## A note on the generalized Hamming weights of Reed–Muller codes

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#### Abstract

In this note, we give a very simple description of the generalized Hamming weights of Reed–Muller codes. For this purpose, we generalize the well-known Macaulay representation of a nonnegative integer and state some of its basic properties.

Keywords: Reed–Muller code, Macaulay decomposition, generalized Hamming weight. MSC: 11H71, 94B27

### 1 Preliminaries

Let  $\mathbb{F}_q$  be the finite field with q elements and denote by  $\mathbb{A}^m := \mathbb{A}^m(\mathbb{F}_q)$  the *m*-dimensional affine space defined over  $\mathbb{F}_q$ . This space consists of  $q^m$  points  $(a_1, \ldots, a_m)$  with  $a_1, \ldots, a_m \in \mathbb{F}_q$ . Let  $T(m) := \mathbb{F}_q[x_1, \ldots, x_m]$  denote the ring of polynomials in m variables and coefficients in  $\mathbb{F}_q$ . Further let  $T_{\leq d}(m)$  be the set of polynomials in T(m) of total degree at most d. A monomial  $X_1^{\alpha_1} \cdots X_m^{\alpha_m}$  is called reduced if  $(\alpha_1, \ldots, \alpha_m) \in \{0, 1, \ldots, q-1\}^m$ . Similarly a polynomial  $f \in T(m)$  is called reduced if it is an  $\mathbb{F}_q$ -linear combination of reduced monomials. We denote the set of reduced polynomials by  $T^{\text{red}}(m)$  and define  $T_{\leq d}^{\text{red}}(m) :=_{\leq d}^T (m) \cap T^{\text{red}}(m)$ .

One reason for considering reduced polynomials comes from coding theory. Indeed Reed–Muller codes are obtained by evaluating certain polynomials in the points of  $\mathbb{A}^m$ , but the evaluation map

 $\operatorname{Ev}: T(m) \to \mathbb{F}_q^{q^m}$ , defined by  $\operatorname{Ev}(f) = (f(P))_{P \in \mathbb{A}}$ 

is not injective. However, its restriction to  $T^{\text{red}}(m)$  is. In fact the kernel of Ev consists precisely of the ideal  $I \subset T(m)$  generated by the polynomials  $x_i^q - x_i$   $(1 \le i \le m)$ . Working with reduced polynomials is simply a convenient way to take this into account, since for two reduced polynomials  $f_1, f_2 \in T(m)$  the equality  $f_1 + I = f_2 + I$  holds if and only if  $f_1 = f_2$ .

The Reed-Muller code  $\operatorname{RM}_q(d,m)$  is the set of vectors from  $\mathbb{F}_q^{m}$  obtained by evaluating polynomials of total degree up to d in the  $q^m$  points of  $\mathbb{A}^m$ , that is to say:

$$RM_q(d,m) := \{ (f(P))_{P \in \mathbb{A}^m} : f \in T_{\leq d}(m) \}.$$

By the above, we also have  $\operatorname{RM}_q(d,m) := \{(f(P))_{P \in \mathbb{A}^m} : f \in T^{\operatorname{red}}_{\leq d}(m)\}$  and moreover, we have

$$\dim \operatorname{RM}_q(d, m) = \dim T_{\leq d}^{\operatorname{red}}(m).$$
(1)

Reed-Muller codes  $\operatorname{RM}_q(d, m)$  have been studied extensively for their elegant algebraic properties. Their generalized Hamming weights  $d_r(\operatorname{RM}_q(d, m))$  have been determined in [4] by Heijnen and Pellikaan. For a general linear code  $C \subseteq \mathbb{F}_q^n$  these are defined as follows:

$$d_r(C) := \min_{D \subseteq C: \dim D = r} |\operatorname{supp}(D)|,$$

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where the minimum is taken over all r-dimensional  $\mathbb{F}_q$ -linear subspaces D of C and where  $\operatorname{supp}(D)$  denotes the support size of D, that is to say

$$supp(D) := \#\{i : \exists (c_1, \dots, c_n) \in D, c_i \neq 0\}.$$

In case of Reed–Muller codes, there is a direct relation between generalized Hamming weights and the number of common solutions to systems of polynomial equations. Indeed, if  $D \subset \operatorname{RM}_q(d,m)$  is spanned by  $(f_i(P))_{P \in \mathbb{A}}$  for  $f_1, \ldots, f_r \in T_{\leq d}^{\operatorname{red}}(m)$ , then  $\operatorname{supp}(D) = q^m - \# Z(f_1, \ldots, f_r)$  where  $Z(f_1, \ldots, f_r) := \{P \in \mathbb{A}^m : f_1(P) = \cdots = f_r(P) = 0\}$  denotes the set of common zeros of  $f_1, \ldots, f_r$  in the *m*-dimensional affine space  $\mathbb{A}^m$  over  $\mathbb{F}_q$ . Therefore, if we define

$$\bar{e}_r^{\mathbb{A}}(d,m) := \max\left\{ |\mathsf{Z}(f_1,\ldots,f_r)| : f_1,\ldots,f_r \in T_{\leq d}^{\mathrm{red}}(m) \text{ linearly independent} \right\}, \quad (2)$$

then  $d_r(\operatorname{RM}_q(d,m)) = q^m - \bar{e}_r^{\mathbb{A}}(d,m)$ . Note that  $T^{\operatorname{red}}(m)$  is a vector space over  $\mathbb{F}_q$  of dimension  $q^m$  and that a reduced polynomial has total degree at most m(q-1). Therefore  $T^{\operatorname{red}}(m) = T_{\leq m(q-1)}^{\operatorname{red}}(m)$ . This implies in particular that  $\operatorname{RM}_q(d,m) = \mathbb{F}_q^{m}$  for  $d \geq m(q-1)$ . Therefore, we will always assume that  $d \leq m(q-1)$ .

The result of Heijnen–Pellikaan in [4] on the value of  $d_r(\mathrm{RM}_q(d, m))$  can now be restated as follows, see for example [2].

$$\bar{e}_r^{\mathbb{A}}(d,m) = \sum_{i=1}^m \mu_i q^{m-i},\tag{3}$$

where  $(\mu_1, \ldots, \mu_m)$  is the *r*-th *m*-tuple in descending lexicographic order among all *m*-tuples  $(\beta_1, \ldots, \beta_m) \in \{0, 1, \ldots, q-1\}^m$  satisfying  $\beta_1 + \cdots + \beta_m \leq d$ .

Following the notation in [4], we denote with  $\rho_q(d, m)$  the dimension of  $\operatorname{RM}_q(d, m)$ . Equation (1) implies that  $\rho_q(d, m) = \dim(T_{\leq d}^{\operatorname{red}}(m))$ . In particular, we have

$$\rho_q(d,m) = \dim(T_{\leq d}(m)) = \binom{m+d}{d}, \text{ if } d \leq q-1, \tag{4}$$

since  $T_{\leq d}(m) = T_{\leq d}^{\text{red}}(m)$  if d < q. Here as well as later on we use the convention that  $\binom{a}{b} = 0$  if a < b. In particular we have  $\rho_q(d, m) = 0$  if d < 0. As shown in [1, §5.4], for the general case  $d \leq m(q-1)$ , we have

$$\rho_q(d,m) = \dim(T_{\le d}^{\text{red}}(m)) = \sum_{i=0}^d \sum_{j=0}^m (-1)^j \binom{m}{j} \binom{m-1+i-qj}{m-1}.$$
(5)

In this note, we will present an easy-to-obtain expression for  $\bar{e}_r^{\mathbb{A}}(d,m)$  involving a certain representation of the number  $\rho_q(d,m) - r$  that we introduce in the next section.

# 2 The *d*-th Macaulay representation with respect to q

Let d be a positive integer. The d-th Macaulay (or d-binomial) representation, of a non-negative integer N is a way to write N as sum as certain binomial coefficients. To be precise

$$N = \sum_{i=1}^d \binom{s_i}{i},$$

where the  $s_i$  integers satisfying  $s_d > s_{d-1} > \cdots > s_1 \ge 0$ . The usual convention that  $\binom{a}{b} = 0$  if a < b, is used. For example, the *d*-th Macaulay representation of 0 is given

by  $0 = \sum_{i=1}^{d} {\binom{i-1}{i}}$ . Given d and N the integers  $s_i$  exist and are unique. The Macaulay representation is among other things used for the study of Hilbert functions of graded modules, see for example [3]. It is well known (see for example [3]) that if N and M are two nonnegative integers with Macaulay representations given by  $(k_d, \ldots, k_1)$  and  $(\ell_d, \ldots, \ell_1)$  then  $N \leq M$  if and only if  $(k_d, \ldots, k_1) \preccurlyeq (\ell_d, \ldots, \ell_1)$ , where  $\preccurlyeq$  denotes the lexicographic order.

For our purposes it is more convenient to define  $m_i := s_i - i$ . We then obtain

$$N = \sum_{i=1}^{d} \binom{m_i + i}{i},\tag{6}$$

where  $m_i$  are integers satisfying  $m_d \ge m_{d-1} \ge \cdots \ge m_1 \ge -1$ . The reason for this is that for  $d \le q-1$  we have  $\rho_q(d,m) = \binom{m+d}{d}$ . Therefore, we can interpret Equation (6) as a statement concerning dimensions of the Reed–Muller codes  $\operatorname{RM}_q(i,m_i)$ . For a suitable choice of N, it turns out that the  $m_i$  completely determine the value of  $\bar{e}_r^{\mathbb{A}}(d,m)$  if  $d \le q-1$ . For  $d \ge q$ , even though the dimension  $\rho_q(d,m)$  is not longer given by  $\binom{m+d}{d}$ , there exists a variant of the usual d-th Macaulay representation that turns out to be equally meaningful for Reed–Muller codes. Before stating this representation, we give a lemma.

**Lemma 2.1.** Let  $m \ge 1$  be an integer. We have

$$\rho_q(d,m) = \sum_{i=0}^{\min\{d,q-1\}} \rho_q(d-i,m-1).$$

*Proof.* Any polynomial  $f \in T(m)$  can be seen as a polynomial in the variable  $X_m$  with coefficients in T(m-1). This implies that  $T(m) = \sum_{i\geq 0} X_m^i T(m)$ , where the sum is a direct sum. Similarly we can write

$$T_{\leq d}^{\rm red}(m) = \sum_{i=0}^{\min\{d,q-1\}} X_m^i T_{\leq d-i}^{\rm red}(m-1).$$

The result now follows.

A consequence of this lemma is the following.

**Corollary 2.2.** Let d = a(q-1) + b for integers a and b satisfying  $a \ge 0$  and  $1 \le b \le q-1$ . Further suppose that  $m \ge a$ . Then

$$\rho_q(d,m) - 1 = \sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_q(d-j(q-1)-\ell, m-j-1) + \sum_{i=1}^{b} \rho_q(i, m-a-1).$$

*Proof.* This follows using Lemma 2.1 repeatedly. First applying the lemma to each sum within the double summation on the right-hand side, we see that

$$\sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_q (d-j(q-1)-\ell, m-j-1) = \sum_{j=0}^{a-1} \left( \rho_q (d-j(q-1), m-j) - \rho_q (d-(j+1)(q-1), m-j-1) \right) = \rho_q (d,m) - \rho_q (d-a(q-1), m-a) = \rho_q (d,m) - \rho_q (b,m-a).$$

Using the same lemma to rewrite the single summation on the right-hand side in Equation (9) we see that if m > a

$$\sum_{i=1}^{b} \rho_q(i, m-a-1) = \rho_q(b, m-a) - \rho_q(0, m-a-1) = \rho_q(b, m-a) - 1,$$

while if m = a, the single summation equals 0 and the double summation simplifies to  $\rho_q(d,m) - 1$ . In either case, we obtain the desired result

We can now show the following.

**Theorem 2.3.** Let  $N \ge 0$  and  $d \ge 1$  be integers and q a prime power. Then there exist uniquely determined integers  $m_1, \ldots, m_d$  satisfying

- 1.  $N = \sum_{i=1}^{d} \rho_q(i, m_i),$
- $2. -1 \le m_1 \le \cdots \le m_d,$
- 3. for all *i* satisfying  $1 \le i \le d q + 1$ , either  $m_{i+q-1} > m_i$  or  $m_{i+q-1} = m_i = -1$ .

Proof. We start by showing uniqueness. Suppose that

$$N = \sum_{i=1}^{d} \rho_q(i, m_i) = \sum_{i=1}^{d} \rho_q(i, n_i)$$
(7)

and the integers  $n_1, \ldots, n_d$  and  $m_1, \ldots, m_d$  satisfy the conditions from the theorem. First of all, if  $m_d = -1$  or  $n_d = -1$  then N = 0. Either assumption implies that  $(m_d, \ldots, m_1) =$  $(-1, \ldots, -1) = (n_d, \ldots, n_1)$ . Indeed  $n_i \ge 0$  or  $m_i \ge 0$  for some *i* directly implies that N > 0. Therefore we from now on assume that  $m_d \ge 0$  and  $n_d \ge 0$ . To arrive at a contradiction, we may assume without loss of generality that  $n_d \le m_d - 1$ .

Define e to be the smallest integer such that  $n_e \ge 0$ . Equation (7) can then be rewritten as

$$N = \sum_{i=1}^{d} \rho_q(i, m_i) = \sum_{i=e}^{d} \rho_q(i, n_i)$$
(8)

Condition 3 from the theorem implies that  $n_{i-q+1} < n_i$  for all *i* satisfying  $e \le i \le d$ . Now write d - e + 1 = a(q-1) + b for integers *a* and *b* satisfying  $a \ge 0$  and  $1 \le b \le q-1$ . With this notation, we obtain that for any  $0 \le j \le a-1$  and  $0 \le \ell \le q-2$  we have that

$$n_{d-j(q-1)-\ell} \le n_d - j \le m_d - j - 1.$$

In particular choosing j = a - 1 and  $\ell = 0$ , this implies that  $m_d \ge a + n_{q-1+b} \ge a + 1 + n_b \ge a$ . Using these observations, we obtain from Equation (7) that

$$\rho_q(d, m_d) \le N = \sum_{i=e}^d \rho_q(i, n_i) \le \sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_q(d-j(q-1)-\ell, m_d-j-1) + \sum_{i=1}^b \rho_q(e+i-1, m_d-a-1)$$
(9)

Applying the same technique as in the proof of Corollary 2.2, we derive that

$$\sum_{j=0}^{a-1} \sum_{\ell=0}^{q-2} \rho_q(d-j(q-1)-\ell, m_d-j-1) = \rho_q(d, m_d) - \rho_q(b+e-1, m_d-a)$$

and Equation (9) can be simplified to

$$\rho_q(d, m_d) \le \rho_q(d, m_d) - \rho_q(b + e - 1, m_d - a) + \sum_{i=1}^b \rho_q(e + i - 1, m_d - a - 1).$$
(10)

For  $m_d = a$  the right-hand side equals  $\rho_q(d, m_d) - 1$ , leading to a contradiction. If  $m_d > q$ , Equation (10) implies

$$\rho_q(b+e-1, m_d-a) \leq \sum_{i=1}^{b} \rho_q(e+i-1, m_d-a-1) \\
= \sum_{j=0}^{b-1} \rho_q(e+b-1-j, m_d-a-1) \\
< \sum_{j=0}^{\min\{e+b-1, q-1\}} \rho_q(e+b-1-j, m_d-a-1) \\
= \rho_q(b+e-1, m_d-a),$$

where in the last equality we used Lemma 2.1. Again we arrive at a contradiction. This completes the proof of uniqueness of the d-th Macaulay representation with respect to q.

Now we show existence. Let d, N and q be given. We will proceed with induction on d. For d = 1, note that  $\rho_q(1, m) = m + 1$  for any  $m \ge -1$ . Therefore, for a given  $N \ge 0$ , we can write  $N = \rho_q(1, N - 1)$ .

Now assume the theorem for d-1. There exists  $m_d \ge -1$  such that

$$\rho_q(d, m_d) \le N < \rho_q(d, m_d + 1).$$
(11)

Applying the induction hypothesis on  $N - \rho_q(d, m_d)$ , we can find  $m_{d-1}, \ldots, m_1$  satisfying the conditions of the theorem for d-1. In particular we have that

- 1.  $N \rho_q(d, m_d) = \sum_{i=1}^{d-1} \rho_q(i, m_i),$
- 2.  $-1 \le m_1 \le \cdots \le m_{d-1}$ ,
- 3.  $m_{i+(q-1)} > m_i$  for all  $1 \le i \le d-q$ .

Clearly this implies that  $N = \sum_{i=1}^{d} \rho_q(i, m_i)$ , but it is not clear a priori that  $m_1, \ldots, m_d$ satisfy conditions 2 and 3 as well. Conditions 2 and 3 would follow once we show that  $m_d \ge m_{d-1}$  and either  $m_d > m_{d-q+1}$  or  $m_d = m_{d-q+1} = -1$ . First of all, if  $m_d = -1$ , then N = 0 and  $(m_d, \ldots, m_1) = (-1, \ldots, -1)$ . Hence there is nothing to prove in that case. Assume  $m_d \ge 0$ . From Equation (11) and Lemma 2.1 we see that

$$N - \rho_q(d, m_d) < \rho_q(d, m_d + 1) - \rho_q(d, m_d) = \sum_{i=1}^{\min\{d, q-1\}} \rho_q(d-i, m_d).$$
(12)

First suppose that  $d \leq q - 1$ . First of all, Condition 3 is empty in that setting. Further, Equation (12) implies

$$N - \rho_q(d, m_d) < \sum_{i=1}^d \rho_q(d-i, m_d) = \sum_{i=1}^{d-1} \rho_q(d-i, m_d) + 1$$

and hence

$$N - \rho_q(d, m_d) \le \sum_{i=1}^{d-1} \rho_q(d-i, m_d) = \sum_{j=0}^{d-2} \rho_q(d-1-j, m_d) < \rho_q(d-1, m_d+1).$$

This shows that  $m_{d-1} \leq m_d$  as desired.

Now suppose that  $d \ge q$ . In this situation Equation (12) implies

$$N - \rho_q(d, m_d) < \sum_{i=1}^{q-1} \rho_q(d-i, m_d) = \sum_{j=0}^{q-2} \rho_q(d-1-j, m_d) < \rho_q(d-1, m_d+1).$$

Hence  $m_{d-1} \leq m_d$  as before. Finally assume that  $m_d \leq m_{d-q+1}$ . Then by the previous and Condition 2, we have  $m_d = m_{d-1} = \cdots = m_{d-q+1}$ . Hence  $N \geq \sum_{i=0}^{q-1} \rho_q(d-i, m_d) = \rho_q(d, m_d + 1)$  which is in contradiction with Equation (11). This concludes the induction step and hence the proof of existence.

We call the representation of N in the above theorem the d-th Macaulay representation of N with respect to q. One retrieves the usual d-th Macaulay representation letting q tend to infinity. We refer to  $(m_d, \ldots, m_1)$  as the coefficient tuple of this representation. A direct corollary of the above is the following.

**Corollary 2.4.** The coefficient tuple  $(m_d, \ldots, m_1)$  of the d-th Macaulay representation with respect to q of a nonnegative integer N can be computed using the following greedy algorithm: The coefficient  $m_{d-i}$  can be computed recursively (starting with i = 0) as the unique integer  $m_{d-i} \ge -1$  such that

$$\rho_q(d-i, m_{d-i}) \le N - \sum_{j=d-i+1}^d \rho_q(j, m_j) < \rho_q(d-i, m_{d-i}+1).$$

*Proof.* From the existence-part of the proof of Theorem 2.3 it follows directly that the given greedy algorithm finds the desired coefficients.  $\Box$ 

A further corollary is the following. As before  $\leq$  denotes the lexicographic order.

**Corollary 2.5.** Suppose the N and M are two nonnegative integers whose respective coefficient tuples are  $(n_d, \ldots, n_1)$  and  $(m_d, \ldots, m_1)$ . Then

$$N \leq M$$
 if and only if  $(n_d, \ldots, n_1) \leq (m_d, \ldots, m_1)$ .

*Proof.* Assume  $(n_d, \ldots, n_1) \preceq (m_d, \ldots, m_1)$ . It is enough to show the corollary in case  $n_d < m_d$ . We know from the previous corollary that  $n_d$  and  $m_d$  may be determined using the given greedy algorithm. In particular this implies that  $n_d < m_d$  implies

$$N < \rho_q(d, n_d + 1) \le \rho_q(d, m_d) \le M$$

Assume that  $N \leq M$ . We use induction on d. The induction basis is trivial: If d = 1, then  $m_1 = M - 1$  and  $n_1 = N - 1$ . For the induction step, note that  $N \leq M < \rho_q(d, m_d + 1)$  implies by the greedy algorithm that  $n_d \leq m_d$ . If  $n_d < m_d$ , we are done. If  $n_d = m_d$ , we replace N with  $N - \rho_q(d, m_d)$  and M with  $M - \rho_q(d, m_d)$  and use the induction hypothesis to conclude that  $(n_d, \ldots, n_1) \leq (m_d, \ldots, m_1)$ .

## 3 A simple expression for $\bar{e}_r^{\mathbb{A}}(d,m)$

We are now ready to state and prove the relation between the Macaulay representation with respect to q and  $\bar{e}^{A}_{\pi}(d, m)$ .

**Theorem 3.1.** For  $1 \le r \le \rho_q(d,m)$ , let the d-th Macaulay representation of  $\rho_q(d,m) - r$  with respect to q be given by

$$\rho_q(d,m) - r = \sum_{i=1}^d \rho_q(i,m_i).$$

Denoting the floor function as  $\lfloor \cdot \rfloor$ , we have

$$\bar{e}_r^{\mathbb{A}}(d,m) = \sum_{i=1}^d \lfloor q^{m_i} \rfloor.$$

*Proof.* We know from Equation (3) that we need to show that

$$\sum_{i=1}^{d} \lfloor q^{m_i} \rfloor = \sum_{i=1}^{m} \mu_i q^{m-i},$$

with  $(\mu_1, \ldots, \mu_m)$  is the r-th element in descending lexicographic order among all m-tuples  $(\beta_1, \ldots, \beta_m)$  in  $\{0, 1, \ldots, q-1\}^m$  satisfying  $\beta_1 + \cdots + \beta_m \leq d$ . First of all note that since  $r \geq 1$ , we have  $\rho_q(d, m) - r < \rho_q(d, m)$ . In particular this implies that  $m_d \leq m-1$ . Therefore the coefficients of the d-tuple  $(m_d, \ldots, m_1)$  are in  $\{-1, 0, \ldots, m-1\}$ . Now for  $1 \leq i \leq m+1$  define  $\mu_i := |\{j : m_j = m - i\}|$ . Since the d-tuple  $(m_d, \ldots, m_1)$  is nonincreasing by Condition 2 from Theorem 2.3, we can reconstruct it uniquely from the (m+1)-tuple  $(\mu_1, \mu_2, \ldots, \mu_{m+1})$ . Moreover, Condition 3 from Theorem2.3, implies that  $(\mu_1, \ldots, \mu_m) \in \{0, 1, \ldots, q-1\}^m$ , but note that  $\mu_{m+1}$  could be strictly larger than q-1. Further by construction we have  $\mu_1 + \cdots + \mu_m + \mu_{m+1} = d$ , implying that  $\mu_1 + \cdots + \mu_m \leq d$ . Note that  $\mu_{m+1}$  is determined uniquely by  $(\mu_1, \ldots, \mu_m)$ , since  $\mu_0 = d - \mu_1 - \cdots - \mu_m$ . Therefore the correspondence between the d-tuples  $(m_d, \ldots, m_1)$  of coefficients of the d-th Macaulay representations with

respect to q of integers  $0 \le N < \rho_q(d, m)$  and the m-tuples  $(\mu_1, \ldots, \mu_m) \in \{0, 1, \ldots, q-1\}^m$  satisfying  $\mu_1 + \cdots + \mu_m \le d$ , is a bijection. Moreover by construction we have

$$\sum_{i=1}^{d} \lfloor q^{m_i} \rfloor = \sum_{j=1}^{m+1} \mu_j \lfloor q^{m-j} \rfloor = \sum_{j=1}^{m} \mu_j q^{m-j}.$$

What remains to be shown is that the constructed *m*-tuple coming from the integer  $\rho_q(d,m) - r$  is in fact the *r*-th in descending lexicographic order. First of all, by Corollary 2.2 we see that for r = 1 and d = aq + b that the *m*-tuple associated to  $\rho_q(d,m) - 1$  equals  $(q - 1, \ldots, q - 1, b, 0, \ldots, 0)$ , which under the lexicographic order is the maximal *m*-tuple among all *m*-tuples  $(\beta_1, \ldots, \beta_m) \in \{0, 1, \ldots, q - 1\}^m$  satisfying  $\beta_1 + \cdots + \beta_m \leq d$ . Next we show that the conversion between *d*-tuples  $(m_d, \ldots, m_1)$  to *m*-tuples  $(\mu_1, \ldots, \mu_m)$  preserves the lexicographic order. Suppose therefore that  $1 \leq r \leq s \leq \rho_q(d,m)$ . We write  $N := \rho_q(d,m) - s$  and  $M := \rho_q(d,m) - r$ . and denote their Macaulay coefficient tuples with  $(n_d, \ldots, n_1)$  and  $(m_d, \ldots, m_1)$ . Since  $N \leq M$ , Corollary 2.5 implies that  $(n_d, \ldots, n_1) \preceq (m_d, \ldots, m_1)$ . Also, since these *d*-tuples are nonincreasing, this implies that their associated *m*-tuples  $(\nu_1, \ldots, \nu_m)$  and  $(\mu_1, \ldots, \mu_m)$  satisfy  $(\nu_1, \ldots, \nu_m) \preceq (\mu_1, \ldots, \mu_m)$ . Indeed assuming without loss of generality that  $\nu_1 < \mu_1$  we see that  $m_i = n_i = m - 1$  for  $d - \nu_1 \leq i \leq d$  but  $n_i < m_i = m - 1$  for  $i = \nu_1 + 1$ . Now the desired result follows immediately.

Combining this theorem with the greedy algorithm in Corollary 2.4, it is very simple to compute values of  $\bar{e}_r^{\mathbb{A}}(d,m)$  or equivalently of  $d_r(\mathrm{RM}_q(d,m))$ . We illustrate this in the two following examples. The parameters in these example also occur in examples from [4].

**Example 3.2.** Let q = 4, r = 8, d = m = 3. Since  $d \le q - 1$ , we may work with the usual Macaulay representation when applying Theorem 3.1. We have  $\rho_q(d,m) = \binom{6}{3} = 20$  and hence

$$\rho_q(d,m) - r = 12 = \binom{5}{3} + \binom{2}{2} + \binom{1}{1} = \rho_4(3,2) + \rho_4(2,0) + \rho_4(1,0)$$

is the 3-rd Macaulay representation of 12. Theorem 3.1 implies that  $\bar{e}_8^{\mathbb{A}}(3,3) = 4^2 + 4^0 + 4^0 = 18$  and hence  $d_8(\mathrm{RM}_4(3,3)) = 64 - 18 = 46$  in accordance with Example 6.10 in [4].

**Example 3.3.** Let q = 2, r = 10, d = 3 and m = 5. We have  $\rho_2(3,5) = 26$  by Equation (5) and hence applying the greedy algorithm from Corollary 2.4, we compute that

$$\rho_q(d,m) - r = 16 = 15 + 1 + 0 = \rho_2(3,4) + \rho_2(2,0) + \rho_2(1,-1)$$

is the 3rd Macaulay representation of 16 with respect to 2. Theorem 3.1 implies that  $\bar{e}_{10}^{\mathbb{A}}(3,3) = 2^4 + 2^0 = 17$  and hence  $d_8(\mathrm{RM}_2(3,5)) = 32 - 17 = 15$  in accordance with Example 6.12 in [4].

**Remark 3.4.** Theorem 3.1 is somewhat similar in spirit as Theorem 6.8 from [4] in the sense that in both theorems a certain representation in terms of dimensions of Reed–Muller codes is used to give an expression for  $d_r(\mathrm{RM}_q(d,m))$ . Where we studied decompositions of  $\rho_q(d,m) - r$ , in [4] the focus was on r itself. This suggest there may exist a duality between the two approaches, but the similarities seem to stop there. The representation in [4] is not the Macaulay representation with respect to q that we have used here. For us it is for example very important that each degree i between 1 and d occurs once in Theorem 2.3 (implying that the greedy algorithm terminates after at most d iterations), while this is not the case in Theorem 6.8 [4]. It could be interesting future work to determine if a deeper lying relationship between the two approaches exists.

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