$, leading up to the top row situation. To contract the edge $[1, 2]$, the link condition must be fulfilled, requiring the inclusion of $[1, 2, 5]$ (middle row). The situation after contraction is shown in the bottom row.

value shared by multiple cohomologous simplices. The root of each tree in the forest points to the actual annotation value of the simplices pointing to nodes in that tree.

The annotation values themselves are also kept referenced in a dictionary (keyed on the annotation values) for easy access and updating as used in the algorithm outlined earlier.

3 Persistent homology of sequences of covers

In the following we assume that all covering sets are subsets of some metric space and that every cover is finite. In particular, this means that all our spaces are paracompact. Moreover, the constructions in this section can be seen as special cases of the much more general construction of a homotopy colimit of a diagram of topological spaces.

To any open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of an open set U we assign a topological space $\Delta U_{\mathcal{U}} \subset |N\mathcal{U}| \times U$ defined as the disjoint union

$$\bigsqcup_{S \in N\mathcal{U}} |S| \times \bigcap_{i \in S} U_i$$

under the equivalence relation $(s, x) \sim (t, x)$ if $s \in |S|, t \in |T|, S \subseteq T$ and $s = t$. This construction comes equipped with continuous projection maps $\pi_1 : \Delta U_{\mathcal{U}} \rightarrow |N\mathcal{U}|$ and $\pi_2 : \Delta U_{\mathcal{U}} \rightarrow U$ given by projecting onto the first and second factor, respectively.

Lemma 11. *The fiber projecting map $\pi_2 : \Delta U_{\mathcal{U}} \rightarrow U$ is a homotopy equivalence.*

sketch. As U is assumed to be paracompact we can choose a partition of unity $\{\phi_i\}_{i \in I}$ subordinate to \mathcal{U} and define $g : U \rightarrow \Delta U_{\mathcal{U}}$ by

$$g(x) = \sum_{i \in I} (\phi_i(x) v_i, x),$$

where v_i is the vertex corresponding to U_i . Then $\pi_2 \circ g = \text{id}_U$ and it is not difficult to show that $g \circ \pi_2 \simeq \text{id}_{\Delta(\mathcal{U})}$. For a complete proof see [23]. \square

Now let $\mathcal{V} = \{V_j \mid j \in J\}$ be a finite cover of $V \supseteq U$ and $F : I \rightarrow J$ a map of covers. Recall that $|F| : |N\mathcal{U}| \rightarrow |N\mathcal{V}|$ denotes the continuous map defined on the vertices by the induced simplicial map between the nerves. If we let $\text{inc}_U^V : U \hookrightarrow V$ denote the inclusion of U into V we get the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\text{inc}_U^V} & V \\ \pi_2 \uparrow \simeq & & \simeq \uparrow \pi_2 \\ \Delta U_{\mathcal{U}} & \xrightarrow{|F| \times \text{inc}_U^V} & \Delta V_{\mathcal{V}} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ |N\mathcal{U}| & \xrightarrow{|F|} & |N\mathcal{V}| \end{array} \quad .$$

By passing to (singular) homology and using that π_2 is a homotopy equivalence we can reverse arrows to find the following commutative diagram:

$$\begin{array}{ccc} H_*(U) & \xrightarrow{(\text{inc}_U^V)_*} & H_*(V) \\ (\pi_1)_* \circ (\pi_2)_*^{-1} \downarrow & & \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} \\ H_*(|N\mathcal{U}|) & \xrightarrow{|F|_*} & H_*(|N\mathcal{V}|) \end{array} \quad (1)$$

Example 12. Note that Diagram (1) does not commute on the level of spaces: let $\mathcal{U} = \{U\}$ and $\mathcal{V} = \{U, V\}$ where $U \cap V \neq \emptyset$. If $x \in U \cap V$ then $(|F| \circ \pi_1 \circ g)(x)$ is a point in $|N\mathcal{U}|$ whereas $(\pi_1 \circ g \circ \text{inc})(x)$ can be any point along the edge $[[U, V]]$, depending on the choice of partition of unity. See Figure 2.

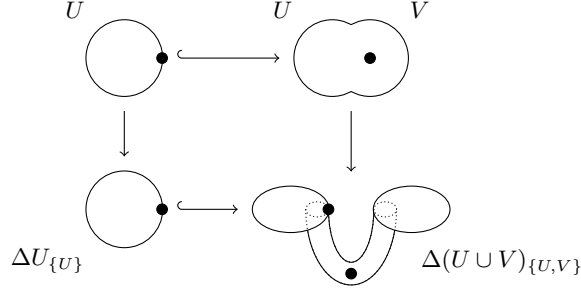


Figure 2: This diagram is an example of the diagram from Example 12 not commuting on the level of spaces.

Definition 13. A cover is said to be **good** if every finite intersection of its sets is contractible.

The following theorem is one of the great pillars of computational algebraic topology. It allows us to use discrete information to capture the topology of a continuous space. For a proof see Section 4.G. of [23].

Theorem 14. *If \mathcal{U} is a good cover, then the base projection map $\pi_1 : \Delta U_{\mathcal{U}} \rightarrow |N\mathcal{U}|$ is a homotopy equivalence.*

Corollary 15. *If \mathcal{U} is a good cover, then the composition $(\pi_1)_* \circ (\pi_2)_*^{-1}$ is an isomorphism.*

3.1 A sandwich theorem for sequences of covers

We will use the results from the previous section to prove a sandwich type theorem for sequences of covers. The idea is that if a sequence of covers can be sandwiched between two sequences of good covers, then the persistence module associated to the middle sequence approximates the persistence modules associated to the good covers.

Let $(\mathcal{U}, F_{\mathcal{U}})$, $(\mathcal{V}, F_{\mathcal{V}})$ and $(\mathcal{W}, F_{\mathcal{W}})$ be sequences of covers satisfying

$$U(\alpha) \subseteq V(\alpha) \subseteq W(\alpha) \subseteq U(c\alpha)$$

together with maps of covers

$$F_{\mathcal{V}, \mathcal{W}}^{\alpha, \alpha'} : \mathcal{V}(\alpha) \rightarrow \mathcal{W}(\alpha') \quad F_{\mathcal{W}, \mathcal{V}}^{\alpha, c\alpha'} : \mathcal{W}(\alpha) \rightarrow \mathcal{V}(c\alpha')$$

for all $\alpha' \geq \alpha$ and a fixed constant $c \geq 1$. Moreover, we assume that the maps of covers satisfy the following coherence relations:

$$F_{\mathcal{W}}^{\alpha', \alpha''} \circ F_{\mathcal{V}, \mathcal{W}}^{\alpha, \alpha'} = F_{\mathcal{V}, \mathcal{W}}^{\alpha', \alpha''} \circ F_{\mathcal{V}}^{\alpha, \alpha'} \quad F_{\mathcal{W}, \mathcal{V}}^{\alpha/c, c\alpha'} \circ F_{\mathcal{V}, \mathcal{W}}^{\alpha/c, \alpha/c} = F_{\mathcal{V}}^{\alpha/c, c\alpha'} \quad (2)$$

for all $\alpha'' \geq \alpha' \geq \alpha$.

For notational simplicity we let $\eta_{\mathcal{U}, \mathcal{V}}^{\alpha, \alpha'} = |F_{\mathcal{U}, \mathcal{V}}^{\alpha, \alpha'}|_*$ and accordingly for the other maps of covers above. From Corollary 15 we know that if $(\mathcal{U}, F_{\mathcal{U}})$ and $(\mathcal{W}, F_{\mathcal{W}})$ are sequences

of good covers, then there exist unique linear maps $\eta_{\mathcal{U},\mathcal{V}}^{\alpha,\alpha'}$, $\eta_{\mathcal{U},\mathcal{W}}^{\alpha,\alpha'}$ and $\eta_{\mathcal{W},\mathcal{U}}^{\alpha,c\alpha'}$ making the following diagrams commute:

$$\begin{array}{ccc}
H_p(U(\alpha)) & \xrightarrow{(\text{inc}_{U(\alpha)}^{V(\alpha')})_*} & H_p(V(\alpha')) \\
\cong \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} & & \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} \cong \\
H_p(|N\mathcal{U}(\alpha)|) & \xrightarrow{\eta_{\mathcal{U},\mathcal{V}}^{\alpha,\alpha'}} & H_p(|N\mathcal{V}(\alpha')|)
\end{array}
\quad
\begin{array}{ccc}
H_p(U(\alpha)) & \xrightarrow{(\text{inc}_{U(\alpha)}^{W(\alpha')})_*} & H_p(W(\alpha')) \\
\cong \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} & & \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} \cong \\
H_p(|N\mathcal{U}(\alpha)|) & \xrightarrow{\eta_{\mathcal{U},\mathcal{W}}^{\alpha,\alpha'}} & H_p(|N\mathcal{W}(\alpha')|)
\end{array}$$

$$\begin{array}{ccc}
H_p(W(\alpha)) & \xrightarrow{(\text{inc}_{W(\alpha)}^{U(c\alpha')})_*} & H_p(U(c\alpha')) \\
\cong \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} & & \downarrow (\pi_1)_* \circ (\pi_2)_*^{-1} \cong \\
H_p(|N\mathcal{W}(\alpha)|) & \xrightarrow{\eta_{\mathcal{W},\mathcal{U}}^{\alpha,c\alpha'}} & H_p(|N\mathcal{U}(c\alpha')|)
\end{array}$$

Hence, there are well-defined linear maps

$$\begin{aligned}
\phi_\alpha &= \eta_{\mathcal{U},\mathcal{V}}^{\alpha,c\alpha} : H_p(|N\mathcal{U}(\alpha)|) \rightarrow H_p(|N\mathcal{V}(c\alpha)|) \\
\psi_\alpha &= \eta_{\mathcal{W},\mathcal{U}}^{\alpha,c\alpha} \circ \eta_{\mathcal{V},\mathcal{W}}^{\alpha,\alpha} : H_p(|N\mathcal{V}(\alpha)|) \rightarrow H_p(|N\mathcal{U}(c\alpha)|)
\end{aligned} \tag{3}$$

Also, note that the map $\eta_{\mathcal{W},\mathcal{V}}^{\alpha,c\alpha'}$ is the unique map that makes Diagram (1) commute.

Theorem 16. *If $(\mathcal{U}, F_{\mathcal{U}})$ and $(\mathcal{W}, F_{\mathcal{W}})$ are sequences of good covers, then the families of homomorphisms $\{\phi_\alpha\}_{\alpha \in [0,\infty)}$ and $\{\psi_\alpha\}_{\alpha \in [0,\infty)}$ defined in Equation (3) satisfy the diagrams of Definition 3. In particular, the persistence modules*

$$(H_p(|N\mathcal{U}|), \eta_{\mathcal{U}}) \text{ and } (H_p(|N\mathcal{V}|), \eta_{\mathcal{V}})$$

are c -approximate.

Proof. We need to show that the following four relations in Definition 3 are satisfied for all $\alpha \leq \alpha'$:

$$\begin{aligned}
\psi_{\alpha'} \circ \eta_{\mathcal{V}}^{\alpha,\alpha'} \circ \phi_{\alpha/c} &= \eta_{\mathcal{U}}^{\alpha/c,c\alpha'} & \psi_{\alpha'} \circ \eta_{\mathcal{V}}^{\alpha,\alpha'} &= \eta_{\mathcal{U}}^{c\alpha,c\alpha'} \circ \psi_\alpha \\
\phi_{\alpha'} \circ \eta_{\mathcal{U}}^{\alpha,\alpha'} \circ \psi_{\alpha/c} &= \eta_{\mathcal{V}}^{\alpha/c,c\alpha'} & \phi_{\alpha'} \circ \eta_{\mathcal{U}}^{\alpha,\alpha'} &= \eta_{\mathcal{V}}^{c\alpha,c\alpha'} \circ \phi_\alpha
\end{aligned} \tag{4}$$

It follows from the uniqueness of the above linear maps, and the associativity of the maps in a sequence of covers, that any map composed out of the maps

$$\eta_{\mathcal{U}}^{\bar{\cdot},\bar{\cdot}}, \eta_{\mathcal{V}}^{\bar{\cdot},\bar{\cdot}}, \eta_{\mathcal{W}}^{\bar{\cdot},\bar{\cdot}}, \eta_{\mathcal{U},\mathcal{W}}^{\bar{\cdot},\bar{\cdot}}, \eta_{\mathcal{W},\mathcal{U}}^{\bar{\cdot},\bar{\cdot}}, \eta_{\mathcal{W},\mathcal{V}}^{\bar{\cdot},\bar{\cdot}} \text{ and } \eta_{\mathcal{U},\mathcal{V}}^{\bar{\cdot},\bar{\cdot}} \tag{5}$$

is uniquely defined by its domain and co-domain. That, together with the coherence relations of Equation (2), will prove the theorem. We will do the top left case of Equation (4) in full detail whereas we will refer to uniqueness arguments in the other three cases.

Top left:

$$\begin{aligned}
& \psi_{\alpha'} \circ \eta_{\mathcal{V}}^{\alpha, \alpha'} \circ \phi_{\alpha/c} \\
&= \eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha', \alpha'} \circ \eta_{\mathcal{V}}^{\alpha, \alpha'} \circ \eta_{\mathcal{U}, \mathcal{V}}^{\alpha/c, \alpha} \\
&= \eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha, \alpha'} \circ \eta_{\mathcal{U}, \mathcal{V}}^{\alpha/c, \alpha} \\
&= \eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha, \alpha'} \circ (\pi_1 \circ \pi_2^{-1})_* \circ \left(\text{inc}_{U(\alpha/c)}^{V(\alpha)} \right)_* \circ (\pi_1 \circ \pi_2^{-1})_*^{-1} \\
&= \eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ (\pi_1 \circ \pi_2^{-1})_* \circ \left(\text{inc}_{V(\alpha)}^{W(\alpha')} \right)_* \left(\text{inc}_{U(\alpha/c)}^{V(\alpha)} \right)_* \circ (\pi_1 \circ \pi_2^{-1})_*^{-1} \\
&= (\pi_1 \circ \pi_2^{-1})_* \circ \left(\text{inc}_{U(\alpha/c)}^{U(c\alpha')} \right)_* \circ (\pi_1 \circ \pi_2^{-1})_*^{-1} \\
&= \eta_{\mathcal{U}}^{\alpha/c, c\alpha'}
\end{aligned}$$

The second equality follows from the coherence relations.

Bottom left: By definition, $\phi_{\alpha'} \circ \eta_{\mathcal{U}}^{\alpha, \alpha'} \circ \psi_{\alpha/c} = \eta_{\mathcal{U}, \mathcal{V}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{U}}^{\alpha, \alpha'} \circ \eta_{\mathcal{W}, \mathcal{U}}^{\alpha/c, \alpha} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha/c, \alpha/c}$. Using that the composition of the three leftmost maps has same domain and co-domain as $\eta_{\mathcal{W}, \mathcal{V}}^{\alpha/c, c\alpha'}$, we are left with $\eta_{\mathcal{W}, \mathcal{V}}^{\alpha/c, c\alpha'} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha/c, \alpha/c} = \eta_{\mathcal{V}}^{\alpha/c, c\alpha'}$. Here the last equality follows from Equation (2).

Top right: From the coherence relations in Equation (2) we find $\psi_{\alpha'} \circ \eta_{\mathcal{V}}^{\alpha, \alpha'} = \eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{V}, \mathcal{W}}^{\alpha, \alpha'} \circ \eta_{\mathcal{V}}^{\alpha, \alpha'}$. Lastly, we note that $\eta_{\mathcal{W}, \mathcal{U}}^{\alpha', c\alpha'} \circ \eta_{\mathcal{V}}^{\alpha, \alpha'}$ and $\eta_{\mathcal{U}}^{c\alpha, c\alpha'} \circ \eta_{\mathcal{W}, \mathcal{U}}^{\alpha, c\alpha}$ are equal by uniqueness.

Bottom right: Both sides of the equation are composed out of maps from (5). □

The following is a corollary of the proof.

Corollary 17. *Any two of the persistence modules*

$$(H_p(|N\mathcal{U}|), \eta_{\mathcal{U}}), (H_p(|N\mathcal{V}|), \eta_{\mathcal{V}}) \text{ and } (H_p(|N\mathcal{W}|), \eta_{\mathcal{W}})$$

are c-approximate.

Note that we do not require the covers in the sequence $(\mathcal{V}, F_{\mathcal{V}})$ to be good. One application of this, which will be pursued in the next section, is the following. Let $(\mathcal{U}, F_{\mathcal{U}})$ be a sequence of good covers and $(\mathcal{V}, F_{\mathcal{V}})$ another sequence of covers where each open set in $\mathcal{V}(\alpha)$ is a union of open sets in $\mathcal{U}(\alpha)$. Thus, we have a map of covers $\mathcal{U}(\alpha) \rightarrow \mathcal{V}(\alpha')$, and more interestingly, a linear map $H_p(|N\mathcal{U}(\alpha)|) \rightarrow H_p(|N\mathcal{V}(\alpha')|)$. However, we do not have a map of covers the other way around, so it is a priori not clear how to define the interleaving map in the opposite direction. This is illustrated in Figure 3. The previous theorem tells us that such a map can be constructed and gives an upper bound on the bottleneck distance between the associated persistence modules.

4 Approximating the Čech complex in Euclidean space

In this section we construct two different approximation schemes for the Čech persistence module built on a finite set of points in Euclidean space.

It is clear that for any c -approximation of the Čech persistence module, a $\sqrt{2}c$ -approximation can be had via the Vietoris–Rips complex built on its 1-skeleton. For a treatment of approximate Vietoris–Rips complexes in general metric spaces see [15, 28].

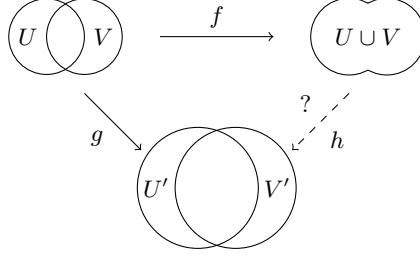


Figure 3: The map of covers f is defined as sending a ball to the union it belongs to, and g as the obvious map of covers arising from $U \subseteq U'$ and $V \subseteq V'$. There is no map of covers h making the diagram commute on the level of covers.

4.1 Linear-size approximation of the Čech persistence module

This section is an adaption of the work in [15] to the case of Čech complexes in Euclidean space. Throughout this section, $P \subseteq \mathbb{R}^n$.

Definition 18. For a set of points P , we say that $P' \subseteq P$ is a δ -**net** of P if

1. for every $p \in P$ there exists a $p' \in P'$ such that $\|p - p'\| \leq \delta$
2. for any $p, q \in P'$, $\|p - q\| > \delta$.

Choose parameters $\alpha_0, \epsilon \geq 0$ and define a sequence of point sets P_k for $k = 0, 1, \dots, m$ such that $P_0 = P$ and P_{k+1} is an $\alpha_0 \epsilon^2 (1 + \epsilon)^{k-1}$ -net of P_k . We refer to such a collection P_0, \dots, P_m as a **net-tree**. Furthermore, let $\mathcal{C}(P_k; \alpha)$ be the Čech complex at scale α built upon the vertex set P_k , and $U(P_k; \alpha)$ the union of open balls of radius α centered at each point in P_k . We clearly have maps $\pi_k : P_k \rightarrow P_{k+1}$ which send a vertex $p \in P_k$ to its most nearby vertex in P_{k+1} .

Lemma 19. For every $k = 0, \dots, m-1$ we have inclusions

$$U(P; \alpha_0(1 + \epsilon)^k) \subseteq U(P_{k+1}; \alpha_0(1 + \epsilon)^{k+1}).$$

Proof. Let $p \in P = P_0$ and $x \in \mathbb{R}^n$ be any point such that $\|p - x\| < \alpha_0(1 + \epsilon)^k$. Since P_1 is an $\alpha_0 \epsilon^2 (1 + \epsilon)^{-1}$ -net of P we can find $\pi_0(p) \in P_1$ such that $\|\pi_0(p) - p\| \leq \alpha_0 \epsilon^2 (1 + \epsilon)^{-1}$. Similarly, we can find $p' = (\pi_k \circ \dots \circ \pi_0)(p) \in P_{k+1}$ such that

$$\begin{aligned} \|p' - x\| &= \|\pi_k \circ \dots \circ \pi_0(p) - x\| \\ &\leq \|p - x\| + \sum_{i=0}^k \alpha_0 \epsilon^2 (1 + \epsilon)^{i-1} \\ &\leq \|p - x\| + \frac{\alpha_0 \epsilon^2}{1 + \epsilon} \cdot \frac{(1 + \epsilon)^{k+1} - 1}{\epsilon} \\ &< \alpha_0(1 + \epsilon)^k + \alpha_0 \epsilon (1 + \epsilon)^k = \alpha_0(1 + \epsilon)^{k+1}. \end{aligned}$$

□

In particular, for $p \in P_k$ we have that $B(p; \alpha_0(1 + \epsilon)^k) \subseteq B(\pi_k(p); \alpha_0(1 + \epsilon)^{k+1})$, and thus $\pi_k : P_k \rightarrow P_{k+1}$ is a map of covers

$$\pi_k : \mathcal{U}(P_k; \alpha_0(1 + \epsilon)^k) \rightarrow \mathcal{U}(P_{k+1}; \alpha_0(1 + \epsilon)^{k+1}).$$

Using this we define a sequence of covers associated to the net tree by defining

$$\mathcal{U}^{\text{net}}(P; \alpha) = \mathcal{U}(P_k; \alpha_0(1 + \epsilon)^k)$$

where k is the greatest integer such that $\alpha_0(1 + \epsilon)^k \leq \alpha$. The maps between the covers are given by compositions of π_k 's. We will denote the induced sequence of nerves by $\mathcal{C}^{\text{net}}(P)$ and the associated persistence module by $(H_p(\mathcal{C}^{\text{net}}(P)), \pi_*)$. Recall that with this notation we have that

$$U^{\text{net}}(P; \alpha) = U(P_k; \alpha_0(1 + \epsilon)^k) = \bigcup_{p \in P_k} B(p; \alpha_0(1 + \epsilon)^k)$$

Proposition 20. *The persistence modules $(H_p(\mathcal{C}^{\text{net}}(P)), \pi_*)$ and $(H_p(\mathcal{C}(P)), \text{id}_*)$ are $(1 + \epsilon)^2$ -approximate.*

Proof. Using that $U^{\text{net}}(P; \alpha) = U(P_k; \alpha_0(1 + \epsilon)^k)$ together with Lemma 19 we have the chain of inclusions

$$\begin{aligned} U^{\text{net}}(P; \alpha) &\subseteq U(P; \alpha) \subseteq U(P; \alpha_0(1 + \epsilon)^{k+1}) \subseteq U(P_{k+2}; \alpha_0(1 + \epsilon)^{k+2}) \\ &= U^{\text{net}}(P; \alpha(1 + \epsilon)^2). \end{aligned}$$

The rest of the proof follows by applying Theorem 16 with $\mathcal{U} = \mathcal{U}^{\text{net}}(P)$ and $\mathcal{V} = \mathcal{W} = \mathcal{U}(P)$. □

Proposition 21. *Let $P \subseteq \mathbb{R}^n$ be a set of m points. Then the number of p -simplices in $\mathcal{C}^{\text{net}}(P; \alpha_0(1 + \epsilon)^k)$ is $\mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{\mathcal{O}(np)} m\right)$.*

Proof. This is Theorem 6.3 in [15] together with the fact that the doubling dimension of \mathbb{R}^n is $\mathcal{O}(n)$. □

The net-tree construction exhibits great theoretical properties both with regards to approximating the Čech persistence module and in terms of size complexity. In practice however, as we shall see in Section 5, the complex often grows too large to be stored. Not doing a single collapse between scale $\alpha_0(1 + \epsilon)^k$ and scale $\alpha_0(1 + \epsilon)^{k+1}$ will in many situations introduce too many new simplices. To mend this we introduce a complex which allows for more numerous collapses, at the expense of computation time and poorer error bounds.

4.2 Approximations through non-good covers

We propose a general framework to approximate persistence modules associated to sequences of good covers. Using this framework we give an explicit approximation of the Čech persistence module in Euclidean space.

Let (\mathcal{U}, F) be a sequence of covers with index sets $\{I(\alpha)\}_{\alpha \geq 0}$ and $J(I(\alpha))$ a partition of $I(\alpha)$. We make the following assumption on the partitions: if $J \in J(I(\alpha))$ then for all $\alpha' \geq \alpha$ there exists $J' \in J(I(\alpha'))$ such that $J \subseteq J'$. In other words, if two elements are partitioned together at some scale α , they will be partitioned together at all scales $\alpha' \geq \alpha$. Moreover, if $J \in J(I(\alpha))$ then $F^{\alpha, \alpha'}(J)$ denotes the set $J' \in J(I(\alpha'))$ such that $J \subseteq J'$.

Lemma 22. *For each $\alpha \geq 0$, let $J(I(\alpha))$ be a partition of $I(\alpha)$ as described above. Then the pair $(\tilde{\mathcal{U}}, F)$ with*

$$\tilde{\mathcal{U}}(\alpha) = \left\{ \tilde{U}_J(\alpha) = \bigcup_{j \in J} U_j(\alpha) \mid J \in J(I(\alpha)) \right\}$$

is a sequence of covers.

Proof. This follows from that $J \subseteq F^{\alpha, \alpha'}(J)$ for all $J \in J(I(\alpha))$. \square

For such a choice of partitions we say that $(\tilde{\mathcal{U}}, F)$ is **coarsening** of (\mathcal{U}, F) .

Let $(\tilde{\mathcal{U}}(P), \text{id})$ be any coarsening of the Čech sequence of covers $\mathcal{U}(P)$ on a finite point set $P \subseteq \mathbb{R}^n$. Furthermore, define an associated sequence of good covers $(\text{CH}\tilde{\mathcal{U}}, \text{id})$ where

$$\text{CH}\tilde{\mathcal{U}}(\alpha) = \left\{ \text{CH}\left(\tilde{U}_k(\alpha)\right) \mid \tilde{U}_k(\alpha) \in \tilde{\mathcal{U}}(\alpha) \right\},$$

and $\text{CH}(-)$ denotes the convex hull. In the following proposition $(\tilde{\mathcal{C}}(P), \text{id})$ denotes the induced sequence of nerves of $(\tilde{\mathcal{U}}(P), \text{id})$.

Proposition 23. *If there exists a constant $c \geq 1$ such that $\text{CH}\left(\tilde{U}_J(\alpha)\right) \subseteq \bigcup_{j \in J} U_j(c\alpha)$ for all $\alpha \geq 0$ and all $J \in J(\alpha)$, then the persistence modules $(H_p(\mathcal{C}(P)), \text{id}_*)$ and $(H_p(\tilde{\mathcal{C}}(P)), \text{id}_*)$ are c -approximate.*

Proof. We will use Theorem 16. We see that the inclusion condition is satisfied by assumption:

$$U(\alpha) \subseteq U(\alpha) \subseteq \bigcup_{J \in J(\alpha)} \text{CH}\left(\tilde{U}_J(\alpha)\right) \subseteq U(c\alpha).$$

Moreover, $\tilde{\mathcal{U}}(P; \alpha)$ and $\text{CH}\tilde{\mathcal{U}}(P; \alpha)$ have the same indexing set, so the coherence relations of Equation (2) are trivially satisfied. \square

We see that every time we make our cover coarser, the number of 0-simplices in the nerve is reduced, and hence so is the size of the simplicial complex.

4.2.1 An explicit approximation

In the previous section we provided a general framework for constructing c -approximations to the Čech persistence module. We now give an explicit construction using Proposition 23.

Lemma 24. *Let $P = \{p_0, p_1, \dots, p_k\} \subset \mathbb{R}^n$ where $p_0 = 0$ and $\|p_i\| \leq \alpha$ for all i . Then for any point $x \in \text{CH}(P)$ there exists $p_i \in P$ such that $\|x - p_i\| \leq \alpha/\sqrt{2}$.*

Proof. By definition of p_0 we may assume without loss of generality that $x = (x_1, 0, \dots, 0)$ where $x_1 > \alpha/\sqrt{2}$. Let $p_i = (p_{i,1}, p_{i,2}, \dots, p_{i,n})$ be a point in P such that $p_{i,1} \geq p_{j,1}$ for every other j , and assume that $\|x - p_i\| > \alpha/\sqrt{2}$. Using the law of cosines:

$$\begin{aligned} \alpha^2 &\geq \|p_i\|^2 = \|(p_i - x) + x\|^2 = \|p_i - x\|^2 + \|x\|^2 - 2\|p_i - x\| \cdot \|x\| \cos(\angle p_0 x p_i) \\ &> \frac{\alpha^2}{2} + \frac{\alpha^2}{2} - 2\|p_i - x\| \cdot \|x\| \cos(\angle p_0 x p_i) \\ &= \alpha^2 - 2\|p_i - x\| \cdot \|x\| \cos(\angle p_0 x p_i) \end{aligned}$$

implying that $\cos(\angle p_0 x p_i) > 0$. By application of the Euclidean inner product we find

$$(p_i - x) \cdot (-x) = -p_{i,1} \cdot x_1 + x_1^2 = \|p_i - x\| \cdot \|x\| \cdot \cos(\angle p_0 x p_i) > 0$$

and therefore $p_{i,1} < x_1$, contradicting that x was enclosed in the convex hull of P . \square

Figure 4 shows an extreme case of the previous Lemma.

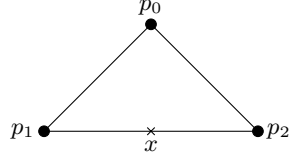


Figure 4: The vertices p_0, p_1, p_2 of an isosceles triangle $T = \text{CH}\{p_0, p_1, p_2\}$ with legs of length α and base of length $\sqrt{2}\alpha$ form an extreme case of Lemma 24 as $x \in T$ lies a distance $\alpha/\sqrt{2}$ from every vertex.

Proposition 25. *Let $\alpha \geq 0$ and $\epsilon \geq 0$. If $P = \{p_0, p_1, \dots, p_k\} \subset \mathbb{R}^n$ is a set of points such that $\|p_i - p_j\| \leq \epsilon\alpha$, then the following relation holds:*

$$\text{CH} \left(\bigcup_{0 \leq i \leq k} B(p_i; \alpha) \right) \subseteq \bigcup_{0 \leq i \leq k} B \left(p_i; \alpha \sqrt{1 + \frac{\epsilon^2}{2}} \right).$$

Proof. First, observe that we have the equality

$$\text{CH} \left(\bigcup_{0 \leq i \leq k} B(p_i; \alpha) \right) = \{x \in \mathbb{R}^n \mid \exists y \in \text{CH}(P), \|x - y\| < \alpha\}.$$

Any point $x \in \text{CH}(P)$ is contained in the union $\bigcup_{0 \leq i \leq k} B(p_i; \epsilon\alpha/\sqrt{2})$ by Lemma 24. Thus, what remains to be shown is that the proposition holds true for any $x \in \mathbb{R}^n$ for which there is a $p \in \text{CH}\{p_{i_0}, \dots, p_{i_k}\}$, $k \leq n-1$, such that $\|x - p\| < \alpha$. The last inequality follows since x is in the exterior of the convex hull and the most nearby point cannot be strictly inside an n -simplex.

Denote by x' the orthogonal projection of x down on the affine space spanned by $\{p_{i_0}, \dots, p_{i_k}\}$. If $x' \in \text{CH}\{p_{i_0}, \dots, p_{i_k}\}$ it follows from Lemma 24 that there exists a p_{i_j} such that

$$\|p_{i_j} - x\|^2 = \|p_{i_j} - x'\|^2 + \|x - x'\|^2 \leq \frac{\epsilon^2 \alpha^2}{2} + \alpha^2 = \alpha^2 \left(1 + \frac{\epsilon^2}{2} \right).$$

If $x' \notin \text{CH}\{p_{i_0}, \dots, p_{i_k}\}$ it implies the existence of a point p' on the boundary of $\text{CH}\{p_{i_0}, \dots, p_{i_k}\}$ such that $\|x - p'\| \leq \|x - p\| < \alpha$ and we can repeat the process for that point. This completes the proof as the proposition is trivially true if $k = 0$. \square



Figure 5: Left: the convex hull of a union of three balls. Right: By increasing the radii of the balls their union eventually covers the convex hull.

The previous proposition is illustrated in Figure 5. By combining Propositions 23 and 25 we have shown the following.

Proposition 26. *Let $\epsilon \geq 0$. Suppose $\tilde{\mathcal{U}}(P)$ is a coarsening of $\mathcal{U}(P)$ with the property that for every $\alpha \geq 0$ and every pair of indices $i, j \in J \in J(I(\alpha))$, the inequality $\|p_i - p_j\| \leq \alpha \cdot \epsilon$ holds. Then $H_p(\tilde{\mathcal{C}}(P), \text{id}_*)$ is a $\sqrt{1 + \epsilon^2/2}$ -approximation of the Čech persistence module built on P .*

The previous proposition allows us to build good approximations to the Čech persistence module with far fewer simplices. A problem with this approach is that such a memory efficient construction comes at the expense of computing weights of simplices. As an example, if $J(I(\alpha))$ consists of k partitions, each with m elements, then computing the smallest α at which they have a k -intersection has time complexity $\mathcal{O}(m^k)$. To mend this we seek methods to approximate this persistence module by ones that are less computationally expensive. The next section details one method for doing so.

4.2.2 Choosing a representative

Let $(\tilde{\mathcal{U}}(P), \text{id})$ be a coarsening of the Čech sequence of covers and for every $\alpha \geq 0$ and every $J \in J(\alpha)$ choose a representative $p_j \in P$, where $j \in J$. Denote the set of representatives at scale α by P_α . For every $\alpha \geq 0$ we define the subcomplex $\mathcal{C}^{\text{rep}}(P; \alpha) \subseteq \tilde{\mathcal{C}}(P; \alpha)$ to be the smallest simplicial complex such that:

1. $\mathcal{C}(P_\alpha; \alpha) \subseteq \mathcal{C}^{\text{rep}}(P; \alpha)$
2. $\text{id}^{\alpha, \alpha'} : \tilde{\mathcal{C}}(P; \alpha) \rightarrow \tilde{\mathcal{C}}(P; \alpha')$ restricts to a simplicial map $\mathcal{C}^{\text{rep}}(P; \alpha) \rightarrow \mathcal{C}^{\text{rep}}(P; \alpha')$

The idea is to choose a set of representatives, one for each element $J(I(\alpha))$, and use those representatives to approximate the persistent homology computation. However, to get a well-defined sequence of simplicial complexes and simplicial maps, we need to make sure that the image of a simplex spanned by one set of representatives is a simplex at a later filtration time, where the set of representatives may be different. Thus, our approximate complex contains the simplicial complex built on the set of representatives and, in addition, the images of simplices spanned by representatives at earlier filtration times.

Proposition 27. *The persistence modules $(H_p(\tilde{\mathcal{C}}(P)), \text{id}_*)$ and $(H_p(\mathcal{C}^{\text{rep}}(P)), \text{id}_*)$ are $\frac{1}{1-\epsilon}$ -approximate.*

Proof. The simplicial complexes $\mathcal{C}^{\text{rep}}(P_\alpha; \alpha)$ and $\tilde{\mathcal{C}}(P; \alpha)$ are defined over the same indexing set $J(I(\alpha))$ for every $\alpha \geq 0$. This follows from having chosen one representative for each covering set of $\tilde{\mathcal{U}}(\alpha)$. Now choose $x \in U_J \in \tilde{\mathcal{U}}(\alpha)$, where $\|x - p_j\| < \alpha$ for some $j \in J$, and let p be the representative of $\text{id}^{\alpha, \alpha/(1-\epsilon)}(J) \in J(I(\alpha/(1-\epsilon)))$. Then

$$\|p - x\| \leq \|p - p_j\| + \|x - p_j\| < \frac{\alpha\epsilon}{1-\epsilon} + \alpha = \frac{\alpha}{1-\epsilon}.$$

Hence, we have a map of covers $\tilde{\mathcal{U}}(P; \alpha) \rightarrow \mathcal{U}(P_{\alpha/(1-\epsilon)}; \frac{\alpha}{1-\epsilon})$ which induces the first map of the composition

$$\tilde{\mathcal{C}}(P; \alpha) \rightarrow \mathcal{C}\left(P_{\alpha/(1-\epsilon)}; \frac{\alpha}{1-\epsilon}\right) \subseteq \mathcal{C}^{\text{rep}}\left(P_{\alpha/(1-\epsilon)}; \frac{\alpha}{1-\epsilon}\right) \subseteq \tilde{\mathcal{C}}\left(P; \frac{\alpha}{1-\epsilon}\right)$$

The proof follows from application of Definition 3 and Theorem 4. \square

4.3 Relationship to graph induced complexes

We conclude this section by briefly discussing a related construction introduced in [16] by Dey *et al.*

Definition 28. Let $G(V)$ be a graph with vertex set V and let $\nu : V \rightarrow V'$ be a vertex map where $\nu(V) = V' \subseteq V$. The **graph induced complex** $\mathcal{G}(V, V', \nu)$ is defined as the simplicial complex where a k -simplex $[v'_1, \dots, v'_{k+1}]$ is in $\mathcal{G}(V, V', \nu)$ if and only if there exists a $(k+1)$ -clique $\{v_1, \dots, v_{k+1}\} \subseteq V$ such that $\nu(v_i) = v'_i$ for each $i \in \{1, \dots, k+1\}$.

First we note that a coarsening of a cover as defined at the beginning of Section 4.2 induces a graph induced complex. Indeed, just choose a representative for each partition and let ν be the map which takes a vertex to its representative. This, together with a net-tree construction as in Section 4.1, is utilized in [15] to construct a linear-size approximation to the Vietoris–Rips persistence module. Constructing the analogue Čech approximation is straightforward and it can be shown that it enjoys error bounds similar to what we proved in Section 4.1. In fact, the analogue Čech construction is nothing more than forming a coarsening of the Čech sequence of covers where the process of partitioning covering sets is determined by a net-tree. Unfortunately, as discussed at the end of Section 4.2.1, computing the k -intersections needed for this construction is very time consuming.

5 Computational experiments

This section details our implementation of the approximation schemes described above, as well as some computational examples examining their efficacy and practical applicability.

5.1 Implementation

We realize an implementation of the approximation schemes detailed in Section 4.2 as a C++ program in the following way.

The program takes as parameters $\epsilon \geq 0$ (as in Section 4.2.1), a maximal scale $\alpha_{\max} > 0$ (as usual when computing persistence), a maximal simplex dimension $D > 0$ (as usual) and $L \in \mathbb{N}$ (to be explained later). Given an input point cloud $P = \{p_1, \dots, p_N\} \subseteq \mathbb{R}^d$, we first use Müllner’s *fastcluster* [25] to compute its hierarchical clustering $\text{HC}(P)$ with the *complete* linkage criterion. This is considered a preprocessing step.

A **cluster** is a pair (p, X) with $p \in X \subseteq P$, wherein p will be called the cluster’s **representative** and X its **members**. At initialization time, we begin with N clusters

$$c_1^0 = (p_1, \{p_1\}), c_2^0 = (p_2, \{p_2\}), \dots, c_N^0 = (p_N, \{p_N\}).$$

and denote their enumeration by $C^0 = \{1, \dots, N\}$.

We shall regard $\text{HC}(P)$ as the data of a series of **linkage events** of the form $(s, i, j) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N}$ ordered by the first component, and (arbitrarily) with the convention that $i < j$. An event like this signifies the linking of clusters $c_i^l = (p_i^l, X_i^l)$ and $c_j^l = (p_j^l, X_j^l)$ at scale s , from which we form a new cluster $c_{i+j}^{l+1} = (p_{i+j}^{l+1}, X_i^l \cup X_j^l)$ where $p_{i+j}^{l+1} \in X_i^l \cup X_j^l$; in principle the new representative p_{i+j}^{l+1} can be chosen arbitrarily from $X_i^l \cup X_j^l$, but for heuristic reasons we pick the point in the member set $X_i^l \cup X_j^l$ closest to that set’s centroid.

We maintain a priority queue Q of simplices prioritized by their persistence time. At initialization, the queue contains the 0-simplices $[1], \dots, [N]$ all at persistence time 0. A simplex tree, along with associated annotations and other data structures as described in Section 2.3, are also initialized empty. These data structures that track homology will

jointly be referred to as PH below, and we shall abuse language and speak of a simplex as “belonging to PH” when the simplex is present in the simplicial complex. We also initialize $\alpha' = 0$ and $l = 0$ to begin with.

The implementation code then proceeds in the following steps:

1. If Q is empty, we are done and go to step 6. If not, pop a simplex σ and its persistence scale α from the front, and continue.
2. If $\alpha > \alpha_{\max}$, we are done and go to step 6. Otherwise continue.
3. If σ is not already in PH, add it according to Section 2.3. In both cases, continue.
4. If $\dim \sigma > D$, go to step 5. Otherwise, for each simplex $\tau \in \{\sigma \cup \{i\} \mid i \in C^l\}$: compute³ the radius r_τ of the smallest enclosing ball of the set $\{p_i^l \mid i \in \tau\} \subseteq P$, and add τ to Q at persistence scale r_τ . Go to step 5.
5. If at least L simplices have been added to PH since the last time this step was reached, we (possibly) perform a simplification by going to step 5a. Otherwise go to step 1.
 - (a) For each linkage event $(s, i, j) \in \text{HC}(P)$ for which $s \in [\alpha', \epsilon\alpha)$, perform the edge contraction $[i, j] \mapsto [i]$ according to Section 2.3, taking care to adjust persistence times to reflect a (possible) series of inclusions to satisfy the link condition. If there were no linkage events in the given interval, go to step 1. Otherwise, denote the clusters present after handling the linkage event, as explained earlier in this section, by

$$\{C_{i_1}^{l+1}, \dots, C_{i_{N_{l+1}}}^{l+1}\} \subseteq \{C_{j_1}^l, \dots, C_{j_{N_l}}^l\}$$

and go to step 5b.

- (b) Clear Q and reset it to contain the 0-simplices $[i_1], \dots, [i_{N_l}]$, all at persistence scale 0. Update l to $l + 1$ and α' to $\epsilon\alpha$, and go to step 1.
6. We are done. Any persistent homology generators not yet killed off are recorded as on the form (b, ∞) .

The algorithm above may be summarized as follows: Compute Čech persistence until the underlying simplicial complex has at least L simplices. When that is the case, walk up the complete linkage dendrogram of the point cloud until scale $\epsilon\alpha$ is reached, where α is the persistence scale. Any linkage event encountered corresponds to an edge contraction, which is performed. After that, computation of Čech persistence resumes as before, albeit on a reduced and changed point cloud, and collapses may happen again when L more simplices have been added. We terminate upon reaching α_{\max} , and ignore simplices of dimension above D (thus computing homology in dimensions $0, \dots, D - 1$).

Note that L is merely a parameter to reduce computational overhead involved in the collapses, as a higher value postpones contractions until the simplicial complex is denser. In principle, L can be thought of as zero. Also observe that $\epsilon = 0$ corresponds to computing ordinary Čech persistence.

5.2 Experiments

This section describes three experiments designed to test the feasibility of our implementation.

³Our implementation uses Gärtner’s *Miniball* [21] for this computation.

A calculation ranging from scale 0 to scale α_{\max} will have its resulting persistence diagram drawn as the region above the diagonal in $[0, \alpha_{\max}]^2$. Generators still alive at α_{\max} will be referred to as on the form (b, ∞) and plotted as triangles, while generators of the form (b, d) with $d \leq \alpha_{\max}$ will be plotted as dots. See Figure 7 for an example of drawing conventions.

5.2.1 Wedge of six circles enclosing each other

We produced a point cloud by randomly (uniformly) sampling 100 points from a circle of radius 1 centered at $(0, 1)$, 200 points from a circle of radius 2 centered at $(0, 2)$ and so forth up to 600 points from a circle of radius 6 centered at $(0, 6)$. Each point in the circle of radius r was perturbed by radial noise sampled from the uniform distribution on $[(1 - 0.02)r, (1 + 0.02)r]$. The very dense region near the origin where all the circles meet (see Figure 6) contributes nothing to homology, but significantly adds to the number of simplices if no collapse is done.

Running to $\alpha_{\max} = 2$, our implementation clearly limited the number of simplices — see Figure 8 and note especially the rapid increases between collapses, the regimes where the ordinary Čech filtration is formed — while producing a highly correct persistence diagram, as is shown in Figure 7.

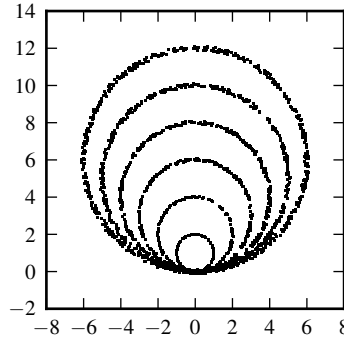


Figure 6: The point cloud from the example in Section 5.2.1.

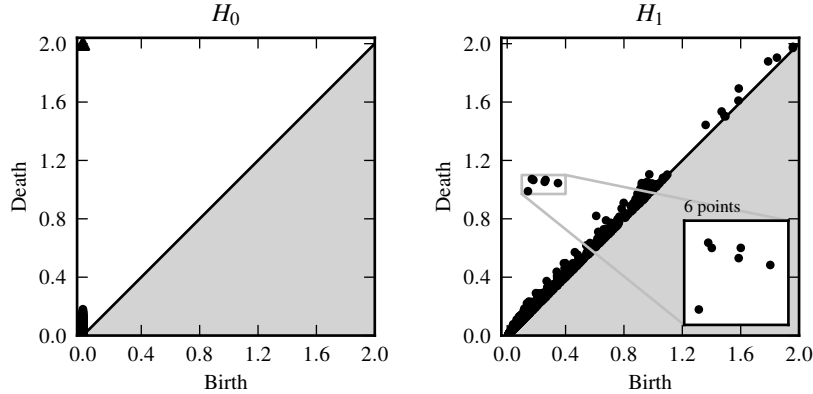


Figure 7: Persistence diagrams of the (noisy) wedge of six circles in Section 5.2.1 with $\epsilon = 3/4$ and $\alpha_{\max} = 2$.

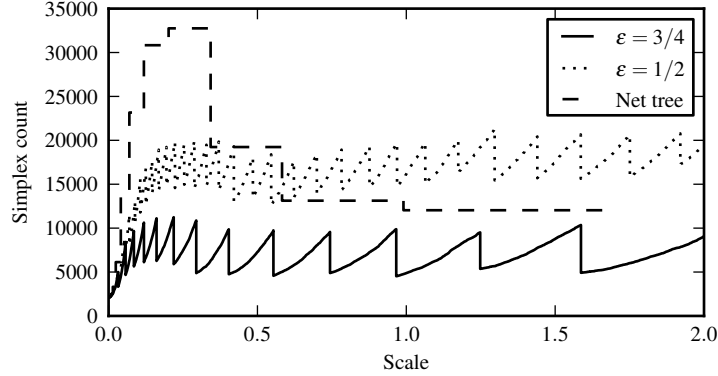


Figure 8: The simplex count while computing persistence for the example in Section 5.2.1. The net tree computations were run with $\alpha_0 = 10^{-3}$ and $\epsilon = 0.7$ in the notation of Section 4.1.

5.2.2 The real projective plane

We sampled $\mathbb{R}P^2$ by randomly selecting 5000 points on \mathbb{S}^2 and embedding them in \mathbb{R}^4 under $(x, y, z) \mapsto (xy, xz, y^2 - z^2, 2yz)$ as a test of how well our scheme handles higher dimensions. Figure 9 shows that the expected persistence diagram resulted when computing to $\alpha_{\max} = 0.54$ at $\epsilon = 1.0$.

Figure 10 compares our scheme (at $\epsilon = 1$) with the very beginning an ordinary Čech filtration. Our implementation keeps the number of simplices manageable, peaking at just above $3 \cdot 10^5$ simplices near the end (scale 0.54), while still recovering the correct persistence diagram. The figure also shows the simplex count for the net tree construction; notice that we were unable to correctly choose α_0 and ϵ so as to make computations with it feasible, unlike for the example in Section 5.2.1.

5.2.3 Time-delay embedding

We solved the Lorenz system (with parameters $\sigma = 10$, $r = 28$, $b = 8/3$ in the notation of [20]) and created a time series $y \in \mathbb{R}^{15000}$ by adding together all three of the solution's coordinates at each of 15000 points in time. Let $A(i)$ denote the (discrete) correlation of y and y shifted i places to the right. The first local minimum of A occurs at 15, so that was used as delay to embed y in \mathbb{R}^3 by delay-embedding. The resulting point cloud, with $15000 - (3 - 1) \cdot 15 = 14970$ points, reconstructs [29] the Lorenz attractor as seen in Figure 11. Observe that there are regions that have a very high density of points.

Our implementation computes the expected persistence diagram (Figure 12) while keeping the number of simplices low (Figure 13).

6 Conclusions and future work

We have presented two approximation schemes for the Čech filtration in Euclidean space. One construction uses a net-tree to build the Čech complex at fewer and fewer simplices as we increase the scale parameter. The other approach forms a coarsening of the Čech filtration by using covering sets formed by unions of open balls. Computing k -intersections of such covering sets is computationally expensive, so we approximated the persistence module by choosing a representative at each scale. In practice we experienced far better results with this method than the net-tree approach. This contrasts with the

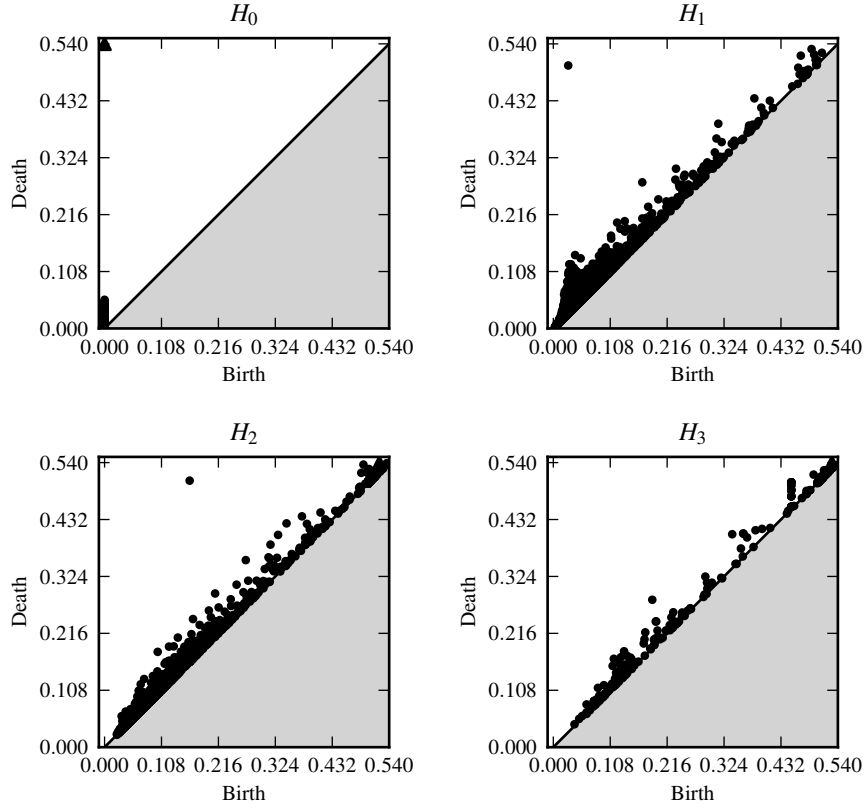


Figure 9: Persistence diagrams for the 5000 point random sample of $\mathbb{R}P^2$ embedded in \mathbb{R}^4 as described in Section 5.2.2, with $\epsilon = 1.0$ and $\alpha_{\max} = 0.54$.

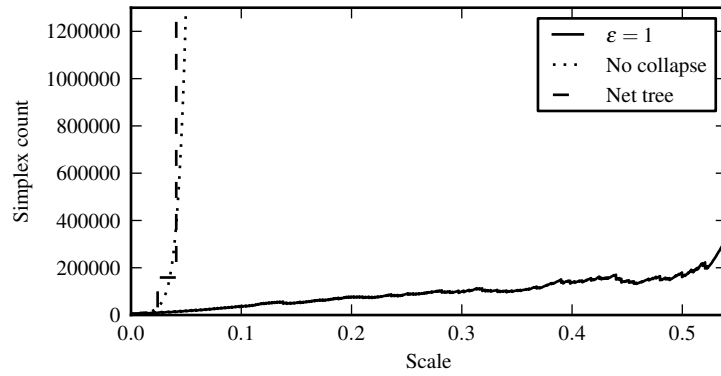


Figure 10: The simplex count for the $\mathbb{R}P^2$ example from Section 5.2.2 compared to that of an ordinary Čech filtration and the net tree approach (with $\alpha_0 = 10^{-3}$ and $\epsilon = 0.7$ in the notation of Section 4.1).

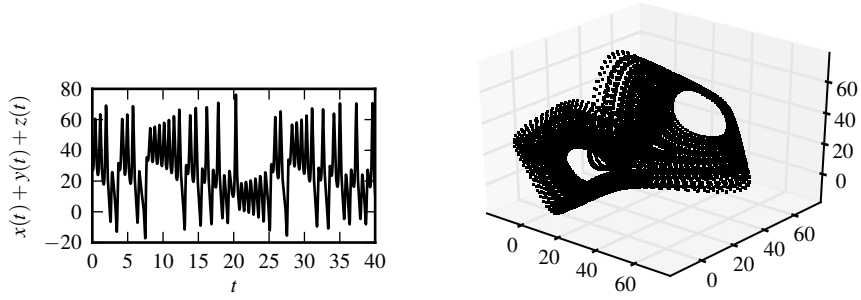


Figure 11: Lorenz system scalar measurements (parts shown on the left) and delay-embedding reconstructed attractor (right), as detailed in Section 5.2.3.

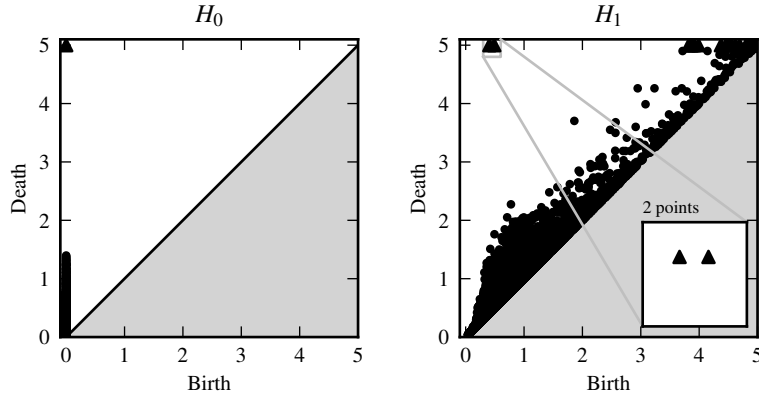


Figure 12: Persistence diagrams for the Lorenz attractor described in Section 5.2.3.

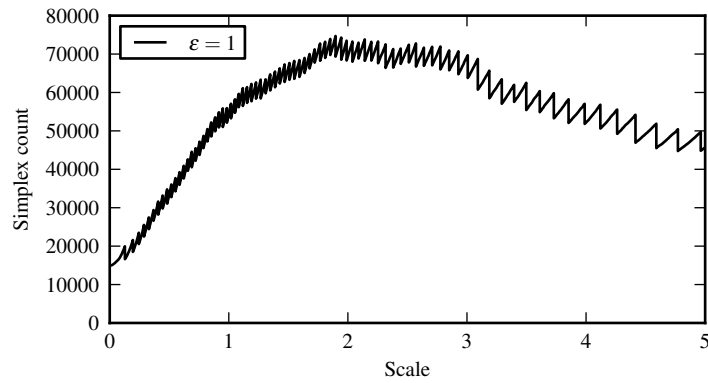


Figure 13: Simplex count for the Lorenz attractor computations described in Section 5.2.3.

superior theoretical guarantees enjoyed by the net-tree construction. By approximating the Čech filtration through representatives we lose much of the theoretical guarantees, but the frequent collapses allow for much greater maximum scales.

We believe that an interesting direction for future work is to find other approximations than choosing a representative for each covering set. This could be done either by choosing multiple representative points, or by using the embedding to approximate the covering sets by sets for which computing k -intersections is tractable.

The proofs in this paper also rely heavily on the notion of good covers. In general metric spaces a cover by a union of balls may fail to be good, and the Nerve lemma is lost. It would be interesting to see if there are similar results without this precondition. We believe it should be so, as the net-tree construction for the Vietoris–Rips filtration extends to general metric spaces.

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