# A note on the generalized Hamming weights of Reed-Muller codes 

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#### Abstract

In this note, we give a very simple description of the generalized Hamming weights of Reed-Muller codes. For this purpose, we generalize the well-known Macaulay representation of a nonnegative integer and state some of its basic properties. Keywords: Reed-Muller code, Macaulay decomposition, generalized Hamming weight. MSC: 11H71, 94B27


## 1 Preliminaries

Let $\mathbb{F}_{q}$ be the finite field with $q$ elements and denote by $\mathbb{A}^{m}:=\mathbb{A}^{m}\left(\mathbb{F}_{q}\right)$ the $m$-dimensional affine space defined over $\mathbb{F}_{q}$. This space consists of $q^{m}$ points $\left(a_{1}, \ldots, a_{m}\right)$ with $a_{1}, \ldots, a_{m} \in$ $\mathbb{F}_{q}$. Let $T(m):=\mathbb{F}_{q}\left[x_{1}, \ldots, x_{m}\right]$ denote the ring of polynomials in $m$ variables and coefficients in $\mathbb{F}_{q}$. Further let $T_{\leq d}(m)$ be the set of polynomials in $T(m)$ of total degree at most $d$. A monomial $X_{1}^{\alpha_{1}} \cdots X_{m}^{\alpha_{m}}$ is called reduced if $\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in\{0,1, \ldots, q-1\}^{m}$. Similarly a polynomial $f \in T(m)$ is called reduced if it is an $\mathbb{F}_{q}$-linear combination of reduced monomials. We denote the set of reduced polynomials by $T^{\mathrm{red}}(m)$ and define $T_{\leq d}^{\mathrm{red}}(m):={ }_{\leq d}^{T}(m) \cap$ $T^{\mathrm{red}}(m)$.

One reason for considering reduced polynomials comes from coding theory. Indeed ReedMuller codes are obtained by evaluating certain polynomials in the points of $\mathbb{A}^{m}$, but the evaluation map

$$
\operatorname{Ev}: T(m) \rightarrow \mathbb{F}_{q}^{q^{m}}, \text { defined by } \operatorname{Ev}(f)=(f(P))_{P \in \mathbb{A}}
$$

is not injective. However, its restriction to $T^{\mathrm{red}}(m)$ is. In fact the kernel of Ev consists precisely of the ideal $I \subset T(m)$ generated by the polynomials $x_{i}^{q}-x_{i}(1 \leq i \leq m)$. Working with reduced polynomials is simply a convenient way to take this into account, since for two reduced polynomials $f_{1}, f_{2} \in T(m)$ the equality $f_{1}+I=f_{2}+I$ holds if and only if $f_{1}=f_{2}$.

The Reed-Muller code $\mathrm{RM}_{q}(d, m)$ is the set of vectors from $\mathbb{F}_{q}^{q^{m}}$ obtained by evaluating polynomials of total degree up to $d$ in the $q^{m}$ points of $\mathbb{A}^{m}$, that is to say:

$$
\operatorname{RM}_{q}(d, m):=\left\{(f(P))_{P \in \mathbb{A}^{m}}: f \in T_{\leq d}(m)\right\} .
$$

By the above, we also have $\operatorname{RM}_{q}(d, m):=\left\{(f(P))_{P \in \mathbb{A}^{m}}: f \in T_{\leq d}^{\mathrm{red}}(m)\right\}$ and moreover, we have

$$
\begin{equation*}
\operatorname{dim} \mathrm{RM}_{q}(d, m)=\operatorname{dim} T_{\leq d}^{\mathrm{red}}(m) \tag{1}
\end{equation*}
$$

Reed-Muller codes $\mathrm{RM}_{q}(d, m)$ have been studied extensively for their elegant algebraic properties. Their generalized Hamming weights $d_{r}\left(\mathrm{RM}_{q}(d, m)\right)$ have been determined in 4] by Heijnen and Pellikaan. For a general linear code $C \subseteq \mathbb{F}_{q}^{n}$ these are defined as follows:

$$
d_{r}(C):=\min _{D \subseteq C: \operatorname{dim} D=r}|\operatorname{supp}(D)|,
$$

[^0]where the minimum is taken over all $r$-dimensional $\mathbb{F}_{q}$-linear subspaces $D$ of $C$ and where $\operatorname{supp}(D)$ denotes the support size of $D$, that is to say
$$
\operatorname{supp}(D):=\#\left\{i: \exists\left(c_{1}, \ldots, c_{n}\right) \in D, c_{i} \neq 0\right\} .
$$

In case of Reed-Muller codes, there is a direct relation between generalized Hamming weights and the number of common solutions to systems of polynomial equations. Indeed, if $D \subset \mathrm{RM}_{q}(d, m)$ is spanned by $\left(f_{i}(P)\right)_{P \in \mathbb{A}}$ for $f_{1}, \ldots, f_{r} \in T_{\leq d}^{\mathrm{red}}(m)$, then $\operatorname{supp}(D)=$ $q^{m}-\# \mathbf{Z}\left(f_{1}, \ldots, f_{r}\right)$ where $\mathbf{Z}\left(f_{1}, \ldots, f_{r}\right):=\left\{P \in \mathbb{A}^{m}: f_{1}(P)=\cdots=f_{r}(P)=0\right\}$ denotes the set of common zeros of $f_{1}, \ldots, f_{r}$ in the $m$-dimensional affine space $\mathbb{A}^{m}$ over $\mathbb{F}_{q}$. Therefore, if we define

$$
\begin{equation*}
\bar{e}_{r}^{\mathbb{A}}(d, m):=\max \left\{\left|\mathrm{Z}\left(f_{1}, \ldots, f_{r}\right)\right|: f_{1}, \ldots, f_{r} \in T_{\leq d}^{\mathrm{red}}(m) \text { linearly independent }\right\} \tag{2}
\end{equation*}
$$

then $d_{r}\left(\mathrm{RM}_{q}(d, m)\right)=q^{m}-\bar{e}_{r}^{\mathbb{A}}(d, m)$. Note that $T^{\text {red }}(m)$ is a vector space over $\mathbb{F}_{q}$ of dimension $q^{m}$ and that a reduced polynomial has total degree at most $m(q-1)$. Therefore $T^{\text {red }}(m)=T_{\leq m(q-1)}^{\mathrm{red}}(m)$. This implies in particular that $\mathrm{RM}_{q}(d, m)=\mathbb{F}_{q}^{q^{m}}$ for $d \geq m(q-1)$. Therefore, we will always assume that $d \leq m(q-1)$.

The result of Heijnen-Pellikaan in [4] on the value of $d_{r}\left(\mathrm{RM}_{q}(d, m)\right)$ can now be restated as follows, see for example 2].

$$
\begin{equation*}
\bar{e}_{r}^{\mathbb{A}}(d, m)=\sum_{i=1}^{m} \mu_{i} q^{m-i}, \tag{3}
\end{equation*}
$$

where $\left(\mu_{1}, \ldots, \mu_{m}\right)$ is the $r$-th $m$-tuple in descending lexicographic order among all $m$-tuples $\left(\beta_{1}, \ldots, \beta_{m}\right) \in\{0,1, \ldots, q-1\}^{m}$ satisfying $\beta_{1}+\cdots+\beta_{m} \leq d$.

Following the notation in 4, we denote with $\rho_{q}(d, m)$ the dimension of $\mathrm{RM}_{q}(d, m)$. Equation (11) implies that $\rho_{q}(d, m)=\operatorname{dim}\left(T_{\leq d}^{\mathrm{red}}(m)\right)$. In particular, we have

$$
\begin{equation*}
\rho_{q}(d, m)=\operatorname{dim}\left(T_{\leq d}(m)\right)=\binom{m+d}{d}, \text { if } d \leq q-1, \tag{4}
\end{equation*}
$$

since $T_{\leq d}(m)=T_{\leq d}^{\mathrm{red}}(m)$ if $d<q$. Here as well as later on we use the convention that $\binom{a}{b}=0$ if $a<b$. In particular we have $\rho_{q}(d, m)=0$ if $d<0$. As shown in [1, §5.4], for the general case $d \leq m(q-1)$, we have

$$
\begin{equation*}
\rho_{q}(d, m)=\operatorname{dim}\left(T_{\leq d}^{\mathrm{red}}(m)\right)=\sum_{i=0}^{d} \sum_{j=0}^{m}(-1)^{j}\binom{m}{j}\binom{m-1+i-q j}{m-1} . \tag{5}
\end{equation*}
$$

In this note, we will present an easy-to-obtain expression for $\bar{e}_{r}^{\mathbb{A}}(d, m)$ involving a certain representation of the number $\rho_{q}(d, m)-r$ that we introduce in the next section.

## 2 The $d$-th Macaulay representation with respect to $q$

Let $d$ be a positive integer. The $d$-th Macaulay (or $d$-binomial) representation, of a nonnegative integer $N$ is a way to write $N$ as sum as certain binomial coefficients. To be precise

$$
N=\sum_{i=1}^{d}\binom{s_{i}}{i},
$$

where the $s_{i}$ integers satisfying $s_{d}>s_{d-1}>\cdots>s_{1} \geq 0$. The usual convention that $\binom{a}{b}=0$ if $a<b$, is used. For example, the $d$-th Macaulay representation of 0 is given
by $0=\sum_{i=1}^{d}\binom{i-1}{i}$. Given $d$ and $N$ the integers $s_{i}$ exist and are unique. The Macaulay representation is among other things used for the study of Hilbert functions of graded modules, see for example [3. It is well known (see for example 3]) that if $N$ and $M$ are two nonnegative integers with Macaulay representations given by $\left(k_{d}, \ldots, k_{1}\right)$ and $\left(\ell_{d}, \ldots, \ell_{1}\right)$ then $N \leq M$ if and only if $\left(k_{d}, \ldots, k_{1}\right) \preccurlyeq\left(\ell_{d}, \ldots, \ell_{1}\right)$, where $\preccurlyeq$ denotes the lexicographic order.

For our purposes it is more convenient to define $m_{i}:=s_{i}-i$. We then obtain

$$
\begin{equation*}
N=\sum_{i=1}^{d}\binom{m_{i}+i}{i} \tag{6}
\end{equation*}
$$

where $m_{i}$ are integers satisfying $m_{d} \geq m_{d-1} \geq \cdots \geq m_{1} \geq-1$. The reason for this is that for $d \leq q-1$ we have $\rho_{q}(d, m)=\binom{m+d}{d}$. Therefore, we can interpret Equation (6) as a statement concerning dimensions of the Reed-Muller codes $\mathrm{RM}_{q}\left(i, m_{i}\right)$. For a suitable choice of $N$, it turns out that the $m_{i}$ completely determine the value of $\bar{e}_{r}^{\mathbb{A}}(d, m)$ if $d \leq q-1$. For $d \geq q$, even though the dimension $\rho_{q}(d, m)$ is not longer given by $\binom{m+d}{d}$, there exists a variant of the usual $d$-th Macaulay representation that turns out to be equally meaningful for Reed-Muller codes. Before stating this representation, we give a lemma.
Lemma 2.1. Let $m \geq 1$ be an integer. We have

$$
\rho_{q}(d, m)=\sum_{i=0}^{\min \{d, q-1\}} \rho_{q}(d-i, m-1) .
$$

Proof. Any polynomial $f \in T(m)$ can be seen as a polynomial in the variable $X_{m}$ with coefficients in $T(m-1)$. This implies that $T(m)=\sum_{i \geq 0} X_{m}^{i} T(m)$, where the sum is a direct sum. Similarly we can write

$$
T_{\leq d}^{\mathrm{red}}(m)=\sum_{i=0}^{\min \{d, q-1\}} X_{m}^{i} T_{\leq d-i}^{\mathrm{red}}(m-1) .
$$

The result now follows.
A consequence of this lemma is the following.
Corollary 2.2. Let $d=a(q-1)+b$ for integers $a$ and $b$ satisfying $a \geq 0$ and $1 \leq b \leq q-1$. Further suppose that $m \geq a$. Then

$$
\rho_{q}(d, m)-1=\sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_{q}(d-j(q-1)-\ell, m-j-1)+\sum_{i=1}^{b} \rho_{q}(i, m-a-1) .
$$

Proof. This follows using Lemma 2.1 repeatedly. First applying the lemma to each sum within the double summation on the right-hand side, we see that

$$
\begin{aligned}
& \sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_{q}(d-j(q-1)-\ell, m-j-1)= \\
& \sum_{j=0}^{a-1}\left(\rho_{q}(d-j(q-1), m-j)-\rho_{q}(d-(j+1)(q-1), m-j-1)\right)= \\
& \quad \rho_{q}(d, m)-\rho_{q}(d-a(q-1), m-a)=\rho_{q}(d, m)-\rho_{q}(b, m-a) .
\end{aligned}
$$

Using the same lemma to rewrite the single summation on the right-hand side in Equation (9) we see that if $m>a$

$$
\sum_{i=1}^{b} \rho_{q}(i, m-a-1)=\rho_{q}(b, m-a)-\rho_{q}(0, m-a-1)=\rho_{q}(b, m-a)-1,
$$

while if $m=a$, the single summation equals 0 and the double summation simplifies to $\rho_{q}(d, m)-1$. In either case, we obtain the desired result

We can now show the following.
Theorem 2.3. Let $N \geq 0$ and $d \geq 1$ be integers and $q$ a prime power. Then there exist uniquely determined integers $m_{1}, \ldots, m_{d}$ satisfying

1. $N=\sum_{i=1}^{d} \rho_{q}\left(i, m_{i}\right)$,
2. $-1 \leq m_{1} \leq \cdots \leq m_{d}$,
3. for all $i$ satisfying $1 \leq i \leq d-q+1$, either $m_{i+q-1}>m_{i}$ or $m_{i+q-1}=m_{i}=-1$.

Proof. We start by showing uniqueness. Suppose that

$$
\begin{equation*}
N=\sum_{i=1}^{d} \rho_{q}\left(i, m_{i}\right)=\sum_{i=1}^{d} \rho_{q}\left(i, n_{i}\right) \tag{7}
\end{equation*}
$$

and the integers $n_{1}, \ldots, n_{d}$ and $m_{1}, \ldots m_{d}$ satisfy the conditions from the theorem. First of all, if $m_{d}=-1$ or $n_{d}=-1$ then $N=0$. Either assumption implies that $\left(m_{d}, \ldots, m_{1}\right)=$ $(-1, \ldots,-1)=\left(n_{d}, \ldots, n_{1}\right)$. Indeed $n_{i} \geq 0$ or $m_{i} \geq 0$ for some $i$ directly implies that $N>0$. Therefore we from now on assume that $m_{d} \geq 0$ and $n_{d} \geq 0$. To arrive at a contradiction, we may assume without loss of generality that $n_{d} \leq m_{d}-1$.

Define $e$ to be the smallest integer such that $n_{e} \geq 0$. Equation (7) can then be rewritten as

$$
\begin{equation*}
N=\sum_{i=1}^{d} \rho_{q}\left(i, m_{i}\right)=\sum_{i=e}^{d} \rho_{q}\left(i, n_{i}\right) \tag{8}
\end{equation*}
$$

Condition 3 from the theorem implies that $n_{i-q+1}<n_{i}$ for all $i$ satisfying $e \leq i \leq d$. Now write $d-e+1=a(q-1)+b$ for integers $a$ and $b$ satisfying $a \geq 0$ and $1 \leq b \leq q-1$. With this notation, we obtain that for any $0 \leq j \leq a-1$ and $0 \leq \ell \leq q-2$ we have that

$$
n_{d-j(q-1)-\ell} \leq n_{d}-j \leq m_{d}-j-1
$$

In particular choosing $j=a-1$ and $\ell=0$, this implies that $m_{d} \geq a+n_{q-1+b} \geq$ $a+1+n_{b} \geq a$. Using these observations, we obtain from Equation (7) that

$$
\begin{equation*}
\rho_{q}\left(d, m_{d}\right) \leq N=\sum_{i=e}^{d} \rho_{q}\left(i, n_{i}\right) \leq \sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_{q}\left(d-j(q-1)-\ell, m_{d}-j-1\right)+\sum_{i=1}^{b} \rho_{q}\left(e+i-1, m_{d}-a-1\right) . \tag{9}
\end{equation*}
$$

Applying the same technique as in the proof of Corollary 2.2 we derive that

$$
\sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_{q}\left(d-j(q-1)-\ell, m_{d}-j-1\right)=\rho_{q}\left(d, m_{d}\right)-\rho_{q}\left(b+e-1, m_{d}-a\right)
$$

and Equation (9) can be simplified to

$$
\begin{equation*}
\rho_{q}\left(d, m_{d}\right) \leq \rho_{q}\left(d, m_{d}\right)-\rho_{q}\left(b+e-1, m_{d}-a\right)+\sum_{i=1}^{b} \rho_{q}\left(e+i-1, m_{d}-a-1\right) . \tag{10}
\end{equation*}
$$

For $m_{d}=a$ the right-hand side equals $\rho_{q}\left(d, m_{d}\right)-1$, leading to a contradiction. If $m_{d}>q$,
Equation (10) implies

$$
\begin{aligned}
\rho_{q}\left(b+e-1, m_{d}-a\right) & \leq \sum_{i=1}^{b} \rho_{q}\left(e+i-1, m_{d}-a-1\right) \\
& =\sum_{j=0}^{b-1} \rho_{q}\left(e+b-1-j, m_{d}-a-1\right) \\
& <\sum_{j=0}^{\min \{e+b-1, q-1\}} \rho_{q}\left(e+b-1-j, m_{d}-a-1\right) \\
& =\rho_{q}\left(b+e-1, m_{d}-a\right),
\end{aligned}
$$

where in the last equality we used Lemma [2.1] Again we arrive at a contradiction. This completes the proof of uniqueness of the $d$-th Macaulay representation with respect to $q$.

Now we show existence. Let $d, N$ and $q$ be given. We will proceed with induction on $d$. For $d=1$, note that $\rho_{q}(1, m)=m+1$ for any $m \geq-1$. Therefore, for a given $N \geq 0$, we can write $N=\rho_{q}(1, N-1)$.

Now assume the theorem for $d-1$. There exists $m_{d} \geq-1$ such that

$$
\begin{equation*}
\rho_{q}\left(d, m_{d}\right) \leq N<\rho_{q}\left(d, m_{d}+1\right) . \tag{11}
\end{equation*}
$$

Applying the induction hypothesis on $N-\rho_{q}\left(d, m_{d}\right)$, we can find $m_{d-1}, \ldots, m_{1}$ satisfying the conditions of the theorem for $d-1$. In particular we have that

1. $N-\rho_{q}\left(d, m_{d}\right)=\sum_{i=1}^{d-1} \rho_{q}\left(i, m_{i}\right)$,
2. $-1 \leq m_{1} \leq \cdots \leq m_{d-1}$,
3. $m_{i+(q-1)}>m_{i}$ for all $1 \leq i \leq d-q$.

Clearly this implies that $N=\sum_{i=1}^{d} \rho_{q}\left(i, m_{i}\right)$, but it is not clear a priori that $m_{1}, \ldots, m_{d}$ satisfy conditions 2 and 3 as well. Conditions 2 and 3 would follow once we show that $m_{d} \geq m_{d-1}$ and either $m_{d}>m_{d-q+1}$ or $m_{d}=m_{d-q+1}=-1$. First of all, if $m_{d}=-1$, then $N=0$ and $\left(m_{d}, \ldots, m_{1}\right)=(-1, \ldots,-1)$. Hence there is nothing to prove in that case. Assume $m_{d} \geq 0$. From Equation (11) and Lemma 2.1 we see that

$$
\begin{equation*}
N-\rho_{q}\left(d, m_{d}\right)<\rho_{q}\left(d, m_{d}+1\right)-\rho_{q}\left(d, m_{d}\right)=\sum_{i=1}^{\min \{d, q-1\}} \rho_{q}\left(d-i, m_{d}\right) . \tag{12}
\end{equation*}
$$

First suppose that $d \leq q-1$. First of all, Condition 3 is empty in that setting. Further, Equation (12) implies

$$
N-\rho_{q}\left(d, m_{d}\right)<\sum_{i=1}^{d} \rho_{q}\left(d-i, m_{d}\right)=\sum_{i=1}^{d-1} \rho_{q}\left(d-i, m_{d}\right)+1
$$

and hence

$$
N-\rho_{q}\left(d, m_{d}\right) \leq \sum_{i=1}^{d-1} \rho_{q}\left(d-i, m_{d}\right)=\sum_{j=0}^{d-2} \rho_{q}\left(d-1-j, m_{d}\right)<\rho_{q}\left(d-1, m_{d}+1\right)
$$

This shows that $m_{d-1} \leq m_{d}$ as desired.
Now suppose that $d \geq q$. In this situation Equation (12) implies

$$
N-\rho_{q}\left(d, m_{d}\right)<\sum_{i=1}^{q-1} \rho_{q}\left(d-i, m_{d}\right)=\sum_{j=0}^{q-2} \rho_{q}\left(d-1-j, m_{d}\right)<\rho_{q}\left(d-1, m_{d}+1\right) .
$$

Hence $m_{d-1} \leq m_{d}$ as before. Finally assume that $m_{d} \leq m_{d-q+1}$. Then by the previous and Condition 2, we have $m_{d}=m_{d-1}=\cdots=m_{d-q+1}$. Hence $N \geq \sum_{i=0}^{q-1} \rho_{q}\left(d-i, m_{d}\right)=$ $\rho_{q}\left(d, m_{d}+1\right)$ which is in contradiction with Equation (11). This concludes the induction step and hence the proof of existence.

We call the representation of $N$ in the above theorem the $d$-th Macaulay representation of $N$ with respect to $q$. One retrieves the usual $d$-th Macaulay representation letting $q$ tend to infinity. We refer to $\left(m_{d}, \ldots, m_{1}\right)$ as the coefficient tuple of this representation. A direct corollary of the above is the following.
Corollary 2.4. The coefficient tuple $\left(m_{d}, \ldots, m_{1}\right)$ of the $d$-th Macaulay representation with respect to $q$ of a nonnegative integer $N$ can be computed using the following greedy algorithm: The coefficient $m_{d-i}$ can be computed recursively (starting with $i=0$ ) as the unique integer $m_{d-i} \geq-1$ such that

$$
\rho_{q}\left(d-i, m_{d-i}\right) \leq N-\sum_{j=d-i+1}^{d} \rho_{q}\left(j, m_{j}\right)<\rho_{q}\left(d-i, m_{d-i}+1\right) .
$$

Proof. From the existence-part of the proof of Theorem 2.3 it follows directly that the given greedy algorithm finds the desired coefficients.

A further corollary is the following. As before $\preceq$ denotes the lexicographic order.
Corollary 2.5. Suppose the $N$ and $M$ are two nonnegative integers whose respective coefficient tuples are $\left(n_{d}, \ldots, n_{1}\right)$ and $\left(m_{d}, \ldots, m_{1}\right)$. Then

$$
N \leq M \text { if and only if }\left(n_{d}, \ldots, n_{1}\right) \preceq\left(m_{d}, \ldots, m_{1}\right) .
$$

Proof. Assume $\left(n_{d}, \ldots, n_{1}\right) \preceq\left(m_{d}, \ldots, m_{1}\right)$. It is enough to show the corollary in case $n_{d}<m_{d}$. We know from the previous corollary that $n_{d}$ and $m_{d}$ may be determined using the given greedy algorithm. In particular this implies that $n_{d}<m_{d}$ implies

$$
N<\rho_{q}\left(d, n_{d}+1\right) \leq \rho_{q}\left(d, m_{d}\right) \leq M
$$

Assume that $N \leq M$. We use induction on $d$. The induction basis is trivial: If $d=1$, then $m_{1}=M-1$ and $n_{1}=N-1$. For the induction step, note that $N \leq M<\rho_{q}\left(d, m_{d}+1\right)$ implies by the greedy algorithm that $n_{d} \leq m_{d}$. If $n_{d}<m_{d}$, we are done. If $n_{d}=m_{d}$, we replace $N$ with $N-\rho_{q}\left(d, m_{d}\right)$ and $M$ with $M-\rho_{q}\left(d, m_{d}\right)$ and use the induction hypothesis to conclude that $\left(n_{d}, \ldots, n_{1}\right) \preceq\left(m_{d}, \ldots, m_{1}\right)$.

## 3 A simple expression for $\bar{e}_{r}^{\mathbb{A}}(d, m)$

We are now ready to state and prove the relation between the Macaulay representation with respect to $q$ and $\bar{e}_{r}^{\mathbb{A}}(d, m)$.
Theorem 3.1. For $1 \leq r \leq \rho_{q}(d, m)$, let the $d$-th Macaulay representation of $\rho_{q}(d, m)-r$ with respect to $q$ be given by

$$
\rho_{q}(d, m)-r=\sum_{i=1}^{d} \rho_{q}\left(i, m_{i}\right) .
$$

Denoting the floor function as $\lfloor\cdot\rfloor$, we have

$$
\bar{e}_{r}^{\mathbb{A}}(d, m)=\sum_{i=1}^{d}\left\lfloor q^{m_{i}}\right\rfloor .
$$

Proof. We know from Equation (3) that we need to show that

$$
\sum_{i=1}^{d}\left\lfloor q^{m_{i}}\right\rfloor=\sum_{i=1}^{m} \mu_{i} q^{m-i}
$$

with $\left(\mu_{1}, \ldots, \mu_{m}\right)$ is the $r$-th element in descending lexicographic order among all $m$-tuples $\left(\beta_{1}, \ldots, \beta_{m}\right)$ in $\{0,1, \ldots, q-1\}^{m}$ satisfying $\beta_{1}+\cdots+\beta_{m} \leq d$. First of all note that since $r \geq 1$, we have $\rho_{q}(d, m)-r<\rho_{q}(d, m)$. In particular this implies that $m_{d} \leq m-1$. Therefore the coefficients of the $d$-tuple $\left(m_{d}, \ldots, m_{1}\right)$ are in $\{-1,0, \ldots, m-1\}$. Now for $1 \leq i \leq m+1$ define $\mu_{i}:=\left|\left\{j: m_{j}=m-i\right\}\right|$. Since the $d$-tuple $\left(m_{d}, \ldots, m_{1}\right)$ is nonincreasing by Condition 2 from Theorem 2.3 we can reconstruct it uniquely from the ( $m+1$ )-tuple $\left(\mu_{1}, \mu_{2}, \ldots, \mu_{m+1}\right)$. Moreover, Condition 3 from Theorem 2.3 implies that $\left(\mu_{1}, \ldots, \mu_{m}\right) \in\{0,1, \ldots, q-1\}^{m}$, but note that $\mu_{m+1}$ could be strictly larger than $q-1$. Further by construction we have $\mu_{1}+\cdots+\mu_{m}+\mu_{m+1}=d$, implying that $\mu_{1}+\cdots+\mu_{m} \leq d$. Note that $\mu_{m+1}$ is determined uniquely by $\left(\mu_{1}, \ldots, \mu_{m}\right)$, since $\mu_{0}=d-\mu_{1}-\cdots-\mu_{m}$. Therefore the correspondence between the $d$-tuples $\left(m_{d}, \ldots, m_{1}\right)$ of coefficients of the $d$-th Macaulay representations with
respect to $q$ of integers $0 \leq N<\rho_{q}(d, m)$ and the $m$-tuples $\left(\mu_{1}, \ldots, \mu_{m}\right) \in\{0,1, \ldots, q-1\}^{m}$ satisfying $\mu_{1}+\cdots+\mu_{m} \leq d$, is a bijection. Moreover by construction we have

$$
\sum_{i=1}^{d}\left\lfloor q^{m_{i}}\right\rfloor=\sum_{j=1}^{m+1} \mu_{j}\left\lfloor q^{m-j}\right\rfloor=\sum_{j=1}^{m} \mu_{j} q^{m-j} .
$$

What remains to be shown is that the constructed $m$-tuple coming from the integer $\rho_{q}(d, m)-r$ is in fact the $r$-th in descending lexicographic order. First of all, by Corollary 2.2 we see that for $r=1$ and $d=a q+b$ that the $m$-tuple associated to $\rho_{q}(d, m)-1$ equals $(q-1, \ldots, q-1, b, 0, \ldots, 0)$, which under the lexicographic order is the maximal $m$ tuple among all $m$-tuples $\left(\beta_{1}, \ldots, \beta_{m}\right) \in\{0,1, \ldots, q-1\}^{m}$ satisfying $\beta_{1}+\cdots+\beta_{m} \leq d$. Next we show that the conversion between $d$-tuples $\left(m_{d}, \ldots, m_{1}\right)$ to $m$-tuples ( $\mu_{1}, \ldots, \mu_{m}$ ) preserves the lexicographic order. Suppose therefore that $1 \leq r \leq s \leq \rho_{q}(d, m)$. We write $N:=\rho_{q}(d, m)-s$ and $M:=\rho_{q}(d, m)-r$. and denote their Macaulay coefficient tuples with $\left(n_{d}, \ldots, n_{1}\right)$ and $\left(m_{d}, \ldots, m_{1}\right)$. Since $N \leq M$, Corollary 2.5 implies that $\left(n_{d}, \ldots, n_{1}\right) \preceq$ $\left(m_{d}, \ldots, m_{1}\right)$. Also, since these $d$-tuples are nonincreasing, this implies that their associated $m$-tuples $\left(\nu_{1}, \ldots, \nu_{m}\right)$ and $\left(\mu_{1}, \ldots, \mu_{m}\right)$ satisfy $\left(\nu_{1}, \ldots, \nu_{m}\right) \preceq\left(\mu_{1}, \ldots, \mu_{m}\right)$. Indeed assuming without loss of generality that $\nu_{1}<\mu_{1}$ we see that $m_{i}=n_{i}=m-1$ for $d-\nu_{1} \leq i \leq d$ but $n_{i}<m_{i}=m-1$ for $i=\nu_{1}+1$. Now the desired result follows immediately.

Combining this theorem with the greedy algorithm in Corollary 2.4 it is very simple to compute values of $\bar{e}_{r}^{\mathbb{A}}(d, m)$ or equivalently of $d_{r}\left(\mathrm{RM}_{q}(d, m)\right)$. We illustrate this in the two following examples. The parameters in these example also occur in examples from 44 .
Example 3.2. Let $q=4, r=8, d=m=3$. Since $d \leq q-1$, we may work with the usual Macaulay representation when applying Theorem 3.1. We have $\rho_{q}(d, m)=\binom{6}{3}=20$ and hence

$$
\rho_{q}(d, m)-r=12=\binom{5}{3}+\binom{2}{2}+\binom{1}{1}=\rho_{4}(3,2)+\rho_{4}(2,0)+\rho_{4}(1,0)
$$

is the 3-rd Macaulay representation of 12 . Theorem 3.1 implies that $\bar{e}_{8}^{\mathbb{A}}(3,3)=4^{2}+4^{0}+4^{0}=$ 18 and hence $d_{8}\left(\mathrm{RM}_{4}(3,3)\right)=64-18=46$ in accordance with Example 6.10 in [4].
Example 3.3. Let $q=2, r=10, d=3$ and $m=5$. We have $\rho_{2}(3,5)=26$ by Equation (5) and hence applying the greedy algorithm from Corollary 2.4. we compute that

$$
\rho_{q}(d, m)-r=16=15+1+0=\rho_{2}(3,4)+\rho_{2}(2,0)+\rho_{2}(1,-1)
$$

is the 3rd Macaulay representation of 16 with respect to 2 . Theorem 3.1 implies that $\bar{e}_{10}^{\mathrm{A}}(3,3)=2^{4}+2^{0}=17$ and hence $d_{8}\left(\mathrm{RM}_{2}(3,5)\right)=32-17=15$ in accordance with Example 6.12 in (4)
Remark 3.4. Theorem 3.1 is somewhat similar in spirit as Theorem 6.8 from [4] in the sense that in both theorems a certain representation in terms of dimensions of Reed-Muller codes is used to give an expression for $d_{r}\left(\operatorname{RM}_{q}(d, m)\right)$. Where we studied decompositions of $\rho_{q}(d, m)-r$, in [4] the focus was on $r$ itself. This suggest there may exist a duality between the two approaches, but the similarities seem to stop there. The representation in [4] is not the Macaulay representation with respect to $q$ that we have used here. For us it is for example very important that each degree $i$ between 1 and $d$ occurs once in Theorem 2.3 (implying that the greedy algorithm terminates after at most d iterations), while this is not the case in Theorem 6.8 [4]. It could be interesting future work to determine if a deeper lying relationship between the two approaches exists.

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