

ON THE WEIGHTS OF DUAL CODES ARISING FROM THE GK CURVE

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ABSTRACT. In this paper we investigate some dual algebraic-geometric codes associated with the Giulietti-Korchmáros maximal curve. We compute the minimum distance and the minimum weight codewords of such codes and we investigate the generalized hamming weights of such codes.

Keywords: Giulietti-Korchmáros curve - algebraic-geometric codes - weight distribution

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1. INTRODUCTION

Let \mathcal{X} be an algebraic curve defined over the finite field \mathbb{F}_q of order q . We recall that a curve \mathcal{X} is called \mathbb{F}_q -maximal if its number of rational points over \mathbb{F}_q attains the Hasse-Weil upper bound

$$|\mathcal{X}(\mathbb{F}_q)| = q + 1 + 2g(\mathcal{X})q^{1/2},$$

where $g(\mathcal{X})$ is the genus of \mathcal{X} .

Since codes with good parameters can be constructed from these curves, many authors studied their properties, see [7, 9, 14, 15, 16, 19, 20]. Most of the known examples have been shown to be subcovers of the Hermitian curve \mathcal{H} , which is defined over \mathbb{F}_{q^2} by the equation $Y^{q+1} = X^q + X$. This led to the question whether every maximal curve is a subcover of the Hermitian curve or not. This question has a negative answer: in [13], Giulietti and Korchmáros introduced an infinity family of curves \mathcal{C}' , the so called GK curve, which is maximal over \mathbb{F}_{q^6} .

Codes from the GK curve have been widely investigate, see for example [6, 7, 10, 12]

In most cases, the weight distribution of a given code is hard to be computed. Even the problem of computing codewords of minimum weight can be a difficult task apart from specific cases. In [6], following the approach of [11], the authors compute the number of minimum weight codewords of certain dual AG codes arising from the GK curve. For this purpose, they provide a useful algebraic-geometric description for codewords with a given weight which belong to a fixed affine-variety code. These techniques are widely used in literature, see [1, 3, 4, 2].

In this paper we investigate, using algebraic geometry techniques, codes arising from the GK maximal curve and we give tools to compute the number of minimum weight codewords of such codes. We also investigate some different construction of codes deriving from the ones in [6] and we study the generalized hamming weights of some codes arising from the GK curve, which are another important pattern for a linear code.

2. PRELIMINARIES

We recall the following results (see [3, Theorem 1]).

Lemma 2.1. *Fix integers $r \geq 2$, $m > 0$ and $e > 0$. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\deg(Z) \leq 3m + r - 3$. If $r \geq 3$ assume that Z spans \mathbb{P}^r . We have $h^1(\mathcal{I}_Z(m)) \geq e$ if and only if there is $W \subseteq Z$ occurring in this list:*

- (a) $\deg(W) = m + 1 + e$ and W is contained in a line;
- (b) $\deg(W) = 2m + 1 + e$ and W is contained in a reduced plane conic;
- (c) $r \geq 3$, $e \geq 2$, and there are an integer $f \in \{1, \dots, e - 1\}$ and lines L_1, L_2 , such that $L_1 \cap L_2 = \emptyset$, $\deg(L_1 \cap Z) = m + 1 + f$ and $\deg(L_2 \cap Z) = m + 1 + e - f$.

Lemma 2.2. *Let \mathbb{F} be any field and let \mathbb{P}^n denote the projective space of dimension n on \mathbb{F} . Let $C \subseteq \mathbb{P}^r$ be a smooth plane curve which is a complete intersection. Fix an integer $d > 0$, a zero-dimensional scheme $E \subseteq C$ and a finite subset $B \subseteq C$ such that $B \cap E_{red} = \emptyset^1$. Denote by \mathcal{C} the code obtained evaluating the vector space $H^0(C, \mathcal{O}_C(d)(-E))$ at the points of B . Set $c = \deg(C)$, $n = |B|$ and assume $|B| > dc - \deg(E)$. The following facts hold.*

- (1) *The code \mathcal{C}^\perp has length n and dimension $k = h^0(C, \mathcal{O}_C(d)) - \deg(E) + h^1(\mathbb{P}^n, \mathcal{I}_E(d))$.*
- (2) *The minimum distance of \mathcal{C}^\perp is the minimal cardinality, say z , of a subset of $S \subseteq B$ such that*

$$h^1(\mathbb{P}^n, \mathcal{I}_{S \cup E}(d)) > h^1(\mathbb{P}^n, \mathcal{I}_E(d)).$$

- (3) *A codeword of \mathcal{C}^\perp has weight z if and only if it is supported by a subset $S \subseteq B$ such that*
 - (a) $|S| = z$,
 - (b) $h^1(\mathbb{P}^n, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^n, \mathcal{I}_E(d))$,
 - (c) $h^1(\mathbb{P}^n, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^n, \mathcal{I}_{E \cup S'}(d))$ for any $S' \subsetneq S$.

A zero-dimensional scheme $Z \subset \mathbb{P}^r$ is said to be *curvilinear* if at each $P \in Z_{red}$ the Zariski tangent space of Z has dimension ≤ 1 . A zero-dimensional scheme is contained in a smooth curve (easy). A zero-dimensional scheme is curvilinear if and only if it has finitely many subschemes (for the “only if” part use that it is contained in a smooth curve, for the “if” part use that a non-curvilinear subscheme has infinitely many subschemes with degree 2). In this note we point out the following partial extension of [1], Theorem 1, to the case of non-reduced, but curvilinear subschemes.

We recall the following results ([4, Theorem 1]).

Theorem 2.3. *Fix an integer $m \geq 3$. Let $Z \subset \mathbb{P}^r$, $r \geq 3$, be a curvilinear zero-dimensional scheme spanning \mathbb{P}^r . If $r = 3$, then assume $\deg(Z) < 3m$. If $r \geq 4$, then assume $\deg(Z) \leq 4m + r - 5$ and $\deg(Z \cap M) < 3m$ for all 3-dimensional linear subspaces $M \subset \mathbb{P}^r$. We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if either there is a line D with $\deg(D \cap Z) \geq m + 2$ or there is a conic D' with $\deg(D' \cap Z) \geq 2m + 2$.*

With minimal modifications of the proof of [1] we get the following result

Lemma 2.4. *Let $Y \subset \mathbb{P}^3$ be a smooth and connected projective curve defined over an algebraically closed field. Fix a zero-dimensional scheme $A \subset Y$ and a finite set $B \subset Y$ such that $A \cap B = \emptyset$. Set $Z := A \cup B$. Assume $\deg(A)$ Assume*

¹Here E_{red} denotes the reduction of the scheme E .

$\deg(A) < 3m$, $\deg(Z) \leq 4m + 2$ and $\deg(Z \cap H) \leq 4m - 5$ for each plane $H \subseteq \mathbb{P}^r$. We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if there is $W \subseteq Z$ as in one of the following cases:

- (a) $\deg(W) = m + 2$ and W is contained in a line;
- (b) $\deg(W) = 2m + 2$ and W is contained in a plane conic;
- (c) $\deg(W) = 3m$ and W is the complete intersection of a degree 3 plane curve and a degree m surface;
- (d) $\deg(W) \geq 3m + 1$ and W is contained in a degree 3 plane curve;
- (e) $\deg(W) = 3m + 2$ and W is contained in a reduced and connected degree 3 curve spanning \mathbb{P}^3 .

3. GK CURVE

Denote by $PG(3, q^6)$ the three dimensional projective space over the field \mathbb{F}_{q^6} with q^6 element. The Giulietti-Korchmáros curve \mathcal{GK} is a non-singular curve in $PG(3, q^6)$ defined by the affine equations

$$(1) \quad \begin{cases} Z^{q^2-q+1} = Y^{q^2} - Y \\ Y^{q+1} = X^q + X \end{cases}.$$

Arbitrary complete intersections in \mathbb{P}^r are defined and studied in [17, Ex. II.8.4 and III.5.5]. We always consider the case of smooth space curves, complete intersection of a surface S of degree a and a surface of degree $b \geq a$ (for GK we have $a = q + 1$ and $b = q^2$). We have $h^1(\mathbb{P}^3, \mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{Z}$ and hence (for a smooth curve) and any zero-dimensional scheme $Z \subset C$ we have $h^1(C, \mathcal{O}_C(t)(-Z)) = h^1(\mathcal{O}_C(t)) + h^1(\mathbb{P}^3, \mathcal{I}_Z(t))$. We have the exact sequences

$$(2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(t-a) \rightarrow \mathcal{O}_{\mathbb{P}^3}(t) \rightarrow \mathcal{O}_S(t) \rightarrow 0$$

$$(3) \quad 0 \rightarrow \mathcal{O}_S(t-b) \rightarrow \mathcal{O}_S(t) \rightarrow \mathcal{O}_C(t) \rightarrow 0$$

We have $h^0(\mathcal{O}_{\mathbb{P}^3}(t)) = \binom{t+3}{3}$ for all $t \geq 0$. From (2) and (3) we get $h^0(\mathcal{O}_C(t)) = \binom{t+3}{3}$ for all $t < a$ and $h^0(\mathcal{O}_C(t)) = \binom{t+3}{3} - \binom{t-a+3}{3}$ if $a \leq t < b$ (see below the proof of the case $t \geq b$). From (2) and the fact that $h^i(\mathcal{O}_{\mathbb{P}^3}(x)) = 0$ for $i = 1, 2$ and all $x \in \mathbb{Z}$ we get $h^1(\mathcal{O}_S(x)) = 0$ for all $x \in \mathbb{Z}$ and $h^0(\mathcal{O}_C(t)) = h^0(\mathcal{O}_S(t))$ if $t < b$ and $h^0(\mathcal{O}_C(t)) = h^0(\mathcal{O}_S(t)) - h^0(\mathcal{O}_S(t-b))$ for all $t \geq b$. Thus for all $t \geq b$ we have $h^0(\mathcal{O}_C(t)) = \binom{t+3}{3} - \binom{t-a+3}{3} + \binom{t-b}{3} - \varepsilon$, where $\varepsilon = 0$ if $t < b + a$ and $\varepsilon = \binom{t-b-a}{3}$ if $t \geq b + a$.

Proposition 3.1. *Let L be a tangent to \mathcal{GK} at a point P . Then $I(L, \mathcal{GK}, P) = q^2 - q + 1$ or $I(L, \mathcal{GK}, P) = q + 1$.*

Proof. We know that the tangent to an affine point of this curve $P = (x_0, y_0, z_0)$ has equation

$$\begin{cases} (Y - y_0) + z_0^{q^2-q}(Z - z_0) = 0 \\ -(X - x_0) + y_0^q(Y - y_0) = 0 \end{cases}$$

The parametric equation of this line is, for $z_0 \neq 0$

$$\begin{cases} X = x_0 + y_0^q t - y_0^{q+1} \\ Y = t \\ Z = \frac{-t + y_0 + z_0^{q^2-q+1}}{z_0^{q^2-q}} = \frac{t + y_0^{q^2}}{z_0^{q^2-q}} \end{cases}$$

while for $z_0 = 0$ it is

$$\begin{cases} X = x_0 \\ Y = y_0 \\ Z = t \end{cases}$$

and the solution corresponding to P is $t = y_0$ and $t = z_0$ respectively.

Suppose $z_0 \neq 0$.

Substituting the equation of the affine equation of the GK curve gives us

$$(4) \quad \begin{cases} \left(\frac{-t+y_0^{q^2}}{z_0^{q^2-q}} \right)^{q^2-q+1} - t^{q^2} + t = 0 \\ -t^{q+1} + (x_0 + y_0^q t - y_0^{q+1})^q + x_0 + y_0^q t - y_0^{q+1} = 0 \end{cases}$$

The first equation becomes

$$\begin{aligned} 0 &= (-t + y_0^{q^2})^{q^2-q+1} - (t^{q^2} - t) z_0^{(q^2-q)(q^2-q+1)} \\ &= (y_0^{q^2} - t)^{q^2-q+1} - (t^{q^2} - t)(y_0^{q^2} - y_0)^{q^2-q} \end{aligned}$$

which has $t = y_0$ as a root, its derivative is

$$-(y_0^{q^2} - t)^{q^2-q} + (y_0^{q^2} - y_0)^{q^2-q} = -(y_0^{q^2} - t)^{q-1} + (y_0^{q^2} - y_0)^{q-1}$$

and since $t = y_0$ is a root of $-(y_0^{q^2} - t)^{q-1} + (y_0^{q^2} - y_0)^{q-1}$ we have that $t = y_0$ is a root of (4) with multiplicity at least $q + 1$.

By direct computations the second equation becomes

$$\begin{aligned} 0 &= -t^{q+1} + x_0^q + y_0^{q^2} t^q - y_0^{q^2+q} + x_0 + y_0^q t - y_0^{q+1} = -t^{q+1} + y_0^{q^2} t^q - y_0^{q^2+q} + y_0^q t \\ &= t^q(-t + y_0^{q^2}) - y_0^q(-t + y_0^{q^2}) = (-t + y_0^{q^2})(t - y_0)^q \end{aligned}$$

and from this we get that $t = y_0$ is a root with multiplicity $q + 1$ if $y_0 \in \mathbb{F}_{q^2}$ or q is $y_0 \notin \mathbb{F}_{q^2}$.

Now we deal with the remaining case $z_0 = 0$. Substituting the equation of the affine equation of the GK curve gives us

$$\begin{cases} t^{q^2-q+1} = y_0^{q^2} - y_0 \\ y_0^{q+1} = x_0^q + x_0 \end{cases}$$

where the second is not an equation in t but just a compatibility condition. So, if this holds we get that

$$t^{q^2-q+1} = 0$$

In this case the tangent in P is a $q^2 - q + 1$ -secant.

The last case we have to study is the case $P_\infty = (1 : 0 : 0 : 0)$, the homogenized equations of the curve are

$$(5) \quad \begin{cases} Z^{q^2-q+1} T^{q-1} = Y^{q^2} - Y T^{q^2-1} \\ Y^{q+1} = X^q T + X T^q \end{cases}.$$

the equation of the tangent line will be then

$$\begin{cases} X = 1 \\ Y = 0 \end{cases}$$

and the multiplicity intersection at this point with the tangent is $q^2 - q + 1$. \square

4. CODES FROM THE GK CURVE

We recall results for the intersections of algebraic curves and \mathcal{GK} and for the minimum distance of the one point AG code $C(D, G_m)^\perp$, where $G_m = m(q^3+1)P_\infty$, $P_\infty = (1 : 0 : 0 : 0)$, and $D = \sum_{P \in \mathcal{GK}(\mathbb{F}_{q^6}) \setminus \{P_\infty\}} P$, see [6] for details.

Proposition 4.1. *Let $r \subset PG(3, q^6)$ be a line. Then*

$$|r \cap \mathcal{GK}| \leq q^2 - q + 1.$$

Also, any $(q^2 - q + 1)$ -secant is parallel to the z -axis and all the $(q^2 - q + 1)$ common points are not \mathbb{F}_{q^2} -rational.

Proposition 4.2. *The total number of $(q^2 - q + 1)$ -secants of the \mathcal{GK} is $(q+1)(q^5 - q^3)$.*

Remember that each point lies in exactly one of such secants.

Proposition 4.3. *Let \mathcal{X} be a curve of degree $\alpha \leq q$ in $PG(3, q^6)$. Then the size $|\mathcal{X} \cap \mathcal{GK}(\mathbb{F}_{q^6})|$ is at most*

$$\begin{cases} \alpha(q^2 - q + 1), & \text{if } \mathcal{X} \text{ is reducible,} \\ \alpha(q + 1), & \text{if } \mathcal{X} \text{ is absolutely irreducible.} \end{cases}$$

Proposition 4.4. *Let $d^* \leq d$ be the designed Goppa minimum distance of $C(D, G_m)^\perp$, $m \geq 2$. Then*

- (1) $d = m + 2$ when $m \leq q^2 - q - 1$;
- (2) $d = 2m + 2$ when $m = q^2 - q$;
- (3) $d = 3m$ when $m = q^2 - q + 1$;
- (4) $d \geq 3m + 1$ when $q^2 - q + 1 < m \leq q^2 - 1$;
- (5) $d \geq d^*$ when $m > q^2 - 1$.

4.1. The family \overline{C}_S . Consider now a set $S \subset \mathcal{GK}(\mathbb{F}_{q^6})$ and the corresponding divisor

$$D_S = D - \sum_{P \in S} P$$

and call

$$S_1 = \{P \in S : P \in \mathcal{GK}(\mathbb{F}_{q^2})\}, \quad S_2 = S \setminus S_1.$$

The following result comes from a straight application of 4.2.

Proposition 4.5. *Let $q + 1 \leq m \leq 2(q + 1)$ and D_S defined as before. Consider the code $\overline{C}_S = C(D_S, (q^3 + 1)mP_\infty)^\perp$, if*

$$|S_2| < (q^2 - q + 1 - m)(q + 1)(q^5 - q)$$

then \overline{C}_S is a $[n - |S|, \ell(D_S) - \ell(D_S - G_m), m + 2]_q$ -code. Moreover, if $S = S_1$ then the number of minimum weight codewords of \overline{C}_S is given by

$$A_{m+2}(\overline{C}_S) = (\ell + 1)(\ell^5 - \ell^3)(\ell^6 - 1) \binom{\ell^2 - \ell + 1}{m + 2}.$$

Proof. Following the proof of Proposition (4.4) and noticing that if $|S_2| < (q^2 - q + 1 - m)(q + 1)(q^5 - q)$ there is at least a $(m + 2)$ -secant line the result holds. \square

Remark 4.6. Actually this bound can be improved depending on the composition of S_2 , i.e. if the points in S_2 are chosen in a way such that at least $m+2$ of them lie in the intersection between one line and \mathcal{GK} , then all the other ones can be taken leaving the minimum distance unchanged (while decreasing the dimension of the code, so improving the code itself).

4.2. Three-point codes.

Theorem 4.7. *Fix any three distinct points $P_1, P_2, P_3 \in (GK)(\mathbb{F}_{q^6}) \setminus \{P_\infty\}$ and assume P_1, P_2 and P_3 to span \mathbb{P}^2 and be such that their connecting line is not parallel to the z axis. Set $B := (GK)(\mathbb{F}_{q^6}) \setminus \{P_1, P_2, P_3\}$. Fix an integer $d \geq 5$ such that $1 \leq d \leq q-1$ and integers $a_1, a_2, a_3 \in \{1, \dots, d\}$ such that $a_1 + a_2 + a_3 \leq 3d-5$ and $a_i = d$ for at most one index $i \in \{1, 2, 3\}$. Set $E := a_1P_1 + a_2P_2 + a_3P_3$. Let $\mathcal{C} := \mathcal{C}(B, d, -E)$ be the code obtained evaluating the vector space $H^0((GK), \mathcal{O}(GK)(d)(-E))$ on the set B . Then \mathcal{C} is a code of length $n = |B| = q^8 - q^6 + q^5 - 2$ and dimension $k = \binom{d+3}{3} - a_1 - a_2 - a_3$. For any $i \in \{1, 2, 3\}$ let L_i denote the line spanned by P_j and P_h with $\{i, j, h\} = \{1, 2, 3\}$. Then \mathcal{C}^\perp has minimum distance d and its minimum-weight codewords are exactly the ones whose support is formed by d points of $B \cap L_i$ for some $i \in \{1, 2, 3\}$.*

Proof. The length of \mathcal{C} is obviously $n = q^8 - q^6 + q^5 - 2$. From what we said previously we have $h^0((GK), \mathcal{O}(GK)(d)) = \binom{d+3}{3}$. If, say, $a_1 \geq a_2 \geq a_3$, the assumptions $a_1 \leq d$ and $a_1 + a_2 + a_3 \leq 3d-1$ give $a_i \leq d+2-i$ for all i . Hence our previous computations tell us that $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$ and so $h^0((GK), \mathcal{O}(GK)(d)(-E)) = \binom{d+3}{3} - a_1 - a_2 - a_3 = k$.

Since $|B| > d \cdot \deg((GK))$, there is not a non-zero element of $H^0((GK), \mathcal{O}(GK)(d))$ vanishes at all the points of B . Hence \mathcal{C} has dimension k . By Lemma 2.2 it is sufficient to prove the following two facts.

- (a) $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup A}(d)) = 0$ for all $A \subseteq B$ such that $|A| \leq d-1$.
- (b) For any $S \subseteq B$ such that $|S| = d$ we have $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup S}(d)) > 0$ if and only if $S \subseteq L_i$ for some $i \in \{1, 2, 3\}$.

Each line L_i contains at most $q-1$ points of B while $\deg(E \cap L_i) = 2$. Hence for any $S \subseteq L_i \cap B$ with $|S| = d$ we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$ from Lemma 2.2.

Let $E_i := a_i P_i$, clearly $E = E_1 + E_2 + E_3$ (seen as a divisor).

Fix a set $S \subseteq B$ such that $|S| \leq d$ and assume $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup S}(d)) > 0$. We have $S \cap \{P_1, P_2, P_3\} = \emptyset$ and $\deg(E \cup S) = a_1 + a_2 + a_3 + |S|$. Since $a_1 + a_2 + a_3 + |S| \leq 4d-5$, we may apply Proposition 2.4 to the scheme $E \cup S$.

Let $T \subseteq \mathbb{P}^n$ be the curve arising from the statement of the lemma. Set $x := \deg(T) \in \{1, 2, 3\}$ and $e_i := \deg(T \cap E_i)$ for $i \in \{1, 2, 3\}$. We have $0 \leq e_i \leq a_i$.

If $e_i \geq x+1$ then we have that the tangent at P_i is $L_{(GK), P_i} \subseteq T$. Assume $e_i \leq x$ for all $i \in \{1, 2, 3\}$. For $x=2$ we get $\deg(T \cap (E \cup S)) \leq 2d+1$. For $x=3$ we get $\deg(T \cap (E \cup S)) \leq 3d-1$. Finally, for $x=1$ we may have $e_i > 0$ only for at most two indices, say $i=1, 2$. Since $|S| \leq d$, we get $|S| + e_1 + e_2 \geq d+2$ and $|S| + e_1 + e_2 = d+2$ if and only if $T = L_3$, $S \subseteq L_3 \cap B$ and $|S| = d$.

Now assume that T contains one of the lines $L_{(GK), P_i}$, say $L_{(GK), P_1}$. Let T' be the curve whose equation is obtained dividing an equation of T by an equation of $L_{(GK), P_1}$. We have $\deg(T') = x-1$, $T' + L_{(GK), P_1} = T$ (as divisors of \mathbb{P}^2) and $T = L_{(GK), P_1} \cup T'$ (as sets). Since $L_{(GK), P_1} \cap B = \emptyset$, we have $T \cap S = T' \cap S$ and $\deg(T \cap (E \cup S)) = \deg(T' \cap (E_2 \cup E_3 \cup S))$.

- (i) If $x = 1$, we get $T \cap S = \emptyset$ and $\deg(T \cap E) = a_1 \leq d$, a contradiction.
- (ii) Assume $x = 2$. The curve T' must be a line such that $\deg(T' \cap (E_2 \cup E_3 \cup S)) \geq 2d + 2 - a_1$. If either $T' = L_{(GK), P_2}$, or $T' = L_{(GK), P_3}$, we get $T' \cap S = \emptyset$ and $\deg(T' \cap (E_2 \cup E_3 \cup S)) \leq \max\{e_2, e_3\} \leq d$, a contradiction. If neither $T' = L_{(GK), P_2}$, nor $T' = L_{(GK), P_3}$, then $\deg(T' \cap E_2) \leq 1$, $\deg(T' \cap E_3) \leq 1$ and $\deg(T' \cap (E_2 \cup E_3)) = 2$ if and only if $T' = L_1$. Since $|S| \leq d$ we deduce $\deg(T \cap (E \cup S)) \leq a_1 + 2 + |S|$. Moreover, the equality holds if and only if $T' = L_1$ and $S \subseteq L_1$. Since $\deg(T \cap (E \cup S)) \geq 2d + 2$ by assumption, $|S| = d$ and $S \subseteq L_1$, as claimed.
- (iii) Now assume $x = 3$. We get $\deg(T' \cap (E_2 \cup E_3 \cup S)) \geq 3d - a_1$ and T' is a conic. If neither $L_{(GK), P_2}$, nor $L_{(GK), P_3}$, is a component of T then $e_2 \leq 2$ and $e_3 \leq 2$ and so $|T' \cap S| \geq 3d - 4 - a_1 \geq 2d - 4 > d$. If, say, T' contains $L_{(GK), P_2}$ and T'' is the line with $T' = T'' + L_{(GK), P_2}$, then we get $|(S \cup E_3) \cap T''| \geq 3d - a_1 - a_2$. Since $a_1 + a_2 \leq 2d - 1$ we deduce $\deg(T'' \cap (E_3 \cup S)) \geq d + 1$. Since $\deg(T'' \cap E_3) \leq 1$, we get $a_1 + a_2 = 2d - 1$, say $a_1 = d$, $a_2 = d - 1$ and that S is formed by d points on a line T'' through P_3 . If either $T'' = L_1$ or $T'' = L_3$, then we are done. In any case it is sufficient to prove that $E_1 \cup E_2 \cup \{P_3\} \cup S$ is not the complete intersection of $T = L_{(GK), P_1} \cup L_{(GK), P_2} \cup T''$ and a degree d curve, say C_d . Since $a_2 = d - 1$, E_2 is not the complete intersection of $L_{(GK), P_2}$ and C_d , while $L_{(GK), P_2} \cap (\{P_3\} \cup S) = \emptyset$, a contradiction.

□

Corollary 4.8. *Using the notation of the previous theorem, the number of minimum weight codewords of the code above is given by*

$$A_d(\mathcal{C}) = (q^6 - 1) \sum_{i=1}^3 \binom{|L_i \cap \mathcal{GK}|}{d} \leq (q^6 - 1) 3 \binom{p+1}{d}.$$

where the binomial coefficient is meant to be zero if $d > |L_i \cap \mathcal{GK}|$ for some i .

Example 4.9. Let $q = 7$ and consider the affine equation of \mathcal{GK} over the field \mathbb{F}_{q^6}

$$\begin{cases} Z^{43} = Y^{49} - Y \\ Y^8 = X^7 + X \end{cases}.$$

Consider $P_1 = (0 : 0 : 0 : 1)$, $P_2 = (1 : 3 : 0 : 1)$ and $P_3 = (1 : 4 : 0 : 1)$. The three points are in general position and their connecting line are not parallel to the Z axis, so the conditions of the previous theorem are satisfied. Moreover, by direct computations, the three lines L_1 , L_2 and L_3 are 8-secants of \mathcal{GK} . Consider now $d = 6$ and $a_1 = 6$, $a_2 = a_3 = 3$ and call $\mathcal{C} = \mathcal{C}(B, 6, 6P_1 + 3P_2 + 3P_3)$. From Theorem 4.7 we have that the minimum distance of \mathcal{C} is $d = 6$ and the minimum weight codewords are exactly

$$A_6(\mathcal{C}) = (7^6 - 1) 3 \binom{8}{6} = (7^6 - 1) 84$$

5. GENERALIZED HAMMING WEIGHTS OF CODES ARISING FROM THE GK CURVE

Let $\mathbb{K} = \mathbb{F}_q$ a finite field with q elements. Let $C \subset \mathbb{K}^n$ be a linear $[n, k]$ code over \mathbb{K} . We recall that the *support* of C is defined as follows

$$\text{supp}(C) = \{i \mid c_i \neq 0 \text{ for some } c \in C\}.$$

So $\#supp(C)$ is the number of nonzero columns in a generator matrix for C . Moreover, for any $1 \leq v \leq k$, the v -th generalized Hamming weight of C

$$d_v(C) = \min\{\#supp(\mathcal{D}) \mid \mathcal{D} \text{ is a linear subcode of } C \text{ with } \dim(\mathcal{D}) = v\}.$$

In other words, for any integer $1 \leq v \leq k$, $d_v(C)$ is the v -th minimum support weights, i.e. the minimal integer t such that there are an $[n, v]$ subcode \mathcal{D} of C and a subset $S \subset \{1, \dots, n\}$ such that $\#(S) = t$ and each codeword of \mathcal{D} has zero coordinates outside S . The sequence $d_1(C), \dots, d_k(C)$ of generalized Hamming weights (also called *weight hierarchy* of C) is strictly increasing (see Theorem 7.10.1 of [18]). Note that $d_1(C)$ is the minimum distance of the code C .

Lemma 5.1. *Let $S \subset B$ be the support of a codeword of C^\perp . Assume that there exists a surface $T \subset \mathbb{P}^3$ such that $h^1(\mathbb{P}^3, \mathcal{I}_{Res_T(E \cup S)}(d-k)) = 0$, where $k = \deg(T)$. Then $S \subset T$.*

Proof. Let W (resp. W') be the subcode of C^\perp formed by the codewords whose support is contained in S (resp. $S \cap T$). Clearly $W' \subseteq W$. From Proposition 2.1 we get $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup S}(d)) = h^1(\mathbb{P}^3, \mathcal{I}_{T \cap (E \cup S)}(d))$. From this we obtain $W = W'$, which means that the thesis is proved. \square

Theorem 5.2. *Fix a positive integer $d \leq \deg(\mathcal{GK}) - 1$, a zero dimensional scheme $E \subseteq \mathcal{X}$ defined over \mathbb{K} and a set $B \subseteq \mathcal{GK}(\mathbb{K}) \setminus E_{red}$ such that $\deg(E) \leq d+1$ and set $C := C(B, \mathcal{O}_{\mathcal{GK}(d)(-E)})$. Assuming that each line is such that $\deg(L \cap (E \cup B)) \leq d+1$ and that there exists a conic such that*

- (i) $\deg(D \cap E) + |B \cap D| \geq 2d+2$;
- (ii) *for each conic \mathcal{C} such that $T \neq D$ we have $\deg(T \cap (E \cup B)) \leq 2d+1$.*

For any integer s such that $2d+2 - \deg(D \cap E) \leq s \leq |B \cap D|$ and for each integer h with $1 \leq h \leq \min\{|B \cap D| - 2d - 2 + \deg(D \cap E), d - 2 - \deg(E)\}$ each h -dimensional linear subspace of C^\perp computing $d_h(C^\perp)$ is supported by some $S \subset \Sigma(2d+h-1+\deg(D \cap E))$ and each element of $\Sigma(d+h-1+\deg(D \cap E))$ is in the support of a h -dimensional linear subspace.

Proof. Fix an integer $e \geq 1$ and any $S \subseteq B$. Lemma 2.2 tell us that S contains the support of an e -dimensional subspace of C^\perp if and only if $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup S}(d)) \geq e$. Fix $S \subseteq B$ such that it is the support of a codeword of C^\perp with weight $\leq 3d+1-\deg(E)$, hence $\deg(E \cup S) \leq 3d-1$ and Lemma 2.1 tells us the value of $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d))$. Let t be the minimal integer such that $h^1(\mathbb{P}^3, \mathcal{I}_{E \cup S}(d)) \geq 0$, clearly $t \leq d$. If $\deg(T \supset S) \leq 2d+1$ then there is a line L with $\deg(L \cap (E \cup B)) \geq \deg(L \cap (E \cup S)) \geq d+2$, but this cannot happen since we excluded this possibility. For this reason we can assume that $\deg(E \cup S) \geq 2d+2$. Since $d \geq 4$, we have that there is a line L such that $\deg(L \cap (E \cup S)) \geq d$ scheme-theoretic base locus of the set $|\mathcal{I}_D(2)|$ of all quadric surfaces containing D . Take any quadric $T \supset D$. Since $\deg(Res_T(E \cup B)) \leq 3d-1-(2d+2) \leq d-1$, we have $h^1(\mathbb{P}^3, \mathcal{I}_{Res_T(E \cup B)}(d-2)) = 0$. Thus Lemma 5.1 gives $E \cup B \subset T$. Since D is scheme-theoretically the base locus of $|\mathcal{I}_D(2)|$, we get $E \cup B \subset D$. \square

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