# Shifts on a finite qubit string: A class of quantum baker's maps 

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#### Abstract

We present two complementary ways in which Saraceno's symmetric version of the quantum baker's map can be written as a shift map on a string of quantum bits. One of these representations leads naturally to a family of quantizations of the baker's map.


The main subject of the theory of quantum chaos is the investigation of quantum signatures of chaos [1], such as characteristic eigenvalue statistics [1] or hypersensitivity to perturbation [2]. In contrast to the situation in classical chaos theory, many results in the theory of quantum chaos are based on numerical simulations, not on rigorous proofs. A major part of recent work on quantum chaos has been the analysis of quantum maps [3], quantized versions of classically chaotic maps.

Classically chaotic maps are, under very general conditions, equivalent to Bernoulli shifts on bi-infinite strings of symbols taken from some finite alphabet. This fact is the basis of the powerful method of symbolic dynamics [⿴囗 results in classical chaos theory. In the present short paper, we study shift maps on strings of quantum bits and thus take the first step towards generalizing the method of symbolic dynamics to the quantum case.

The quantum baker's map [5] is a particularly simple map on the quantized unit square [6]. It has recently been shown [7, 8] to have an experimental realization on present-day

[^0]quantum computers [9, 10]. Here we investigate finite shift-map representations of the quantum baker's map. Shifts on infinite quantum spin chains in the context of quantum chaos have been discussed in [1]. A related symbolic description of the quantum baker's map is given in (12.

The classical baker's transformation [13], which maps the unit square $0 \leq q, p \leq 1$ onto itself, has a simple description in terms of its symbolic dynamics 4. Each point in phase space is represented by a symbolic string

$$
\begin{equation*}
s=\cdots s_{-2} s_{-1} s_{0} \cdot s_{1} s_{2} \cdots \tag{1}
\end{equation*}
$$

where $s_{k}=0$ or 1 . The string $s$ is identified with a point $(q, p)$ in the unit square by setting

$$
\begin{equation*}
q=\sum_{k=1}^{\infty} s_{k} 2^{-k} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
p=\sum_{k=0}^{\infty} s_{-k} 2^{-k-1} \tag{3}
\end{equation*}
$$

The action of the baker's map on a symbolic string $s$ is given by the shift map $U$ defined by $U s=s^{\prime}$, where $s_{k}^{\prime}=s_{k+1}$. This means that, at each time step, the entire string is shifted one place to the left while the dot remains fixed. Geometrically, if $q$ labels the horizontal direction and $p$ labels the vertical, the baker's map on the unit square is equivalent to stretching the $q$ direction and squeezing the $p$ direction, each by a factor of two, and then stacking the right half on top of the left.

We now quantize the unit square as in [6, 14]. To represent the unit square in $D$ dimensional Hilbert space, we start with unitary "displacement" operators $\hat{U}$ and $\hat{V}$, which produce displacements in the "momentum" and "position" directions, respectively, and which obey the commutation relation (6]

$$
\begin{equation*}
\hat{U} \hat{V}=\hat{V} \hat{U} \epsilon \tag{4}
\end{equation*}
$$

where $\epsilon^{D}=1$. We choose $\epsilon=e^{2 \pi i / D}$. For consistency of units, we let the quantum scale on "phase space" be $2 \pi \hbar=1 / D$. We further assume that $D=2^{N}$, which is the dimension of the Hilbert space of $N$ qubits (i.e., $N$ two-state systems), and that $\hat{V}^{D}=\hat{U}^{D}=-\hat{1}$. The latter choice enforces antiperiodic boundary conditions; it is motivated by the fact [14] that for an even dimension $D$, antiperiodic boundary conditions guarantee that the classical and quantized maps have similar symmetry properties. For an alternative quantization using periodic boundary conditions, see [15]. It follows [6, 14] that the operators $\hat{U}$ and $\hat{V}$ can be written as

$$
\begin{equation*}
\hat{U}=e^{(i / \hbar) \hat{q} / D}=e^{2 \pi i \hat{q}} \quad \text { and } \quad \hat{V}=e^{-(i / \hbar) \hat{p} / D}=e^{-2 \pi i \hat{p}}, \tag{5}
\end{equation*}
$$

where the "position" operator $\hat{q}$ has eigenvalues $q_{j}=\left(j+\frac{1}{2}\right) / D, j=0, \ldots, D-1$, and likewise the "momentum" operator $\hat{p}$ has eigenvalues $p_{k}=\left(k+\frac{1}{2}\right) / D, k=0, \ldots, D-1$.

The $D=2^{N}$ dimensional Hilbert space modeling the unit square can be realized as the product space of $N$ qubits in such a way that

$$
\begin{equation*}
\left|q_{j}\right\rangle=\left|x_{1}\right\rangle \otimes\left|x_{2}\right\rangle \otimes \cdots \otimes\left|x_{N}\right\rangle \tag{6}
\end{equation*}
$$

where $j=\sum_{l=1}^{N} x_{l} 2^{N-l}, x_{l} \in\{0,1\}$, and where each qubit has basis states $|0\rangle$ and $|1\rangle$. It follows that, written as binary numbers, $j=x_{1} x_{2} \ldots x_{N}$ and $q_{j}=0 . x_{1} x_{2} \ldots x_{N} 1$. We define the notation

$$
\begin{equation*}
\left|\cdot x_{1} x_{2} \ldots x_{N}\right\rangle=e^{i \pi / 2}\left|q_{j}\right\rangle \tag{7}
\end{equation*}
$$

which is closely analogous to Eq. (11), where the bits to the right of the dot specify the position variable; the reason for the phase shift $e^{-i \pi / 2}$ becomes apparent below.

Momentum and position eigenstates are related through the quantum Fourier transform operator $\hat{F}$ [14], defined by $\hat{F}\left|q_{k}\right\rangle \equiv\left|p_{k}\right\rangle$, where

$$
\begin{equation*}
\left|p_{k}\right\rangle=\frac{1}{\sqrt{D}} \sum_{j=0}^{D-1}\left|q_{j}\right\rangle e^{(i / \hbar) p_{k} q_{j}}=\frac{1}{\sqrt{2^{N}}} \sum_{x_{1}, \ldots, x_{N}}\left|x_{1}\right\rangle \otimes \ldots \otimes\left|x_{N}\right\rangle e^{2 \pi i a x / 2^{N}} \tag{8}
\end{equation*}
$$

In this expression $a=k+\frac{1}{2}=a_{1} \ldots a_{N} \cdot 1=2^{N} p_{k}$, and $x=j+\frac{1}{2}=x_{1} \ldots x_{N} \cdot 1=2^{N} q_{j}$. We now define the notation

$$
\begin{equation*}
\left|a_{N} \ldots a_{1} .\right\rangle=\left|p_{k}\right\rangle \tag{9}
\end{equation*}
$$

which is again analogous to Eq. (11), where the bits to the left of the dot, read backwards, specify the momentum variable.

It will be useful to define a partial Fourier transform, $\hat{G}_{n}$, which acts on the $N-n$ least significant bits of a state,

$$
\begin{align*}
& \hat{G}_{n}\left|x_{1}\right\rangle \otimes \ldots \otimes\left|x_{n}\right\rangle \otimes\left|a_{1}\right\rangle \otimes \ldots \otimes\left|a_{N-n}\right\rangle \\
& \quad=\left|x_{1}\right\rangle \otimes \cdots \otimes\left|x_{n}\right\rangle \otimes \frac{1}{\sqrt{2^{N-n}}} \sum_{x_{n+1}, \ldots, x_{N}}\left|x_{n+1}\right\rangle \otimes \cdots \otimes\left|x_{N}\right\rangle e^{2 \pi i a x / 2^{N-n}} \tag{10}
\end{align*}
$$

where now $a$ and $x$ are defined by the binary expansions $a=a_{1} \ldots a_{N-n} .1$ and $x=$ $x_{n+1} \ldots x_{N}$.1. Again in close analogy to Eq. (11), we define the notation

$$
\begin{equation*}
\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle=\hat{G}_{n}\left|x_{1}\right\rangle \otimes \ldots \otimes\left|x_{n}\right\rangle \otimes\left|a_{1}\right\rangle \otimes \ldots \otimes\left|a_{N-n}\right\rangle \tag{11}
\end{equation*}
$$

Notice that had we instead used $x=x_{1} \ldots x_{N} .1$ in Eq. (10), the only difference would have been to multiply $\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle$ by a phase $e^{i \pi x_{n}}$. The operator $\hat{G}_{n}$ is unitary, and the states $\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle$ form an orthonormal basis. As our notation requires, for $n=0$ the states $\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle$ reduce to the momentum eigenstates (i.e., $\hat{G}_{0}=\hat{F}$ ), and for $n=N$ they reduce to $e^{i \pi / 2}\left|x_{1}\right\rangle \ldots\left|x_{N}\right\rangle=\left|x_{1} \ldots x_{N}\right\rangle$ (i.e., $\hat{G}_{N}=i \hat{1}$ ). The phase shift for $n=N$ is the reason for the $\pi / 2$ phase shift in Eq. (7); it is a consequence of the antiperiodic boundary conditions.

The state $\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle$ is localized in both position and momentum: it is strictly localized within a position region of width $1 / 2^{n}$, centered at position $q=0 . x_{1} \ldots x_{n} 1$, and it is crudely localized within a momentum region of width $1 / 2^{N-n}$, centered at momentum $p=0 . a_{1} \ldots a_{N-n} 1$. Using the notation of Eq. (1) for phase-space points, we can say that the states $\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle$ are localized near the points $1 a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n} 1$, with position and momentum widths determined by this lattice of points.

The quantum baker's map as defined in [14] is now given by [7]

$$
\begin{equation*}
\hat{B}=\hat{G}_{0} \circ \hat{G}_{1}^{-1} \tag{12}
\end{equation*}
$$

By noting that

$$
\begin{equation*}
\hat{G}_{0}\left|x_{1}\right\rangle \otimes\left|a_{1}\right\rangle \otimes \ldots \otimes\left|a_{N-1}\right\rangle=\left|a_{N-1} \ldots a_{1} x_{1} .\right\rangle \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{G}_{1}\left|x_{1}\right\rangle \otimes\left|a_{1}\right\rangle \otimes \ldots \otimes\left|a_{N-1}\right\rangle=\left|a_{N-1} \ldots a_{1} \cdot x_{1}\right\rangle \tag{14}
\end{equation*}
$$

one sees that the action of the baker's map is equivalent to shifting the dot in the symbolic representation, i.e.,

$$
\begin{equation*}
\hat{B}\left|a_{N-1} \ldots a_{1} \cdot x_{1}\right\rangle=\left|a_{N-1} \ldots a_{1} x_{1} \cdot\right\rangle \tag{15}
\end{equation*}
$$

similar to the classical symbolic dynamics (11). Motivated by this form for the quantum baker's map and by the symbolic representation of Eq. (11), we can define a whole class of quantum baker's maps, $\left\{\hat{B}_{n} \mid n=1, \ldots, N\right\}$, through

$$
\begin{equation*}
\hat{B}_{n}\left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle=\left|a_{N-n} \ldots a_{1} x_{1} \cdot x_{2} \ldots x_{n}\right\rangle \tag{16}
\end{equation*}
$$

In phase-space language, the map $\hat{B}_{n}$ takes a state localized at $1 a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n} 1$ to a state localized at $1 a_{N-n} \ldots a_{1} x_{1} . x_{2} \ldots x_{n} 1$, while it stretches the state by a factor of two in the $q$ direction and squeezes it by a factor of two in the $p$ direction.

The classical shift map acting on the symbolic string (11) can be regarded equivalently as either a right-shift of the dot or a left-shift of the infinite string of bits. We now show that, complementary to the dot-shifting representation (16), there is a representation of the quantum baker's map $\hat{B}_{n}$ as a shift of the qubits. Following [16], we write the partial Fourier transform (10) as a product state

$$
\begin{align*}
& \left|a_{N-n} \ldots a_{1} \cdot x_{1} \ldots x_{n}\right\rangle \\
& =\left|x_{1}\right\rangle \otimes\left|x_{2}\right\rangle \otimes \cdots \otimes\left|x_{n}\right\rangle \\
& \quad \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{N-n} 1\right)}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{N-n-1} a_{N-n} 1\right)}|1\rangle\right) \\
& \quad \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{1} \ldots a_{N-n} 1\right)}|1\rangle\right) e^{i \pi\left(0 . a_{1} \ldots a_{N-n} 1\right)} \tag{17}
\end{align*}
$$

Similarly, we can write

$$
\begin{align*}
\mid a_{N-n} \ldots & \left.a_{1} x_{1} \cdot x_{2} \ldots x_{n}\right\rangle \\
=\left|x_{2}\right\rangle & \otimes \cdots \otimes\left|x_{n}\right\rangle \\
& \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{N-n} 1\right)}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{N-n-1} a_{N-n} 1\right)}|1\rangle\right) \\
& \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . a_{1} \ldots a_{N-n} 1\right)}|1\rangle\right) \\
& \otimes e^{i \pi\left(0 . x_{1} a_{1} \ldots a_{N-n} 1\right)} \frac{1}{\sqrt{2}}\left(|0\rangle+e^{2 \pi i\left(0 . x_{1} a_{1} \ldots a_{N-n} 1\right)}|1\rangle\right) . \tag{18}
\end{align*}
$$

Since the quantum baker's map $\hat{B}_{n}$ maps the state (17) to the state (18), it can be seen that it shifts the states of all the qubits to the left, except the state of the leftmost, most
significant qubit. The state $\left|x_{1}\right\rangle$ of the leftmost qubit can be thought as being shifted to the rightmost qubit, where it suffers controlled phase changes that are determined by the state parameters $a_{1} \ldots a_{N}$ for the original "momentum qubits." The quantum baker's map can thus be written as a shift map on a finite string of qubits, followed by controlled phase changes on the least significant qubit.

An important special case arises for $n=N$, for then there are no momentum qubits on which to condition the phase changes of the least significant qubit. Working either from Eq. (10) or from Eqs. (17) and (18), one can show that

$$
\begin{equation*}
\hat{B}_{N}\left|x_{1}\right\rangle \otimes \cdots \otimes\left|x_{N}\right\rangle=\left|x_{2}\right\rangle \otimes \cdots \otimes\left|x_{N}\right\rangle \otimes \frac{e^{i \pi x_{1}}}{\sqrt{2}}\left(e^{-i \pi / 4} e^{-i \pi x_{1} / 2}|0\rangle+e^{i \pi / 4} e^{i \pi x_{1} / 2}|1\rangle\right) \tag{19}
\end{equation*}
$$

The state $\left|x_{1}\right\rangle$ of the leftmost qubit is shifted to the rightmost qubit, where it undergoes a single-qubit transformation, not controlled by the state parameters of the other qubits. As a result, this incarnation of the quantum baker's map, unlike the others, does not entangle initial product states.

In conclusion, we have given a symbolic representation of the states of $N$ qubits that leads naturally to a class of quantum baker's maps, defined as shift maps with respect to the symbolic representation. For each of the maps in this class, there is a product basis such that the action of the map on an arbitrary basis state is equivalent to a shift of the string of qubits to the left, followed by controlled phase changes on the rightmost qubit. This result is a potential starting point for a generalization of the method of classical symbolic dynamics to chaotic quantum maps.

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