# On Division Polynomial PIT and Supersingularity

### Javad Doliskani<sup>1</sup>

<sup>1</sup>Institute for Quantum Computing, University of Waterloo

#### **Abstract**

For an elliptic curve E over a finite field  $\mathbb{F}_q$ , where q is a prime power, we propose new algorithms for testing the supersingularity of E. Our algorithms are based on the Polynomial Identity Testing (PIT) problem for the p-th division polynomial of E. In particular, an efficient algorithm using points of high order on E is given.

Key words: Division polynomials; Polynomial Identity Testing; Elliptic curves

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### 1 Introduction

Recent cryptographic initiatives based on supersingular elliptic curves have attracted considerable attention [7, 8, 3]. In particular, the underlying security assumption in such constructions is the hardness of computing isogenies between two curves. Various attempts on solving the Supersingular Isogeny Problem has led to a more extensive study of supersingular elliptic curves. One of the interesting algorithmic questions about an elliptic curve is to efficiently decide whether it is ordinary or supersingular. In this paper, we propose efficient solutions for this question.

An elliptic curve  $E/\mathbb{F}_q$  is supersingular if its Hasse invariant is zero, and ordinary otherwise. The Hasse invariant is said to be zero if the map  $\pi^*: H^1(E, \mathscr{O}_E) \to H^1(E, \mathscr{O}_E)$  induced by the Frobenius  $\pi: E \to E$  is zero. Other equivalent definitions of supersingularity can be derived from this one. For example, given the short Weierstrass equation  $y^2 = x^3 + ax + b$  of E, the Hasse invariant is zero if and only if the coefficient of  $x^{p-1}$  in  $(x^3 + ax + b)^{(p-1)/2}$  is zero. One of the more usual definitions of supersingularity is based on the p-torsion subgroup E[p] of E. If E[p] = 0 (equivalently  $E[p^n] = 0$ ) then E is supersingular. See [6] for an introduction on elliptic curves and other definitions for supersingularity.

It can be shown that the j-invariants of all supersingular elliptic curves  $E/\overline{\mathbb{F}}_p$  reside in the quadratic extension  $\mathbb{F}_{p^2}$ . Since isomorphism classes of elliptic curves are uniquely determined by their j-invariant, it follows that every supersigular elliptic curve can be defined over  $\mathbb{F}_{p^2}$ . So for the rest of this paper we assume  $\mathbb{F}_q = \mathbb{F}_{p^2}$ . Note that an elliptic curve E with j(E) = 0 (resp. j(E) = 1728) is supersingular if and only if  $p = 2 \mod 3$  (resp.  $p = 3 \mod 4$ ) [16, Theorem 4.1.c]. Since there is only one supersingular curve for p = 2 and p = 3, which has j-invariant j = 0, we assume that p > 3.

Denote by M(n) the cost of multiplying two polynomials of degree n over  $\mathbb{F}_q$  where the unit cost is operations in  $\mathbb{F}_q$ . We can take  $M(n) = O(n \log n \log \log n)$  [22]. We also assume that two n-bit integers can be multiplied using  $O(n \log n \log \log n)$  bit operations [22]. Since  $\mathbb{F}_q/\mathbb{F}_p$  is only a quadratic extension, multiplication in  $\mathbb{F}_q$  is performed using a few multiplications in  $\mathbb{F}_p$ . Therefore,

polynomial multiplication over  $\mathbb{F}_q$  has the same asymptotic cost as that over  $\mathbb{F}_p$ , and we shall always count the number of operations in  $\mathbb{F}_p$ .

Given  $E/\mathbb{F}_q$ , an obvious algorithm to determine whether E is ordinary or supersingular is to do the exponentiation above and check the coefficient of  $x^{p-1}$ . The exponentiations algorithm then takes O(M(p)) operations in  $\mathbb{F}_q$ . A slightly better complexity O(p) can be obtained by using the Deuring polynomial  $H_p(x)$  [16, Theorem 4.1.b]. These algorithms are exponential in  $\log p$ , and we wish to have algorithms with complexity at most polynomial in  $\log p$ .

The Frobenius endomorphism  $\pi$  of E satisfies  $\pi^2 - t\pi + p^2 = 0$  in the endomorphism ring  $\operatorname{End}(E)$ . The number t is called the trace of the Frobenius. One can show that E is supersingular if and only if  $t \equiv 0 \pmod{p}$ . An immediate algorithm that comes to mind is then to compute t by counting the number of points on E, and test the above congruence. An efficient deterministic point counting algorithm is due to [12], which takes  $\tilde{O}(\log^5 p)$  bit operations. The notation  $\tilde{O}$  means ignoring logarithmic factors in the main complexity parameter. A Las Vegas variant of Schoof's algorithm, called SEA, was proposed by Elkies and Atkin. Under some heuristic assumption, the SEA algorithm runs in an expected  $\tilde{O}(\log^4 p)$  bit operations.

If one is content with a high-probability output then a simple *Monte Carlo* test can be performed as follows. The curve E is supersingular if and only if  $E(\mathbb{F}_q) \cong (\mathbb{Z}/(p\pm 1)\mathbb{Z})^2$ . Therefore, we can pick a random point  $P \in E(\mathbb{F}_q)$  and check whether  $(p\pm 1)P = 0$ . If so, then E is supersingular with probability at least (p-1)/p [18, Prop. 2], which is very close to 1 when p is large. The cost of this test is  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$  or  $\tilde{O}(\log^2 p)$  bit operations.

The best known deterministic algorithm, up to the author's knowledge, is due to [18]. The algorithm is based on traversing the isogeny graph of E, and it runs in  $O(\log^3 p \log \log^2 p)$  bit operations. In fact, the algorithm exploits an interesting structural difference between isogeny graphs of ordinary and supersingular curves. For more details on the algorithm and on isogeny graphs we refer the reader to [18, 9]. In this paper, we propose an algorithm that can efficiently check the supersingularity of E using points of high order on E. Our main result can be summarized as follows:

**Theorem 1.** Given an elliptic curve  $E/\mathbb{F}_q$ , there exists a Las Vegas algorithm that can efficiently decide whether E is supersingular or ordinary. On input an ordinary curve, the algorithm runs in an expected  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$ . On input a supersingular curve, under the generalized Riemann hypothesis, the algorithm runs in an expected  $\tilde{O}(\log^2 p)$  operations in  $\mathbb{F}_p$ .

Note that, ignoring the logarithmic factors, our result does not asymptotically improve on the state-of-the-art algorithm [18]. However, as experiments in Section 5 show, our algorithm attains better runtimes in practice. There is a conjecture of Poonen, discussed in Section 4, about orders of points in subvarieties of semiabelian varieties. If the conjecture holds, our algorithm always runs in an expected  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$ . This is a significant improvement on the algorithm of [18] both in theory and in practice.

**Polynomial identity testing (PIT)** Given a field K, an arithmetic circuit with n variables over K is a directed acyclic graph with the leaves considered as input variables and the root as output. The operations on the input, which are implemented by the internal nodes (gates), consist of only addition and multiplication in K. Here, the edges act as wires. Therefore, a circuit implements a polynomial function in  $K[x_1, \ldots, x_n]$ . The size of a circuit C is defined as the number of gates in C. The polynomial identity testing can be formally stated as:

PIT Problem: Let  $f \in K[x_1, ..., x_n]$  be a polynomial given by the arithmetic circuit C. Find a deterministic algorithm with complexity poly(size(C)) operations in K that tests if f is identically

zero.

The above is equivalent to testing  $f_1 = f_2$  for two given polynomials  $f_1, f_2 \in K[x_1, \dots, x_n]$ . A very efficient probabilistic algorithm is derived from the following theorem [13, 24].

**Lemma 2** (Schwartz-Zippel). Let  $f \in K[x_1, ..., x_n]$  be of degree  $d \ge 0$ . Let  $S \subseteq K$  be a finite subset, and let  $\mathbf{s} \in S^n$  be a point with coordinates chosen independently and uniformly at random. Then  $\Pr[f(\mathbf{s}) = 0] \le d/|S|$ .

If the degree d is small compared to |S| then, by the theorem, evaluating f at a random point tells if f = 0 with high probability. We will use this theorem in Section 3. For a survey on polynomial identity testing see [11].

## 2 Division polynomials

The *n*-th division polynomial  $\psi_n$  of E is an element of the function field K(E) of E with divisor  $(\psi_n) = [n]^* \infty - n^2 \infty$ . The map  $[n]^*$  is induced by the multiplication-by-n endomorphism on the divisor class group Div(E). Division polynomials can be defined using recursive relations as follows. Given  $E: y^2 = x^3 + ax + b$  we have

$$\psi_{0} = 0 
\psi_{1} = 1 
\psi_{2} = 2y 
\psi_{3} = 3x^{4} + 6ax^{2} + 12bx - a^{2} 
\psi_{4} = 4y(x^{6} + 5ax^{4} + 20bx^{3} - 5a^{2}x^{2} - 4abx - 8b^{2} - a^{3}) 
\psi_{2m+1} = \psi_{m+2}\psi_{m}^{3} - \psi_{m-1}\psi_{m+1}^{3} \text{ for } m \ge 2 
\psi_{2m} = (2y)^{-1}(\psi_{m+2}\psi_{m-1}^{2} - \psi_{m-2}\psi_{m+1}^{2})\psi_{m} \text{ for } m \ge 3.$$
(1)

We also define the following polynomials which will be used in the subsequent sections.

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1} 
\omega_m = (4y)^{-1}(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2)$$
(2)

It follows from the definition that a point  $P \in E(\overline{\mathbb{F}}_p)$  is an *n*-torsion if and only if  $\psi_n(P) = 0$ . In other words, the division polynomial  $\psi_n$  exactly encodes the whole *n*-torsion E[n]. Division polynomials play an important role in the theory of elliptic curves. They are also heavily used in implementations of the point counting algorithm of [12].

#### 2.1 Computing division polynomials

In this subsection, we briefly review an algorithm that efficiently computes the division polynomials. To compute  $\psi_n$ , the idea is to simply use the recursive relations (1) to achieve a double-and-add scheme on the subscript n. First, we should note that it is possible to characterize non-2-torsion points with univariate versions of the  $\psi_n$ . This makes computations much easier. Define

$$f_m = \begin{cases} \psi_m & m \text{ odd} \\ \psi_m/\psi_2 & m \text{ even.} \end{cases}$$

Then for non-2-torsion  $P \in E$  we have  $P \in E[n]$  if and only if  $f_n(P) = 0$ . Let  $F = \psi_2^2 = 4(x^3 + ax + b)$ . Then the following relations for the  $f_m$  can be derived from (1).

$$f_{0} = 0$$

$$f_{1} = 1$$

$$f_{2} = 1$$

$$f_{3} = \psi_{3}$$

$$f_{4} = \psi_{4}/\psi_{2}$$

$$f_{2m+1} = \begin{cases} f_{m+2}f_{m}^{3} - F^{2}f_{m-1}f_{m+1}^{3} & m \text{ odd, } m \geq 3 \\ F^{2}f_{m+2}f_{m}^{3} - f_{m-1}f_{m+1}^{3} & m \text{ even, } m \geq 2 \end{cases}$$

$$f_{2m} = (f_{m+2}f_{m-1}^{2} - f_{m-2}f_{m+1}^{2})f_{m} \text{ for } m \geq 3.$$

$$(3)$$

From the indices involved in the above relations it is immediate that given  $f_{i-3}, \ldots, f_{i+5}$ , one can compute the polynomials  $f_{2i-3}, \ldots, f_{2i+5}$ , or the polynomials  $f_{2(i+1)-3}, \ldots, f_{2(i+1)+5}$ . We can save some multiplications by introducing  $S_i = f_{i-1}f_{i+1}$ ,  $T_i = f_i^2$  and rewriting (3) as

$$f_{2m+1} = \begin{cases} T_m S_{m+1} - F^2 T_{m+1} S_m & m \text{ odd, } m \ge 3 \\ F^2 T_m S_{m+1} - T_{m+1} S_m & m \text{ even, } m \ge 2 \end{cases}$$

$$f_{2m} = T_{m-1} S_{m+1} - T_{m+1} S_{m-1} \text{ for } m \ge 3.$$

$$(4)$$

We shall only need modular computation of division polynomials in this paper. Therefore, Algorithm 1 performs computations mod f for a given polynomial  $f \in \mathbb{F}_q[x]$ .

### Algorithm 1 Division polynomial computation

```
Input: Integer m \geq 1, and polynomial f(x) \in \mathbb{F}_q[x] of degree n

Output: The division polynomials f_{m-3}, \ldots, f_{m+5} \pmod{f}

1: Let b_k b_{k-1} \cdots b_0 be the binary representation of m

2: Set r, s as follows

3: r = 3, s = 2 if b_k b_{k-1} = 11

4: r = 4, s = 3 if b_k b_{k-1} b_{k-2} = 100

5: r = 5, s = 3 if b_k b_{k-1} b_{k-2} = 101

6: Set j = r, and compute f_{j-3}, \ldots, f_{j+5} \pmod{f}

7: for i = k - s down to 0 do

8: Compute f_{2j+b_i-3}, \ldots, f_{2j+b_i+5} \pmod{f} from f_{j-3}, \ldots, f_{j+5} using relations (4)

9: j \leftarrow 2j + b_i

10: end for

11: return f_{m-3}, \ldots, f_{m+5}
```

The correctness of the algorithm follows from the previous remarks. The runtime is dominated by the for-loop at Step 7. The number of iterations is  $O(\log m)$ , and at each iteration a few polynomial multiplications of degree n is done at the cost of  $O(\mathsf{M}(n))$  operations in  $\mathbb{F}_p$ . Therefore, the total runtime is  $O(\mathsf{M}(n)\log m)$  operations in  $\mathbb{F}_p$ .

# 3 PIT for the p-th division polynomial

Given an elliptic curve  $E/\mathbb{F}_q$ , we have  $E[p] \cong 0$  (resp.  $E[p] \cong \mathbb{Z}/p\mathbb{Z}$ ) when E is supersingular (resp. ordinary). Therefore, the p-th division polynomial  $\psi_p$  of E is a constant  $c \in \mathbb{F}_q$  when E

is supersingular, and non-constant otherwise. This means solving the polynomial identity testing problem  $\psi_p = c$  will give an answer to the question of whether E is ordinary or supersingular.

First, we investigate the shape of  $\psi_p$  in both cases. For the general polynomial  $\psi_n$  one has

$$\psi_n = \begin{cases} y(nx^{(n^2-4)/2} + \cdots) & n \text{ even} \\ nx^{(n^2-1)/2} + \cdots & n \text{ odd,} \end{cases}$$

see [23] for more details. It follows that for  $\psi_p$ , which is a univariate polynomial, deg  $\psi_p < (p^2-1)/2$ . In fact, for ordinary curves we have

**Lemma 3.** Let  $E/\mathbb{F}_q$  be an ordinary elliptic curve and let r be the order of trace t of the Frobenius in  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then  $\deg \psi_p = p(p-1)/2$ . Also

$$\psi_p = f_1^p f_2^p \cdots f_{(p-1)/2r}^p$$

is the factorization of  $\psi_p$  over  $\mathbb{F}_q$  where all  $f_i$  are of the same degree r.

*Proof.* We have  $\pi \circ \hat{\pi} = [p]$  where  $\hat{\pi}$  is the dual<sup>1</sup> of  $\pi$ . Let  $\hat{\pi}(x,y) = (F,G)$  where F,G are rational functions in x,y. Then

$$(F^p, G^p) = \pi(F, G) = [p](x, y) = \left(\frac{\phi_p}{\psi_p^2}, \frac{\omega_p}{\psi_p^3}\right).$$
 (5)

The last equality is the formula for multiplication by p, where  $\phi_p$  and  $\omega_p$  are defined using (2). It follows that  $\psi_p$  is a p-th power. Since  $\psi_p$  has roots exactly the abscissas of the nonzero p-torsion points it must have degree p(p-1)/2, which proves the first part.

For the second part, note that the action of the Frobenius on E[p] is just multiplication by the trace t. By the first part  $\psi_p = \tilde{\psi}_p^p$  where  $\tilde{\psi}_p$  splits into factors of degree r over  $\mathbb{F}_q$ .

For supersingular elliptic curves we have

**Lemma 4.** Let  $E/\mathbb{F}_q$  be a supersingular elliptic curve with  $j(E) \neq 0,1728$ . Then  $\psi_p = \pm 1$ .

*Proof.* When E is supersingular we have  $t=\pm 2p$  so that the characteristic polynomial of the Frobenius factorizes as  $(X\pm p)^2$ . Therefore,  $\pi(x,y)=\pm [p](x,y)$ . Comparing the first coordinates and considering the multiplication-by-p formula (5) gives  $x^{p^2}=\phi_p(x)/\psi_p(x)^2$ . Since  $\phi_p(x)=x^{p^2}+\cdots$  (see [23]), it follows that  $\psi_p(x)=\pm 1$ .

It is not hard to distinguish the cases  $\psi_p = 1$  and  $\psi_p = -1$  in Lemma 4, but we are not concerned with that. In fact, it is easily done in practice by computing  $\psi_p(0)$ . Without loss of generality, we only consider the case  $\psi_p(x) = 1$ . The remainder of this section is devoted to two algorithms derived from the above results. An efficient algorithm based on points of high order is discussed in Section 4.

A probabilistic algorithm. Lemmas 3, 4 and Lemma 2 give a probabilistic algorithm for the PIT  $\psi_p(x) = 1$ : select a random element  $a \in S = \mathbb{F}_q$  and compute  $\psi_p(a)$ . If  $\psi_p(a) = 1$  then the algorithm outputs "supersingular", otherwise it outputs "ordinary". If E is ordinary, then the output is "supersingular" with probability

$$P[\psi_p(a) = 1] = P[\tilde{\psi}_p(a) = 1] \le (p-1)/2p^2 < 1/2p,$$

where  $\psi_p = \tilde{\psi}_p^p$  as in Lemma 3. The runtime of this algorithm is  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$ , which is the same as the Monte Carlo algorithm given in Section 1.

<sup>&</sup>lt;sup>1</sup>The dual of an isogeny  $\phi: E_1 \to E_2$  of degree m is a unique isogeny  $\hat{\phi}: E_2 \to E_1$  such that  $\phi \circ \hat{\phi} = [m]$ , see [16, III.6].

A Schoof-like algorithm To see whether  $E/\mathbb{F}_q$  is supersingular one can check either of the identities  $\psi_p(x) = 1$  or  $\pi = \pm [p]$ . Therefore, solving the former PIT is equivalent to testing the latter identity. This can be done using arithmetic modulo division polynomials as we show in the following. See [18, Section 2.2] for a similar algorithm.

Let  $S = \{2, 3, ..., \ell\}$  be a set of primes such that  $\prod_{r \in S} r \geq 4p$ . For this to be true we only need  $\ell = O(\log p)$ . We know that E is supersingular if and only if  $t = \pm 2p$ . Therefore, it follows from the Chinese Remainder Theorem that E is supersingular if and only if  $t \equiv \pm 2p \pmod{r}$  for all  $r \in S$ . Let  $P \in E$  be a point of prime order r. Evaluating the characteristic polynomial of  $\pi$  at P gives

$$\pi^{2}(P) - t\pi(P) + [p^{2}]P = 0.$$

If also  $\pi(P) = \pm [p]P$  then  $t \equiv \pm 2p \pmod{r}$ . So, by the above, E is supersingular if and only if  $\pi(P) = \pm [p]P$  for a point P of order r for all  $r \in \mathcal{S}$ .

The last condition can be checked using division polynomials. More precisely, it is equivalent to checking the identities

$$x^{p^2} = x - \frac{\phi_s(x)}{\psi_s^2(x)} \mod \psi_r(x), \quad \text{for all } r \in \mathcal{S},$$
(6)

where  $s = p \mod r$ . The polynomials  $\{\psi_i\}_{i \leq |S|}$  can be computed in negligible time. Since the degree of  $\phi_r(x)$  is  $r^2 = O(\log^2 p)$ , computing  $x^{p^2} \mod \psi_r$  takes  $\tilde{O}(\log^3 p)$  operations in  $\mathbb{F}_p$ . So the total cost of these checks is  $\tilde{O}(\log^4 p)$  operations in  $\mathbb{F}_p$ .

## 4 PIT using points of high order

From the identity  $\psi_p(x) = 1$  for a supersingular curve  $E/\mathbb{F}_q$  and the multiplication-by-p formula (5) we get the stronger condition

$$\psi_p(x) = 1 \psi_{p-1}\psi_{p+1}(x) = x - x^{p^2}$$
(7)

for supersingularity. Note that in this case the univariate polynomial  $\psi_{p-1}\psi_{p+1}(x)$  splits completely over  $\mathbb{F}_q$ . If these identities are not satisfied when evaluated at any point  $P \in E$  then E is ordinary. Otherwise, not much can be said about E unless the point P is chosen more carefully. More precisely, we have

**Proposition 5.** Let  $P \in E$  be a point of order r, with (r,p) = 1, that satisfies (7). Then  $\psi_p(kP) = 1$  for all odd  $1 \le k \le r - 1$ . If moreover  $\psi_p(2P) = 1$  then  $\psi_p(kP) = 1$  for all  $1 \le k \le r - 1$ .

*Proof.* For all positive integers m, n we have

$$\psi_{mn} = (\psi_m \circ [n])\psi_n^{m^2}. \tag{8}$$

This follows from comparing divisors on both sides. When k < r is odd,  $\psi_k$  is a univariate polynomial and so  $\psi_k(P)$  only involves the x-coordinate of P. Since  $x_{pP} = \phi_p/\psi_p^2$  where  $\phi_p$  is defined by (2), it follows that if  $P \in E$  satisfies (7) then  $x_{pP} = x_{\pi P}$  and hence

$$\psi_k(P)^{p^2} = \psi_k(\pi(P)) = \psi_k(pP). \tag{9}$$

Therefore, for (m, n) = (p, k) and (m, n) = (k, p) identity (8) gives

$$\psi_{kp}(P) = \psi_p(P)^{k^2} \psi_k(pP)$$

$$= \psi_k(pP),$$

$$\psi_{kp}(P) = \psi_k(P)^{p^2} \psi_p(kP)$$

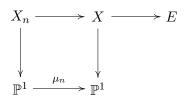
$$= \psi_k(pP) \psi_p(kP),$$

where  $\psi_k(pP) \neq 0$  since P is of order r > k. Comparing the above identities proves the first part. For the second claim we show that (9) holds for any k, and the rest of the proof is the same. Using (8) with (m,n) = (2,p) and (m,n) = (p,2) we get  $2y_{\pi P} = \psi_2(\pi(P)) = \psi_2(pP) = 2y_{pP}$  which implies  $\pi(P) = pP$ , and hence  $\psi_k(\pi(P)) = \psi_k(pP)$  for any k.

The above result enables us to check the PIT (7) using a high order point as follows. Let  $P \in E$  be a point of order r > 2p + 2. If P satisfies (7) then  $\tilde{\psi}_p(kP)^p = \psi_p(kP) = 1$  hence  $\tilde{\psi}_p(kP) = 1$  for at least p + 1 values of k. This means  $\tilde{\psi}_p(x_{kP}) = 1$  for at least (p + 1)/2 distinct abscissas of the points kP. But the univariate polynomial  $\tilde{\psi}_p$  has degree at most (p - 1)/2 so it is uniquely determined by (p + 1)/2 pairs  $(a, \tilde{\psi}_p(a))$ . It follows that  $\tilde{\psi}_p^p = 1$ , and hence  $\psi_p = 1$ . It only remains to efficiently find a point of high order on E. For this, we adapt the approach of Voloch [20, 21].

Let  $\mathbb{G}_m$  be the multiplicative group over  $\mathbb{F}_q$ . In [21], elements of high order in  $\mathbb{F}_q^*$  are obtained using points on a curve contained in the fibered product  $E \times \mathbb{G}_m$ . We can use the same technique to obtain points of high order on E. The proofs remain essentially the same except for some parts which we explain in the following. We fix an embedding  $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$  and assume all curves are projective.

Let X be an absolutely irreducible curve contained in  $E \times \mathbb{P}^1$ . Also assume that X has non-constant projections to both factors and denote by D the degree of  $X \to \mathbb{P}^1$ . Consider the pullback diagram



where  $X_n$  is the fibered product  $X \times \mathbb{P}^1$ . The bottom morphism corresponds to the field extension  $\mathbb{F}_q(u)/\mathbb{F}_q(t)$  with  $u^n = t$ . If n is coprime to Dp then the morphism  $X_n \to X$  is separable of degree n, and  $X_n$  is also absolutely irreducible. The morphism  $X_n \to E$  obtained by composing the top two morphisms determines an element  $y_n$  in the function field  $K_n = K(X_n)$ . Assume that all such  $y_n$  are elements of some fixed algebraic closure of K(X). Then Lemma 2.2 in [21] becomes

**Lemma 6.** The functions  $y_n$ , considered as morphisms  $X_n \to E$  with (n, Dp) = 1, are  $\mathbb{Z}$ -linearly independent.

*Proof.* Given  $\{y_{n_i}\}_{1\leq i\leq s}$ , let L be the compositum of the function fields  $K_{n_i}$ , and let  $C_L$  be the smooth curve with function field L. We have an isomorphism

$$E(L) \cong \operatorname{Hom}_{\mathbb{F}_q}(C_L, E),$$
 (10)

where the right hand side is the group of morphisms of k-schemes. This isomorphism holds if  $(L, C_L)$  are replaced by any  $(K_{n_i}, X_{n_i})$ . To linearize the addition of the  $y_{n_i}$  on E to addition of

differentials in  $\Omega_{C_L}$  we consider the pullbacks  $\omega_{n_i} = y_{n_i}^*(\omega_E)$  of the invariant differential  $\omega_E$  of E. Since  $C_L \to X_{n_i}$  is separable for all  $1 \le i \le s$ , the induced morphisms

$$y_{n_i}^*: \Omega_{X_{n_i}} \longrightarrow \Omega_{C_L}$$
 (11)

are injective. Now it follows from (10) and (11) that the  $\omega_{n_i}$  reside in different extensions inside  $\Omega_{C_L}$  so they must be  $\mathbb{Z}$ -linearly independent.

We need Lemma 2 from [20] which we state here for convenience.

**Lemma 7.** Let  $m, q \geq 2$  be fixed integers and  $\epsilon > 0$  a real number. For an integer  $r \geq 2$  with (r, mq) = 1, if d is the order of  $q \mod r$ , then, given N < d, there is a coset  $\Gamma$  of  $\langle q \rangle \subset (\mathbb{Z}/r)^*$  with

$$\#\{n \mid 1 \le n \le N, (n, m) = 1, n \mod r \in \Gamma\} \gg Nd^{1-\epsilon}/r - r^{\epsilon}.$$

Following the notation of [21], for a function field  $L/\mathbb{F}_q$  we define  $\deg_L z$  to be the degree of the divisor of zeros of z in L. So  $\deg_{K_n} y_n \ll n$ .

**Theorem 8.** Let  $X \subset E \times \mathbb{G}_m$  be an absolutely irreducible curve over  $\mathbb{F}_q$  with non-constant projections to both factors. Given  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $(P, b) \in X$  satisfies

- (i)  $d = [\mathbb{F}_q(b) : \mathbb{F}_q]$  sufficiently large,
- (ii)  $\langle P \rangle$  invariant under the action of  $\pi$ ,
- (iii) the order r of b satisfies  $r < d^{2-\epsilon}$

then P has order at least  $\exp(\delta(\log d)^2)$ .

Proof. A slight modification of the proof of [21, Theorem 1.1] works here. Let  $N = [d^{1-\epsilon}]$ . Then using the parameters in the conditions of the theorem we get a coset  $\Gamma = \gamma \langle q \rangle$  from Lemma 7. Let  $c \in \overline{\mathbb{F}}_q$  be an r-th root of unity such that  $c^{\gamma} = b$ . For an integer  $n \leq N$  with (n,q) = 1 and  $n \mod r \in \Gamma$  we have, by construction,  $n \equiv \gamma q^j \mod r$  for some j. Let J be the set of j's obtained for all such n. For simplicity, denote also by  $\pi$  the  $\mathbb{F}_q$ -Frobenius map on  $E \times \mathbb{G}_m$ . Then for  $j \in J$  we have

$$\pi^{j}(P,b) = (\pi^{j}(P), \pi^{j}(b)) = (\pi^{j}(P), b^{q^{j}}) = (\pi^{j}(P), c^{n_{j}})$$

where  $n_j$  corresponds to j, that is  $n_j \leq N$ ,  $(n_j, q) = 1$ ,  $n_j \mod r \in \Gamma$  and  $n_j \equiv \gamma q^j \mod r$ . This means that there is a place of  $K_{n_j}$  above c where  $y_{n_j}$  takes the value  $\pi^j(P)$ .

For a subset  $I \subset J$  define  $P_I = \sum_{j \in I} \pi^j(P)$ . Note that  $P_I$  is in  $\langle P \rangle$  since  $\langle P \rangle$  is invariant under  $\pi$ . Let  $T = [\eta \log d]$  for some real parameter  $\eta > 0$ . We show that for distinct  $I \subset J$  with  $|I| \leq T$  the points  $P_I$  are distinct. Assume  $P_I = P_{I'}$  for  $I \neq I'$ . Define

$$z = \sum_{j \in I} y_{n_j} - \sum_{j \in I'} y_{n_j}$$

where the addition occurs on E. Let L be the compositum of  $\{K_{n_j}\}_{j\in I\cup I'}$ . Then z vanishes at a place of L above c. But we have

$$\deg_L z \le \sum_{j \in I \cup I'} \deg_L y_j = \sum_{j \in I \cup I'} [L : K_{n_j}] \deg_{K_{n_j}} y_{n_j} \ll TD^{2T} N$$

which can be made smaller than  $d = [\mathbb{F}_q(c) : \mathbb{F}_q]$  for some small  $\eta$  and all sufficiently large d. This is not possible unless z = 0 and so the  $\{y_{n_i}\}_{j \in I \cup I'}$  are  $\mathbb{Z}$ -linearly dependent, which contradicts

Lemma 6. Therefore, the number of distinct points  $P_I$  is at least  $\binom{|J|}{T}$ . Setting  $\epsilon \leftarrow \epsilon/3$  in Lemma 7 we get the same bound as in [20]:

$$|J| \gg d^{2-\epsilon/3}/r - r^{\epsilon/3} \gg d^{2\epsilon/3}$$

and 
$$\binom{|J|}{T} \ge (|J|/T - 1)^T \gg \exp(\delta(\log d)^2)$$
 for some  $\delta > 0$ .

The logarithmic term  $\log d$  in the exponent in Theorem 8 is forced by the exponent 2T in the bound  $TD^{2T}N$  obtained in the proof. For special cases of the irreducible curve X, e.g. an open subset of the graph of a morphism  $f: E \to \mathbb{P}^1$ , we can obtain much better bounds on the order of P.

**Theorem 9.** Let  $E/\mathbb{F}_q$  be an elliptic curve and let  $f: E \to \mathbb{P}^1$  be the projection to the first coordinate. Then with the assumptions of Theorem 8 and with X as an open subset of the graph of f, the point P has order at least  $\exp(d^{\delta})$ .

*Proof.* Let X be an open subset of the graph of a morphism  $f: E \to \mathbb{P}^1$  defined by projection to the first coordinate, that is  $f(P) = x_P$  for any point P on E. A point  $(P, x_P)$  is in  $X_n$  if  $(P, x_P^n)$  is in X. Therefore, it is implied by the commutative digram

$$K_n \longleftarrow K(X) \longleftarrow K(E)$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \downarrow$$

$$K(\mathbb{P}^1) \longleftarrow^{\mu_n^*} K(\mathbb{P}^1)$$

that  $y_n = (x^n, y)$ . So, following the proof of Theorem 8, a much smaller bound  $\deg_L z \ll TDN$  is obtained. This means we can choose a larger value of T, say  $T = [d^{\eta}]$  following the notation of [21]. Now the calculation of Theorem 8 implies that P has order at least  $\exp(d^{\delta})$  for some suitable  $\delta > 0$ .

One concludes from the above theorems and the ones in [21] that given a point  $P \in E$  and a function f on E, under some conditions, one of P, f(P) has large order in its respective group. Therefore, forcing f(P) to be of small order in  $\mathbb{F}_q^*$  yields P of large order in E. For appropriate choices of  $\epsilon$  in Theorem 9, one gets large enough  $\delta$ , and hence P of large order, without having to choose d very large. Experiments show, however, that the lower bound of the theorem is very far from optimal.

From the proof of Theorem 8 we see that the point P has order at least  $\binom{|J|}{T}$  where  $|J| \gg d^{2-\epsilon/3}/r - r^{\epsilon/3}$  and  $T = [d^{\eta}]$  for some  $\eta > 0$ . The bound  $\exp(d^{\delta})$  is just an approximation of the binomial coefficient. In practice, we could ignore this approximation and use the value of the binomial coefficient directly. Here,  $\eta$  is a function of  $\epsilon$  and can be calculated from the inequality  $\deg_L(\sum_{j\in I} y_{n_j} - \sum_{j\in I'} y_{n_j}) \ll TDN$ . From this we obtain the rough estimate  $T \approx d^{2\epsilon}/3$  which gives  $\eta \approx 2\epsilon$ .

Let r be a prime such that p is a generator of  $(\mathbb{Z}/r\mathbb{Z})^*$ . Then  $q=p^2$  has order d=(r-1)/2, and  $d^{2-\epsilon}>r$  for a wide range of values of  $0<\epsilon<1$ . Now the r-th cyclotomic polynomial  $\Phi_r(T)$  splits into two irreducible factors of degree d over  $\mathbb{F}_q$ . Let g(T) be one of the factors so that  $K=\mathbb{F}_q[T]/g(T)$  is a field, and let t be the image of T in K. A point  $P\in E$  with first coordinate t lies in E(F) where  $[F:K]\leq 2$ . If P satisfies (7) then  $\langle P\rangle$  is invariant under the action of  $\pi$ . Then, by the above, the order of P is at least  $\binom{|J|}{T}$ . We need to choose a suitable r and optimize for  $\epsilon$  so that

$$\binom{|J|}{T} \gg \binom{d^{2-\epsilon/3}/r - r^{\epsilon/3}}{d^{2\epsilon}} \approx \binom{\lfloor r/2 \rfloor^{1-\epsilon/3} - r^{\epsilon/3}}{\lfloor r/2 \rfloor^{2\epsilon}} \ge 2p + 2.$$
 (12)

It suffices to take  $r \in O(\log p)$  and an appropriate  $\epsilon$  to obtain the bound in (12). Then P will have order large enough to imply that  $\psi_p = 1$ .

### Algorithm 2 Testing supersingularity

```
Input: An elliptic curve E over \mathbb{F}_q
Output: True if E is supersingular, and false otherwise
 1: if j(E) = 0 then
        If p = 2 \mod 3 then return true, otherwise return false
 3: end if
 4: if j(E) = 1728 then
        If p = 3 \mod 4 then return true, otherwise return false
 6: end if
 7: Compute \psi_p(a) for a random a \in \mathbb{F}_q.
 8: if \psi(a) \neq \pm 1 then
        return false
10: end if
11: Find r \in O(\log p) such that p generates (\mathbb{Z}/r\mathbb{Z})^* and such that (12) holds
12: Obtain an irreducible factor g(x) of the r-th cyclotomic polynomial \Phi_r(x) over \mathbb{F}_q
13: Compute x^{p^2}, f_p, f_{p-1}, f_{p+1} \mod g(x) using Algorithm 1
14: if (7) holds then
        return true
15:
16: else
17:
        return false
18: end if
```

To analyze Algorithm 2 we need the following theorem [5, 10, 4].

**Theorem 10.** Let S(p,x) be the number of primes  $r \leq x$  such that  $p \mod r$  is a generator of  $(\mathbb{Z}/r\mathbb{Z})^*$ . Assuming the generalized Riemann hypothesis, we have

$$S(p,x) \approx C(p) \frac{x}{\log(x)}$$

where C(p) is a constant depending on p.

The constant C(p) in Theorem 10 can be explicitly written as  $C(p) = (1 + 1/(p^2 - p - 1))C_{Artin}$  where  $C_{Artin} = 0.3739558136...$  is called Artin's constant. The theorem simply states that the density of primes for which a given prime is a primitive root is roughly C(p). Theorem 1 follows from the following proposition.

**Proposition 11.** Algorithm 2 is correct, and on input an ordinary curve, runs in an expected  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$ . On input a supersingular curve, assuming the generalized Riemann hypothesis, the algorithm runs in an expected  $\tilde{O}(\log^2 p)$  operations in  $\mathbb{F}_p$ .

*Proof.* Step 7 is done using  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$  using Algorithm 1. Since most curves  $E/\mathbb{F}_q$  are ordinary and they almost always fail to satisfy the condition  $\psi(a) \neq \pm 1$ , this will be the average-case complexity of the algorithm. According to Theorem 10, the integer r in Step 11 always exists and it can be computed in negligible time.

The cyclotomic polynomial  $\Phi_r(x)$  in Step 12 can be factored using  $\tilde{O}(\log^2 p)$  operations in  $\mathbb{F}_p$  [14]. Step 13 is performed at the cost of  $\tilde{O}(\log^2 p)$  operations in  $\mathbb{F}_p$  using Algorithm 1. Therefore, the worst-case complexity of Algorithm 2 is  $\tilde{O}(\log^2 p)$  operations in  $\mathbb{F}_p$ .

		Ordinary		Supersingular	
$\#\mathbb{F}_p$ (bits)	Sage pt-cnt	Alg 2	IsoGr	Alg 2	$\operatorname{IsoGr}$
33	0.035000	0.004	0.003	0.118	0.180
65	0.396000	0.007	0.006	0.566	0.595
129	5.185000	0.007	0.006	1.074	2.783
257	141.3410	0.019	0.012	5.609	12.59
385	1839.875	0.030	0.024	19.87	32.47
513	3820.591	0.049	0.039	35.87	67.66
641	32393.92	0.074	0.055	102.1	140.2
769	49644.34	0.135	0.084	157.2	256.4
897	96446.71	0.162	0.110	224.2	384.8
1025	169138.5	0.219	0.134	322.1	556.9

Table 1: Experiments (times are in seconds)

**Remark.** Our experiments have been better estimated by the following stronger lower bound conjectured by Poonen<sup>2</sup>.

Conjecture 12 (Poonen). Let  $X/\mathbb{F}_q$  be a semiabelian variety and let  $Y \subset X$  be a closed subvariety. Let Z be the union of all translates of positive-dimensional semiabelian varieties  $X'/\overline{\mathbb{F}}_q$  contained in X. Then there is a constant c > 0 such that for every nonzero  $x \in (X - Z)(\overline{\mathbb{F}}_q)$ , x has order at least  $(\#\mathbb{F}_q(x))^c$ .

Chang et al.[2] have obtained strong results indirectly confirming the above conjecture for general varieties. In our context, this implies that if  $P \in E(\mathbb{F}_q)$  does not lie in any subfield, and f is a non-constant function on E, then either P or f(P) has order at least  $(\#\mathbb{F}_q(P))^c$ .

The point  $P \in E(K)$  used in Algorithm 2 also satisfies the hypothesis of Conjecture 12. If the conjecture holds, then P has order  $\geq q^{rc} = p^{2rc}$ . This means we only need to take  $r \approx 1/2c$ . In this case, Algorithm 2 always runs in  $\tilde{O}(\log p)$  operations in  $\mathbb{F}_p$ . Although, according to our experiments, values of r obtained this way are small, in theory we do not know of any explicit bounds on c. This does not allow us to use Conjecture 12 for testing the supersingularity of E.

# 5 Experiments

We have implemented Algorithm 2 of this paper and Algorithm 2 of [18] both in C++. The arithmetic in  $\mathbb{F}_{p^2}[x]$  is done using the NTL library [15]. The timings are obtained on a single core of an AMD FX(tm)-8120 at 1.4GHz on a Linux machine. Table 1 compares the runtimes for different sizes of the base field  $\mathbb{F}_p$ . The first column is the size of a randomly selected prime p in bits.

For each prime, we have generated 10 random ordinary and 10 random supersingular curves using Sage [17]. Since most of the elliptic curves  $E/\mathbb{F}_q$  are ordinary, generating random ordinary curves amounts to simply choosing random coefficients a, b for the Weierstrass equation. Generating supersingular curves can be efficiently done using the Complex Multiplication method in [1]. As pointed out in [18], one can start from a curve constructed by the method of [1] and take a random walk in the 2-isogeny graph to get a random supersingular curve.

The average times for each set of curves are listed in columns "Ordinary" and "Supersingular". Columns "Alg. 2" and "IsoGr" refer to Algorithm 2 of Section 4 and the one in [18], respectively.

<sup>&</sup>lt;sup>2</sup>See [20].

As the timings suggest, complexities of the two algorithms differ only by a constant factor, which confirms the theory. Also detecting ordinary curves is substantially faster on average than detecting supersingular curves, which again confirms the complexities claimed in Proposition 11.

The second column shows timings for the point counting algorithm in Sage 7.5.1. The supersingularity test in Sage is done using a call to the point counting subroutine. Since Sage performs naive point counting over the extension  $\mathbb{F}_q$ , we have used the more efficient underlying subroutine \_pari\_().ellsea() from PARI [19].

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#### References

- [1] Reinier Bröker. Constructing supersingular elliptic curves. J. Comb. Number Theory, 1(3):269–273, 2009.
- [2] Mei-Chu Chang, Bryce Kerr, Igor E Shparlinski, and Umberto Zannier. Elements of large order on varieties over prime finite fields. *Journal de Théorie des Nombres de Bordeaux*, 26(3):579–593, 2014.
- [3] Denis X Charles, Kristin E Lauter, and Eyal Z Goren. Cryptographic hash functions from expander graphs. *Journal of Cryptology*, 22(1):93–113, 2009.
- [4] Steven R Finch. Mathematical constants, volume 93. Cambridge university press, 2003.
- [5] Christopher Hooley. On artins conjecture. J. reine angew. Math, 225(209-220):248, 1967.
- [6] D. Husemöller. Elliptic curves, volume 111 of Graduate texts in Mathematics. Springer, 1987.
- [7] David Jao and Luca De Feo. Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies. In *International Workshop on Post-Quantum Cryptography*, pages 19–34. Springer, 2011.
- [8] David Jao and Vladimir Soukharev. Isogeny-based quantum-resistant undeniable signatures. In *International Workshop on Post-Quantum Cryptography*, pages 160–179. Springer, 2014.
- [9] David Kohel. Endomorphism rings of elliptic curves over finite fields. PhD thesis, University of California at Berkeley, 1996.
- [10] Keith R Matthews. A generalisation of artin's conjecture for primitive roots. *Acta arithmetica*, 29:113–146, 1976.
- [11] Nitin Saxena. Progress on polynomial identity testing. Bulletin of the EATCS, 99:49–79, 2009.
- [12] René Schoof. Elliptic curves over finite fields and the computation of square roots mod p.  $Math.\ Comp.,\ 44(170):483-494,\ 1985.$
- [13] Jacob T Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *Journal of the ACM (JACM)*, 27(4):701–717, 1980.

- [14] Victor Shoup. Fast construction of irreducible polynomials over finite fields. *Journal of Symbolic Computation*, 17(5):371–391, 1994.
- [15] Victor Shoup et al. NTL: A library for doing number theory, 2016.
- [16] Joseph H Silverman. The arithmetic of elliptic curves, volume 106. Springer Science & Business Media, 2009.
- [17] William Stein et al. Sage: Open source mathematical software. 7 December 2009, 2016.
- [18] Andrew V Sutherland. Identifying supersingular elliptic curves. *LMS Journal of Computation and Mathematics*, 15:317–325, 2012.
- [19] The PARI Group, Bordeaux. PARI/GP, version 2.8.0, 2016.
- [20] José Felipe Voloch. On the order of points on curves over finite fields. *Integers: Electronic Journal Of Combinatorial Number Theory*, 7(A49):1, 2007.
- [21] José Felipe Voloch. Elements of high order on finite fields from elliptic curves. Bulletin of the Australian Mathematical Society, 81(03):425–429, 2010.
- [22] Joachim von zur Gathen and Jurgen Gerhard. *Modern Computer Algebra*. Cambridge University Press, New York, NY, USA, 1999.
- [23] Lawrence C Washington. Elliptic curves: number theory and cryptography. CRC press, 2008.
- [24] Richard Zippel. Probabilistic algorithms for sparse polynomials. Springer, 1979.