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Parametric Approximation of Isotropic and Anisotropic Elastic Flow for Closed and Open Curves

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Abstract

Deckelnick and Dziuk (2009) proved a stability bound for a continuous-in-time semidiscrete parametric finite element approximation of Willmore flow/elastic flow of closed curves in $\mathbb{R}^d$, $d \geq 2$. We extend these ideas in considering an alternative finite element approximation of the same flow that retains some of the features of the formulations in Barrett, Garcke, and Nürnberg (2007b, 2008b, 2010b), in particular an equidistribution mesh property. For this new approximation, we obtain also a stability bound for a continuous-in-time semidiscrete scheme. Apart from the isotropic situation, we also consider the case of an anisotropic elastic energy. In addition to the evolution of closed curves, we also consider the isotropic and anisotropic elastic flow of a single open curve in the plane and in higher codimension that satisfies various boundary conditions.

Key words. elastic flow, Willmore flow, Navier boundary conditions, clamped boundary conditions, parametric finite elements, tangential movement, anisotropy

AMS subject classifications. 65M60, 65M12, 35K55, 53C44, 74E10

1 Introduction

In this paper, we study gradient flows of elastic energies for curves. Elastic energies, which are based on the integral of the squared curvature of a curve, have been considered as early as 1738 by D. Bernoulli. A first definitive analysis of curvature energies using methods of calculus of variations is due to Euler in 1743, see Euler (1952). Elastic energies have many applications e.g. in rod theories, and in the theory of splines. For a discussion of classical results and classical applications, we refer to Truesdell (1983).

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Recent applications include the modelling of DNA rings, see Goyal *et al.* (2005); Tu and Ou-Yang (2008), edge completion in computer vision, see Mio *et al.* (2004), and in theoretical efforts to understand curved nanostructures, see Tu and Ou-Yang (2008). The simplest evolution problem related to curvature energies is the corresponding L^2-gradient flow, which we will numerically study in this paper for various curvature energies both in $2d$ and for curves with higher codimension. We will also focus on the case of open curves, which gives the additional feature that boundary conditions have to be prescribed at the ends. We point out that boundary value problems for Euler–Lagrange equations or gradient flows related to curvature energies are notoriously difficult, and only a few results are known so far, see e.g. Deckelnick and Grunau (2007, 2009); Schätzle (2010) and the references therein. In addition, we also generalize our numerical approaches for elastic flows to anisotropic situations. Below we state the problems under consideration in detail.

Let $(\Gamma(t))_{t\in[0,T]}$ be a family of closed curves in $\mathbb{R}^d$, $d \geq 2$, parameterized by $\vec{x}(\rho,t) : I \times [0,T] \to \mathbb{R}^d$, where $I := \mathbb{R}/\mathbb{Z}$. Introducing the arclength s of the curve, i.e. $\partial_s = |\vec{x}_\rho|^{-1}\,\partial_\rho$ on $\Gamma(t) \equiv \vec{x}(I,t)$, then

$$\vec{\varkappa} := \vec{x}_{ss} \qquad \Rightarrow \qquad \vec{\varkappa}\,.\,\vec{x}_s = 0 \tag{1.1}$$

denotes the usual curvature vector of Γ. In the case $d = 2$, we can introduce curvature via $\vec{\varkappa} = \varkappa\,\vec{\nu}$ with $\vec{\nu} := -\vec{x}_s^{\perp}$ and $\cdot^{\perp}$ acting on a vector in $\mathbb{R}^2$ denoting a clockwise rotation through $90°$.

For a given $\lambda \in \mathbb{R}$, we will consider the following energy

$$E_\lambda(\Gamma,\vec{\varkappa}) := \int_\Gamma \left[\tfrac{1}{2}\,|\vec{\varkappa}|^2 + \lambda\right] \mathrm{d}s\,, \tag{1.2}$$

where $\int_\Gamma f \;\mathrm{d}s := \int_I f\,|\vec{x}_\rho|\;\mathrm{d}\rho$ for $f : I \to \mathbb{R}$. Analogously to the case of two-dimensional hypersurfaces, the bending energy $E_0(\Gamma,\vec{\varkappa})$, i.e. (1.2) with $\lambda = 0$, is often also called the Willmore energy of the curve Γ. In this paper we will refer to (1.2) as the elastic energy of Γ. The inclusion of the parameter λ either penalises growth ($\lambda > 0$) or encourages growth ($\lambda < 0$) in the length of the curve. Another reason for the inclusion of λ is that as a time-dependent parameter it can act as a Lagrange multiplier for a length preservation constraint. Historically the minimization of an elastic energy under a length constraint has received particular attention, see e.g. Euler (1952); and this has applications in e.g. rod theory. In this paper we want to derive finite element approximations of the L^2-gradient flow of (1.2). This flow is of interest as a means to find stable critical points of (1.2). Here we recall that critical points of (1.2) are called elasticae. Hence the L^2-gradient flow of (1.2), i.e.

$$\vec{x}_t = -\vec{\nabla}_s^2\,\vec{\varkappa} - \tfrac{1}{2}\,|\vec{\varkappa}|^2\,\vec{\varkappa} + \lambda\,\vec{\varkappa}\,, \tag{1.3}$$

is commonly called elastic flow of curves, or Willmore flow of curves. Here $\vec{\nabla}_s^2\,\cdot := \vec{\nabla}_s\,(\vec{\nabla}_s\,\cdot)$ and $\vec{\nabla}_s\,\vec{\eta} := \vec{P}\,\vec{\eta}_s$ is the normal component of $\vec{\eta}_s$, where $\vec{P} := \vec{Id} - \vec{x}_s \otimes \vec{x}_s$ is the projection onto the part normal to Γ and $\vec{Id}$ is the identity operator/function on $\mathbb{R}^d$. We note that the velocity $\vec{x}_t$ in (1.3) has no tangential component, i.e. $\vec{x}_t\,.\,\vec{x}_s = 0$.

For $d = 2$ and $\lambda > 0$ global existence of smooth solutions for (1.3) was proved in Polden (1996), while the corresponding result for curves in arbitrary codimension and $\lambda \geq 0$ was obtained in Dziuk *et al.* (2002). The latter paper also suggests a finite element approximation for (1.3). The first error analysis for a numerical approximation of (1.3), including a stability result for a continuous-in-time semidiscrete finite element approximation, was recently presented in Deckelnick and Dziuk (2009). It is the aim of this paper to extend this stability analysis to an alternative finite element approximation, which retains some of the features of the schemes presented in previous work by the authors, see Barrett, Garcke, and Nürnberg (2007b, 2008b, 2010b); notably an equidistribution property.

The starting point for our schemes in the planar case $d = 2$ in Barrett, Garcke, and Nürnberg (2007b, 2008b) was to allow for a nonzero tangential velocity in (1.3). Since such a velocity has no influence on the geometric evolution of $\Gamma(t)$, a valid alternative L^2-gradient flow formulation is

$$\vec{x}_t \,.\, \vec{\nu} = -\varkappa_{ss} - \tfrac{1}{2}\varkappa^3 + \lambda\,\varkappa\,, \tag{1.4}$$

where, in the case $d = 2$ as $\vec{\varkappa} = \varkappa\,\nu$, we note that $E_\lambda(\Gamma, \vec{\varkappa}) \equiv E_\lambda(\Gamma, \varkappa)$ and that (1.3) can equivalently be written as $\vec{x}_t = (-\varkappa_{ss} - \frac{1}{2}\,\varkappa^3 + \lambda\,\varkappa)\,\vec{\nu}$. For the planar flow (1.4) the present authors introduced fully practical finite element approximations in Barrett, Garcke, and Nürnberg (2007b) and Barrett, Garcke, and Nürnberg (2008b), where the schemes in the latter paper naturally generalize to the Willmore flow of two-dimensional surfaces in $\mathbb{R}^3$. The schemes in Barrett, Garcke, and Nürnberg (2007b, 2008b) have in common that an equidistribution property can be shown for the corresponding continuous-in-time semidiscrete approximations. The case of closed curves in arbitrary codimension was considered in Barrett, Garcke, and Nürnberg (2010b). In order to recover the desired equidistribution under discretization, it is once again essential to allow for a nonzero tangential velocity. Hence in Barrett, Garcke, and Nürnberg (2010b) the present authors introduced the following L^2-gradient flow formulation for (1.2):

$$\vec{P}\,\vec{x}_t = -(\vec{\nabla}_s\,\vec{\varkappa})_s - \tfrac{1}{2}\,(|\vec{\varkappa}|^2\,\vec{x}_s)_s + \lambda\,\vec{\varkappa} \equiv -\vec{\nabla}_s^2\,\vec{\varkappa} - \tfrac{1}{2}\,|\vec{\varkappa}|^2\,\vec{\varkappa} + \lambda\,\vec{\varkappa}\,, \tag{1.5}$$

and presented a fully practical finite element approximation based on the corresponding weak formulation. The approximation is also generalized to the anisotropic elastic flow in higher codimension, where the curvature energy (1.2) is replaced by an anisotropic equivalent. We stress that it appears that for none of the mentioned schemes in Barrett, Garcke, and Nürnberg (2007b, 2008b, 2010b) a stability analysis seems to be possible.

It is the aim of this paper to combine the techniques in Deckelnick and Dziuk (2009) and Barrett, Garcke, and Nürnberg (2010b) in order to introduce fully practical approximations of (1.5), for which the continuous-in-time semidiscrete variants can be shown to be stable and to have an equidistribution property. Moreover, we want to extend these new approximations to the anisotropic elastic flow.

In the case of planar curves, i.e. $d = 2$, one could consider the energy

$$\widetilde{E}_\beta(\Gamma, \varkappa) := \tfrac{1}{2}\int_\Gamma (\varkappa - \beta)^2 \,\mathrm{d}s\,, \tag{1.6}$$

where $\beta \in \mathbb{R}$ is a given so-called spontaneous curvature. Clearly, $\widetilde{E}_0(\Gamma, \varkappa) \equiv E_0(\Gamma, \varkappa)$. However, it follows from the Gauß–Bonnet theorem that $\int_\Gamma \varkappa \, \mathrm{d}s = 2\,\pi\, m(\Gamma)$, where $m(\Gamma) \in \mathbb{Z}$ denotes the turning number of Γ. Here, the turning number of a planar curve is defined as the winding number of the normal around zero, which is well-defined for continuous piecewise smooth curves. Noting this, we have that

$$\int_\Gamma (\varkappa - \beta)^2 \, \mathrm{d}s = \int_\Gamma \left[\varkappa^2 + \beta^2\right] \mathrm{d}s - 2\,\beta \int_\Gamma \varkappa \, \mathrm{d}s = \int_\Gamma \left[\varkappa^2 + \beta^2\right] \mathrm{d}s - 4\,\beta\,\pi\, m(\Gamma)\,. \quad (1.7)$$

As $m(\Gamma)$ is invariant for smooth flows, then (1.7) yields that such gradient flows of $\widetilde{E}_\beta(\Gamma, \varkappa)$ and $E_\lambda(\Gamma, \varkappa)$ are equivalent for the choice $\lambda = \frac{1}{2}\,\beta^2$. This also generalises to the anisotropic case, see Lemma 2.1 below. Similarly for $d \geq 2$ and a given $\vec\beta \in \mathbb{R}^d$, one could consider the energy

$$\widetilde{E}_{\vec\beta}(\Gamma, \vec\varkappa) := \tfrac{1}{2} \int_\Gamma |\vec\varkappa - \vec\beta|^2 \, \mathrm{d}s\,.$$

Clearly, $\widetilde{E}_{\vec 0}(\Gamma, \vec\varkappa) \equiv E_0(\Gamma, \vec\varkappa)$. Noting, on applying integration by parts, that

$$\int_\Gamma \vec\beta \,.\, \vec\varkappa \, \mathrm{d}s = \int_\Gamma \vec\beta \,.\, \vec x_{ss} \, \mathrm{d}s = 0\,, \quad (1.8)$$

we have that $\widetilde{E}_{\vec\beta}(\Gamma, \vec\varkappa) = E_\lambda(\Gamma, \vec\varkappa)$ for the choice $\lambda = \frac{1}{2}\,|\vec\beta|^2$. Hence for closed curves, there is no advantage in considering the energies $\widetilde{E}_{\vec\beta}(\Gamma, \vec\varkappa)$, for $d \geq 2$, and $\widetilde{E}_\beta(\Gamma, \varkappa)$, for $d = 2$, over $E_\lambda(\Gamma, \vec\varkappa)$ and $E_\lambda(\Gamma, \varkappa) \equiv E_\lambda(\Gamma, \vec\varkappa)$, respectively. Moreover, it follows from the above analysis in the case $d = 2$ that the unique global minimizers of $E_\lambda(\Gamma, \vec\varkappa)$ for $\lambda > 0$ are given by circles of radius $(2\,\lambda)^{-\frac{1}{2}}$. It is possible to generalize this result to $d \geq 2$, on recalling the fundamental theorem that $\int_\Gamma |\vec\varkappa| \, \mathrm{d}s \geq 2\,\pi$, with equality only for convex curves in a plane, from Fenchel (1929); see also Chai and Kim (2000). In particular, it then follows from the bounds

$$2\,\pi \leq \int_\Gamma |\vec\varkappa| \, \mathrm{d}s \leq |\Gamma|^{\frac{1}{2}} \left(\int_\Gamma |\vec\varkappa|^2 \, \mathrm{d}s\right)^{\frac{1}{2}} \leq (2\,\lambda)^{-\frac{1}{2}}\, E_\lambda(\Gamma, \vec\varkappa)\,,$$

where $|\Gamma|$ denotes the length of Γ, and the fact that the right hand side equals to $2\,\pi$ for circles of radius $(2\,\lambda)^{-\frac{1}{2}}$, that such circles are the the unique global minimizers of $E_\lambda(\Gamma, \vec\varkappa)$ for $\lambda > 0$ in $\mathbb{R}^d$, $d \geq 2$.

Moreover, a further approach that will lead to similar, yet different, evolution equations is to consider the energy $\widetilde{E}_\beta(\Gamma, k)$ in $\mathbb{R}^3$, where

$$k := |\vec\varkappa| = |\vec x_{ss}| \quad (1.9)$$

denotes the curvature of Γ in the Frenet–Serret frame, sometimes also called absolute curvature. Here the parameter $\beta \in \mathbb{R}_{\geq 0}$ prescribes an intrinsic curvature value, see e.g. Goyal *et al.* (2005); Garrivier and Fourcade (2000); Swigon (2009); Lin and Schwetlick (2004) for more details. In order to generalize our approach to full rod theory, the torsion of the curve has to be considered. It is possible, building on the techniques introduced in this paper, to consider finite element approximations of the corresponding flows, and this will form part of our future research in this area.

Another aspect of the present paper is the study of the elastic flow for open curves, where now $\vec{x}(\rho,t) : [0,1] \times [0,T] \to \mathbb{R}^d$ parameterizes $(\Gamma(t))_{t\in[0,T]}$. As the elastic flow (1.5) is of fourth order, we need to prescribe two sets of boundary conditions at the two endpoints of the curve. In this paper, we will consider the clamped boundary conditions

$$\vec{x}(0,t) = \vec{\alpha}_0\,,\ \vec{x}(1,t) = \vec{\alpha}_1 \quad \text{and} \quad \vec{x}_s(0,t) = \vec{\zeta}_0\,,\ \vec{x}_s(1,t) = \vec{\zeta}_1\,,$$

where $\vec{\alpha}_i \in \mathbb{R}^d$ and $\vec{\zeta}_i \in \mathbb{S}^{d-1} := \{\vec{p} \in \mathbb{R}^d : |\vec{p}| = 1\}$, $i = 0 \to 1$; and, in the case $d = 2$, the symmetric Navier boundary conditions

$$\vec{x}(0,t) = \vec{\alpha}_0\,,\ \vec{x}(1,t) = \vec{\alpha}_1 \quad \text{and} \quad \varkappa(0,t) = \beta\,,\ \varkappa(1,t) = \beta\,, \tag{1.10}$$

where $\beta \in \mathbb{R}$. For these Navier boundary conditions, it is essential to consider the energy

$$\widehat{E}_\beta(\Gamma,\varkappa) := \int_\Gamma \left[\tfrac{1}{2}\,\varkappa^2 - \beta\,\varkappa\right] \mathrm{d}s \tag{1.11}$$

in order to recover the flow (1.4) together with (1.10). We remark that considering the natural higher codimension analogue of (1.11), i.e. $\widehat{E}_{\vec{\beta}}(\Gamma,\vec{\varkappa}) := \int_\Gamma [\frac{1}{2}\,|\vec{\varkappa}|^2 - \vec{\beta}\,.\,\vec{\varkappa}]\,\mathrm{d}s$ for a given $\vec{\beta} \in \mathbb{R}^d$ leads to a flow satisfying the boundary conditions

$$\vec{x}(0,t) = \vec{\alpha}_0\,,\ \vec{x}(1,t) = \vec{\alpha}_1 \quad \text{and} \quad \vec{\varkappa}(0,t) = \vec{P}(0,t)\,\vec{\beta}\,,\ \vec{\varkappa}(1,t) = \vec{P}(1,t)\,\vec{\beta}\,. \tag{1.12}$$

As the conditions (1.12) do not appear to have a natural physical interpretation, we do not pursue this in detail in this paper. In the case of clamped boundary conditions, on the other hand, we show that, as for closed curves, the gradient flows of (a) $\widehat{E}_{\vec{\beta}}(\Gamma,\vec{\varkappa})$ and $E_0(\Gamma,\vec{\varkappa})$, and (b) $\widehat{E}_\beta(\Gamma,\varkappa)$ and $E_0(\Gamma,\varkappa)$ are equivalent. Once again we are able to introduce fully practical finite element approximations of these open curve problems, for which the continuous-in-time semidiscrete variants can be shown to be stable and to have an equidistribution property. Moreover, we extend these new approximations to the anisotropic case. To our knowledge, the finite element approximations introduced here are the first numerical approximations of such initial boundary value problems in the literature. We note that solutions to the corresponding stationary problems have recently been analysed in the graph case in Deckelnick and Grunau (2007, 2009).

The layout of this paper is the following. In the next section we introduce our new variational formulations of these elastic flow problems. In Section 3 we introduce continuous-in-time semidiscrete finite element approximations of these problems. We show that such approximations satisfy a stability bound and a mesh equidistribution property. In Section 4 we introduce the corresponding fully discrete versions of the semidiscrete approximations derived in the previous section. At every time level, a linear system has to be solved for the approximations in the case that λ is a fixed given parameter. We show the well-posedness of these linear systems under very mild restrictions on the mesh, and in Section 5 we discuss our approach for solving these linear systems. In situations where λ acts as a Lagrange multiplier for a length constraint, we obtain a nonlinear system of equations at every time level. Finally, in Section 6 we report on numerous numerical experiments, which demonstrate the effectiveness of our fully discrete approximations.

2 Variational formulations

2.1 Isotropic elastic flow

On defining the test function space

$$\underline{V}_{0,\vec{\tau}} := \{\vec{\eta} \in \underline{V}_0 : \vec{\eta}\,.\,\vec{x}_s = 0\}\,, \qquad \text{where } \underline{V}_0 := H^1(I,\mathbb{R}^d) \text{ and } V_0 := H^1(I,\mathbb{R})\,,$$

the present authors in Barrett, Garcke, and Nürnberg (2010b) obtained the following weak formulation of (1.5): Given $\Gamma(0) = \vec{x}(I,0)$, with $\vec{x}(0) \in \underline{V}_0$, for all $t \in (0,T]$ find $\Gamma(t) = \vec{x}(I,t)$, where $\vec{x}(t) \in \underline{V}_0$, and $\vec{\varkappa}(t) \in \underline{V}_{0,\vec{\tau}}$ such that

$$\langle \vec{P}\,\vec{x}_t - \lambda\,\vec{\varkappa}, \vec{\chi}\rangle_\Gamma - \langle \vec{\nabla}_s\,\vec{\varkappa}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle |\vec{\varkappa}|^2\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\chi} \in \underline{V}_{0,\vec{\tau}}\,, \tag{2.1a}$$

$$\langle \vec{\varkappa}, \vec{\eta}\rangle_\Gamma + \langle \vec{x}_s, \vec{\eta}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0\,. \tag{2.1b}$$

Here, and throughout, $\langle \cdot,\cdot\rangle_\Gamma$ denotes the L^2-inner product on Γ; that is, $\langle u, v\rangle_\Gamma := \int_I u\,.\,v\,|\vec{x}_\rho|\,\mathrm{d}\rho$.

In this paper, we will derive an approximation based on an alternative formulation of (1.5). We consider the L^2-gradient flow of (1.2) for $\Gamma(t) = \vec{x}(I,t)$, with $\vec{x} \in \underline{V}_0$ and $\vec{\varkappa} \in \underline{V}_0$, subject to the side constraints

$$\langle \vec{\varkappa}, \vec{\eta}\rangle_\Gamma + \langle \vec{x}_s, \vec{\eta}_s\rangle_\Gamma = 0 \qquad \forall\ \vec{\eta} \in \underline{V}_0 \tag{2.2a}$$

and

$$\langle \vec{\varkappa}\,.\,\vec{x}_s, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in U_0\,, \tag{2.2b}$$

where $U_0 := L^2(I,\mathbb{R})$. Here we should stress that the finite element discretization of the constraints (2.2a,b), building on the ideas published in the series of papers Barrett, Garcke, and Nürnberg (2007b,a, 2008b, 2010b,a), will lead to an induced tangential motion that gives rise to an equidistribution property in the semidiscrete setting. Of course, on the continuous level the side constraint (2.2b) is redundant, recall (1.1). We now consider E_λ as a functional in $\vec{x}$ and $\vec{\varkappa}$. We want to compute its derivative, taking the constraints (2.2a,b) into account. Using the formal calculus of PDE constrained optimization, see e.g. Tröltzsch (2010), we now introduce the Lagrange multipliers $\vec{y} \in \underline{V}_0$ and $z \in U_0$ for (2.2a,b), and define the Lagrangian

$$\mathcal{L}(\vec{x}, \vec{\varkappa}, \vec{y}, z) := \tfrac{1}{2}\,\langle \vec{\varkappa}, \vec{\varkappa}\rangle_\Gamma + \lambda\,|\Gamma| - \langle \vec{\varkappa}, \vec{y}\rangle_\Gamma - \langle \vec{x}_s, \vec{y}_s\rangle_\Gamma + \langle \vec{\varkappa}\,.\,\vec{x}_s, z\rangle_\Gamma\,, \tag{2.3}$$

where $|\Gamma| := \langle 1, 1\rangle_\Gamma$ is the length of Γ. Hence we obtain, on taking variations $[\frac{\delta}{\delta \vec{x}}\,\mathcal{L}](\vec{\chi})$, $[\frac{\delta}{\delta \vec{\varkappa}}\,\mathcal{L}](\vec{\xi})$, $[\frac{\delta}{\delta \vec{y}}\,\mathcal{L}](\vec{\eta})$ and $[\frac{\delta}{\delta z}\,\mathcal{L}](\chi)$, that the direction of steepest descent of E_λ under the constraints (2.2a,b) is given by $-[\frac{\delta}{\delta \vec{x}}\,\mathcal{L}](\vec{\chi})$, with the remaining variations of $\mathcal{L}$ set to zero. In particular, we obtain the gradient flow

$$\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma = \langle \vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle (|\vec{\varkappa}|^2 - 2\,\vec{\varkappa}\,.\,\vec{y} + 2\,\lambda)\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma - \langle z\,\vec{\varkappa}, \vec{\chi}_s\rangle_\Gamma \quad \forall\ \vec{\chi} \in \underline{V}_0\,, \tag{2.4a}$$

$$\langle \vec{\varkappa} + z\,\vec{x}_s - \vec{y}, \vec{\xi}\rangle_\Gamma = 0 \qquad \forall\ \vec{\xi} \in \underline{V}_0\,, \tag{2.4b}$$

$$\langle \vec{\varkappa}, \vec{\eta}\rangle_\Gamma + \langle \vec{x}_s, \vec{\eta}_s\rangle_\Gamma = 0 \qquad \forall\ \vec{\eta} \in \underline{V}_0\,, \tag{2.4c}$$

$$\langle \vec{\varkappa}\,.\,\vec{x}_s, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in U_0\,. \tag{2.4d}$$

It follows from (2.4b,d) that $\vec{P}\,\vec{y} = \vec{\varkappa}$ and $z = \vec{y}\,.\,\vec{x}_s$. Hence the normal part of the Lagrange multiplier $\vec{y}$ agrees with the curvature vector, but in addition it may have a nonzero tangential component z. Overall our formal weak formulation of the L^2-gradient flow for (1.2) subject to (2.2a,b) can now be formulated as: Given $\Gamma(0) = \vec{x}(I, 0)$, with $\vec{x}(0) \in \underline{V}_0$, for all $t \in (0, T]$ find $\Gamma(t) = \vec{x}(I, t)$, where $\vec{x}(t) \in \underline{V}_0$, and $\vec{y}(t) \in \underline{V}_0$ such that

$$\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma - \langle \vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle (|\vec{P}\,\vec{y}|^2 - 2\,\lambda)\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma + \langle (\vec{y}\,.\,\vec{x}_s)\,\vec{P}\,\vec{y}, \vec{\chi}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\chi} \in \underline{V}_0\,, \tag{2.5a}$$

$$\langle \vec{P}\,\vec{y}, \vec{\eta}\rangle_\Gamma + \langle \vec{x}_s, \vec{\eta}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0\,. \tag{2.5b}$$

An important property of the formulation (2.5a,b), which will have repercussions on the discrete level, is that it is independent of the tangential part, $\vec{y}\,.\,\vec{x}_s$, of the Lagrange multiplier $\vec{y}$. To see this, note that it immediately follows from (2.4b,c) that

$$-\langle \vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma + \langle (\vec{y}\,.\,\vec{x}_s)\,\vec{P}\,\vec{y}, \vec{\chi}_s\rangle_\Gamma = -\langle \vec{\nabla}_s\,\vec{\varkappa}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma \quad \forall\ \vec{\chi} \in \underline{V}_0\,, \tag{2.6}$$

which yields that (2.5a,b) is independent of $\vec{y}\,.\,\vec{x}_s$. The identity (2.6) also gives an insight into the connection between (2.5a,b) and (2.1a,b). In particular, this means that enforcing (2.1a) also for test functions $\vec{\chi} \in \underline{V}_0 \setminus \underline{V}_{0,\vec{\tau}}$ yields no further conditions on $\vec{x}$, as can be seen on choosing $\vec{\chi} = \chi\,\vec{x}_s$ in (2.1a) for all $\chi \in V_0$. Moreover, in common with similar formulations of general geometric evolution equations in the series of papers Barrett, Garcke, and Nürnberg (2007b,a, 2008a, 2010b), the tangential part $(\vec{Id} - \vec{P})\,\vec{x}_t$, of the velocity vector $\vec{x}_t$ is not prescribed in (2.5a,b). Hence there does not exist a unique solution to (2.5a,b). Under spatial discretization, the tangential part of the discrete approximation to $\vec{x}_t$ will be intrinsically fixed, and this choice will lead to an equidistribution property. The nonuniqueness of $\vec{y}\,.\,\vec{x}_s$, on the other hand, appears to persist on the discrete level. However, under appropriate spatial and temporal discretization uniqueness of a fully discrete solution is ensured.

Comparing (2.1a,b) and (2.5a,b) we observe that the main difference is that in the latter, the Lagrange multiplier $\vec{y}$ is allowed to have a nonzero tangential component, while in the former the curvature vector $\vec{\varkappa}$ is always normal. As a result, the test spaces for the first equations differ and, moreover, an additional fourth term is introduced in (2.5a). In particular, in the new formulation it is possible to test (2.5a) with $\vec{\chi} = \vec{x}_t$; whereas (2.1a) only allows for testing with $\vec{\chi} = \vec{P}\vec{x}_t$, which does not lead to straightforward estimates on the discrete level. Apart from these differences, the two formulations (2.1a,b) and (2.5a,b) are very close.

Remark. 2.1. *In applications it is often natural to look at flows with a constraint on the total length of the curve. A time-dependent $\lambda(t) \in \mathbb{R}$ can also be interpreted as a Lagrange multiplier for a side constraint on $|\Gamma(t)|$. For the length preserving flow version of* (2.5a,b)*, we would choose*

$$\lambda(t) = \frac{\langle \vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{y}\rangle_\Gamma + \frac{1}{2}\,\langle (|\vec{P}\,\vec{y}|^2\,\vec{x}_s, \vec{y}_s\rangle_\Gamma - \langle (\vec{y}\,.\,\vec{x}_s)\,\vec{P}\,\vec{y}, \vec{y}_s\rangle_\Gamma}{\langle \vec{x}_s, \vec{y}_s\rangle_\Gamma} \tag{2.7}$$

to yield $|\Gamma(t)|_t = \langle \vec{x}_s, (\vec{x}_t)_s\rangle_\Gamma = -\langle \vec{P}\,\vec{y}, \vec{x}_t\rangle_\Gamma = -\langle \vec{y}, \vec{P}\,\vec{x}_t\rangle_\Gamma = 0$, where we note that (2.7) *is well-defined because $\langle \vec{x}_s, \vec{y}_s\rangle_\Gamma = -\langle \vec{P}\,\vec{y}, \vec{y}\rangle_\Gamma = -\langle \vec{P}\,\vec{y}, \vec{P}\,\vec{y}\rangle_\Gamma = -\langle \vec{\varkappa}, \vec{\varkappa}\rangle_\Gamma$.*

2.2 Anisotropic elastic flow

Here we generalize the previously introduced geometric evolution equations to the case of anisotropic curve energy densities.

In many applications the energy of a curve in $\mathbb{R}^d$ depends locally on the orientation in space. For the case of curves in $\mathbb{R}^d$, $d \geq 2$, the local orientation is given by the unit tangent $\vec{x}_s$, cf. also Pozzi (2007). Hence we introduce an anisotropic curve energy of the form

$$|\Gamma|_\phi := \langle \phi(\vec{x}_s), 1 \rangle_\Gamma = \int_I \phi(\vec{x}_\rho)\, \mathrm{d}\rho\,,$$

where $\phi \in C^2(\mathbb{R}^d \setminus \{\vec{0}\}, \mathbb{R}_{>0}) \cap C(\mathbb{R}^d, \mathbb{R}_{\geq 0})$ is a given anisotropic energy density, which is positively homogeneous of degree one, i.e.

$$\phi(\lambda\, \vec{p}) = \lambda\, \phi(\vec{p}) \quad \forall\, \vec{p} \in \mathbb{R}^d\,,\ \forall\, \lambda \in \mathbb{R}_{\geq 0}\,. \tag{2.8}$$

The one-homogeneity immediately implies that

$$\phi'(\vec{p})\,.\,\vec{p} = \phi(\vec{p}) \quad \text{and} \quad \phi''(\vec{p})\,\vec{p} = \vec{0} \qquad \forall\, \vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}\,, \tag{2.9}$$

where ϕ' denotes the gradient and ϕ'' the matrix of second derivatives of ϕ. In addition, we assume that ϕ is strictly convex in the sense that there exists a positive constant c_0 such that

$$\vec{q}\,.\,\phi''(\vec{p})\,\vec{q} \geq c_0 \qquad \forall\, \vec{p},\, \vec{q} \in \mathbb{S}^{d-1} \text{ with } \vec{p}\,.\,\vec{q} = 0\,. \tag{2.10}$$

In the isotropic case it holds that

$$\phi(\vec{p}) = |\vec{p}| \quad \forall\, \vec{p} \in \mathbb{R}^d \qquad \Rightarrow \qquad \phi'(\vec{q}) = \vec{q}\,,\ \phi''(\vec{q}) = \vec{Id} - \vec{q} \otimes \vec{q} \quad \forall\, \vec{q} \in \mathbb{S}^{d-1}\,, \tag{2.11}$$

and so $\phi(\vec{x}_s) = 1$, which means that $|\Gamma|_\phi$ reduces to $|\Gamma|$, the length of Γ. A given anisotropy can be visualized by its Wulff shape, Wulff (1901), which for $d = 2$ can be defined as

$$\mathcal{W}_\phi := \{\vec{q} \in \mathbb{R}^2 : \phi^*(\vec{q}^{\perp}) \leq 1\}\,, \tag{2.12}$$

where the dual of ϕ is defined by $\phi^*(\vec{q}) = \sup_{\vec{p} \in \mathbb{R}^2 \setminus \{\vec{0}\}} \frac{\vec{p}\,.\,\vec{q}}{\phi(\vec{p})}$; see e.g. Pozzi (2007). It is known that among curves in $\mathbb{R}^2$ enclosing the same area, the boundary of a scaled Wulff shape has the smallest weighted length $|\cdot|_\phi$, see Fonseca and Müller (1991). Similarly, it can be shown that the boundary of the scaled Wulff shape $(2\,\lambda)^{-\frac{1}{2}}\,\mathcal{W}_\phi$ minimizes $E_\lambda(\Gamma, \varkappa_\phi)$ for $\lambda > 0$; see (2.24) below. For an introduction to anisotropic curve energies in general, and to Wulff shapes in particular, we refer to Giga (2006), Deckelnick, Dziuk, and Elliott (2005) and the references therein.

The first variation of $|\Gamma|_\phi$ is given by

$$[\frac{\delta}{\delta \vec{x}}\, |\Gamma|_\phi](\vec{\eta}) = \langle \phi'(\vec{x}_s), \vec{\eta}_s \rangle_\Gamma = -\langle [\phi'(\vec{x}_s)]_s, \vec{\eta} \rangle_\Gamma\,.$$

The quantity

$$\vec{\varkappa}_\phi := [\phi'(\vec{x}_s)]_s = \phi''(\vec{x}_s)\,\vec{\varkappa} \in \underline{V}_{0,\vec{\tau}} \tag{2.13}$$

can be viewed as an anisotropic curvature vector; where, in deriving the inclusion, we have recalled from (2.9) that $\phi''(\vec{x}_s)\,\vec{x}_s = \vec{0}$. For later use we remark that in the planar case, $d = 2$, the following anisotropic version of the well-known planar Gauß–Bonnet theorem holds.

LEMMA. 2.1. *Let $d = 2$, let $\Gamma = \vec{x}(I)$, with $\vec{x} \in C^2(I, \mathbb{R}^2)$, be a closed curve in $\mathbb{R}^2$ and let*

$$\varkappa_\phi := \vec{\varkappa}_\phi \,.\, \vec{\nu} = \vec{\nu} \,.\, \phi''(\vec{x}_s)\,\vec{x}_{ss} = (\vec{\nu} \,.\, \phi''(\vec{x}_s)\,\vec{\nu})\,\varkappa \tag{2.14}$$

denote the anisotropic curvature of Γ. Then it holds that

$$\langle \varkappa_\phi, 1 \rangle_\Gamma = m(\Gamma) \int_0^{2\pi} \phi((\cos u, \sin u)^T)\,\mathrm{d}u\,, \tag{2.15}$$

where $m(\Gamma)$ is the turning number of Γ.

Proof. Let $\theta \in C^1([0,1], \mathbb{R})$ denote, for each $\rho \in [0,1]$, an angle such that $\vec{x}_s = (\cos\theta, \sin\theta)^T$. It follows from (1.1) and our sign convention on the normal of Γ that $\varkappa = \theta_s = |\vec{x}_\rho|^{-1}\,\theta_\rho$. In addition, we have that $\theta(1) - \theta(0) = 2\,\pi\,m(\Gamma)$. We define the function $\widehat{\phi} \in C^2(\mathbb{R})$ by setting

$$\widehat{\phi}(u) := \phi((\cos u, \sin u)^T) \qquad \forall\ u \in \mathbb{R}\,, \tag{2.16}$$

so that $\widehat{\phi}$ is periodic with period $2\,\pi$. Then, on recalling that $\vec{\nu} = -\vec{x}_s^{\perp} = (-\sin\theta, \cos\theta)^T$, it follows from (2.9) that

$$\vec{\nu} \,.\, \phi''(\vec{x}_s)\,\vec{\nu} = \widehat{\phi}''(\theta) + \widehat{\phi}(\theta)\,. \tag{2.17}$$

Hence, we have that

$$\begin{aligned}
\langle \varkappa_\phi, 1 \rangle_\Gamma &= \langle \vec{\nu} \,.\, \phi''(\vec{x}_s)\,\vec{\nu}, \varkappa \rangle_\Gamma = \int_I [\widehat{\phi}''(\theta) + \widehat{\phi}(\theta)]\,\theta_\rho\,\mathrm{d}\rho = \int_0^1 [\widehat{\phi}'(\theta) + \widehat{\Phi}(\theta)]_\rho\,\mathrm{d}\rho \\
&= \left[\widehat{\phi}'(\theta(\rho)) + \widehat{\Phi}(\theta(\rho))\right]_0^1 = \left[\widehat{\Phi}(\theta(\rho))\right]_0^1 = \int_{\theta(0)}^{\theta(1)} \widehat{\Phi}'(u)\,\mathrm{d}u = \int_{\theta(0)}^{\theta(1)} \widehat{\phi}(u)\,\mathrm{d}u \\
&= m(\Gamma) \int_0^{2\pi} \widehat{\phi}(u)\,\mathrm{d}u\,,
\end{aligned}$$

where $\widehat{\Phi}(u) = \int_0^u \widehat{\phi}(r)\,\mathrm{d}r$. ∎

In Barrett, Garcke, and Nürnberg (2010b) the authors derived the L^2-gradient flow of the anisotropic elastic energy $E_0(\Gamma, \vec{\varkappa}_\phi)$, recall (1.2), as

$$\vec{P}\,\vec{x}_t = -(\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s)_s - \tfrac{1}{2}\,(|\vec{\varkappa}_\phi|^2\,\vec{x}_s)_s \equiv -\vec{\nabla}_s(\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s) - \tfrac{1}{2}\,|\vec{\varkappa}_\phi|^2\,\vec{\varkappa}\,, \tag{2.18}$$

where, in obtaining the equivalence above, we have noted from (2.13) and (2.9) that

$$\begin{aligned}
\tfrac{1}{2}\,(|\vec{\varkappa}_\phi|^2)_s &= (\phi''(\vec{x}_s)\,\vec{\varkappa}) \,.\, (\vec{\varkappa}_\phi)_s = [\,[\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s] \,.\, \vec{x}_s]_s - [\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s]_s \,.\, \vec{x}_s \\
&= -[\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s]_s \,.\, \vec{x}_s\,.
\end{aligned}$$

In the case $d = 2$, (2.18) can be equivalently formulated, see Theorem 2.3 below, as

$$\vec{x}_t \,.\, \vec{\nu} = -([\vec{\nu} \,.\, \phi''(\vec{x}_s)\, \vec{\nu}]\, (\varkappa_\phi)_s)_s - \tfrac{1}{2}\, \varkappa_\phi^2\, \varkappa\,, \tag{2.19}$$

which in the isotropic case (2.11) clearly collapses to (1.4) with $\lambda = 0$. For later use, we state a similarity solution of (2.19), which is the analogue for the anisotropic elastic flow of the solution found in Soner (1993) for the anisotropic curve shortening flow.

THEOREM. 2.1. *Let $d = 2$ and let (2.10) hold. Then the weighted anisotropic elastic flow*

$$[\sigma(\vec{x}_s)]^{-1}\, \vec{x}_t \,.\, \vec{\nu} = -([\vec{\nu} \,.\, \phi''(\vec{x}_s)\, \vec{\nu}]\, (\varkappa_\phi)_s)_s - \tfrac{1}{2}\, \varkappa_\phi^2\, \varkappa\,, \tag{2.20}$$

where $\sigma(\vec{q}) := \phi(\vec{q})\, (\vec{q}^{\perp} \,.\, \phi''(\vec{q})\, \vec{q}^{\perp}) > 0$ for $\vec{q} \in \mathbb{S}^1$, has a similarity solution of the form

$$\Gamma(t) = \{\vec{q} \in \mathbb{R}^2 : \phi^*(\vec{q}^{\perp}) = (1 + 2\,t)^{\frac{1}{4}}\}\,, \tag{2.21}$$

i.e. the expanding boundary of the Wulff shape corresponding to the anisotropy ϕ.

Proof. As $\partial\mathcal{W}_\phi$ is by definition convex, w.l.o.g. we can consider a parameterization $\vec{x}_o(\theta) : \mathbb{R}/(2\,\pi\,\mathbb{Z}) \to \partial\mathcal{W}_\phi$ such that $\vec{\tau}_o := (\vec{x}_o)_s = (\cos\theta, \sin\theta)^T$. Let $\vec{\nu}_o(\theta) := (-\sin\theta, \cos\theta)^T$ and recall from (2.16) the definition of $\widehat{\phi}$. Then it holds that

$$\vec{x}_o(\theta) = \widehat{\phi}(\theta)\, \vec{\nu}_o(\theta) - \widehat{\phi}'(\theta)\, \vec{\tau}_o(\theta)\,, \tag{2.22}$$

with curvature $\varkappa_o(\theta) = -[\widehat{\phi}''(\theta) + \widehat{\phi}(\theta)]^{-1}$, see e.g. Gurtin (1993, (1.10), (7.43)).

Now let $\vec{x}(\theta, t) = z(t)\, \vec{x}_o(\theta)$. Then it follows that the curvature is given by $\varkappa(\theta, t) = [z(t)]^{-1}\, \varkappa_o(\theta) = -[z(t)\, (\widehat{\phi}''(\theta) + \widehat{\phi}(\theta))]^{-1}$, which on combining with (2.14) and (2.17) yields that $\varkappa_\phi(\theta, t) = -[z(t)]^{-1}$, and so $(\varkappa_\phi(\theta, t))_s = 0$ for all θ and t. Moreover, it follows from (2.22) that

$$\vec{x}_t(\theta, t) \,.\, \vec{\nu}(\theta, t) = \vec{x}_t(\theta, t) \,.\, \vec{\nu}_o(\theta) = z'(t)\, \vec{x}_o(\theta) \,.\, \vec{\nu}_o(\theta) = z'(t)\, \widehat{\phi}(\theta)\,.$$

It follows for $\vec{x}(\theta, t) = z(t)\, \vec{x}_o(\theta)$ that the flow (2.20) collapses to

$$[\widehat{\phi}\, (\widehat{\phi}'' + \widehat{\phi})]^{-1}\, z'\, \widehat{\phi} = \tfrac{1}{2}\, [z^3\, (\widehat{\phi}'' + \widehat{\phi})]^{-1}\,,$$

i.e. $z' = \frac{1}{2}\, z^{-3}$, which together with $z(0) = 1$ is solved by $z(t) = (1 + 2\,t)^{\frac{1}{4}}$. The desired result now follows from (2.12). $\square$

The present authors in Barrett, Garcke, and Nürnberg (2010b) introduced a finite element approximation of (2.18) based on the following weak formulation: Given $\Gamma(0) = \vec{x}(I, 0)$, with $\vec{x}(0) \in \underline{V}_0$, for all $t \in (0, T]$ find $\Gamma(t) = \vec{x}(I, t)$, where $\vec{x}(t) \in \underline{V}_0$, and $\vec{\varkappa}_\phi(t) \in \underline{V}_{0,\vec{\tau}}$ such that

$$\langle \vec{P}\, \vec{x}_t, \vec{\chi} \rangle_\Gamma - \langle \phi''(\vec{x}_s)\, (\vec{\varkappa}_\phi)_s, \vec{\chi}_s \rangle_\Gamma - \tfrac{1}{2}\, \langle |\vec{\varkappa}_\phi|^2\, \vec{x}_s, \vec{\chi}_s \rangle_\Gamma = 0 \quad \forall\, \vec{\chi} \in \underline{V}_{0,\vec{\tau}}\,, \tag{2.23a}$$

$$\langle \vec{\varkappa}_\phi, \vec{\eta} \rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s \rangle_\Gamma = 0 \quad \forall\, \vec{\eta} \in \underline{V}_0\,. \tag{2.23b}$$

On recalling that in the isotropic case (2.11) it holds that $\phi''(\vec{x}_s) = \vec{P}$, we see that (2.18) and (2.23a,b) are the natural anisotropic analogues of (1.5) and (2.1a,b), respectively, in the case $\lambda = 0$.

Similarly to (1.8), it follows from (2.13) that for closed curves $\widetilde{E}_{\vec{\beta}}(\Gamma, \vec{\varkappa}_\phi) = E_\lambda(\Gamma, \vec{\varkappa}_\phi)$ for the choice $\lambda = \frac{1}{2}\,|\vec{\beta}|^2$. Hence, from now on we consider $E_\lambda(\Gamma, \vec{\varkappa}_\phi)$ for a general $\lambda \in \mathbb{R}$. In addition, in the case $d = 2$, similarly to (1.7), it follows from Lemma 2.1 that

$$\widetilde{E}_\beta(\Gamma, \varkappa_\phi) = E_{\frac{1}{2}\,\beta^2}(\Gamma, \varkappa_\phi) - \beta\, m(\Gamma) \int_0^{2\,\pi} \phi((\cos u, \sin u)^T)\; \mathrm{d}u\,; \tag{2.24}$$

and hence, gradient flows of $\widetilde{E}_\beta(\Gamma, \varkappa_\phi)$ are equivalent to gradient flows of $E_\lambda(\Gamma, \varkappa_\phi) \equiv E_\lambda(\Gamma, \vec{\varkappa}_\phi)$ for the choice $\lambda = \frac{1}{2}\,\beta^2$. Moreover, the above analysis shows that the scaled Wulff shapes $(2\,\lambda)^{-\frac{1}{2}}\,\mathcal{W}_\phi$ having $\varkappa_\phi = -(2\,\lambda)^{\frac{1}{2}}$, recall the proof of Theorem 2.1, are the unique global minimizers of $E_\lambda(\Gamma, \varkappa_\phi)$ for $\lambda > 0$. In view of all this, for closed curves we will just consider the energy $E_\lambda(\Gamma, \vec{\varkappa}_\phi)$ for $d \geq 2$, and not consider the case $d = 2$ separately.

Analogously to the isotropic case treated beforehand, we consider the L^2-gradient flow for $E_\lambda(\Gamma, \vec{\varkappa}_\phi)$ subject to

$$\langle \vec{\varkappa}_\phi, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = 0 \qquad \forall\ \vec{\eta} \in \underline{V}_0 \tag{2.25a}$$

$$\text{and} \qquad \langle \vec{\varkappa}_\phi \,.\, \vec{x}_s, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in U_0\,. \tag{2.25b}$$

Introducing the Lagrange multipliers $\vec{y} \in \underline{V}_0$ and $z \in U_0$ for (2.25a,b), we define the Lagrangian

$$\mathcal{L}(\vec{x}, \vec{\varkappa}_\phi, \vec{y}, z) := \tfrac{1}{2}\,\langle \vec{\varkappa}_\phi, \vec{\varkappa}_\phi\rangle_\Gamma + \lambda\,|\Gamma| - \langle \vec{\varkappa}_\phi, \vec{y}\rangle_\Gamma - \langle \phi'(\vec{x}_s), \vec{y}_s\rangle_\Gamma + \langle \vec{\varkappa}_\phi \,.\, \vec{x}_s, z\rangle_\Gamma \tag{2.26}$$

and hence we obtain, on taking variations and on setting $\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma = -[\frac{\delta}{\delta \vec{x}}\,\mathcal{L}](\vec{\chi})$, that

$$\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma = \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle (|\vec{\varkappa}_\phi|^2 - 2\,\vec{\varkappa}_\phi \,.\, \vec{y} + 2\,\lambda)\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma - \langle z\,\vec{\varkappa}_\phi, \vec{\chi}_s\rangle_\Gamma \qquad \forall\ \vec{\chi} \in \underline{V}_0\,, \tag{2.27a}$$

$$\langle \vec{\varkappa}_\phi + z\,\vec{x}_s - \vec{y}, \vec{\xi}\rangle_\Gamma = 0 \qquad \forall\ \vec{\xi} \in \underline{V}_0\,, \tag{2.27b}$$

$$\langle \vec{\varkappa}_\phi, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0\,, \tag{2.27c}$$

$$\langle \vec{\varkappa}_\phi \,.\, \vec{x}_s, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in U_0\,. \tag{2.27d}$$

Clearly, it follows from (2.27b,d) that

$$\vec{P}\,\vec{y} = \vec{\varkappa}_\phi \qquad \text{and} \qquad z = \vec{y} \,.\, \vec{x}_s\,. \tag{2.28}$$

Overall, and similarly to (2.5a,b), our weak formulation for the gradient flow is then given by

$$\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma - \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle (|\vec{P}\,\vec{y}|^2 - 2\,\lambda)\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma + \langle (\vec{y} \,.\, \vec{x}_s)\,\vec{P}\,\vec{y}, \vec{\chi}_s\rangle_\Gamma = 0 \qquad \forall\ \vec{\chi} \in \underline{V}_0\,, \tag{2.29a}$$

$$\langle \vec{P}\,\vec{y}, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0\,. \tag{2.29b}$$

Of course, in the isotropic case (2.29a,b) collapses to (2.5a,b), recall (2.11). And analogously to the isotropic case, the system (2.29a,b) with (2.28) is equivalent to (2.27a–d). The following theorem establishes that (2.29a,b) with (2.28) is indeed a weak formulation for the L^2-gradient flow of the energy $E_\lambda(\Gamma, \vec\varkappa_\phi)$.

THEOREM. 2.2. *Let $(\vec x(t), \vec y(t))_{t\in(0,T]}$ be a solution to* (2.29a,b)*. Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda(\Gamma, \vec P\,\vec y) = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma \leq 0\,, \tag{2.30}$$

where $\vec P\,\vec y = \vec\varkappa_\phi$ is the anisotropic curvature vector.

Proof. Differentiating (2.29b) with respect to t yields, on noting that

$$(\vec x_s)_t = \vec P\,\vec x_{t,s} = \vec\nabla_s\,\vec x_t\,, \tag{2.31}$$

that

$$\langle (\vec P\,\vec y)_t, \vec\eta\rangle_\Gamma + \langle (\vec P\,\vec y)\,.\,\vec\eta, \vec x_s\,.\,\vec x_{t,s}\rangle_\Gamma + \langle \phi''(\vec x_s)\,\vec\nabla_s\,\vec x_t, \vec\nabla_s\,\vec\eta\rangle_\Gamma = 0 \qquad \forall\ \vec\eta \in \underline{V}_0\,. \tag{2.32}$$

On choosing $\vec\eta = \vec y$ in (2.32), we obtain that

$$\langle (\vec P\,\vec y)_t, \vec y\rangle_\Gamma + \langle |\vec P\,\vec y|^2, \vec x_s\,.\,\vec x_{t,s}\rangle_\Gamma + \langle \phi''(\vec x_s)\,\vec\nabla_s\,\vec x_t, \vec\nabla_s\,\vec y\rangle_\Gamma = 0\,. \tag{2.33}$$

Combining (2.33) and (2.29a) with $\vec\chi = \vec x_t$ yields that

$$\langle (\vec P\,\vec y)_t, \vec y\rangle_\Gamma + \tfrac12\,\langle |\vec P\,\vec y|^2 + 2\,\lambda, \vec x_s\,.\,\vec x_{t,s}\rangle_\Gamma + \langle (\vec y\,.\,\vec x_s)\,\vec P\,\vec y, \vec x_{t,s}\rangle_\Gamma = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma\,. \tag{2.34}$$

The desired result (2.30) then follows from (2.34) and (2.28), on noting that

$$\begin{aligned}\langle (\vec P\,\vec y)_t, (\vec{Id} - \vec P)\,\vec y\rangle_\Gamma &= \langle (\vec P\,\vec y)_t, (\vec y\,.\,\vec x_s)\,\vec x_s\rangle_\Gamma = -\langle \vec P\,\vec y, (\vec y\,.\,\vec x_s)\,(\vec x_s)_t\rangle_\Gamma \\ &= -\langle (\vec y\,.\,\vec x_s)\,\vec P\,\vec y, \vec x_{t,s}\rangle_\Gamma\,.\end{aligned}$$

□

COROLLARY. 2.1. *Let $(\vec x(t), \vec y(t))_{t\in(0,T]}$ be a solution to* (2.5a,b) *with $\lambda(t) \in \mathbb{R}$ chosen as in* (2.7)*. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Gamma(t)| = 0 \qquad \textit{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} E_0(\Gamma, \vec P\,\vec y) = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma \leq 0\,, \tag{2.35}$$

where $\vec P\,\vec y = \vec\varkappa$ is the isotropic curvature vector.

Proof. As noted in Remark 2.1, choosing $\lambda(t)$ as in (2.7) yields that $|\Gamma(t)|_t = \langle \vec x_s, (\vec x_t)_s\rangle_\Gamma = 0$; and so the term involving λ in (2.34) vanishes. □

2.3 Initial boundary value problems

In this section, we want to study the elastic flow (1.5) for a single open curve. Here $(\Gamma(t))_{t\in[0,T]}$ is given by a parameterization $\vec{x}(\rho,t):[0,1]\times[0,T]\to\mathbb{R}^d$. As there is no particular difficulty in allowing an anisotropic curve energy density, we will consider the anisotropic elastic flow (2.18) throughout. As mentioned in the introduction, suitable boundary conditions need to be considered at the two endpoints of the curve $\Gamma(t)$, and one can either fix position and angle(s) (clamped conditions), or fix position and curvature (Navier conditions), see e.g. Deckelnick and Grunau (2007, 2009).

In what follows, we will derive the elastic flow (2.18), supplemented with various suitable boundary conditions, in each case as an L^2-gradient flow of an appropriately chosen curvature integral.

Lemma. 2.2. *Let $\beta\in\mathbb{R}$, $\vec{\alpha}_0,\vec{\alpha}_1,\vec{\beta}\in\mathbb{R}^d$ and let $\vec{x}(\rho,t):[0,1]\times[0,T]\to\mathbb{R}^d$ be such that $\Gamma(t)=\vec{x}([0,1],t)$ with $\vec{x}(0,t)=\vec{\alpha}_0$ and $\vec{x}(1,t)=\vec{\alpha}_1$. We compute the time derivative of $\widehat{E}_{\vec{\beta}}(\Gamma,\vec{\varkappa}_\phi)$ as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tfrac12\,|\vec{\varkappa}_\phi|^2-\vec{\beta}\,.\,\vec{\varkappa}_\phi,1\rangle_\Gamma=\langle\vec{\nabla}_s(\phi''(\vec{x}_s)\,(\vec{\varkappa}_\phi)_s)+\tfrac12\,|\vec{\varkappa}_\phi|^2\,\vec{\varkappa},\vec{\mathcal{V}}\rangle_\Gamma+\Big[(\vec{\varkappa}_\phi-\vec{\beta})\,.\,\phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{\mathcal{V}}\Big]_0^1\,, \tag{2.36a}$$

where $\vec{\mathcal{V}}:=\vec{P}\,\vec{x}_t$. Similarly, in the case $d=2$ we compute the time derivative of $\widehat{E}_\beta(\Gamma,\varkappa_\phi)$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac12\,\varkappa_\phi^2-\beta\,\varkappa_\phi,1\rangle_\Gamma=\langle([\vec{\nu}\,.\,\phi''(\vec{x}_s)\,\vec{\nu}]\,(\varkappa_\phi)_s)_s+\tfrac12\,\varkappa_\phi^2\,\varkappa,\mathcal{V}\rangle_\Gamma+[(\varkappa_\phi-\beta)\,(\vec{\nu}\,.\,\phi''(\vec{x}_s)\,\vec{\nu})\,\mathcal{V}_s]_0^1\,, \tag{2.36b}$$

where $\mathcal{V}:=\vec{x}_t\,.\,\vec{\nu}$.

Proof. The fixed boundary conditions imply that $\vec{x}_t(0,t)=\vec{x}_t(1,t)=0$ for all $t\in(0,T)$. Setting $\vec{\tau}=\vec{x}_s$ we can write $\vec{x}_t=\vec{\mathcal{V}}+v\,\vec{\tau}$. Hence we have for all $t\in(0,T)$ that

$$v(0,t)=v(1,t)=0\qquad\text{and}\qquad\vec{\mathcal{V}}(0,t)=\vec{\mathcal{V}}(1,t)=\vec{0}\,. \tag{2.37}$$

The following results are easily derived, see e.g. Dziuk *et al.* (2002, Lemma 2.1):

$$|\vec{x}_\rho|_t=(v_s-\vec{\varkappa}\,.\,\vec{\mathcal{V}})\,|\vec{x}_\rho|,\quad \partial_t\,\partial_s-\partial_s\,\partial_t=(\vec{\varkappa}\,.\,\vec{\mathcal{V}}-v_s)\,\partial_s\quad\text{and}\quad\vec{\tau}_t=\vec{\nabla}_s\,\vec{\mathcal{V}}+v\,\vec{\varkappa}\,, \tag{2.38}$$

where the last equality is just a rewrite of (2.31). On noting (2.13), (2.38) and (2.9), it follows that

$$\begin{aligned}(\vec{\varkappa}_\phi)_t&=([\phi'(\vec{x}_s)]_s)_t=([\phi'(\vec{x}_s)]_t)_s+(\vec{\varkappa}\,.\,\vec{\mathcal{V}}-v_s)\,[\phi'(\vec{x}_s)]_s\\&=(\phi''(\vec{x}_s)\,\vec{\tau}_t)_s+(\vec{\varkappa}\,.\,\vec{\mathcal{V}}-v_s)\,\vec{\varkappa}_\phi=(\phi''(\vec{x}_s)\,(\vec{\nabla}_s\,\vec{\mathcal{V}}+v\,\vec{\varkappa}))_s+(\vec{\varkappa}\,.\,\vec{\mathcal{V}}-v_s)\,\vec{\varkappa}_\phi\\&=(\phi''(\vec{x}_s)\,\vec{\mathcal{V}}_s)_s+(\vec{\varkappa}\,.\,\vec{\mathcal{V}})\,\vec{\varkappa}_\phi+v\,(\vec{\varkappa}_\phi)_s\,.\end{aligned} \tag{2.39}$$

On noting (2.38), (2.39), (2.37) and (2.9), we then compute

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \tfrac12\,|\vec\varkappa_\phi|^2 - \vec\beta\,.\,\vec\varkappa_\phi, 1\rangle_\Gamma \\
&\quad= \langle \vec\varkappa_\phi - \vec\beta, (\vec\varkappa_\phi)_t\rangle_\Gamma + \langle \tfrac12\,|\vec\varkappa_\phi|^2 - \vec\beta\,.\,\vec\varkappa_\phi, v_s - \vec\varkappa\,.\,\vec{\mathcal{V}}\rangle_\Gamma \\
&\quad= \langle \vec\varkappa_\phi - \vec\beta, (\phi''(\vec x_s)\,\vec{\mathcal{V}}_s)_s\rangle_\Gamma + \tfrac12\,\langle |\vec\varkappa_\phi|^2\,\vec\varkappa, \vec{\mathcal{V}}\rangle_\Gamma + \langle [(\tfrac12\,|\vec\varkappa_\phi|^2 - \vec\beta\,.\,\vec\varkappa_\phi)\,v]_s, 1\rangle_\Gamma \\
&\quad= -\langle (\vec\varkappa_\phi)_s, \phi''(\vec x_s)\,\vec{\mathcal{V}}_s\rangle_\Gamma + \tfrac12\,\langle |\vec\varkappa_\phi|^2\,\vec\varkappa, \vec{\mathcal{V}}\rangle_\Gamma + \Big[(\vec\varkappa_\phi - \vec\beta)\,.\,\phi''(\vec x_s)\,\vec\nabla_s\,\vec{\mathcal{V}}\Big]_0^1 \\
&\quad= \langle \vec\nabla_s(\phi''(\vec x_s)\,(\vec\varkappa_\phi)_s) + \tfrac12\,|\vec\varkappa_\phi|^2\,\vec\varkappa, \vec{\mathcal{V}}\rangle_\Gamma + \Big[(\vec\varkappa_\phi - \vec\beta)\,.\,\phi''(\vec x_s)\,\vec\nabla_s\,\vec{\mathcal{V}}\Big]_0^1\,,
\end{aligned}
\tag{2.40}
$$

and hence the desired result (2.36a).

The above is easily adapted to the energy $\widehat E_\beta(\Gamma,\varkappa_\phi)$ in the case $d=2$. Now $\vec{\mathcal{V}} = \mathcal{V}\,\vec\nu$ and $\vec\varkappa_\phi = \varkappa_\phi\,\vec\nu$. It follows from (2.39) and (2.9) that

$$
\begin{aligned}
(\varkappa_\phi)_t = (\vec\varkappa_\phi)_t\,.\,\vec\nu &= (\mathcal{V}_s\,\phi''(\vec x_s)\,\vec\nu)_s\,.\,\vec\nu + \varkappa\,\varkappa_\phi\,\mathcal{V} + v\,(\varkappa_\phi)_s \\
&= (\mathcal{V}_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu)_s + \varkappa\,\varkappa_\phi\,\mathcal{V} + v\,(\varkappa_\phi)_s\,.
\end{aligned}
\tag{2.41}
$$

We then adapt (2.40), using (2.41) in place of (2.39), to obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \tfrac12\,\varkappa_\phi^2 - \beta\,\varkappa_\phi, 1\rangle_\Gamma \\
&\quad= \langle \varkappa_\phi - \beta, (\varkappa_\phi)_t\rangle_\Gamma + \langle \tfrac12\,\varkappa_\phi^2 - \beta\,\varkappa_\phi, v_s - \vec\varkappa\,.\,\vec{\mathcal{V}}\rangle_\Gamma \\
&\quad= \langle \varkappa_\phi - \beta, (\mathcal{V}_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu)_s\rangle_\Gamma + \tfrac12\,\langle \varkappa_\phi^2\,\varkappa, \mathcal{V}\rangle_\Gamma + \langle [(\tfrac12\,\varkappa_\phi^2 - \beta\,\varkappa_\phi)\,v]_s, 1\rangle_\Gamma \\
&\quad= -\langle (\varkappa_\phi)_s, (\mathcal{V}_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu)\rangle_\Gamma + \tfrac12\,\langle \varkappa_\phi^2\,\varkappa, \mathcal{V}\rangle_\Gamma + [(\varkappa_\phi - \beta)\,(\mathcal{V}_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu)]_0^1 \\
&\quad= \langle ((\varkappa_\phi)_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu))_s + \tfrac12\,\varkappa_\phi^2\,\varkappa, \mathcal{V}\rangle_\Gamma + [(\varkappa_\phi - \beta)\,(\mathcal{V}_s\,\vec\nu\,.\,\phi''(\vec x_s)\,\vec\nu)]_0^1\,,
\end{aligned}
$$

and hence the desired result (2.36b). $\square$

We are now in a position to state the strong formulations of gradient flows for the energies

$$
\widehat E_{\vec\beta,\lambda}(\Gamma,\vec\varkappa) := \langle \tfrac12\,|\vec\varkappa|^2 - \vec\beta\,.\,\vec\varkappa, 1\rangle_\Gamma + \lambda\,|\Gamma| \quad\text{and}\quad \widehat E_{\beta,\lambda}(\Gamma,\varkappa) := \langle \tfrac12\,\varkappa^2 - \beta\,\varkappa, 1\rangle_\Gamma + \lambda\,|\Gamma|\,.
$$

In each case, we find that the elastic flow equation is satisfied in the interior of Γ.

Theorem. 2.3. *Let $\lambda,\beta\in\mathbb{R}$, $\vec\beta,\vec\alpha_0,\vec\alpha_1\in\mathbb{R}^d$ and $\vec\zeta_0,\vec\zeta_1\in\mathbb{S}^{d-1}$. Then the flow*

$$
\vec P\,\vec x_t = -\vec\nabla_s(\phi''(\vec x_s)\,(\vec\varkappa_\phi)_s) - \tfrac12\,|\vec\varkappa_\phi|^2\,\vec\varkappa + \lambda\,\vec\varkappa \equiv -(\phi''(\vec x_s)\,(\vec\varkappa_\phi)_s)_s - \tfrac12\,(|\vec\varkappa_\phi|^2\,\vec x_s)_s + \lambda\,\vec\varkappa \quad \text{in } (0,1)\times(0,T)
\tag{2.42}
$$

with the clamped boundary conditions

$$
\vec x(0,t) = \vec\alpha_0\,,\ \vec x(1,t) = \vec\alpha_1 \quad\text{and}\quad \vec x_s(0,t) = \vec\zeta_0\,,\ \vec x_s(1,t) = \vec\zeta_1
\tag{2.43}
$$

is an L^2-gradient flow of $\widehat E_{\vec\beta,\lambda}$. Moreover, (2.42) *with the boundary conditions*

$$
\vec x(0,t) = \vec\alpha_0\,,\ \vec x(1,t) = \vec\alpha_1 \quad\text{and}\quad \vec\varkappa_\phi(0,t) = \vec P(0,t)\,\vec\beta\,,\ \vec\varkappa_\phi(1,t) = \vec P(1,t)\,\vec\beta
\tag{2.44}
$$

is an L^2*-gradient flow of* $\widehat{E}_{\vec{\beta},\lambda}$*. If* $d=2$*, then the flow*

$$\vec{x}_t \,.\, \vec{\nu} = -([\vec{\nu} \,.\, \phi''(\vec{x}_s)\, \vec{\nu}]\, (\varkappa_\phi)_s)_s - \tfrac{1}{2}\, \varkappa_\phi^2\, \varkappa + \lambda\, \varkappa \qquad in\ (0,1)\times(0,T) \tag{2.45}$$

with the Navier boundary conditions

$$\vec{x}(0,t) = \vec{\alpha}_0\,,\ \vec{x}(1,t) = \vec{\alpha}_1 \quad and \quad \varkappa_\phi(0,t) = \beta\,,\ \varkappa_\phi(1,t) = \beta \tag{2.46}$$

is an L^2*-gradient flow of* $\widehat{E}_{\beta,\lambda}$*.*

Proof. It immediately follows from (2.38) and (2.37) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Gamma| = \langle v_s - \vec{\varkappa}\,.\,\vec{\mathcal{V}}, 1\rangle_\Gamma = -\langle \vec{\varkappa}, \vec{\mathcal{V}}\rangle_\Gamma + [v]_0^1 = -\langle \vec{\varkappa}, \vec{\mathcal{V}}\rangle_\Gamma\,.$$

Hence it remains to show that in each case the boundary terms in (2.36a) and (2.36b), respectively, vanish. For the clamped conditions (2.43) we observe that $\vec{\tau}_t(0,t) = \vec{\tau}_t(1,t) = \vec{0}$ and so $\vec{\nabla}_s\, \mathcal{V}(0,t) = \vec{\nabla}_s\, \mathcal{V}(1,t) = \vec{0}$ for all $t \in (0,T)$, on recalling (2.38) and (2.37). Hence, the boundary term in (2.36a) vanishes. The boundary conditions (2.44), on the other hand, imply that $[\phi''(\vec{x}_s)(\vec{\varkappa}_\phi - \vec{P}\,\vec{\beta})](0,t) = [\phi''(\vec{x}_s)(\vec{\varkappa}_\phi - \vec{P}\,\vec{\beta})](1,t) = \vec{0}$ for all $t \in (0,T)$, on noting (2.9); and hence the desired result. Finally, in the case $d=2$ it immediately follows from the Navier conditions (2.46) that the boundary term in (2.36b) vanishes. □

Corollary. 2.2. *The flow* (2.42) *with the clamped boundary conditions* (2.43) *is an* L^2*-gradient flow of the energy* $E_\lambda(\Gamma, \vec{\varkappa}_\phi)$*.*

Proof. This follows immediately from Theorem 2.3 on noting that (2.42) and (2.43) do not depend on $\vec{\beta}$. □

For later use, we introduce the definitions

$$\underline{V} := H^1((0,1),\mathbb{R}^d) \qquad \text{and} \qquad \underline{W} := H^1_0((0,1),\mathbb{R}^d)$$

and similarly $U := L^2((0,1),\mathbb{R})$, $V := H^1((0,1),\mathbb{R})$, $W := H^1_0((0,1),\mathbb{R})$. We then have the following weak formulations of (2.13) on the open curve $\Gamma(t)$. For the clamped boundary conditions (2.43), in the case $d \geq 2$, we formulate this as $\vec{\varkappa}_\phi \in \underline{V}$ with

$$\langle \vec{\varkappa}_\phi, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = \phi'(\vec{\zeta}_1)\,.\,\vec{\eta}(1) - \phi'(\vec{\zeta}_0)\,.\,\vec{\eta}(0) \qquad \forall\ \vec{\eta} \in \underline{V}\,, \tag{2.47a}$$

$$\text{and} \qquad \langle \vec{\varkappa}_\phi \,.\, \vec{x}_s, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in U\,, \tag{2.47b}$$

while for the Navier boundary conditions (2.46), in the case $d=2$, we use the formulation $(\varkappa_\phi - \beta) \in W$ with

$$\langle \varkappa_\phi\, \vec{\nu}, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = 0 \qquad \forall\ \vec{\eta} \in \underline{W}\,. \tag{2.48}$$

As is standard, and unlike the clamped boundary case above, we use $\underline{W}$ instead of $\underline{V}$ in (2.48) since we have no information about $\vec{x}_s$ at the two endpoints.

2.3.1 Clamped conditions

In view of Corollary 2.2, we have that (2.42) and (2.43) is an L^2-gradient flow for $E_\lambda(\Gamma, \vec\varkappa_\phi)$. Then, similarly to the closed curve case (2.29a,b), our weak formulation of this gradient flow, subject to the side constraints (2.47a,b), is given by: Given $\Gamma(0) = \vec x([0,1],0)$ with $\vec x(0,0) = \vec\alpha_0$ and $\vec x(1,0) = \vec\alpha_1$, for all $t \in (0,T]$ find $\Gamma(t) = \vec x([0,1],t)$, where $\vec x(t) \in \underline{V}$ with $\vec x_t(t) \in \underline{W}$, and $\vec y(t) \in \underline{V}$ such that

$$\langle \vec P\,\vec x_t, \vec\chi\rangle_\Gamma - \langle \phi''(\vec x_s)\,\vec\nabla_s\,\vec y, \vec\nabla_s\,\vec\chi\rangle_\Gamma - \tfrac12\,\langle(|\vec P\,\vec y|^2 - 2\,\lambda)\,\vec x_s, \vec\chi_s\rangle_\Gamma + \langle(\vec y\,.\,\vec x_s)\,\vec P\,\vec y, \vec\chi_s\rangle_\Gamma = 0 \qquad \forall\,\vec\chi \in \underline{W}\,, \tag{2.49a}$$

$$\langle \vec P\,\vec y, \vec\eta\rangle_\Gamma + \langle \phi'(\vec x_s), \vec\eta_s\rangle_\Gamma = \phi'(\vec\zeta_1)\,.\,\vec\eta(1) - \phi'(\vec\zeta_0)\,.\,\vec\eta(0) \quad \forall\,\vec\eta \in \underline{V}\,. \tag{2.49b}$$

Here we observe that in the above formulation the fixed position conditions in (2.43) are enforced strongly through $\vec x_t \in \underline{W}$, while the angle conditions in (2.43) are enforced weakly through (2.49b).

THEOREM. 2.4. *Let $(\vec x(t), \vec y(t))_{t\in(0,T]}$ be a solution to* (2.49a,b)*. Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E_\lambda(\Gamma, \vec P\,\vec y) = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma \le 0\,, \tag{2.50}$$

where $\vec P\,\vec y = \vec\varkappa_\phi$ is the anisotropic curvature vector.

Proof. The proof follows along the same lines as the proof of Theorem 2.2, on noting that $\vec x_t \in \underline{W}$ and $\vec y \in \underline{V}$. □

Moreover, it is not difficult to show that (2.49a,b) with now $\vec y(t) \in \underline{W}$ and the test space $\underline{V}$ for (2.49b) replaced by $\underline{W}$ is a weak formulation of the flow (2.42) with homogeneous boundary conditions (2.44); that is, $\vec\beta = \vec 0$. However, since the boundary conditions (2.44) are non-physical in the case $\vec\beta \neq \vec 0$, we do not consider the flow (2.42) and (2.44) in detail in this paper.

2.3.2 Navier conditions

We derive our weak formulation for (2.45) and (2.46) as an L^2-gradient flow for $\widehat E_{\beta,\lambda}(\Gamma, \varkappa_\phi)$. First, we observe that

$$\widehat E_{\beta,\lambda}(\Gamma, \varkappa_\phi) = \tfrac12\,\langle(\varkappa_\phi - \beta)^2, 1\rangle_\Gamma + \tfrac12\,(2\,\lambda - \beta^2)\,|\Gamma|\,.$$

For a family of open curves $(\Gamma(t))_{t\in[0,T]}$ satisfying the boundary conditions (2.46), we then define the Lagrangian

$$\mathcal{L}(\vec x, \varkappa_\phi, \vec y) := \tfrac12\,\langle \varkappa_\phi - \beta, \varkappa_\phi - \beta\rangle_\Gamma + \tfrac12\,(2\,\lambda - \beta^2)\,|\Gamma| - \langle \varkappa_\phi\,\vec\nu, \vec y\rangle_\Gamma - \langle \phi'(\vec x_s), \vec y_s\rangle_\Gamma\,, \tag{2.51}$$

where $\vec{y}(t) \in \underline{W}$ is a Lagrange multiplier for the side constraint (2.48). Hence we obtain, on taking variations $[\frac{\delta}{\delta \vec{x}}\,\mathcal{L}](\vec{\chi})$, $[\frac{\delta}{\delta \varkappa_\phi}\,\mathcal{L}](\chi)$ and $[\frac{\delta}{\delta \vec{y}}\,\mathcal{L}](\vec{\eta})$, and on setting $\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma = -[\frac{\delta}{\delta \vec{x}}\,\mathcal{L}](\vec{\chi})$, that

$$\begin{aligned}
&\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma = \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma - \tfrac{1}{2}\,\langle[(\varkappa_\phi - \beta)^2 + 2\,\lambda - \beta^2 - 2\,\varkappa_\phi\,(\vec{y}\,.\,\vec{\nu})]\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma \\
&\qquad\qquad - \langle \varkappa_\phi\,\vec{y}, (\vec{\nabla}_s\,\vec{\chi})^\perp\rangle_\Gamma \quad \forall\ \vec{\chi} \in \underline{W}\,, && (2.52a)\\
&\langle \varkappa_\phi - \beta - \vec{y}\,.\,\vec{\nu}, \chi\rangle_\Gamma = 0 \qquad \forall\ \chi \in W\,, && (2.52b)\\
&\langle \varkappa_\phi\,\vec{\nu}, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = 0 \quad \forall\ \vec{\eta} \in \underline{W}\,. && (2.52c)
\end{aligned}$$

Clearly, it follows from (2.52b), $(\varkappa_\phi - \beta) \in W$ and $\vec{y} \in \underline{W}$ that

$$\varkappa_\phi - \beta = \vec{y}\,.\,\vec{\nu}\,. \tag{2.53}$$

Our weak formulation of this gradient flow is then given by: Given $\Gamma(0) = \vec{x}([0,1],0)$ with $\vec{x}(0,0) = \vec{\alpha}_0$ and $\vec{x}(1,0) = \vec{\alpha}_1$, for all $t \in (0,T]$ find $\Gamma(t) = \vec{x}([0,1],t)$, where $\vec{x}(t) \in \underline{V}$ with $\vec{x}_t(t) \in \underline{W}$, and $\vec{y}(t) \in \underline{W}$ such that

$$\begin{aligned}
&\langle \vec{P}\,\vec{x}_t, \vec{\chi}\rangle_\Gamma - \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{y}, \vec{\nabla}_s\,\vec{\chi}\rangle_\Gamma + \tfrac{1}{2}\,\langle[(\vec{y}\,.\,\vec{\nu})^2 + 2\,\lambda - \beta^2]\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma - \langle(\vec{y}\,.\,\vec{\nu} + \beta)\,\vec{y}^\perp, \vec{\chi}_s\rangle_\Gamma \\
&\qquad\qquad = 0 \quad \forall\ \vec{\chi} \in \underline{W}\,, && (2.54a)\\
&\langle \vec{P}\,\vec{y}, \vec{\eta}\rangle_\Gamma + \langle \phi'(\vec{x}_s), \vec{\eta}_s\rangle_\Gamma = -\beta\,\langle \vec{\nu}, \vec{\eta}\rangle_\Gamma \quad \forall\ \vec{\eta} \in \underline{W}\,. && (2.54b)
\end{aligned}$$

We note that in deriving (2.54a) we have observed that

$$-\langle \varkappa_\phi\,(\vec{y}\,.\,\vec{\nu})\,\vec{x}_s, \vec{\chi}_s\rangle_\Gamma + \langle \varkappa_\phi\,\vec{y}, (\vec{\nabla}_s\,\vec{\chi})^\perp\rangle_\Gamma = -\langle \varkappa_\phi\,\vec{y}^\perp, \vec{\chi}_s\rangle_\Gamma\,. \tag{2.55}$$

Once again, it is easy to show that (2.54a,b) with (2.53) is equivalent to (2.52a–c). Moreover, we note that $\vec{x}_t \in \underline{W}$ and $\vec{y} \in \underline{W}$ enforce the boundary conditions (2.46) strongly, recall (2.53).

THEOREM. 2.5. *Let $d = 2$ and let $(\vec{x}(t), \vec{y}(t))_{t\in(0,T]}$ be a solution to (2.54a,b). Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{E}_{\beta,\lambda}(\Gamma, \vec{y}\,.\,\vec{\nu} + \beta) = -\langle \vec{P}\,\vec{x}_t, \vec{P}\,\vec{x}_t\rangle_\Gamma \leq 0\,, \tag{2.56}$$

where $\vec{y}\,.\,\vec{\nu} + \beta = \varkappa_\phi$ is the anisotropic curvature.

Proof. Differentiating (2.54b) with respect to t yields that

$$\begin{aligned}
&\langle(\vec{P}\,\vec{y})_t, \vec{\eta}\rangle_\Gamma + \langle(\vec{P}\,\vec{y})\,.\,\vec{\eta}, \vec{x}_s\,.\,\vec{x}_{t,s}\rangle_\Gamma + \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{x}_t, \vec{\nabla}_s\,\vec{\eta}\rangle_\Gamma = -\beta\,\langle \vec{\nu}_t + (\vec{x}_s\,.\,\vec{x}_{t,s})\,\vec{\nu}, \vec{\eta}\rangle_\Gamma \\
&\qquad\qquad \forall\ \vec{\eta} \in \underline{W}\,. && (2.57)
\end{aligned}$$

On choosing $\vec{\eta} = \vec{y} \in \underline{W}$ in (2.57), we obtain that

$$\langle(\vec{P}\,\vec{y})_t, \vec{y}\rangle_\Gamma + \langle(\vec{y}\,.\,\vec{\nu})^2, \vec{x}_s\,.\,\vec{x}_{t,s}\rangle_\Gamma + \langle \phi''(\vec{x}_s)\,\vec{\nabla}_s\,\vec{x}_t, \vec{\nabla}_s\,\vec{y}\rangle_\Gamma = -\beta\,\langle \vec{\nu}_t + (\vec{x}_s\,.\,\vec{x}_{t,s})\,\vec{\nu}, \vec{y}\rangle_\Gamma\,. \tag{2.58}$$

Combining (2.58) and (2.54a) with $\vec\chi = \vec x_t \in \underline{W}$ yields that

$$\langle (\vec P\,\vec y)_t, \vec y\rangle_\Gamma + \tfrac12 \langle 3\,(\vec y\,.\,\vec\nu)^2 + 2\,\lambda - \beta^2, \vec x_s\,.\,\vec x_{t,s}\rangle_\Gamma - \langle (\vec y\,.\,\vec\nu)\,\vec y^\perp, \vec x_{t,s}\rangle_\Gamma + \langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma$$
$$= -\beta\,\langle (\vec x_{t,s})^\perp + \vec\nu_t + (\vec x_s\,.\,\vec x_{t,s})\,\vec\nu, \vec y\rangle_\Gamma\,. \qquad (2.59)$$

Noting that $\vec\nu_t = -(\vec\nabla_s\,\vec x_t)^\perp$, recall (2.31), and that

$$\vec y^\perp = (\vec y\,.\,\vec\nu)\,\vec x_s - (\vec y\,.\,\vec x_s)\,\vec\nu\,,$$

yields that (2.59) collapses to

$$\langle (\vec P\,\vec y)_t, \vec y\rangle_\Gamma + \tfrac12 \langle (\vec y\,.\,\vec\nu)^2 + 2\,\lambda - \beta^2, \vec x_s\,.\,\vec x_{t,s}\rangle_\Gamma + \langle (\vec y\,.\,\vec x_s)\,\vec P\,\vec y, \vec x_{t,s}\rangle_\Gamma = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma\,. \qquad (2.60)$$

The desired result (2.56) then follows from (2.60) and (2.31), similarly to (2.34), on noting that $\widehat E_{\beta,\lambda}(\Gamma, \vec y\,.\,\vec\nu + \beta) = \frac12\,\langle |\vec P y|^2, 1\rangle_\Gamma + \frac12\,(2\,\lambda - \beta^2)\,|\Gamma| = E_\lambda(\Gamma, \vec P\,\vec y) - \frac12\,\beta^2\,|\Gamma|$. $\square$

REMARK. 2.2. *Also in the case of open curves we can consider a constraint on the total length of the curve, recall* Remark 2.1 *to handle the case of homogeneous Navier boundary conditions in the isotropic case. Testing* (2.54b) *with* $\vec\eta = \vec x_t$ *and* (2.54a) *with* $\vec\chi = \vec y$ *yields an equation for* $\lambda(t)$ *in order to fulfill the length constraint. This equation is not well-defined if and only if* $\Gamma(t)$ *is a straight line, which is a steady state.*

Alternatively, in order to handle a length constraint in any of our variational formulations, including the case of clamped boundary conditions, we introduce the Lagrangian

$$\mathcal{L}(\vec x, \vec\varkappa_\phi, \vec y, z, \lambda) := \tfrac12\,\langle \vec\varkappa_\phi, \vec\varkappa_\phi\rangle_\Gamma + \lambda\,(|\Gamma| - l) - \langle \vec\varkappa_\phi, \vec y\rangle_\Gamma - \langle \phi'(\vec x_s), \vec y_s\rangle_\Gamma + \langle \vec\varkappa_\phi\,.\,\vec x_s, z\rangle_\Gamma$$

where $l > 0$ *is a given length. We now consider* $\lambda \in \mathbb{R}$ *as an unknown and a variation with respect to* λ *gives the additional equation*

$$|\Gamma| = l\,. \qquad (2.61)$$

For example, as the length preserving variant of the elastic flow in the case of clamped boundary conditions we then obtain (2.49a,b) *with the additional unknown* $\lambda(t)$ *and the additional constraint* (2.61) *for* $l := |\Gamma(0)|$. *Similarly to* Theorem 2.4, *it is a simple matter to show that this length preserving flow fulfills* $\frac{\mathrm d}{\mathrm dt}\,E_0(\Gamma, \vec P\,\vec y) = -\langle \vec P\,\vec x_t, \vec P\,\vec x_t\rangle_\Gamma \leq 0$.

3 Semidiscrete finite element approximation

In this section we introduce continuous-in-time semidiscrete finite element approximations of the curvature flows discussed in Section 2. In particular, we will repeat on a discrete level the considerations in Section 2 and, as a consequence, we will derive spatially discrete finite element approximations that are stable and that fulfil an equidistribution property, similarly to the semidiscrete schemes considered in e.g. Barrett, Garcke, and Nürnberg (2007b, 2010b).

We introduce the decomposition $I = \cup_{j=1}^J I_j$, $J \geq 3$ of $I = \mathbb{R}/\mathbb{Z}$ into intervals given by the nodes q_j, $I_j = [q_{j-1}, q_j]$. Let $h_j = |I_j|$ and $h = \max_{j=1\to J} h_j$ be the maximal length of a grid element. Then the necessary finite element spaces are defined as follows

$$\underline{V}^h_0 := \{\vec{\chi} \in C(I, \mathbb{R}^d) : \vec{\chi}|_{I_j} \text{ is linear } \forall\ j = 1 \to J\} =: [V^h_0]^d \subset H^1(I, \mathbb{R}^d),$$

where $V^h_0 \subset H^1(I, \mathbb{R})$ is the space of scalar continuous (periodic) piecewise linear functions, with $\{\chi_j\}_{j=1}^J$ denoting the standard basis of V^h_0. In addition, let $\pi^h : C(I, \mathbb{R}) \to V^h_0$ be the standard Lagrange interpolation operator, and similarly for all the other finite element spaces, e.g. $\pi^h : C(I, \mathbb{R}^d) \to \underline{V}^h_0$. Throughout this paper, we make use of the periodicity of I, i.e. $q_J \equiv q_0$, $q_{J+1} \equiv q_1$ and so on.

From now on we will consider a family $(\Gamma^h(t))_{t\in[0,T]}$ of polygonal curves parameterized by $\vec{X}^h(t) \in \underline{V}^h_0$. Here we make the natural assumption that

$(\mathcal{C}^h_0)$ Let $\vec{X}^h(q_j, t) \neq \vec{X}^h(q_{j+1}, t)$ and $\vec{X}^h(q_{j-1}, t) \neq \vec{X}^h(q_{j+1}, t)$, $j = 1 \to J$, for all $t \in [0, T]$.

In addition, we recall that $\langle u, v\rangle_{\Gamma^h} = \int_I u \,.\, v\, |\vec{X}^h_\rho|\, \mathrm{d}\rho$ and, if u, v are piecewise continuous, with possible jumps at the nodes $\{q_j\}_{j=1}^J$, we define the mass lumped inner product

$$\langle u, v\rangle^h_{\Gamma^h} := \tfrac{1}{2} \sum_{j=1}^J |\vec{X}^h(q_j) - \vec{X}^h(q_{j-1})| \left[(u \,.\, v)(q_j^-) + (u \,.\, v)(q_{j-1}^+)\right], \tag{3.1}$$

where we define $u(q_j^\pm) := \lim_{\varepsilon\searrow 0} u(q_j \pm \varepsilon)$.

For the following considerations it will be crucial to define a discrete analogue of the projection $\vec{P} = \vec{Id} - \vec{x}_s \otimes \vec{x}_s$ on the continuous level. It turns out that replacing $\vec{P}$ with the obvious choice $\vec{P}^h := \vec{Id} - \vec{X}^h_s \otimes \vec{X}^h_s$ is not ideal. Although this leads to a stable scheme, the derived approximation in general will not satisfy an equidistribution property. Ultimately, this will lead to a fully discrete approximation that has inferior properties compared to other choices of discrete projections, see Remark 3.1 below. In particular, replacing $\vec{P}$ with a vertex based projection, see (3.6) below, will give rise to all the desired properties.

To this end, we introduce the following differential operators on Γ^h. Let D_s, $\widehat{D}_s$: $V^h_0 \to V^h_0$ be such that

$$\begin{aligned}(D_s\, \eta)(q_j) &= \frac{|\vec{X}^h(q_j) - \vec{X}^h(q_{j-1})|\, \eta_s(q_j^-) + |\vec{X}^h(q_{j+1}) - \vec{X}^h(q_j)|\, \eta_s(q_j^+)}{|\vec{X}^h(q_j) - \vec{X}^h(q_{j-1})| + |\vec{X}^h(q_{j+1}) - \vec{X}^h(q_j)|} \\ &= \frac{\eta(q_{j+1}) - \eta(q_{j-1})}{|\vec{X}^h(q_j) - \vec{X}^h(q_{j-1})| + |\vec{X}^h(q_{j+1}) - \vec{X}^h(q_j)|}\,, \qquad j = 1 \to J\,,\end{aligned} \tag{3.2a}$$

$$(\widehat{D}_s\, \eta)(q_j) = \frac{(D_s\, \eta)(q_j)}{|(D_s\, \vec{X}^h)(q_j)|} = \frac{\eta(q_{j+1}) - \eta(q_{j-1})}{|\vec{X}^h(q_{j+1}) - \vec{X}^h(q_{j-1})|}\,, \qquad j = 1 \to J\,, \tag{3.2b}$$

where, as usual, $D_s : \underline{V}^h_0 \to \underline{V}^h_0$ is defined component-wise. We then define $\vec\theta^h$, $\vec\omega^h_d \in \underline{V}^h_0$ to be

$$\vec\theta^h = D_s\,\vec X^h \qquad \text{and} \qquad \vec\omega^h_d = \widehat D_s\,\vec X^h = \frac{\vec\theta^h}{|\vec\theta^h|}\,, \tag{3.3}$$

which, on recalling assumption $(\mathcal{C}^h_0)$, are well-defined. Here we recall that the discrete vertex unit tangents $\vec\omega^h_d$, with this notation, have previously been introduced in Barrett, Garcke, and Nürnberg (2010b). We note for future reference that

$$\langle \vec X^h_s, \vec\xi\rangle^h_{\Gamma^h} = \langle \vec\theta^h, \vec\xi\rangle^h_{\Gamma^h} \qquad \forall\ \vec\xi \in \underline{V}^h_0\,; \tag{3.4}$$

and similarly,

$$\langle |\vec\theta^h|^{-1}\,\vec\eta_s, \vec\xi\rangle^h_{\Gamma^h}, = \langle \widehat D_s\,\vec\eta, \vec\xi\rangle^h_{\Gamma^h} \qquad \forall\ \vec\eta,\ \vec\xi \in \underline{V}^h_0\,. \tag{3.5}$$

Let

$$\vec Q^h := \vec{Id} - \vec\omega^h_d \otimes \vec\omega^h_d \ \ : \underline{V}^h_0 \to \underline{V}_0\,. \tag{3.6}$$

Hence for all $\vec\chi \in \underline{V}^h_0$, $(\vec Q^h\,\vec\chi)(q_j)$ is the projection of $\vec\chi(q_j)$ onto the subspace of $\mathbb{R}^d$ normal to $\vec\omega^h_d(q_j)$ for $j = 1 \to J$. For future reference, we note also that for $\vec\eta \in \underline{V}^h_0$ it holds that

$$[\frac{\delta}{\delta \vec X^h}\,\vec\omega^h_d](\vec\eta) = \pi^h\left[\vec Q^h\left(\widehat D_s\,\vec\eta\right)\right] \qquad \text{and hence} \qquad [\vec\omega^h_d]_t = \pi^h\left[\vec Q^h\left(\widehat D_s\,\vec X^h_t\right)\right]. \tag{3.7}$$

3.1 Isotropic elastic flow

In what follows we state a discrete analogue of (2.5a,b), as well as theorems regarding the stability and equidistribution property of this semidiscrete continuous-in-time finite element approximation. In particular, we will show that the scheme is an L^2-gradient flow for the discrete energy

$$E^h_\lambda(\Gamma^h, \vec\kappa^h) := \tfrac12\,\langle \vec\kappa^h, \vec\kappa^h\rangle^h_{\Gamma^h} + \lambda\,|\Gamma^h|\,,$$

where $\vec\kappa^h \in \underline{V}^h_0$ is a discrete curvature vector of Γ^h. Details of its derivations as well as proofs of the theorems, which are very close to the details in the continuous case, will be presented only in the anisotropic setting in §3.2, below.

Given $\Gamma^h(0) = \vec X^h(I, 0)$, with $\vec X^h(0) \in \underline{V}^h_0$, for all $t \in (0, T]$ find $\Gamma^h(t) = \vec X^h(I, t)$ with $\vec X^h(t) \in \underline{V}^h_0$, and $\vec Y^h(t) \in \underline{V}^h_0$ such that

$$\begin{aligned}\langle \vec Q^h\,\vec X^h_t, \vec\chi\rangle^h_{\Gamma^h} - \langle \vec\nabla_s\,\vec Y^h, \vec\nabla_s\,\vec\chi\rangle_{\Gamma^h} - \tfrac12\,\langle (|\vec Q^h\,\vec Y^h|^2 - 2\,\lambda)\,\vec X^h_s, \vec\chi_s\rangle^h_{\Gamma^h}& \\ + \langle |\vec\theta^h|^{-1}\,(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h, \vec\chi_s\rangle^h_{\Gamma^h} = 0 \quad \forall\ \vec\chi \in \underline{V}^h_0\,,& \qquad (3.8a)\\ \langle \vec Q^h\,\vec Y^h, \vec\eta\rangle^h_{\Gamma^h} + \langle \vec X^h_s, \vec\eta_s\rangle_{\Gamma^h} = 0 \quad \forall\ \vec\eta \in \underline{V}^h_0\,.& \qquad (3.8b)\end{aligned}$$

THEOREM. 3.1. *Let* $(\mathcal{C}_0^h)$ *hold and let* $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ *be a solution to* (3.8a,b). *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda^h(\Gamma^h, \vec{Q}^h\,\vec{Y}^h) = -\langle \vec{Q}^h\,\vec{X}_t^h, \vec{Q}^h\,\vec{X}_t^h\rangle_{\Gamma^h}^h \leq 0\,, \tag{3.9}$$

where $\vec{\kappa}^h = \pi^h[\vec{Q}^h\,\vec{Y}^h]$ *is the discrete curvature vector of* Γ^h.

Proof. The result (3.9) follows from Theorem 3.3, below, for the special case $\phi(\cdot) = |\cdot|$. ∎

Next we show an equidistribution property for the scheme (3.8a,b).

THEOREM. 3.2. *Let* $(\mathcal{C}_0^h)$ *hold and let* $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ *denote a solution to* (3.8a,b). *For a fixed time* $t \in (0,T]$ *let* $\vec{a}_{j-\frac12}^h := \vec{X}^h(q_j) - \vec{X}^h(q_{j-1})$, $j = 1 \to J$. *Then it holds for* $j = 1 \to J$ *that*

$$|\vec{a}_{j+\frac12}^h| = |\vec{a}_{j-\frac12}^h| \quad \textit{if} \quad \vec{a}_{j+\frac12}^h \nparallel \vec{a}_{j-\frac12}^h\,. \tag{3.10}$$

Proof. The proof is identical to the proof in Barrett, Garcke, and Nürnberg (2010b, Remark 3.3). Moreover, the result (3.10) directly follows from Theorem 3.4, below, for the special case $\phi(\cdot) = |\cdot|$. ∎

Theorem 3.2 establishes that the scheme (3.8a,b) will always equidistribute the nodes along Γ^h if the corresponding intervals are not locally parallel. Although it does not appear possible to prove an analogue for the fully discrete setting, in practice we see that the nodes are moved tangentially so that they are eventually equidistributed.

REMARK. 3.1. *As an alternative to* (3.8a,b), *one can also consider the following semidiscrete approximation of* (2.5a,b): *Given* $\Gamma^h(0) = \vec{X}^h(I,0)$, *with* $\vec{X}^h(0) \in \underline{V}_0^h$, *for all* $t \in (0,T]$ *find* $\Gamma^h(t) = \vec{X}^h(I,t)$ *with* $\vec{X}^h(t) \in \underline{V}_0^h$, *and* $\vec{Y}^h(t) \in \underline{V}_0^h$ *such that*

$$\langle \vec{P}^h\,\vec{X}_t^h, \vec{\chi}\rangle_{\Gamma^h}^h - \langle \vec{\nabla}_s\,\vec{Y}^h, \vec{\nabla}_s\,\vec{\chi}\rangle_{\Gamma^h} - \tfrac12\,\langle (|\vec{P}^h\,\vec{Y}^h|^2 - 2\,\lambda)\,\vec{X}_s^h, \vec{\chi}_s\rangle_{\Gamma^h}^h + \langle (\vec{Y}^h\,.\,\vec{X}_s^h)\,\vec{P}^h\,\vec{Y}^h, \vec{\chi}_s\rangle_{\Gamma^h}^h = 0 \quad \forall\ \vec{\chi} \in \underline{V}_0^h\,, \tag{3.11a}$$

$$\langle \vec{P}^h\,\vec{Y}^h, \vec{\eta}\rangle_{\Gamma^h}^h + \langle \vec{X}_s^h, \vec{\eta}_s\rangle_{\Gamma^h} = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0^h\,. \tag{3.11b}$$

Similarly to Theorem 3.1, *it is then not difficult to show that* $\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda^h(\Gamma^h, \vec{P}^h\,\vec{Y}^h) \leq 0$. *However, it does not appear possible to relate this stability result to the decrease of the energy in* (3.9) *for some discrete curvature approximation belonging to* $\underline{V}_0^h$, *since the function* $\vec{P}^h\,\vec{Y}^h$ *is discontinuous piecewise linear. In addition, it does not appear possible to prove an equidistribution property, as in* Theorem 3.2, *for* (3.11a,b)*; and this leads to meshes that are not well distributed for a fully discrete variant in practice. Hence, we do not pursue the scheme* (3.11a,b) *any further.*

REMARK. 3.2. *Deckelnick and Dziuk (2009) considered the following semidiscrete approximation of* (1.3): *Given* $\Gamma^h(0) = \vec{X}^h(I,0)$, *with* $\vec{X}^h(0) \in \underline{V}_0^h$, *for all* $t \in (0,T]$ *find* $\Gamma^h(t) = \vec{X}^h(I,t)$ *with* $\vec{X}^h(t) \in \underline{V}_0^h$, *and* $\vec{\kappa}^h(t) \in \underline{V}_0^h$ *such that*

$$\langle \vec{X}_t^h, \vec{\chi}\rangle_{\Gamma^h}^h - \langle \vec{\nabla}_s\,\vec{\kappa}^h, \vec{\nabla}_s\,\vec{\chi}\rangle_{\Gamma^h} - \tfrac12\,\langle (|\vec{\kappa}^h|^2 - 2\,\lambda)\,\vec{X}_s^h, \vec{\chi}_s\rangle_{\Gamma^h}^h = 0 \quad \forall\ \vec{\chi} \in \underline{V}_0^h\,, \tag{3.12a}$$

$$\langle \vec{\kappa}^h, \vec{\eta}\rangle_{\Gamma^h}^h + \langle \vec{X}_s^h, \vec{\eta}_s\rangle_{\Gamma^h} = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0^h\,. \tag{3.12b}$$

They showed that $\frac{\mathrm{d}}{\mathrm{d}t}\,E^h_\lambda(\Gamma^h,\vec\kappa^h)\leq 0$*, and in addition established existence, uniqueness and error bounds for a solution to* (3.12a,b).

3.2 Anisotropic elastic flow

Similarly to the continuous setting in (2.27a–d), we consider the L^2-gradient flow of the discrete energy

$$E^h_\lambda(\Gamma^h,\vec\kappa^h_\phi)=\tfrac12\,\langle\vec\kappa^h_\phi,\vec\kappa^h_\phi\rangle^h_{\Gamma^h}+\lambda\,|\Gamma^h|\,, \tag{3.13}$$

where $\vec\kappa_\phi\in\underline{V}^h_0$, subject to the side constraints

$$\langle\vec\kappa^h_\phi,\vec\eta\rangle^h_{\Gamma^h}+\langle\phi'(\vec X^h_s),\vec\eta_s\rangle_{\Gamma^h}=0\qquad\forall\ \vec\eta\in\underline{V}^h_0 \tag{3.14a}$$

and

$$\langle\vec\kappa^h_\phi\,.\,\vec X^h_s,\chi\rangle^h_{\Gamma^h}=0\qquad\forall\ \chi\in V^h_0\,. \tag{3.14b}$$

Hence $\vec\kappa^h_\phi$ is the natural discrete analogue of the anisotropic curvature vector $\vec\varkappa_\phi$, recall (2.13). Here we remark that it is also natural to consider the mass-lumped inner products $\langle\cdot,\cdot\rangle^h_{\Gamma^h}$ in (3.14a,b), as this gives rise to more practical approximations that in addition satisfy an equidistribution property in the isotropic case (2.11). As a consequence, we also employ numerical integration in the discrete energy (3.13). Introducing the Lagrange multipliers $\vec Y^h\in\underline{V}^h_0$ and $Z^h\in V^h_0$ for (3.14a,b) with $\vec X^h$, $\vec\kappa^h_\phi\in\underline{V}^h_0$, and the discrete analogue of the Lagrangian (2.26); we can derive the following system:

$$\begin{aligned}\langle\vec Q^h\,\vec X^h_t,\vec\chi\rangle^h_{\Gamma^h}&=\langle\phi''(\vec X^h_s)\,\vec\nabla_s\,\vec Y^h,\vec\nabla_s\,\vec\chi\rangle_{\Gamma^h}-\tfrac12\,\langle(|\vec\kappa^h_\phi|^2-2\,\vec\kappa^h_\phi\,.\,\vec Y^h+2\,\lambda)\,\vec X^h_s,\vec\chi_s\rangle^h_{\Gamma^h}\\&\qquad-\langle Z^h\,\vec\kappa^h_\phi,\vec\chi_s\rangle^h_{\Gamma^h}\quad\forall\ \vec\chi\in\underline{V}^h_0\,, &(3.15a)\\\langle\vec\kappa^h_\phi+Z^h\,\vec X^h_s-\vec Y^h,\vec\xi\rangle^h_{\Gamma^h}&=0\qquad\forall\ \vec\xi\in\underline{V}^h_0\,, &(3.15b)\\\langle\vec\kappa^h_\phi,\vec\eta\rangle^h_{\Gamma^h}+\langle\phi'(\vec X^h_s),\vec\eta_s\rangle_{\Gamma^h}&=0\quad\forall\ \vec\eta\in\underline{V}^h_0\,, &(3.15c)\\\langle\vec\kappa^h_\phi\,.\,\vec X^h_s,\chi\rangle^h_{\Gamma^h}&=0\qquad\forall\ \chi\in V^h_0\,. &(3.15d)\end{aligned}$$

It immediately follows from (3.15b,d), (3.3) and (3.4) that

$$\vec\kappa^h_\phi=\pi^h[\vec Q^h\,\vec Y^h]\quad\text{and}\quad Z^h=\pi^h\left[|\vec\theta^h|^{-1}\,\vec Y^h\,.\,\vec\omega^h_d\right]. \tag{3.16}$$

Hence the system (3.15a–d) can be equivalently rewritten as: Given $\Gamma^h(0)=\vec X^h(I,0)$, with $\vec X^h(0)\in\underline{V}^h_0$, for all $t\in(0,T]$ find $\Gamma^h(t)=\vec X^h(I,t)$ with $\vec X^h(t)\in\underline{V}^h_0$, and $\vec Y^h(t)\in\underline{V}^h_0$ such that

$$\begin{aligned}&\langle\vec Q^h\,\vec X^h_t,\vec\chi\rangle^h_{\Gamma^h}-\langle\phi''(\vec X^h_s)\,\vec\nabla_s\,\vec Y^h,\vec\nabla_s\,\vec\chi\rangle_{\Gamma^h}-\tfrac12\,\langle(|\vec Q^h\,\vec Y^h|^2-2\,\lambda)\,\vec X^h_s,\vec\chi_s\rangle^h_{\Gamma^h}\\&\qquad+\langle|\vec\theta^h|^{-1}\,(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h,\vec\chi_s\rangle^h_{\Gamma^h}=0\quad\forall\ \vec\chi\in\underline{V}^h_0\,, &(3.17a)\\&\langle\vec Q^h\,\vec Y^h,\vec\eta\rangle^h_{\Gamma^h}+\langle\phi'(\vec X^h_s),\vec\eta_s\rangle_{\Gamma^h}=0\quad\forall\ \vec\eta\in\underline{V}^h_0\,. &(3.17b)\end{aligned}$$

Clearly (3.17a,b) in the isotropic case (2.11) collapses to the isotropic scheme (3.8a,b). Moreover, we note that (3.17a,b) with (3.16) is equivalent to (3.15a–d).

The following theorem establishes the stability of the above semidiscrete finite element approximation. It is the direct discrete analogue of Theorems 2.2. On recalling (3.16), the theorem establishes that (3.17a,b) with (3.16) formulates an L^2-gradient flow of the discrete energy (3.13).

THEOREM. 3.3. *Let* $(\mathcal{C}_0^h)$ *hold and let* $(\vec X^h(t), \vec Y^h(t))_{t\in(0,T]}$ *be a solution to* (3.17a,b). *Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h_\lambda(\Gamma^h, \vec Q^h\,\vec Y^h) = -\langle \vec Q^h\,\vec X^h_t, \vec Q^h\,\vec X^h_t\rangle^h_{\Gamma^h} \leq 0\,, \tag{3.18}$$

where $\vec\kappa^h_\phi = \pi^h[\vec Q^h\,\vec Y^h]$ *is the discrete anisotropic curvature vector of* Γ^h.

Proof. Differentiating (3.17b) with respect to t yields, on noting that $(\vec X^h_s)_t = \vec P^h\,\vec X^h_{t,s} = \vec\nabla_s\,\vec X^h_t$, that

$$\langle(\vec Q^h\,\vec Y^h)_t, \vec\eta\rangle^h_{\Gamma^h} + \langle(\vec Q^h\,\vec Y^h)\,.\,\vec\eta, \vec X^h_s\,.\,\vec X^h_{t,s}\rangle^h_{\Gamma^h} + \langle\phi''(\vec X^h_s)\,\vec\nabla_s\,\vec X^h_t, \vec\nabla_s\,\vec\eta\rangle_{\Gamma^h} = 0 \quad \forall\ \vec\eta\in\underline{V}^h_0\,. \tag{3.19}$$

On choosing $\vec\eta = \vec Y^h$ in (3.19), we obtain that

$$\langle(\vec Q^h\,\vec Y^h)_t, \vec Y^h\rangle^h_{\Gamma^h} + \langle|\vec Q^h\,\vec Y^h|^2, \vec X^h_s\,.\,\vec X^h_{t,s}\rangle^h_{\Gamma^h} + \langle\phi''(\vec X^h_s)\,\vec\nabla_s\,\vec X^h_t, \vec\nabla_s\,\vec Y^h\rangle_{\Gamma^h} = 0\,. \tag{3.20}$$

Combining (3.20) and (3.17a) with $\vec\chi = \vec X^h_t$ yields that

$$\begin{aligned}\langle(\vec Q^h\,\vec Y^h)_t, \vec Y^h\rangle^h_{\Gamma^h} + \tfrac12\,\langle(|\vec Q^h\,\vec Y^h|^2 + 2\,\lambda), \vec X^h_s\,.\,\vec X^h_{t,s}\rangle^h_{\Gamma^h} + \langle|\vec\theta^h|^{-1}(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h, \vec X^h_{t,s}\rangle^h_{\Gamma^h}&\\ = -\langle\vec Q^h\,\vec X^h_t, \vec Q^h\,\vec X^h_t\rangle^h_{\Gamma^h}\,.&\end{aligned} \tag{3.21}$$

The desired result (3.18) then follows from (3.21) on noting that

$$\begin{aligned}\langle(\vec Q^h\,\vec Y^h)_t, (\vec{Id} - \vec Q^h)\,\vec Y^h\rangle^h_{\Gamma^h} &= \langle(\vec Q^h\,\vec Y^h)_t, (\vec Y^h\,.\,\vec\omega^h_d)\,\vec\omega^h_d\rangle^h_{\Gamma^h} = -\langle\vec Q^h\,\vec Y^h, (\vec Y^h\,.\,\vec\omega^h_d)\,(\vec\omega^h_d)_t\rangle^h_{\Gamma^h}\\ &= -\langle(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h, \widehat D_s\,\vec X^h_t\rangle^h_{\Gamma^h}\\ &= -\langle|\vec\theta^h|^{-1}\,(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h, \vec X^h_{t,s}\rangle^h_{\Gamma^h}\,,\end{aligned}$$

where we have recalled (3.6), (3.7) and (3.5). $\square$

COROLLARY. 3.1. *Let* $(\mathcal{C}_0^h)$ *hold and let* $(\vec X^h(t), \vec Y^h(t))_{t\in(0,T]}$ *be a solution to* (3.8a,b) *with*

$$\lambda(t) = \frac{\langle\vec\nabla_s\,\vec Y^h, \vec\nabla_s\,\vec Y^h\rangle_{\Gamma^h} + \frac12\,\langle|\vec Q^h\,\vec Y^h|^2\,\vec X^h_s, \vec Y^h_s\rangle^h_{\Gamma^h} - \langle|\theta^h|^{-1}(\vec Y^h\,.\,\vec\omega^h_d)\,\vec Q^h\,\vec Y^h, \vec Y^h_s\rangle^h_{\Gamma^h}}{\langle\vec X^h_s, \vec Y^h_s\rangle_{\Gamma^h}}\,. \tag{3.22}$$

Then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Gamma^h(t)| = 0 \qquad \textit{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,E^h_0(\Gamma^h, \vec Q^h\,\vec Y^h) = -\langle\vec Q^h\,\vec X^h_t, \vec Q^h\,\vec X^h_t\rangle^h_{\Gamma^h} \leq 0\,, \tag{3.23}$$

where $\vec\kappa^h = \pi^h[\vec Q^h\,\vec Y^h]$ *is the discrete isotropic curvature vector of* Γ^h.

Proof. The first part of the proof is a discrete analogue of Remark 2.1. We note that the choice of $\lambda(t)$ yields that $|\Gamma(t)|_t = \langle \vec{X}^h_s, (\vec{X}^h_t)_s\rangle_{\Gamma^h} = -\langle \vec{Y}^h, \vec{Q}^h \vec{X}^h_t\rangle^h_{\Gamma^h} = 0$, and $\lambda(t)$ is well-defined because $\langle \vec{X}^h_s, \vec{Y}^h_s\rangle_{\Gamma^h} = -\langle \vec{\kappa}^h, \vec{\kappa}^h\rangle^h_{\Gamma^h} < 0$, on recalling $(\mathcal{C}^h_0)$. The energy inequality in (3.23) then follows similarly to the proof of Corollary 2.1. $\square$

Next we show a weighted equidistribution property for the scheme (3.17a,b), when the anisotropy is of a certain form. To this end, we introduce

$$\phi(\vec{p}) = \sum_{\ell=1}^{L} \phi_\ell(\vec{p}) = \sum_{\ell=1}^{L} [\vec{p}\,.\,\vec{G}_\ell\,\vec{p}]^{\frac{1}{2}} \quad \Rightarrow \quad \phi'(\vec{p}) = \sum_{\ell=1}^{L} [\phi_\ell(\vec{p})]^{-1}\,\vec{G}_\ell\,\vec{p} \quad \forall\ \vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}\,, \tag{3.24}$$

where $\vec{G}_\ell \in \mathbb{R}^{d\times d}$, $\ell = 1 \to L$, are symmetric and positive definite. It is a simple matter to show that ϕ of the form (3.24) satisfies all our earlier assumptions, e.g. (2.8) and (2.10).

THEOREM. 3.4. *Let $(\mathcal{C}^h_0)$ hold and let $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ denote a solution to (3.17a,b). For a fixed time $t \in (0,T]$ let $\vec{a}^h_{j-\frac{1}{2}} := \vec{X}^h(q_j) - \vec{X}^h(q_{j-1})$, $j = 1 \to J$. Then it holds for $j = 1 \to J$ that*

$$\phi(\vec{a}^h_{j+\frac{1}{2}}) - \phi(\vec{a}^h_{j-\frac{1}{2}}) + \phi'(\vec{a}^h_{j+\frac{1}{2}})\,.\,\vec{a}^h_{j-\frac{1}{2}} - \phi'(\vec{a}^h_{j-\frac{1}{2}})\,.\,\vec{a}^h_{j+\frac{1}{2}} = 0 \quad \textit{if} \quad \vec{a}^h_{j+\frac{1}{2}} \nparallel \vec{a}^h_{j-\frac{1}{2}}\,. \tag{3.25a}$$

For ϕ of the form (3.24), this implies that

$$\sum_{\ell=1}^{L} \lambda^{(\ell)}_j\,\phi_\ell(\vec{a}^h_{j+\frac{1}{2}}) = \sum_{\ell=1}^{L} \lambda^{(\ell)}_j\,\phi_\ell(\vec{a}^h_{j-\frac{1}{2}}) \quad \textit{if} \quad \vec{a}^h_{j+\frac{1}{2}} \nparallel \vec{a}^h_{j-\frac{1}{2}}\,, \tag{3.25b}$$

where $\lambda^{(\ell)}_j := 1 - [\phi_\ell(\vec{a}^h_{j+\frac{1}{2}})\,\phi_\ell(\vec{a}^h_{j-\frac{1}{2}})]^{-1}\,(\vec{a}^h_{j+\frac{1}{2}}\,.\,\vec{G}_\ell\,\vec{a}^h_{j-\frac{1}{2}}) \in [0,2]$, $\ell = 1 \to L$, with $\lambda^{(\ell)}_j > 0$ if $\vec{a}^h_{j+\frac{1}{2}} \nparallel \vec{a}^h_{j-\frac{1}{2}}$. In the special case that $L = 1$ this yields that

$$\phi(\vec{a}^h_{j+\frac{1}{2}}) = \phi(\vec{a}^h_{j-\frac{1}{2}}) \quad \textit{if} \quad \vec{a}^h_{j+\frac{1}{2}} \nparallel \vec{a}^h_{j-\frac{1}{2}}\,. \tag{3.25c}$$

Proof. On choosing $\vec{\eta} = \vec{\omega}^h_d(q_j)\,\chi_j$ in (3.17b) and noting that $\pi^h[\vec{Q}^h\,\vec{\omega}^h_d] = \vec{0}$, it follows that

$$\left(\phi'(\vec{a}^h_{j+\frac{1}{2}}) - \phi'(\vec{a}^h_{j-\frac{1}{2}})\right)\,.\,\left(\vec{a}^h_{j+\frac{1}{2}} + \vec{a}^h_{j-\frac{1}{2}}\right) = 0\,, \quad j = 1 \to J\,, \tag{3.26}$$

which yields (3.25a). Similarly to Barrett, Garcke, and Nürnberg (2008a, (2.24)) it follows from (3.26) for ϕ of the form (3.24) that

$$\sum_{\ell=1}^{L} [\,\phi_\ell(\vec{a}^h_{j+\frac{1}{2}}) - \phi_\ell(\vec{a}^h_{j-\frac{1}{2}})\,]\,[1 - \frac{\vec{a}^h_{j+\frac{1}{2}}\,.\,\vec{G}_\ell\,\vec{a}^h_{j-\frac{1}{2}}}{\phi_\ell(\vec{a}^h_{j+\frac{1}{2}})\,\phi_\ell(\vec{a}^h_{j-\frac{1}{2}})}\,] = 0\,,$$

which yields (3.25b). Finally, (3.25c) follows immediately from (3.25b) on recalling that $\lambda^{(\ell)}_j > 0$ if $\vec{a}^h_{j+\frac{1}{2}} \nparallel \vec{a}^h_{j-\frac{1}{2}}$ $\square$

In the special case $L = 1$ we note that Theorem 3.4 gives equidistribution with respect to ϕ, provided that intervals are not locally parallel. In the isotropic case this yields an equidistribution of the vertices, as discussed in Theorem 3.2.

3.3 Initial boundary value problems

Let $[0,1] = \cup_{j=1}^{J} I_j$, be a decomposition of $[0,1]$ into intervals $I_j = [q_{j-1}, q_j]$ based on the nodes $\{q_j\}_{j=0}^{J}$, as before. The appropriate finite element spaces are then defined by

$$\underline{V}^h := \{\vec{\chi} \in C([0,1], \mathbb{R}^d) : \vec{\chi}|_{I_j} \text{ is linear } \forall\, j = 1 \to J\} \subset \underline{V}$$

and similarly for the spaces $\underline{W}^h \subset \underline{W}$, $V^h \subset V$ and $W^h \subset W$.

In this section, we consider a family $(\Gamma^h(t))_{t\in[0,T]}$ of curves parameterized by $\vec{X}^h(t) \in \underline{V}^h$. Here we make the natural assumption that

($\mathcal{C}^h$) Let $\vec{X}^h(q_j, t) \neq \vec{X}^h(q_{j+1}, t)$, $j = 0 \to J-1$, and $\vec{X}^h(q_{j-1}, t) \neq \vec{X}^h(q_{j+1}, t)$, $j = 1 \to J-1$, for all $t \in [0,T]$.

Then we introduce the (obvious) open curve analogues $\vec{\theta}^h$, $\vec{\varpi}_d^h \in \underline{V}^h$ and $\vec{Q}^h$ of (3.3) and (3.6). In particular, we introduce the differential operators D_s, $\widehat{D}_s : \underline{V}^h \to \underline{V}^h$ via the analogues of (3.2a,b) for the interior nodes q_j, $j = 1 \to J-1$, and via setting

$$(D_s\, \eta)(q_0) = (\widehat{D}_s\, \eta)(q_0) = \eta_s\,|_{I_1} \quad \text{and} \quad (D_s\, \eta)(q_J) = (\widehat{D}_s\, \eta)(q_J) = \eta_s\,|_{I_J}\,,$$

for the boundary nodes. Then $\vec{\theta}^h$, $\vec{\varpi}_d^h \in \underline{V}^h$ are defined via (3.3), as before. With these definitions it is easy to see that (3.4), (3.5) and (3.7) still hold with $\underline{V}_0^h$ replaced by $\underline{V}^h$.

In addition, in the planar case, $d = 2$, we define $\vec{\nu}^h := -(\vec{X}_s^h)^\perp$ and $\vec{\omega}^h := -(\vec{\theta}^h)^\perp \in \underline{V}^h$, and set

$$\vec{Q}_\omega^h := \vec{\omega}^h \otimes \vec{\omega}^h \equiv |\vec{\omega}^h|^2\, \vec{Q}^h\,.$$

Here we note that as generally $|\vec{\omega}^h(q_j)| < 1$, the operator $\vec{Q}_\omega^h$ in general is not a projection. We note also for all $\vec{\chi}$, $\vec{\eta} \in \underline{W}^h$ that

$$\langle \vec{\omega}^h, \vec{\eta}\rangle_{\Gamma^h}^h = \langle \vec{\nu}^h, \vec{\eta}\rangle_{\Gamma^h}^h \quad \text{and} \quad \langle \vec{Q}_\omega^h\, \vec{\chi}, \vec{\eta}\rangle_{\Gamma^h}^h = \langle \vec{\chi}\,.\,\vec{\omega}^h, \vec{\eta}\,.\,\vec{\nu}^h\rangle_{\Gamma^h}^h\,. \tag{3.27}$$

3.3.1 Clamped conditions

A spatially discrete variant of (2.49a,b) is given by the following approximation. Given $\Gamma^h(0) = \vec{X}^h([0,1],0)$, with $\vec{X}^h(0) \in \underline{V}^h$ and $\vec{X}^h(0,0) = \vec{\alpha}_0$, $\vec{X}^h(1,0) = \vec{\alpha}_1$, for all $t \in (0,T]$ find $\Gamma^h(t) = \vec{X}^h([0,1],t)$ with $\vec{X}^h(t) \in \underline{V}^h$ and $\vec{X}_t^h(t) \in \underline{W}^h$, and $\vec{Y}^h(t) \in \underline{V}^h$ such that

$$\begin{aligned} &\langle \vec{Q}^h\, \vec{X}_t^h, \vec{\chi}\rangle_{\Gamma^h}^h - \langle \phi''(\vec{X}_s^h)\, \vec{\nabla}_s\, \vec{Y}^h, \vec{\nabla}_s\, \vec{\chi}\rangle_{\Gamma^h} - \tfrac{1}{2}\,\langle (|\vec{Q}^h\, \vec{Y}^h|^2 - 2\,\lambda)\, \vec{X}_s^h, \vec{\chi}_s\rangle_{\Gamma^h}^h \\ &\qquad + \langle |\vec{\theta}^h|^{-1}\, (\vec{Y}^h\,.\,\vec{\varpi}_d^h)\, \vec{Q}^h\, \vec{Y}^h, \vec{\chi}_s\rangle_{\Gamma^h}^h = 0 \quad \forall\, \vec{\chi} \in \underline{W}^h\,, \qquad (3.28a) \\ &\langle \vec{Q}^h\, \vec{Y}^h, \vec{\eta}\rangle_{\Gamma^h}^h + \langle \phi'(\vec{X}_s^h), \vec{\eta}_s\rangle_{\Gamma^h} = \phi'(\vec{\zeta}_1)\,.\,\vec{\eta}(1) - \phi'(\vec{\zeta}_0)\,.\,\vec{\eta}(0) \quad \forall\, \vec{\eta} \in \underline{V}^h\,. \qquad (3.28b) \end{aligned}$$

Similarly to (2.49a,b) we note that in the above approximation the fixed position conditions in (2.43) are enforced strongly, while the angle conditions in (2.43) are approximated weakly. In a complete analogue to the proof of Theorem 3.3, using the $\underline{V}^h$ analogues of (3.5), (3.6) and (3.7), it is then straightforward to prove stability for (3.28a,b), i.e. that a solution satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h_\lambda(\Gamma^h, \vec{Q}^h\,\vec{Y}^h) = -\langle \vec{Q}^h\,\vec{X}^h_t, \vec{Q}^h\,\vec{X}^h_t\rangle^h_{\Gamma^h} \leq 0\,.$$

However, looking ahead to the fully discrete approximation that will be used in practice, in order to prove its well-posedness it will be necessary for the testing procedure to replace the projection $\vec{Q}^h$ in (3.28b) with the modified operator

$$\vec{Q}^h_\star(\rho) := \begin{cases} \vec{Q}^h(\rho) & \rho \in (0,1)\,, \\ \vec{Id} & \rho \in \{0,1\}\,. \end{cases} \tag{3.29}$$

In order to achieve this, we now consider the L^2-gradient flow of $E^h_\lambda(\Gamma^h, \vec{\kappa}^h_\phi)$, where $\vec{\kappa}^h_\phi \in \underline{V}^h$, subject to the side constraints

$$\langle \vec{\kappa}^h_\phi, \vec{\eta}\rangle^h_{\Gamma^h} + \langle \phi'(\vec{X}^h_s), \vec{\eta}_s\rangle_{\Gamma^h} = \phi'(\vec{\zeta}_1)\,.\,\vec{\eta}(1) - \phi'(\vec{\zeta}_0)\,.\,\vec{\eta}(0) \qquad \forall\ \vec{\eta} \in \underline{V}^h \tag{3.30a}$$

and

$$\langle \vec{\kappa}^h_\phi\,.\,\vec{X}^h_s, \chi\rangle^h_{\Gamma^h} = 0 \qquad \forall\ \chi \in W^h\,. \tag{3.30b}$$

Here it is crucial to use the test space W^h in (3.30b), as (a) this will give rise to the desired altered projection $\vec{Q}^h_\star$ in (3.28b) and as (b) the resulting scheme will still satisfy an equidistribution property; see Theorem 3.7 below. Defining the discrete analogue of the Lagrangian (2.26) and taking variations; we can derive the following system for $\vec{X}^h$, $\vec{\kappa}^h_\phi$, $\vec{Y}^h \in \underline{V}^h$ and $Z^h \in W^h$ with $\vec{X}^h_t \in \underline{W}^h$: (3.15a,b,d), now with the test spaces $\underline{W}^h$, $\underline{V}^h$ and W^h, respectively, supplemented with (3.30a). In place of (3.16), on noting the $\underline{V}^h$ analogue of (3.4), we now obtain

$$\vec{\kappa}^h_\phi = \pi^h[\vec{Q}^h_\star\,\vec{Y}^h] \quad \text{and} \quad Z^h = \pi^h_W\left[|\vec{\theta}^h|^{-1}\,\vec{Y}^h\,.\,\vec{\omega}^h_d\right]\,,$$

where $\pi^h_W : C([0,1],\mathbb{R}) \to W^h$ is the standard Lagrange interpolation operator with zero Dirichlet boundary conditions. An equivalent reformulation of the derived conditions is then given as follows. Given $\Gamma^h(0) = \vec{X}^h([0,1],0)$, with $\vec{X}^h(0) \in \underline{V}^h$ and $\vec{X}^h(0,0) = \vec{\alpha}_0$, $\vec{X}^h(1,0) = \vec{\alpha}_1$, for all $t \in (0,T]$ find $\Gamma^h(t) = \vec{X}^h([0,1],t)$ with $\vec{X}^h(t) \in \underline{V}^h$ and $\vec{X}^h_t(t) \in \underline{W}^h$, and $\vec{Y}^h(t) \in \underline{V}^h$ such that

$$\begin{aligned} \langle \vec{Q}^h\,\vec{X}^h_t, \vec{\chi}\rangle^h_{\Gamma^h} - \langle \phi''(\vec{X}^h_s)\,\vec{\nabla}_s\,\vec{Y}^h, \vec{\nabla}_s\,\vec{\chi}\rangle_{\Gamma^h} - \tfrac{1}{2}\,\langle (|\vec{Q}^h_\star\,\vec{Y}^h|^2 - 2\,\lambda)\,\vec{X}^h_s, \vec{\chi}_s\rangle^h_{\Gamma^h} & \\ + \langle \pi^h_W[|\vec{\theta}^h|^{-1}\,(\vec{Y}^h\,.\,\vec{\omega}^h_d)]\,\vec{Q}^h_\star\,\vec{Y}^h, \vec{\chi}_s\rangle^h_{\Gamma^h} = 0 \quad \forall\ \vec{\chi} \in \underline{W}^h\,, & \qquad (3.31a) \\ \langle \vec{Q}^h_\star\,\vec{Y}^h, \vec{\eta}\rangle^h_{\Gamma^h} + \langle \phi'(\vec{X}^h_s), \vec{\eta}_s\rangle_{\Gamma^h} = \phi'(\vec{\zeta}_1)\,.\,\vec{\eta}(1) - \phi'(\vec{\zeta}_0)\,.\,\vec{\eta}(0) \quad \forall\ \vec{\eta} \in \underline{V}^h\,. & \qquad (3.31b) \end{aligned}$$

THEOREM. 3.5. *Let* $(\mathcal{C}^h)$ *hold and let* $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ *be a solution to* (3.31a,b). *Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h_\lambda(\Gamma^h, \vec{Q}^h_\star\,\vec{Y}^h) = -\langle \vec{Q}^h\,\vec{X}^h_t, \vec{Q}^h\,\vec{X}^h_t\rangle^h_{\Gamma^h} \leq 0\,, \tag{3.32}$$

where $\vec{\kappa}^h_\phi = \pi^h[\vec{Q}^h_\star\,\vec{Y}^h]$ *is the discrete anisotropic curvature vector of* Γ^h.

Proof. The proof is an adaption of the proof of Theorem 3.3. Differentiating (3.31b) with respect to t, choosing $\vec\eta = \vec Y^h$ and combining (3.31a) with $\vec\chi = \vec X^h_t$ yields the following analogue of (3.21)

$$\langle (\vec Q^h_\star\, \vec Y^h)_t, \vec Y^h\rangle^h_{\Gamma^h} + \tfrac12\,\langle (|\vec Q^h_\star\, \vec Y^h|^2 + 2\,\lambda), \vec X^h_s \,.\, \vec X^h_{t,s}\rangle^h_{\Gamma^h} + \langle \pi^h_W[\,|\vec\theta^h|^{-1}(\vec Y^h \,.\, \vec\omega^h_d)]\, \vec Q^h_\star\, \vec Y^h, \vec X^h_{t,s}\rangle^h_{\Gamma^h} \\ = -\langle \vec Q^h\, \vec X^h_t, \vec Q^h\, \vec X^h_t\rangle^h_{\Gamma^h} . \qquad (3.33)$$

The desired result (3.32) then follows from (3.33) on noting that

$$\begin{aligned}
&\langle (\vec Q^h_\star\, \vec Y^h)_t, (\vec{Id} - \vec Q^h_\star)\, \vec Y^h\rangle^h_{\Gamma^h} \\
&\quad = \langle (\vec Q^h_\star\, \vec Y^h)_t, \pi^h_W[(\vec Y^h \,.\, \vec\omega^h_d)\, \vec\omega^h_d]\rangle^h_{\Gamma^h} = -\langle \vec Q^h_\star\, \vec Y^h, \pi^h_W[(\vec Y^h \,.\, \vec\omega^h_d)\, (\vec\omega^h_d)_t]\rangle^h_{\Gamma^h} \\
&\quad = -\langle \pi^h_W[\vec Y^h \,.\, \vec\omega^h_d]\, \vec Q^h_\star\, \vec Y^h, \widehat D_s\, \vec X^h_t\rangle^h_{\Gamma^h} = -\langle \pi^h_W[\,|\vec\theta^h|^{-1}\,(\vec Y^h \,.\, \vec\omega^h_d)]\, \vec Q^h_\star\, \vec Y^h, \vec X^h_{t,s}\rangle^h_{\Gamma^h} ,
\end{aligned}$$

where we have recalled (3.29) and the $\underline{V}^h$ versions of (3.7) and (3.5). ∎

We remark that the scheme (3.31a,b) with now $\vec Y^h(t) \in \underline{W}^h$ and the test space $\underline{V}^h$ for (3.31b) replaced by $\underline{W}^h$ is a stable semidiscrete approximation of the flow (2.42) with the homogeneous boundary conditions (2.44), i.e. $\vec\beta = \vec 0$.

3.3.2 Navier conditions

We now consider the planar case, $d = 2$, and derive a stable semidiscrete approximation of the system (2.52a–c). To this end, we consider the L^2-gradient flow of the discrete energy

$$\widehat E^h_{\beta,\lambda}(\Gamma^h, \kappa^h_\phi) := \tfrac12\,\langle \kappa^h_\phi - \beta, \kappa^h_\phi - \beta\rangle^h_{\Gamma^h} + \tfrac12\,(2\,\lambda - \beta^2)\,|\Gamma^h|\,,$$

where $(\kappa^h_\phi - \beta) \in W^h$, subject to the side constraint

$$\langle \kappa^h_\phi\, \vec\nu^h, \vec\eta\rangle^h_{\Gamma^h} + \langle \phi'(\vec X^h_s), \vec\eta_s\rangle_{\Gamma^h} = 0 \qquad \forall\ \vec\eta \in \underline{W}^h . \qquad (3.34)$$

Observe that (3.34) is a discrete analogue of (2.48). Introducing the Lagrange multiplier $\vec Y^h \in \underline{W}^h$ for (3.34), defining the discrete Lagrangian $\mathcal{L}^h$ corresponding to (2.51) and taking variations of $\mathcal{L}^h$, we can derive the following system:

$$\langle \vec Q^h_\omega\, \vec X^h_t, \vec\chi\rangle^h_{\Gamma^h} = \langle \phi''(\vec X^h_s)\, \vec\nabla_s\, \vec Y^h, \vec\nabla_s\, \vec\chi\rangle_{\Gamma^h} - \tfrac12\,\langle [(\kappa^h_\phi - \beta)^2 - 2\,\kappa^h_\phi\, \vec Y^h \,.\, \vec\nu^h]\, \vec X^h_s, \vec\chi_s\rangle^h_{\Gamma^h} \\ - \langle \kappa^h_\phi\, \vec Y^h, (\vec\nabla_s\, \vec\chi)^\perp\rangle^h_{\Gamma^h} \quad \forall\ \vec\chi \in \underline{W}^h , \qquad (3.35a)$$

$$\langle \kappa^h_\phi - \beta - \vec Y^h \,.\, \vec\nu^h, \chi\rangle^h_{\Gamma^h} = 0 \qquad \forall\ \chi \in W^h , \qquad (3.35b)$$

$$\langle \kappa^h_\phi\, \vec\nu^h, \vec\eta\rangle^h_{\Gamma^h} + \langle \phi'(\vec X^h_s), \vec\eta_s\rangle_{\Gamma^h} = 0 \qquad \forall\ \vec\eta \in \underline{W}^h . \qquad (3.35c)$$

Once again, $\vec\nu^h = -(\vec X^h_s)^\perp$, recall (2.55), yields that

$$-\langle \kappa^h_\phi\, (\vec Y^h \,.\, \vec\nu^h)\, \vec X^h_s, \vec\chi_s\rangle^h_{\Gamma^h} + \langle \kappa^h_\phi\, \vec Y^h, (\vec\nabla_s\, \vec\chi)^\perp\rangle^h_{\Gamma^h} = -\langle \kappa^h_\phi\, (\vec Y^h)^\perp, \vec\chi_s\rangle^h_{\Gamma^h} .$$

Moreover, it follows from $(\kappa^h_\phi - \beta) \in W^h$, $\vec Y^h \in \underline{W}^h$, (3.35b) and (3.27) that

$$\kappa^h_\phi - \beta = \pi^h[\vec Y^h \,.\, \vec\omega^h] . \qquad (3.36)$$

Hence we can rewrite (3.35a–c) equivalently as follows. Given $\Gamma^h(0) = \vec{X}^h([0,1],0)$, with $\vec{X}^h(0) \in \underline{V}^h$ and $\vec{X}^h(0,0) = \vec{\alpha}_0$, $\vec{X}^h(1,0) = \vec{\alpha}_1$, for all $t \in (0,T]$ find $\Gamma^h(t) = \vec{X}^h([0,1],t)$ with $\vec{X}^h(t) \in \underline{V}^h$ and $\vec{X}^h_t(t) \in \underline{W}^h$, and $\vec{Y}^h(t) \in \underline{W}^h$ such that

$$\langle \vec{Q}^h_\omega\, \vec{X}^h_t, \vec{\chi}\rangle^h_{\Gamma^h} - \langle \phi''(\vec{X}^h_s)\, \vec{\nabla}_s\, \vec{Y}^h, \vec{\nabla}_s\, \vec{\chi}\rangle_{\Gamma^h} + \tfrac{1}{2}\, \langle [(\vec{Y}^h\,.\,\vec{\omega}^h)^2 + 2\,\lambda - \beta^2]\, \vec{X}^h_s, \vec{\chi}_s\rangle^h_{\Gamma^h}$$
$$- \langle (\vec{Y}^h\,.\,\vec{\omega}^h + \beta)\, (\vec{Y}^h)^\perp, \vec{\chi}_s\rangle^h_{\Gamma^h} = 0 \quad \forall\ \vec{\chi} \in \underline{W}^h\,, \tag{3.37a}$$

$$\langle \vec{Q}^h_\omega\, \vec{Y}^h, \vec{\eta}\rangle^h_{\Gamma^h} + \langle \phi'(\vec{X}^h_s), \vec{\eta}_s\rangle_{\Gamma^h} = -\beta\, \langle \vec{\omega}^h, \vec{\eta}\rangle^h_{\Gamma^h} \quad \forall\ \vec{\eta} \in \underline{W}^h\,. \tag{3.37b}$$

Similarly to (2.54a,b) with (2.53) we note that in the above approximation the boundary conditions in (2.46) are enforced strongly. In particular, it follows from (3.36) that $\kappa^h_\phi(0,t) = \kappa^h_\phi(1,t) = \beta$ for all $t \in (0,T]$, where we recall that $\vec{Y}^h(t) \in \underline{W}^h$.

THEOREM. 3.6. *Let $d = 2$, let $(\mathcal{C}^h)$ hold and let $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ be a solution to* (3.37a,b)*. Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \widehat{E}^h_{\beta,\lambda}(\Gamma^h, \vec{Y}^h\,.\,\vec{\omega}^h + \beta) = -\langle \vec{X}^h_t\,.\,\vec{\omega}^h, \vec{X}^h_t\,.\,\vec{\omega}^h\rangle^h_{\Gamma^h} \leq 0\,, \tag{3.38}$$

where $\kappa^h_\phi = \pi^h[\vec{Y}^h\,.\,\vec{\omega}^h] + \beta$ is the discrete anisotropic curvature of Γ^h.

Proof. Differentiating (3.37b) with respect to t yields, on noting (3.27), that

$$\langle (\vec{Y}^h\,.\,\vec{\omega}^h)_t, \vec{\eta}\,.\,\vec{\nu}^h\rangle^h_{\Gamma^h} + \langle \vec{Y}^h\,.\,\vec{\omega}^h, \vec{\eta}\,.\,\vec{\nu}^h_t\rangle^h_{\Gamma^h} + \langle (\vec{Y}^h\,.\,\vec{\omega}^h)\,(\vec{\eta}\,.\,\vec{\nu}^h), \vec{X}^h_s\,.\,\vec{X}^h_{t,s}\rangle^h_{\Gamma^h}$$
$$+ \langle \phi''(\vec{X}^h_s)\, \vec{\nabla}_s\, \vec{X}^h_t, \vec{\nabla}_s\, \vec{\eta}\rangle_{\Gamma^h} = -\beta\, \langle \vec{\nu}^h_t + (\vec{X}^h_s\,.\,\vec{X}^h_{t,s})\, \vec{\nu}^h, \vec{\eta}\rangle^h_{\Gamma^h} \qquad \forall\ \vec{\eta} \in \underline{W}^h\,. \tag{3.39}$$

On choosing $\vec{\eta} = \vec{Y}^h \in \underline{W}^h$ in (3.39), and recalling (3.27), we obtain that

$$\langle (\vec{Y}^h\,.\,\vec{\omega}^h)_t, \vec{Y}^h\,.\,\vec{\omega}^h\rangle^h_{\Gamma^h} + \langle \vec{Y}^h\,.\,\vec{\omega}^h, \vec{Y}^h\,.\,\vec{\nu}^h_t\rangle^h_{\Gamma^h} + \langle (\vec{Y}^h\,.\,\vec{\omega}^h)\,(\vec{Y}^h\,.\,\vec{\nu}^h), \vec{X}^h_s\,.\,\vec{X}^h_{t,s}\rangle^h_{\Gamma^h}$$
$$+ \langle \phi''(\vec{X}^h_s)\, \vec{\nabla}_s\, \vec{X}^h_t, \vec{\nabla}_s\, \vec{Y}^h\rangle_{\Gamma^h} = -\beta\, \langle \vec{\nu}^h_t + (\vec{X}^h_s\,.\,\vec{X}^h_{t,s})\, \vec{\nu}^h, \vec{Y}^h\rangle^h_{\Gamma^h}\,. \tag{3.40}$$

Combining (3.40) and (3.37a) with $\vec{\chi} = \vec{X}^h_t \in \underline{W}^h$ yields the desired result (3.38), on recalling that $\vec{\nu}^h_t = -(\vec{\nabla}_s\, \vec{X}^h_t)^\perp$ and that $(\vec{Y}^h)^\perp = (\vec{Y}^h\,.\,\vec{\nu}^h)\, \vec{X}^h_s - (\vec{Y}^h\,.\,\vec{X}^h_s)\, \vec{\nu}^h$. □

THEOREM. 3.7. *Let $(\mathcal{C}^h)$ hold and let $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ denote a solution to* (3.31a,b) *or to* (3.37a,b)*. For a fixed time $t \in (0,T]$ let $\vec{a}^h_{j-\frac{1}{2}} := \vec{X}^h(q_j) - \vec{X}^h(q_{j-1})$, $j = 1 \to J$. Then it holds that $\vec{X}^h(t)$ satisfies* (3.25a–c) *for $j = 1 \to J-1$.*

Proof. On choosing $\vec{\eta} = \vec{\omega}^h_d(q_j)\, \chi_j$, for $j = 1 \to J-1$, in (3.31b) or (3.37b), and noting that $\vec{\omega}^h(q_j)\,.\,\vec{\omega}^h_d(q_j) = 0$, for $j = 1 \to J-1$, in the latter case; then the proof follows exactly as the proof of Theorem 3.4. □

We note, as before, for $L = 1$ the above theorem gives equidistribution with respect to ϕ, provided that intervals are not locally parallel; and in the isotropic case this yields an equidistribution of the vertices, as discussed in Theorem 3.2.

REMARK. 3.3. *Replacing the test and trial spaces* $\underline{W}^h$ *in* (3.37a,b) *with the space* $\underline{V}_0^h$ *in the closed curve case, we can immediately introduce the following alternative semidiscrete approximation to* (3.17a,b) *for the flow* (2.45) *in the case of a closed curve in the plane. Given* $\Gamma^h(0) = \vec{X}^h(I,0)$, *with* $\vec{X}^h(0) \in \underline{V}_0^h$, *for all* $t \in (0,T]$ *find* $\Gamma^h(t) = \vec{X}^h(I,t)$ *with* $\vec{X}^h(t) \in \underline{V}_0^h$, *and* $\vec{Y}^h(t) \in \underline{V}_0^h$ *such that*

$$\langle \vec{Q}_\omega^h\, \vec{X}_t^h, \vec{\chi}\rangle_{\Gamma^h}^h - \langle \phi''(\vec{X}_s^h)\, \vec{\nabla}_s\, \vec{Y}^h, \vec{\nabla}_s\, \vec{\chi}\rangle_{\Gamma^h} + \tfrac{1}{2}\, \langle [(\vec{Y}^h \,.\, \vec{\omega}^h)^2 + 2\,\lambda]\, \vec{X}_s^h, \vec{\chi}_s\rangle_{\Gamma^h}^h - \langle \vec{Y}^h \,.\, \vec{\omega}^h\, (\vec{Y}^h)^\perp, \vec{\chi}_s\rangle_{\Gamma^h}^h = 0 \quad \forall\ \vec{\chi} \in \underline{V}_0^h\,, \tag{3.41a}$$

$$\langle \vec{Q}_\omega^h\, \vec{Y}^h, \vec{\eta}\rangle_{\Gamma^h}^h + \langle \phi'(\vec{X}_s^h), \vec{\eta}_s\rangle_{\Gamma^h} = 0 \quad \forall\ \vec{\eta} \in \underline{V}_0^h\,. \tag{3.41b}$$

In addition, we immediately have the following result. A solution $(\vec{X}^h(t), \vec{Y}^h(t))_{t\in(0,T]}$ *to* (3.41a,b) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E_\lambda^h(\Gamma^h, \vec{Y}^h \,.\, \vec{\omega}^h) \leq -\langle \vec{X}_t^h \,.\, \vec{\omega}^h, \vec{X}_t^h \,.\, \vec{\omega}^h\rangle_{\Gamma^h}^h \leq 0\,, \tag{3.42}$$

where $\kappa_\phi^h := \pi^h[\vec{Y}^h \,.\, \vec{\omega}^h]$ *defines the discrete anisotropic curvature of* Γ^h.

REMARK. 3.4. *Similarly to* Corollary 3.1, *we can also consider a constraint on the total length of the curve in the anisotropic case, as well as in the case of open curves. In particular, mimicking on the discrete level the procedure in* Remark 2.2, *we can introduce the discrete Lagrangian* $\mathcal{L}^h(\vec{X}^h, \vec{\kappa}_\phi, \vec{Y}^h, Z^h, \lambda) := \frac{1}{2}\, \langle \vec{\kappa}_\phi, \vec{\kappa}_\phi\rangle_{\Gamma^h}^h + \lambda\, (|\Gamma^h| - l) - \langle \vec{\kappa}_\phi, \vec{Y}^h\rangle_{\Gamma^h}^h - \langle \phi'(\vec{X}_s^h), \vec{Y}_s^h\rangle_{\Gamma^h} + \langle \vec{\kappa}_\phi \,.\, \vec{X}_s^h, Z^h\rangle_{\Gamma^h}^h$, *where* $l > 0$ *is a given length. We now consider* $\lambda \in \mathbb{R}$ *as an unknown and a variation with respect to* λ *gives the additional equation*

$$|\Gamma^h| = l\,. \tag{3.43}$$

For example, as the length preserving approximation of the elastic flow in the case of clamped boundary conditions we then obtain (3.31a,b) *with the additional unknown* $\lambda(t)$ *and the additional constraint* (3.43) *for* $l := |\Gamma^h(0)|$. *Similarly to* Theorem 3.5, *it is then a simple matter to show that this semidiscrete approximation fulfills* $\frac{\mathrm{d}}{\mathrm{d}t}\, |\Gamma^h(t)| = 0$ *and* $\frac{\mathrm{d}}{\mathrm{d}t}\, E_0^h(\Gamma^h, \vec{Q}_\star^h\, \vec{Y}^h) = -\langle \vec{Q}^h\, \vec{X}_t^h, \vec{Q}^h\, \vec{X}_t^h\rangle_{\Gamma^h}^h \leq 0$.

4 Fully discrete finite element approximation

In this section we introduce fully discrete variants of the semidiscrete finite element approximations derived in Section 3.

Let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_m := t_{m+1} - t_m$, $m = 0 \to M-1$. We set $\tau := \max_{m=0\to M-1} \tau_m$. Given $\Gamma^0 = \vec{X}^0(I)$, our fully discrete approximation will define a sequence of polygonal curves Γ^m, $m = 0 \to M$, where $\Gamma^m = \vec{X}^m(I)$ with $\vec{X}^m \in \underline{V}_0^h$. Similarly to (3.1), we define $\langle u, v\rangle_{\Gamma^m} = \int_I u \,.\, v\, |\vec{X}_\rho^m|\ \mathrm{d}\rho$ and, for the case that u, v are piecewise continuous, we also define the mass lumped inner product

$$\langle u, v\rangle^h_{\Gamma^m} := \tfrac{1}{2} \sum_{j=1}^{J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})| \left[(u\,.\,v)(q_j^-) + (u\,.\,v)(q_{j-1}^+)\right].$$

Furthermore, we note that on Γ^m we have almost everywhere that

$$u_s\,.\,v_s = \frac{u_\rho\,.\,v_\rho}{|\vec{X}^m_\rho|^2} \qquad \text{and} \qquad \vec{\nabla}_s\,\vec{u}\,.\,\vec{\nabla}_s\,\vec{v} = \frac{\vec{P}^m\,\vec{u}_\rho\,.\,\vec{P}^m\,\vec{v}_\rho}{|\vec{X}^m_\rho|^2} = \frac{\vec{P}^m\,\vec{u}_\rho\,.\,\vec{v}_\rho}{|\vec{X}^m_\rho|^2},$$

where $\vec{P}^m = \vec{Id} - \vec{X}^m_s \otimes \vec{X}^m_s$. Similarly to (3.3) and (3.6), we introduce the definitions $\vec{\theta}^m$, $\vec{\omega}^m_d \in \underline{V}^h_0$ and $\vec{Q}^m$, which are based on $\vec{X}^m$ in place of $\vec{X}^h(t)$. For the case $d = 2$ we introduce, in addition, $\vec{\nu}^m := -(\vec{X}^m_s)^\perp$, as well as $\vec{\omega}^m \in \underline{V}^h_0$ and $\vec{Q}^m_\omega := \vec{\omega}^m \otimes \vec{\omega}^m$.

Before we introduce our approximations, we have to make the following very weak assumptions.

($\mathcal{C}^m_0$) Let $\vec{X}^m(q_j) \neq \vec{X}^m(q_{j+1})$ and $\vec{X}^m(q_{j-1}) \neq \vec{X}^m(q_{j+1})$, $j = 1 \to J$, and in addition let $\bigcup_{j=1}^J \{\vec{\omega}^m_d(q_j)\}^\perp = \mathbb{R}^d$.

4.1 Isotropic elastic flow

We have the following fully discrete approximation of (3.8a,b). Find $(\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ such that

$$\langle \vec{Q}^m\,\frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \vec{\chi}\rangle^h_{\Gamma^m} - \langle \vec{Y}^{m+1}_s, \vec{\chi}_s\rangle_{\Gamma^m} + \langle (\vec{Id} - \vec{P}^m)\,\vec{Y}^m_s, \vec{\chi}_s\rangle_{\Gamma^m}$$
$$= \tfrac{1}{2}\,\langle (|\vec{Q}^m\,\vec{Y}^m|^2 - 2\,\lambda)\,\vec{X}^m_s, \vec{\chi}_s\rangle^h_{\Gamma^m} - \langle |\vec{\theta}^m|^{-1}\,(\vec{Y}^m\,.\,\vec{\omega}^m_d)\,\vec{Q}^m\,\vec{Y}^m, \vec{\chi}_s\rangle^h_{\Gamma^m} \quad \forall\ \vec{\chi} \in \underline{V}^h_0, \tag{4.1a}$$

$$\langle \vec{Q}^m\,\vec{Y}^{m+1}, \vec{\eta}\rangle^h_{\Gamma^m} + \langle \vec{X}^{m+1}_s, \vec{\eta}_s\rangle_{\Gamma^m} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h_0, \tag{4.1b}$$

where, here and throughout, $\vec{Y}^0$ is a suitably chosen initial value. Of course, for the above we let $\vec{Y}^0 \in \underline{V}^h_0$, while for the open curve schemes in §4.3 we let $\vec{Y}^0 \in \underline{V}^h$.

THEOREM. 4.1. *Let the assumptions ($\mathcal{C}^m_0$) hold. Then there exists a unique solution $(\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ to (4.1a,b).*

Proof. The result follows from Theorem 4.2, below, for the special case $\phi(\cdot) = |\cdot|$. ∎

4.2 Anisotropic elastic flow

A wide class of anisotropies can either be modelled or at least very well approximated by (3.24), see Barrett, Garcke, and Nürnberg (2008c), and for our fully discrete approximations we will restrict ourselves to anisotropies of the form (3.24). Here we recall that the authors in Barrett, Garcke, and Nürnberg (2008a, 2010b) introduced the linearized semi-implicit approximation $\vec\Phi(\vec X^m_s)\,\vec X^{m+1}_s$ of $\phi'(\vec x_s)$, where

$$\vec\Phi(\vec q) = \sum_{\ell=1}^{L} [\phi_\ell(\vec q)]^{-1}\,\vec G_\ell \quad \forall\ \vec q\in\mathbb{S}^{d-1} \qquad \Rightarrow \qquad \vec\Phi(\vec q)\,\vec q = \phi'(\vec q) \quad \forall\ \vec q \in \mathbb{S}^{d-1}\,. \tag{4.2}$$

On employing the above linearization, the authors in Barrett, Garcke, and Nürnberg (2008a, 2010b) introduced parametric finite element approximations for anisotropic geometric evolution equations in the plane and in $\mathbb{R}^d$, respectively; and showed that for the class of anisotropy densities that correspond to the choice (3.24), unconditionally stable fully discrete approximations are obtained for certain gradient flows of the weighted length functional $|\Gamma|_\phi$. For later purposes we also note for all $\vec p\in\mathbb{R}^d\setminus\{\vec 0\}$ that

$$\phi''(\vec p) = \sum_{\ell=1}^{L}[\phi_\ell(\vec p)]^{-1}\left[\vec{Id} - [\phi_\ell(\vec p)]^{-2}\,(\vec G_\ell\,\vec p)\otimes\vec p\right]\vec G_\ell \tag{4.3}$$

is positive semi-definite.

The natural extension of (4.1a,b) to the anisotropic flow (2.18) for anisotropies of the form (3.24) is then: Find $(\vec X^{m+1}, \vec Y^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ such that

$$\begin{aligned}
&\langle \vec Q^m\,\frac{\vec X^{m+1}-\vec X^m}{\tau_m}, \vec\chi\rangle^h_{\Gamma^m} - \langle \vec\Phi(\vec X^m_s)\,\vec Y^{m+1}_s, \vec\chi_s\rangle_{\Gamma^m} + \langle (\vec\Phi(\vec X^m_s) - \phi''(\vec X^m_s))\,\vec Y^m_s, \vec\chi_s\rangle_{\Gamma^m} \\
&\qquad = \tfrac12\,\langle (|\vec Q^m\,\vec Y^m|^2\,\vec X^m_s - 2\,\lambda), \vec\chi_s\rangle^h_{\Gamma^m} - \langle |\vec\theta^m|^{-1}\,(\vec Y^m\,.\,\vec\omega^m_d)\,\vec Q^m\,\vec Y^m, \vec\chi_s\rangle^h_{\Gamma^m} \qquad \forall\ \vec\chi\in\underline{V}^h_0\,,
\end{aligned} \tag{4.4a}$$

$$\langle \vec Q^m\,\vec Y^{m+1}, \vec\eta\rangle^h_{\Gamma^m} + \langle \vec\Phi(\vec X^m_s)\,\vec X^{m+1}_s, \vec\eta_s\rangle_{\Gamma^m} = 0 \qquad \forall\ \vec\eta\in\underline{V}^h_0\,. \tag{4.4b}$$

On recalling (2.9) and (4.2), we note that (4.4a,b) is a fully discrete variant of the semidiscrete scheme (3.17a,b).

THEOREM. 4.2. *Let the assumptions* $(\mathcal{C}^m_0)$ *hold. Then there exists a unique solution* $(\vec X^{m+1}, \vec Y^{m+1}) \in \underline{V}^h_0\times\underline{V}^h_0$ *to* (4.4a,b).

Proof. As (4.4a,b) is a linear system, existence follows from uniqueness. To investigate the latter, we consider the system: Find $(\vec X, \vec Y)\in\underline{V}^h_0\times\underline{V}^h_0$ such that

$$\langle \vec Q^m\,\vec X, \vec\chi\rangle^h_{\Gamma^m} - \tau_m\,\langle \vec\Phi(\vec X^m_s)\,\vec Y_s, \vec\chi_s\rangle_{\Gamma^m} = 0 \qquad \forall\ \vec\chi\in\underline{V}^h_0\,, \tag{4.5a}$$

$$\langle \vec Q^m\,\vec Y, \vec\eta\rangle^h_{\Gamma^m} + \langle \vec\Phi(\vec X^m_s)\,\vec X_s, \vec\eta_s\rangle_{\Gamma^m} = 0 \qquad \forall\ \vec\eta\in\underline{V}^h_0\,. \tag{4.5b}$$

Choosing $\vec{\chi} = \vec{Y}$ in (4.5a) and $\vec{\eta} = \vec{X}$ in (4.5b) yields that

$$\tau_m \sum_{\ell=1}^{L} \langle [\phi_\ell(\vec{X}^m_s)]^{-1}\, \vec{G}_\ell\, \vec{Y}_s, \vec{Y}_s\rangle_{\Gamma^m} + \sum_{\ell=1}^{L} \langle [\phi_\ell(\vec{X}^m_s)]^{-1}\, \vec{G}_\ell\, \vec{X}_s, \vec{X}_s\rangle_{\Gamma^m} = 0\,.$$

It follows from the positive definiteness of $\vec{G}_\ell$, $\ell = 1 \to L$, that $\vec{X} = \vec{X}^c \in \mathbb{R}^d$ and $\vec{Y} = \vec{Y}^c \in \mathbb{R}^d$. Hence it follows from (4.5a,b) that $\pi^h[\vec{Q}^m\,\vec{X}^c] = \pi^h[\vec{Q}^m\,\vec{Y}^c] = \vec{0}$. The assumptions $(\mathcal{C}^m_0)$ then yield that $\vec{X}^c = \vec{Y}^c = \vec{0}$. Hence there exists a unique solution $(\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ to (4.4a,b). $\square$

Remark. 4.1. *We note that the natural semi-implicit fully discrete approximation of* (3.17a,b) *has* $-\langle \phi''(\vec{X}^m_s)\,\vec{Y}^{m+1}_s, \vec{\chi}_s\rangle_{\Gamma^m}$ *instead of the last two terms on the left-hand side of* (4.4a). *However, then existence and uniqueness of the discrete solution* $(\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ *to this modified system is in general no longer guaranteed. In particular, existence and uniqueness can only be guaranteed if*

$$\langle \phi''(\vec{X}^m_s)\,\vec{Y}_s, \vec{Y}_s\rangle_{\Gamma^m} = 0 \tag{4.6}$$

for $\vec{Y} \in \underline{V}^h_0$ *implies that* $\vec{Y}$ *is constant, which is only true if the curve* Γ^m *has nowhere locally parallel segments. To see this, observe that if the two segments meeting at* $\vec{X}^m(q_j)$ *are parallel, then* (4.6), *on recalling* (2.9), *imparts no information on the tangential part of* $\vec{Y}(q_j)$. *In addition, we observed in practice that even for curves* Γ^m *where existence and uniqueness is theoretically guaranteed, the resulting linear system is very ill conditioned, and so numerical blow-up can be observed. Hence our preference for the scheme* (4.4a,b).

Remark. 4.2. *In Barrett, Garcke, and Nürnberg (2010b), the authors introduced the following fully discrete approximation of* (2.23a,b), *where they defined the following subspace of* $\underline{V}^h_0$*:*

$$\underline{V}^{h,m}_{0,\vec{\tau}} := \{\vec{\eta} \in \underline{V}^h_0 : \vec{\eta}(q_j)\,.\,\vec{\omega}^m_d(q_j) = 0\,,\ j = 1 \to J\}\,.$$

Find $(\vec{X}^{m+1}, \vec{\kappa}^{m+1}_\phi) \in \underline{V}^h_0 \times \underline{V}^{h,m}_{0,\vec{\tau}}$*, such that*

$$\langle \vec{Q}^m\,\frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \vec{\chi}\rangle^h_{\Gamma^m} - \langle \phi''(\vec{X}^m_s)\,(\vec{\kappa}^{m+1}_\phi)_s, \vec{\chi}_s\rangle_{\Gamma^m} - \tfrac{1}{2}\,\langle |\vec{\kappa}^m_\phi|^2\,\vec{X}^m_s, \vec{\chi}_s\rangle^h_{\Gamma^m} = 0 \qquad \forall\ \vec{\chi} \in \underline{V}^{h,m}_{0,\vec{\tau}}\,, \tag{4.7a}$$

$$\langle \vec{Q}^m\,\vec{\kappa}^{m+1}_\phi, \vec{\eta}\rangle^h_{\Gamma^m} + \langle \vec{\Phi}(\vec{X}^m_s)\,\vec{X}^{m+1}_s, \vec{\eta}_s\rangle_{\Gamma^m} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h_0\,, \tag{4.7b}$$

where $\vec{\kappa}^0_\phi \in \underline{V}^{h,0}_{0,\vec{\tau}}$ *is suitably chosen. On noting that* $\vec{\chi}$, $\vec{\kappa}^{m+1}_\phi \in \underline{V}^{h,m}_{0,\vec{\tau}}$*, it is easy to see that* (4.7a,b) *is the scheme* (3.20), (3.18b) *from Barrett, Garcke, and Nürnberg (2010b) for the case* $\lambda = 0$. *While the scheme* (4.7a,b) *is fully practical and while existence of a unique solution* $(\vec{X}^{m+1}, \vec{\kappa}^{m+1}_\phi) \in \underline{V}^h_0 \times \underline{V}^{h,m}_{0,\vec{\tau}}$ *is easily established, it does not appear possible to derive a stability result similar to* Theorem 3.3 *for the semidiscrete continuous-in-time version of* (4.7a,b).

In the isotropic situation (2.11), *a fully discrete approximation of the stable semidiscrete scheme* (3.12a,b) *as considered in Deckelnick and Dziuk (2009), is given by: Find*

$(\vec{X}^{m+1}, \vec{\kappa}^{m+1}) \in \underline{V}_0^h \times \underline{V}_0^h$ *such that*

$$\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \vec{\chi} \rangle_{\Gamma^m}^h - \langle \vec{\nabla}_s \, \vec{\kappa}^{m+1}, \vec{\nabla}_s \, \vec{\chi} \rangle_{\Gamma^m} - \tfrac{1}{2} \, \langle (|\vec{\kappa}^m|^2 - 2\,\lambda) \, \vec{X}_s^{m+1}, \vec{\chi}_s \rangle_{\Gamma^m} = 0 \quad \forall \ \vec{\chi} \in \underline{V}_0^h \,, \tag{4.8a}$$

$$\langle \vec{\kappa}^{m+1}, \vec{\eta} \rangle_{\Gamma^m}^h + \langle \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_{\Gamma^m} = 0 \quad \forall \ \vec{\eta} \in \underline{V}_0^h \,. \tag{4.8b}$$

4.3 Initial boundary value problems

Similarly to §3.3, we extend the previously given definitions to $\vec{\theta}^m$, $\vec{\omega}_d^m \in \underline{V}^h$ and, in the case $d = 2$, to $\vec{\omega}^m \in \underline{V}^h$ for the open curve case $\Gamma^m = \vec{X}^m([0,1])$. We make the following assumption for our fully discrete approximations.

($\mathcal{C}^m$) Let $\vec{X}^m(q_j) \neq \vec{X}^m(q_{j+1})$, $j = 0 \to J-1$, and $\vec{X}^m(q_{j-1}) \neq \vec{X}^m(q_{j+1})$, $j = 1 \to J-1$, and let $\bigcup_{j=1}^{J-1} \{\vec{\omega}_d^m(q_j)\}^\perp = \mathbb{R}^d$.

4.3.1 Clamped conditions

We introduce the time discrete analogue $\vec{Q}_\star^m$ of (3.29). Our fully discrete analogue of the semidiscrete approximation (3.31a,b) is then given as follows. Given $\vec{X}^0 \in \underline{V}^h$ with $\vec{X}^0(0) = \vec{\alpha}_0$ and $\vec{X}^0(1) = \vec{\alpha}_1$, for $m = 0 \to M-1$ find $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{W}^h \times \underline{V}^h$, with $\vec{X}^{m+1} = \vec{X}^m + \delta\vec{X}^{m+1}$, such that

$$\begin{aligned}
&\langle \vec{Q}_\star^m \, \frac{\delta\vec{X}^{m+1}}{\tau_m}, \vec{\chi} \rangle_{\Gamma^m}^h - \langle \vec{\Phi}(\vec{X}_s^m) \, \vec{Y}_s^{m+1}, \vec{\chi}_s \rangle_{\Gamma^m} + \langle (\vec{\Phi}(\vec{X}_s^m) - \phi''(\vec{X}_s^m)) \, \vec{Y}_s^m, \vec{\chi}_s \rangle_{\Gamma^m} \\
&\qquad = \tfrac{1}{2} \, \langle (|\vec{Q}_\star^m \, \vec{Y}^m|^2 - 2\,\lambda) \, \vec{X}_s^m, \vec{\chi}_s \rangle_{\Gamma^m}^h - \langle \pi_W^h [|\vec{\theta}^m|^{-1} \, (\vec{Y}^m \, . \, \vec{\omega}_d^m)] \, \vec{Q}_\star^m \, \vec{Y}^m, \vec{\chi}_s \rangle_{\Gamma^m}^h \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \ \vec{\chi} \in \underline{W}^h \,,
\end{aligned} \tag{4.9a}$$

$$\langle \vec{Q}_\star^m \, \vec{Y}^{m+1}, \vec{\eta} \rangle_{\Gamma^m}^h + \langle \vec{\Phi}(\vec{X}_s^m) \, \vec{X}_s^{m+1}, \vec{\eta}_s \rangle_{\Gamma^m} = \phi'(\vec{\zeta}_1) \, . \, \vec{\eta}(1) - \phi'(\vec{\zeta}_0) \, . \, \vec{\eta}(0) \quad \forall \ \vec{\eta} \in \underline{V}^h \,. \tag{4.9b}$$

As $\delta\vec{X}^{m+1} \in \underline{W}^h$ it follows that $\vec{Q}_\star^m$ in the first term in (4.9a) can be replaced by $\vec{Q}^m$, so that (4.9a,b) is indeed a fully discrete variant of the semidiscrete approximation (3.31a,b). We prefer the stated version of (4.9a) as it makes the resulting linear system more symmetric.

THEOREM. 4.3. *Let the first assumptions in* ($\mathcal{C}^m$) *hold, so that* (4.9a,b) *is well-defined. Then there exists a unique solution* $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{W}^h \times \underline{V}^h$ *to* (4.9a,b).

Proof. As (4.9a,b) is a linear system, existence follows from uniqueness. To investigate the latter, we consider the system: Find $(\delta\vec{X}, \vec{Y}) \in \underline{W}^h \times \underline{V}^h$ such that

$$\langle \vec{Q}_\star^m \, \delta\vec{X}, \vec{\chi} \rangle_{\Gamma^m}^h - \tau_m \, \langle \vec{\Phi}(\vec{X}_s^m) \, \vec{Y}_s, \vec{\chi}_s \rangle_{\Gamma^m} = 0 \qquad \forall \ \vec{\chi} \in \underline{W}^h \,, \tag{4.10a}$$

$$\langle \vec{Q}_\star^m \, \vec{Y}, \vec{\eta} \rangle_{\Gamma^m}^h + \langle \vec{\Phi}(\vec{X}_s^m) \, \delta\vec{X}_s, \vec{\eta}_s \rangle_{\Gamma^m} = 0 \qquad \forall \ \vec{\eta} \in \underline{V}^h \,. \tag{4.10b}$$

Choosing $\vec\chi = \delta\vec X$ in (4.10a) and $\vec\eta = \vec Y$ in (4.10b), and combining, yields that $\pi^h[\vec Q^m_\star\,\delta\vec X] = \pi^h[\vec Q^m_\star\,\vec Y] = \vec 0$, and hence that $\vec Y \in \underline{W}^h$. On choosing $\vec\chi = \vec Y$ in (4.10a) and $\vec\eta = \delta\vec X$ in (4.10b), it follows from the positive definiteness of $\vec G_\ell$, $\ell = 1 \to L$, that $\delta\vec X = \vec X^c \in \mathbb{R}^d$ and $\vec Y = \vec Y^c \in \mathbb{R}^d$. Recalling that $\delta\vec X$, $\vec Y \in \underline{W}^h$ then immediately yields that $\vec X^c = \vec Y^c = \vec 0$. Hence there exists a unique solution $(\delta\vec X^{m+1}, \vec Y^{m+1}) \in \underline{W}^h \times \underline{V}^h$ to (4.9a,b). $\square$

Remark. 4.3. *The natural fully discrete analogue of* (3.28a,b) *is given by* (4.9a,b) *with* $\vec Q^m_\star$ *replaced by* $\vec Q^m$ *and with* π^h_W *removed. It is then no longer straightforward to establish existence and uniqueness for this approximation. In particular, it is not possible to infer from* (4.10a,b), *with* $\vec Q^m_\star$ *replaced by* $\vec Q^m$, *that* $\vec Y \in \underline{W}^h$ *and hence that* $\vec X = \vec Y = \vec 0$. *Of course, this problem can be overcome on the fully discrete level by considering instead* (4.9b) *with the right hand side correction term* $\langle(\vec Q^m_\star - \vec Q^m)\,\vec Y^m, \vec\eta\rangle^h_{\Gamma^m}$, *which would yield a different fully discrete analogue of* (3.28a,b). *However, both of these schemes performed badly in practice, and hence our preference for the approximation* (4.9a,b).

We note that the scheme (4.9a,b) with a zero right hand side in (4.9b) and with $\underline{V}^h$ replaced by $\underline{W}^h$ is a fully discrete approximation of the flow (2.42) with the homogeneous boundary conditions (2.44) with $\vec\beta = \vec 0$.

4.3.2 Navier conditions

The fully discrete analogue of the semidiscrete approximation (3.37a,b) is given as follows. Given $\vec X^0 \in \underline{V}^h$ with $\vec X^0(0) = \vec\alpha_0$ and $\vec X^0(1) = \vec\alpha_1$, for $m = 0 \to M-1$ find $(\delta\vec X^{m+1}, \vec Y^{m+1}) \in \underline{W}^h \times \underline{W}^h$, with $\vec X^{m+1} = \vec X^m + \delta\vec X^{m+1}$, such that

$$\begin{aligned}
&\langle \vec Q^m_\omega\,\frac{\delta\vec X^{m+1}}{\tau_m}, \vec\chi\rangle^h_{\Gamma^m} - \langle \vec\Phi(\vec X^m_s)\,\vec Y^{m+1}_s, \vec\chi_s\rangle_{\Gamma^m} + \langle(\vec\Phi(\vec X^m_s) - \phi''(\vec X^m_s))\,\vec Y^m_s, \vec\chi_s\rangle_{\Gamma^m} \\
&\quad = -\tfrac12\,\langle[(\vec Y^m\,.\,\vec\omega^m)^2 + 2\,\lambda - \beta^2]\,\vec X^m_s, \vec\chi_s\rangle^h_{\Gamma^m} + \langle(\vec Y^m\,.\,\vec\omega^m + \beta)\,(\vec Y^m)^\perp, \vec\chi_s\rangle^h_{\Gamma^m} \\
&\qquad\qquad \forall\ \vec\chi \in \underline{W}^h\,, \qquad (4.11a)\\
&\langle \vec Q^m_\omega\,\vec Y^{m+1}, \vec\eta\rangle^h_{\Gamma^m} + \langle \vec\Phi(\vec X^m_s)\,\vec X^{m+1}_s, \vec\eta_s\rangle_{\Gamma^m} = -\beta\,\langle\vec\omega^m, \vec\eta\rangle^h_{\Gamma^m} \quad \forall\ \vec\eta \in \underline{W}^h\,. \qquad (4.11b)
\end{aligned}$$

Theorem. 4.4. *Let the assumptions* ($\mathcal{C}^m$) *hold. Then there exists a unique solution* $(\delta\vec X^{m+1}, \vec Y^{m+1}) \in \underline{W}^h \times \underline{W}^h$ *to* (4.11a,b).

Proof. The proof follows similarly to the proof of Theorem 4.2. $\square$

Remark. 4.4. *Similarly to* (4.11a,b), *a fully discrete variant of* (3.41a,b) *is given by: Find* $(\vec X^{m+1}, \vec Y^{m+1}) \in \underline{V}^h_0 \times \underline{V}^h_0$ *such that*

$$\begin{aligned}
&\langle \vec Q^m_\omega\,\frac{\vec X^{m+1} - \vec X^m}{\tau_m}, \vec\chi\rangle^h_{\Gamma^m} - \langle \vec\Phi(\vec X^m)\,\vec Y^{m+1}_s, \vec\chi_s\rangle_{\Gamma^m} + \langle(\vec\Phi(\vec X^m) - \phi''(\vec X^m))\,\vec Y^m_s, \vec\chi_s\rangle_{\Gamma^m} \\
&\quad = -\tfrac12\,\langle[(\vec Y^m\,.\,\vec\omega^m)^2 + 2\,\lambda]\,\vec X^m_s, \vec\chi_s\rangle^h_{\Gamma^m} + \langle\vec Y^m\,.\,\vec\omega^m\,(\vec Y^m)^\perp, \vec\chi_s\rangle^h_{\Gamma^m} \quad \forall\ \vec\chi \in \underline{V}^h_0\,, \\
&\qquad\qquad (4.12a)\\
&\langle \vec Q^m_\omega\,\vec Y^{m+1}, \vec\eta\rangle^h_{\Gamma^m} + \langle \vec\Phi(\vec X^m)\,\vec X^{m+1}_s, \vec\eta_s\rangle_{\Gamma^m} = 0 \quad \forall\ \vec\eta \in \underline{V}^h_0\,. \qquad (4.12b)
\end{aligned}$$

In the case $d = 2$ the scheme (4.12a,b) *is an alternative to the fully discrete approximation* (4.4a,b). *But as there is no advantage in using* (4.12a,b) *we do not pursue this scheme any further. We remark that both schemes, with the choice $\lambda = \frac{1}{2}\beta^2$, may be used to approximate gradient flows for the spontaneous curvature energy* (1.6) *in the case $d = 2$.*

REMARK. 4.5. *For any of our fully discrete schemes, we can approximate the corresponding length preserving flow by replacing the fixed given $\lambda \in \mathbb{R}$ by a $\lambda^{m+1} \in \mathbb{R}$, where this unknown is chosen so that $|\Gamma^{m+1}| = |\Gamma^0|$. Then we obtain fully discrete versions of the semidiscrete schemes discussed in* Remark 3.4. *In each case, the fully discrete approximation then leads to a nonlinear system of equations at each time level, which can be solved by a root finding method in terms of λ^{m+1}, e.g. the secant method.*

5 Solution of the linear systems

For an arbitrary $n \in \mathbb{N}$, let $\vec{Id}_n \in (\mathbb{R}^{d\times d})^{n\times n}$ be the identity matrix. We introduce also the diagonal matrix $\vec{\mathcal{M}}_Q \in (\mathbb{R}^{d\times d})^{J\times J}$, and the stiffness matrices $A \in \mathbb{R}^{J\times J}$, $\vec{A}$, $\vec{A}_\phi$, $\vec{A}_Q$, $\vec{\mathcal{A}}$, $\vec{\mathcal{A}}_\phi$ $\in (\mathbb{R}^{d\times d})^{J\times J}$ with entries

$$
\begin{gathered}
[\vec{\mathcal{M}}_Q]_{kl} := \langle \chi_k, \chi_l\, \vec{Q}^m \rangle^h_{\Gamma^m}, \quad A_{kl} := \langle [\chi_k]_s, [\chi_l]_s \rangle_{\Gamma^m}, \quad \vec{A}_{kl} := A_{kl}\, \vec{Id}_1, \\
[\vec{A}_\phi]_{kl} := \langle [\chi_k]_s, [\chi_l]_s\, \vec{\Phi}(\vec{X}^m_s) \rangle_{\Gamma^m}\,, \quad [\vec{A}_Q]_{kl} := \tfrac{1}{2}\, \langle |\vec{Q}^m\, \vec{Y}^m|^2\, [\chi_k]_s, [\chi_l]_s \rangle^h_{\Gamma^m}\, \vec{Id}_1, \\
\text{and} \quad \vec{\mathcal{A}}_{kl} := \langle [\chi_k]_s, [\chi_l]_s\, \vec{P}^m \rangle_{\Gamma^m}\,, \quad [\vec{\mathcal{A}}_\phi]_{kl} = \langle [\chi_k]_s, [\chi_l]_s\, \phi''(\vec{X}^m_s) \rangle_{\Gamma^m}\,.
\end{gathered} \tag{5.1}
$$

5.1 Isotropic elastic flow

The linear system for the scheme (4.1a,b) reads as follows. Find $(\vec{Y}^{m+1}, \delta\vec{X}^{m+1}) \in (\mathbb{R}^d)^J \times (\mathbb{R}^d)^J$ such that

$$
\begin{pmatrix} \vec{A} & -\frac{1}{\tau_m}\vec{\mathcal{M}}_Q \\ \vec{\mathcal{M}}_Q & \vec{A} \end{pmatrix} \begin{pmatrix} \vec{Y}^{m+1} \\ \delta\vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} (\vec{A} - \vec{\mathcal{A}})\, \vec{Y}^m + (\lambda\, \vec{A} - \vec{A}_Q)\, \vec{X}^m + \vec{f}_0 \\ -\vec{A}\, \vec{X}^m \end{pmatrix}, \tag{5.2}
$$

where $\vec{f}_0 \in (\mathbb{R}^d)^J$ with $[\vec{f}_0]_j = \langle |\vec{\theta}^m|^{-1}\, (\vec{Y}^m \,.\, \vec{\omega}^m_d)\, \vec{Q}^m\, \vec{Y}^m, [\chi_j]_s \rangle^h_{\Gamma^m}$, $j = 1 \to J$.

5.2 Anisotropic elastic flow

The linear system for the approximation (4.4a,b) is given by: Find $(\vec{Y}^{m+1}, \delta\vec{X}^{m+1}) \in (\mathbb{R}^d)^J \times (\mathbb{R}^d)^J$ such that

$$
\begin{pmatrix} \vec{A}_\phi & -\frac{1}{\tau_m}\vec{\mathcal{M}}_Q \\ \vec{\mathcal{M}}_Q & \vec{A}_\phi \end{pmatrix} \begin{pmatrix} \vec{Y}^{m+1} \\ \delta\vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} (\vec{A}_\phi - \vec{\mathcal{A}}_\phi)\, \vec{Y}^m + (\lambda\, \vec{A} - \vec{A}_Q)\, \vec{X}^m + \vec{f}_0 \\ -\vec{A}_\phi\, \vec{X}^m \end{pmatrix}, \tag{5.3}
$$

where we recall the definition of $\vec{f}_0$ from (5.2).

REMARK. 5.1. *Let $\vec{\mathcal{P}}_{\vec\tau} : (\mathbb{R}^d)^J \to \underline{\mathbb{X}}^m_{0,\vec\tau} := \{\vec z \in (\mathbb{R}^d)^J : \vec z_j \,.\, \vec\omega^m_d(q_j) = 0\,, j = 1 \to J\}$ be the orthogonal projection onto $\underline{\mathbb{X}}^m_{0,\vec\tau}$. In addition, let $\vec A_\kappa$ be defined as $\vec A_Q$ in* (5.1) *with $\vec Q^m\,\vec Y^m$ replaced by $\vec\kappa^m_\phi$. Then the linear system for the approximation* (4.7a,b) *is given by: Find $(\vec\kappa^{m+1}_\phi, \delta\vec X^{m+1}) \in \underline{\mathbb{X}}^m_{0,\vec\tau} \times (\mathbb{R}^d)^J$ such that*

$$\begin{pmatrix} \vec{\mathcal{P}}_{\vec\tau}\,\vec{\mathcal{A}}_\phi\,\vec{\mathcal{P}}_{\vec\tau} & -\frac{1}{\tau_m}\,\vec{\mathcal{M}}_Q \\ \vec{\mathcal{M}}_Q & \vec A_\phi \end{pmatrix} \begin{pmatrix} \vec\kappa^{m+1}_\phi \\ \delta\vec X^{m+1} \end{pmatrix} = \begin{pmatrix} -\vec{\mathcal{P}}_{\vec\tau}\,\vec A_\kappa\,\vec X^m \\ -\vec A_\phi\,\vec X^m \end{pmatrix}, \tag{5.4}$$

where we have noted that $\vec{\mathcal{P}}_{\vec\tau}\,\vec{\mathcal{M}}_Q = \vec{\mathcal{M}}_Q\,\vec{\mathcal{P}}_{\vec\tau} = \vec{\mathcal{M}}_Q$.

5.3 Initial boundary value problems

In addition to the open curve analogues of (5.1), we introduce the matrices $\vec{\mathcal{M}}_\star$, $\vec{\mathcal{M}}_\omega$, $\vec A_{Q_\star}$, $\vec A_\omega \in (\mathbb{R}^{d\times d})^{(J+1)\times(J+1)}$ with entries

$$[\vec{\mathcal{M}}_\star]_{kl} := \langle \chi_k, \chi_l\,\vec Q^m_\star\rangle^h_{\Gamma^m}\,, \qquad [\vec A_{Q_\star}]_{kl} := \tfrac12\,\langle |\vec Q^m_\star\,\vec Y^m|^2\,[\chi_k]_s, [\chi_l]_s\rangle^h_{\Gamma^m}\,\vec{Id}_1\,,$$
$$[\vec{\mathcal{M}}_\omega]_{kl} := \langle \chi_k, \chi_l\,\vec Q^m_\omega\rangle^h_{\Gamma^m}\,, \qquad [\vec A_\omega]_{kl} := \tfrac12\,\langle [(\vec Y^m\,.\,\vec\omega^m)^2 - \beta^2]\,[\chi_k]_s, [\chi_l]_s\rangle^h_{\Gamma^m}\,\vec{Id}_1\,.$$

Moreover, let

$$\vec{\mathcal{P}}_W : (\mathbb{R}^d)^{J+1} \to \underline{\mathbb{W}} := \{\vec z \in (\mathbb{R}^d)^{J+1} : \vec z_0 = \vec z_J = \vec 0\}$$

be the orthogonal projection onto $\underline{\mathbb{W}}$.

5.3.1 Clamped conditions

The linear system for the scheme (4.9a,b) can be formulated as: Find $(\vec Y^{m+1}, \delta\vec X^{m+1}) \in \mathbb{R}^{J+1} \times \underline{\mathbb{W}}$ such that

$$\begin{pmatrix} \vec{\mathcal{P}}_W\,\vec A_\phi & -\frac{1}{\tau_m}\,\vec{\mathcal{P}}_W\,\vec{\mathcal{M}}_\star\,\vec{\mathcal{P}}_W \\ \vec{\mathcal{M}}_\star & \vec A_\phi\,\vec{\mathcal{P}}_W \end{pmatrix} \begin{pmatrix} \vec Y^{m+1} \\ \delta\vec X^{m+1} \end{pmatrix} = \begin{pmatrix} \vec{\mathcal{P}}_W\,[(\vec A_\phi - \vec{\mathcal{A}}_\phi)\,\vec Y^m + (\lambda\,\vec A - \vec A_Q)\,\vec X^m + \vec f] \\ -\vec A_\phi\,\vec X^m + \vec d \end{pmatrix}, \tag{5.5}$$

where $\vec f \in (\mathbb{R}^d)^{J+1}$ with $[\vec f]_j = \langle \pi^h_W[|\vec\theta^m|^{-1}\,(\vec Y^m\,.\,\vec\omega^m_d)]\,\vec Q^m_\star\,\vec Y^m, [\chi_j]_s\rangle^h_{\Gamma^m}$, $j = 0 \to J$; and where $\vec d \in (\mathbb{R}^d)^{J+1}$ with $\vec d_0 = -\phi'(\vec\zeta_0)$, $\vec d_J = \phi'(\vec\zeta_1)$ and $\vec d_i = \vec 0$, $i = 1 \to J-1$.

5.3.2 Navier conditions

The linear system for the scheme (4.11a,b) can be formulated as: Find $(\vec Y^{m+1}, \delta\vec X^{m+1}) \in \underline{\mathbb{W}} \times \underline{\mathbb{W}}$ such that

$$\begin{pmatrix} \vec{\mathcal{P}}_W\,\vec A_\phi\,\vec{\mathcal{P}}_W & -\frac{1}{\tau_m}\,\vec{\mathcal{P}}_W\,\vec{\mathcal{M}}_\omega\,\vec{\mathcal{P}}_W \\ \vec{\mathcal{P}}_W\,\vec{\mathcal{M}}_\omega\,\vec{\mathcal{P}}_W & \vec{\mathcal{P}}_W\,\vec A_\phi\,\vec{\mathcal{P}}_W \end{pmatrix} \begin{pmatrix} \vec Y^{m+1} \\ \delta\vec X^{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} \vec{\mathcal{P}}_W\,[(\vec A_\phi - \vec{\mathcal{A}}_\phi)\,\vec Y^m + (\lambda\,\vec A + \vec A_\omega)\,\vec X^m - \vec b] \\ -\vec{\mathcal{P}}_W\,[\vec A_\phi\,\vec X^m + \vec c] \end{pmatrix}, \tag{5.6}$$

where $\vec{b}$, $\vec{c} \in (\mathbb{R}^2)^{J+1}$ with $\vec{b}_j = \langle (\vec{Y}^m \,.\, \vec{\omega}^m + \beta)\, (\vec{Y}^m)^\perp, [\chi_j]_s \rangle_{\Gamma^m}^h$ and $\vec{c}_j = \beta \,\langle \vec{\omega}^m, \chi_j \rangle_{\Gamma^m}^h$, $j = 0 \to J$.

5.4 Solution methods

All of the stated linear systems in this section can be written as

$$\begin{pmatrix} \vec{\mathcal{P}}_X & 0 \\ 0 & \vec{\mathcal{P}}_Y \end{pmatrix} \begin{pmatrix} \vec{A}_Y & -\frac{1}{\tau_m}\, \vec{M} \\ \vec{M} & \vec{A}_X \end{pmatrix} \begin{pmatrix} \vec{\mathcal{P}}_Y & 0 \\ 0 & \vec{\mathcal{P}}_X \end{pmatrix} \begin{pmatrix} \vec{Y}^{m+1} \\ \delta \vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} \vec{\mathcal{P}}_X & 0 \\ 0 & \vec{\mathcal{P}}_Y \end{pmatrix} \begin{pmatrix} \vec{f}_X \\ \vec{f}_Y \end{pmatrix}, \tag{5.7a}$$

$$\vec{\mathcal{P}}_Y\, \vec{Y}^{m+1} = \vec{Y}^{m+1}, \quad \vec{\mathcal{P}}_X\, \delta \vec{X}^{m+1} = \delta \vec{X}^{m+1}, \tag{5.7b}$$

where, apart from the obvious block matrices and right-hand sides, $\vec{\mathcal{P}}_X$ and $\vec{\mathcal{P}}_Y$ are projections onto the Euclidean solution spaces for $\delta \vec{X}^{m+1}$ and $\vec{Y}^{m+1}$, respectively. Of course, for the closed curve systems in §5.1 and §5.2 it holds that $\vec{\mathcal{P}}_X = \vec{\mathcal{P}}_Y = \vec{Id}_J$, and so (5.7a) can be directly solved with a sparse factorization solver. Here we employ the package UMFPACK, see Davis (2004).

In the open curve case, when the projections $\vec{\mathcal{P}}_X$ or $\vec{\mathcal{P}}_Y$ may have a nontrivial kernel, it is possible to solve the system (5.7a) with a preconditioned BiCGSTAB iterative solver, see e.g. Barrett *et al.* (1994), where a natural preconditioner is

$$\begin{pmatrix} \vec{\mathcal{P}}_Y & 0 \\ 0 & \vec{\mathcal{P}}_X \end{pmatrix} \begin{pmatrix} \vec{A}_Y & -\frac{1}{\tau_m}\, \vec{M} \\ \vec{M} & \vec{A}_X \end{pmatrix}^{-1} \begin{pmatrix} \vec{\mathcal{P}}_X & 0 \\ 0 & \vec{\mathcal{P}}_Y \end{pmatrix} \tag{5.8}$$

if this is well-defined, and otherwise (5.8) with $\vec{A}_X$ and $\vec{A}_Y$ replaced by $\operatorname{diag}(\vec{A}_X)$ and $\operatorname{diag}(\vec{A}_Y)$, respectively. In each case, the inverses in (5.8) may be computed with the help of UMFPACK.

A more efficient solution method, which has the additional advantage that it is guaranteed to find the unique solution to (5.7a,b) even when the system is very badly conditioned, is to find a solution to the underdetermined system (5.7a) over $(\mathbb{R}^d)^{J+1} \times (\mathbb{R}^d)^{J+1}$, and then to orthogonally project that solution back to the solution space with the help of $\vec{\mathcal{P}}_X$ and $\vec{\mathcal{P}}_Y$, where we know that the solution is unique. Finding a solution of the singular system (5.7a) is done with the help of the sparse QR factorization package SuiteSparseQR, see Davis (2011). We used this latter method throughout in our numerical experiments.

6 Numerical results

Throughout the numerical experiments for closed curves we take $\vec{Y}^0 = \vec{\kappa}_\phi^0$, where $\vec{\kappa}_\phi^0 \in \underline{V}_0^h$ is the usual (anisotropic) discrete curvature vector on Γ^0 defined by

$$\langle \vec{\kappa}_\phi^0, \vec{\eta} \rangle_{\Gamma^0}^h + \langle \vec{\Phi}(\vec{X}_s^0)\, \vec{X}_s^0, \vec{\eta}_s \rangle_{\Gamma^0} = 0 \qquad \forall\ \vec{\eta} \in \underline{V}_0^h\,.$$

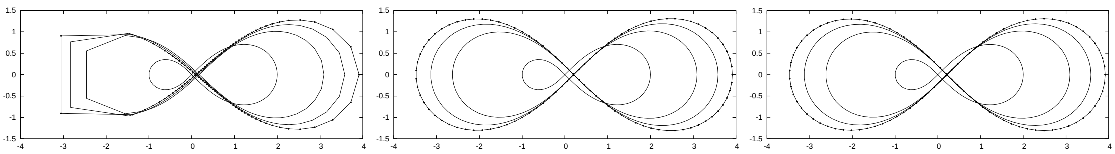

Figure 1: $(d = 2)$ Evolution for (4.8a,b), (4.7a,b) and (4.1a,b). Plots of $\vec{X}(t)$, $t = 0, 2, 4, 6$.

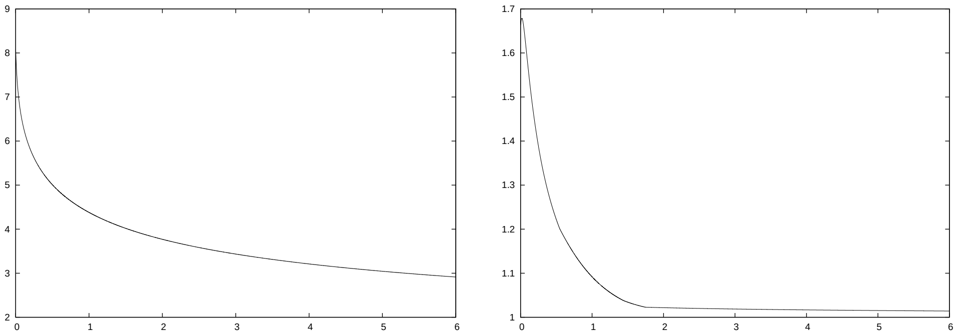

Figure 2: $(d = 2)$ Plots of the energy $E_0^h(\Gamma^m, \vec{Q}^m\,\vec{Y}^{m+1})$ (left) and the element ratio (right) for (4.1a,b).

Unless otherwise stated we fix $\lambda = 0$ throughout, and use the discretization parameters $J = 100$, $\tau = 10^{-3}$. For later purposes, we define

$$\vec{X}(t) := \tfrac{t - t_{m-1}}{\tau}\,\vec{X}^m + \tfrac{t_m - t}{\tau}\,\vec{X}^{m-1} \qquad t \in [t_{m-1}, t_m] \quad m \geq 1.$$

Finally, we stress that no remeshing was used for any of the experiments presented in this section.

6.1 Isotropic elastic flow

First, we compare the existing schemes in the literature; that is (4.8a,b) from Deckelnick and Dziuk (2009) and (4.7a,b) from Barrett, Garcke, and Nürnberg (2010b) to our new approximation (4.1a,b). As the initial curve we choose a $2:1$ lemniscate, and let the time discretization parameters be $\tau = 5 \times 10^{-4}$ and $T = 6$. In Figure 1 we compare the results for the schemes (4.8a,b), (4.7a,b) and (4.1a,b). It is clear that the lack of tangential motion in the scheme (4.8a,b) in this case results in very non-uniform element sizes. On the other hand, the results in Figure 2, where we show a plot of the discrete energy $E_0^h(\Gamma^m, \vec{Q}^m\,\vec{Y}^{m+1})$ and of the $r := h_{\vec{X}^m}/\ell_{\vec{X}^m}$, with $h_{\vec{X}^m} := \max_{j=1\to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|$ and $\ell_{\vec{X}^m} := \min_{j=1\to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|$, over time, show that we get close to equidistribution in practice for the scheme (4.1a,b).

In Figure 3 we present some results for positive values of λ, so that growth in the length of the curve is penalized. In particular, we set $\lambda = \frac{1}{2}$, 2 and 8. On recalling (1.6) and (1.7), we observe that the results in Figure 3 may also be interpreted as approximations for the gradient flow of $\widetilde{E}_\beta(\Gamma, \varkappa)$ with $\beta = 1$, 2 and 4, respectively. In particular, the minimizing steady states are given by circles of radius 1, $\frac{1}{2}$ and $\frac{1}{4}$, respectively.

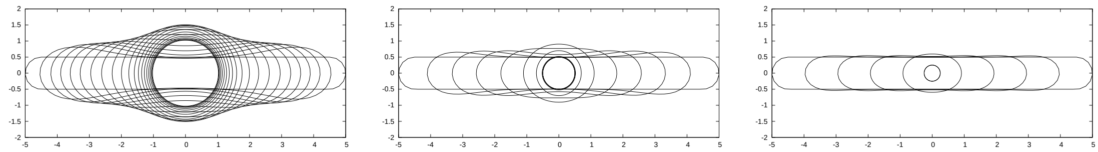

Figure 3: ($d = 2$) Evolutions for $\lambda = \frac{1}{2}, 2, 8$. Plots show $\vec{X}(t)$ at times $t = 0, 0.5, \ldots, 10$, $t = 0, 0.2, \ldots, 2$, and $t = 0, 0.05, \ldots, 0.3$, respectively.

6.2 Anisotropic elastic flow

In this subsection, we show some numerical experiments for our approximation (4.4a,b). We begin with a computation for the similarity solution from Theorem 2.1 in the case $d = 2$. To this end, we fix $\phi(\vec{p}) = [\varepsilon^2\, p_1^2 + p_2^2]^{\frac{1}{2}}$, i.e. ϕ is of the form (3.24) with $L = 1$ and $\vec{G}_1 = \vec{G} := \left(\begin{smallmatrix}\varepsilon^2 & 0\\ 0 & 1\end{smallmatrix}\right)$. Hence the Wulff shape $\mathcal{W}_\phi$ is given by an ellipse. On recalling (4.3), we obtain that

$$\begin{aligned}\sigma(\vec{X}^m_s) &= \phi(\vec{X}^m_s)\, [\vec{\nu}^m \,.\, \phi''(\vec{X}^m_s)\, \vec{\nu}^m] = \vec{\nu}^m \,.\, \left[\vec{I}d - [\phi(\vec{X}^m_s)]^{-2}\, (\vec{G}\,\vec{X}^m_s) \otimes \vec{X}^m_s\right] \vec{G}\,\vec{\nu}^m \\ &= \vec{\nu}^m \,.\, \vec{G}\,\vec{\nu}^m - (\vec{X}^m_s \,.\, \vec{G}\,\vec{X}^m_s)^{-1}\, (\vec{X}^m_s \,.\, \vec{G}\,\vec{\nu}^m)^2\,.\end{aligned}$$

On incorporating the term $[\sigma(\vec{X}^m_s)]^{-1}$ into the first term on the left-hand side of (4.4a), we can approximate the flow (2.20) by this modification of (4.4a,b); and similarly for the scheme (4.12a,b).

We compare the solutions from our approximations (4.4a,b) and (4.12a,b) with the true solution

$$\vec{x}(\rho, t) = (1 + 2\,t)^{\frac{1}{4}}\, (\cos g(\rho), \varepsilon\, \sin g(\rho))^T\,, \tag{6.1}$$

where $g(\rho) = 2\pi\rho + 0.1\, \sin(2\pi\rho)$ in order to make the initial distribution of nodes non-uniform. Here we compute the error $\|\vec{X} - \vec{x}\|_{L^\infty} := \max_{m=1\to M} \|\vec{X}^m - \vec{x}(\cdot, t_m)\|_{L^\infty}$, where $\|\vec{X}^m - \vec{x}(\cdot, t_m)\|_{L^\infty} := \max_{j=1\to J} \min_{\rho\in J} |\vec{X}^m(q_j) - \vec{x}(\rho, t_m)|$, between $\vec{X}$ and the true solution $\vec{x}$ on the interval $[0, T]$ by employing a Newton method. The numbers in Table 1, where we report on the errors for $T = 1$ and $\tau = 0.5\, h^2$, indicate a convergence rate for the errors of $O(h^2)$.

Next, we reproduce the evolutions for the flow (2.18) that were presented in Figures 20, 21 and 22 in Barrett, Garcke, and Nürnberg (2010b) for the scheme (4.7a,b). First we repeat the experiment in Figure 1, but now for our scheme (4.4a,b) and for the anisotropic energy densities $\phi(\vec{p}) = \sqrt{0.25\, p_1^2 + p_2^2}$ and $\phi(\vec{p}) = \sqrt{p_1^2 + 0.25\, p_2^2}$, see Figure 4. We observe that in each case the lemniscate clearly aligns itself with the chosen anisotropy. There are only minor differences between the results shown in Figure 4 and the plots in Figure 20 in Barrett, Garcke, and Nürnberg (2010b). The next experiments are for a trefoil knot in $\mathbb{R}^3$, and in particular the initial curve is given by

$$\vec{x}(\rho, 0) = ((2 + \cos(6\,\pi\,\rho))\cos(4\,\pi\,\rho), (2 + \cos(6\,\pi\,\rho))\sin(4\,\pi\,\rho), \sin(6\,\pi\,\rho))^T\,, \quad \rho \in I\,.$$

	$\varepsilon = 0.5$		$\varepsilon = 0.1$	
J	(4.4a,b)	(4.12a,b)	(4.4a,b)	(4.12a,b)
128	8.8676e-04	9.0700e-04	1.1383e-02	6.8666e-03
256	2.2217e-04	2.2734e-04	2.6458e-03	1.7818e-03
512	5.5621e-05	5.6923e-05	6.5288e-04	4.5071e-04
1024	1.3917e-05	1.4243e-05	1.6286e-04	1.1312e-04

Table 1: Absolute error $\|\vec{X} - \vec{x}\|_{L^\infty}$ for the test problems.

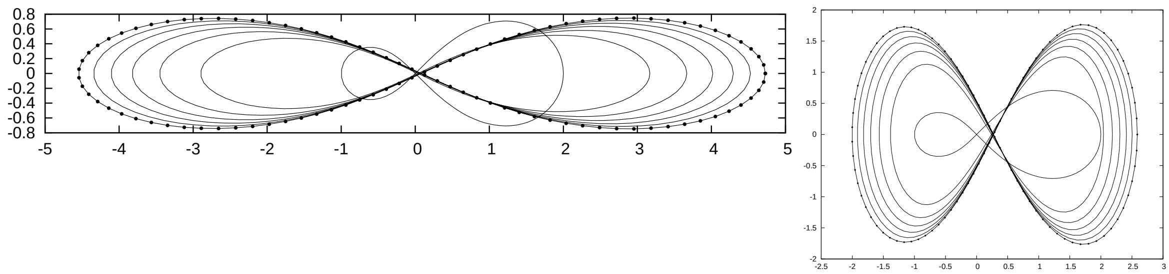

Figure 4: ($d = 2$, $\phi(\vec{p}) = \sqrt{0.25\,p_1^2 + p_2^2}$ (left) and $\phi(\vec{p}) = \sqrt{p_1^2 + 0.25\,p_2^2}$ (right)) Anisotropic elastic flow of a lemniscate. Plots of $\vec{X}(t)$, $t = 0, 1, \ldots, 6$.

We use the anisotropic energy density $\phi(\vec{p}) = \sqrt{0.75\,p_1^2 + p_2^2 + p_3^2}$. The numerical results for our scheme (4.4a,b) are shown in Figure 5. The same computation for the density $\phi(\vec{p}) = \sqrt{p_1^2 + 0.75\,p_2^2 + p_3^2}$ can be seen in Figure 6. The reader will note that the presented results differ significantly from the ones shown in Barrett, Garcke, and Nürnberg (2010b, Figs. 21, 22). In particular, the final coordinate planes in which the solutions settle down are different. It has since come to our attention that the results in Barrett, Garcke, and Nürnberg (2010b, Figs. 20, 21, 22) are based on a faulty numerical implementation of the operator $\vec{\mathcal{P}}_{\vec{\tau}}\,\vec{\mathcal{A}}_\phi\,\vec{\mathcal{P}}_{\vec{\tau}}$ in the anisotropic setting, recall (5.4). The correct implementation of the scheme (4.7a,b) yields numerical results that are graphically indistinguishable from the ones in Figures 4, 5 and 6. Hence the three incorrect plots in Barrett, Garcke, and Nürnberg (2010b) and the last paragraph in §5.6 in that article should be ignored.

In order to complete the picture, we present in Figure 7 the same computation for $\phi(\vec{p}) = \sqrt{p_1^2 + p_2^2 + 0.75\,p_3^2}$.

6.3 Initial boundary value problems

In all of the following experiments we consider an open curve attached to the endpoints of the unit interval $[0, 1]$, i.e. we set $\vec{\alpha}_0 = \vec{0}$ and $\vec{\alpha}_1 = \vec{e}_1$. Unless otherwise stated we fix $d = 2$ and choose the isotropic case (2.11) with $\lambda = 0$. The discretization parameters are $J = 100$ and $\tau = 10^{-3}$ throughout.

We recall that several theoretical results on the steady state solutions of the (isotropic) flows considered in this subsection have been derived in Deckelnick and Grunau (2007).

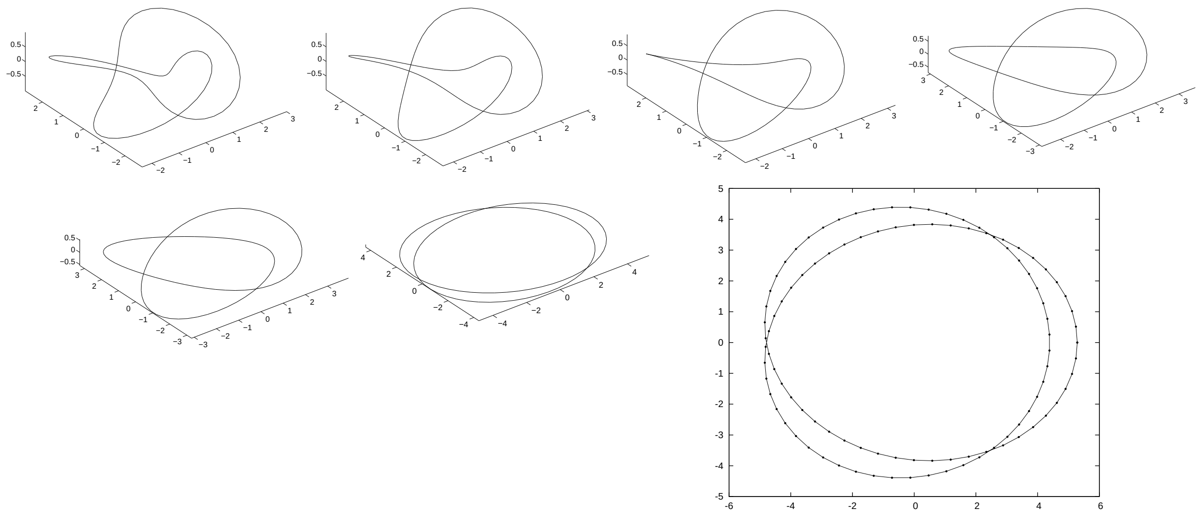

Figure 5: ($\phi(\vec{p}) = \sqrt{0.75\, p_1^2 + p_2^2 + p_3^2}$) Plots of $\vec{X}(t)$ at times $t = 0,\ 0.5,\ 2,\ 10,\ 20,\ 200$. A $2d$ plot of $\vec{X}(T)$ in the $x_1 - x_2$ plane below.

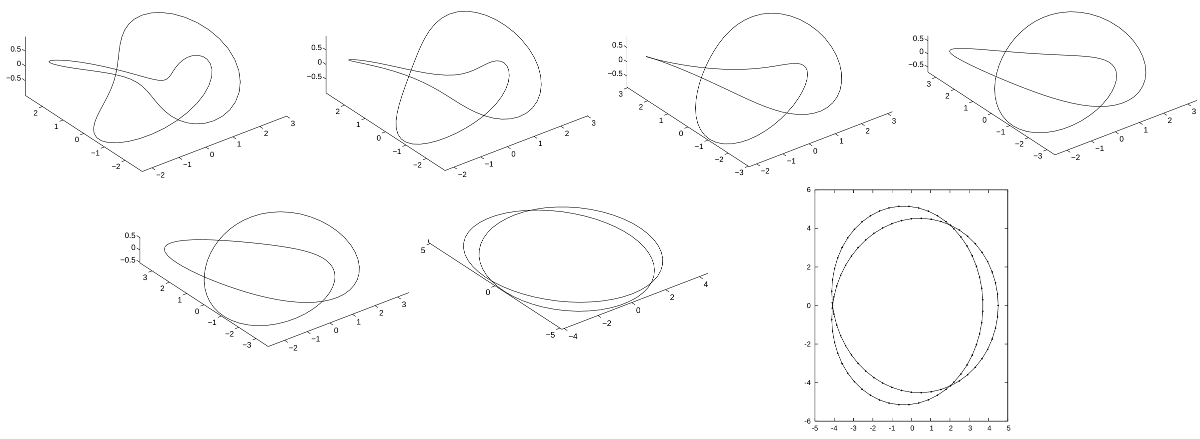

Figure 6: ($\phi(\vec{p}) = \sqrt{p_1^2 + 0.75\, p_2^2 + p_3^2}$) Plots of $\vec{X}(t)$ at times $t = 0,\ 0.5,\ 2,\ 10,\ 20,\ 200$. A $2d$ plot of $\vec{X}(T)$ in the $x_1 - x_2$ plane below.

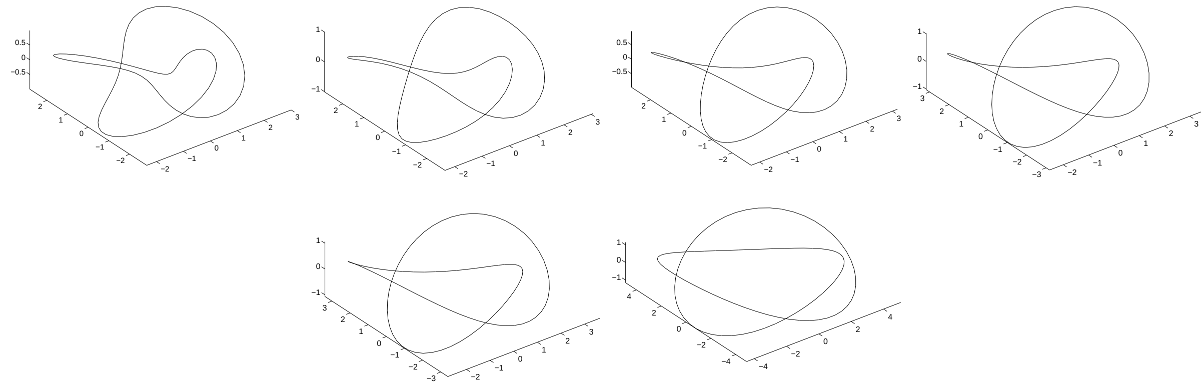

Figure 7: ($\phi(\vec{p}) = \sqrt{p_1^2 + p_2^2 + 0.75\, p_3^2}$) Plots of $\vec{X}(t)$ at times $t = 0,\ 0.5,\ 2,\ 10,\ 20,\ 200$.

In particular, the authors in that paper considered the case where the steady state is a smooth graph solution.

6.3.1 Clamped conditions

As an initial value for $\vec{Y}^0 \in \underline{V}^h$ we choose $\vec{Y}^0 = \pi^h[\vec{Q}^0\, \vec{\kappa}_\phi^0]$, where $\vec{\kappa}_\phi^0 \in \underline{V}^h$ is such that

$$\langle \vec{\kappa}_\phi^0, \vec{\eta} \rangle_{\Gamma^0}^h + \langle \vec{\Phi}(\vec{X}_s^0)\, \vec{X}_s^0, \vec{\eta}_s \rangle_{\Gamma^0} = \phi'(\vec{\zeta}_1)\,.\,\vec{\eta}(1) - \phi'(\vec{\zeta}_0)\,.\,\vec{\eta}(0) \qquad \forall\ \vec{\eta} \in \underline{V}^h\,.$$

For the parameters $\vec{\zeta}_i$, $i = 0 \to 1$, in (2.43) we always set

$$\vec{\zeta}_0 = \begin{pmatrix} \cos\delta_0 \\ \sin\delta_0 \end{pmatrix} \quad \text{and} \quad \vec{\zeta}_1 = \begin{pmatrix} \cos\delta_1 \\ -\sin\delta_1 \end{pmatrix}, \tag{6.2}$$

where $\delta_i \in [0, 2\,\pi)$, $i = 0 \to 1$, describe the contact angles that the curve makes with the x_1-axis at both ends. We remark that for the clamped conditions (6.2) with $\delta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $i = 0 \to 1$, it was shown in Deckelnick and Grunau (2007, Theorem 2) that among all smooth graphs over $[0, 1]$, there exists a unique minimizer of the elastic energy $E_0(\Gamma, \varkappa)$. In all our numerical experiments with $\delta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $i = 0 \to 1$, we did observe a numerical steady state that is a graph.

At first, we consider the symmetric case (6.2) with $\delta_0 = -\delta_1$. For the numerical results in Figure 8, we start with a unit semicircle and use our fully discrete scheme (4.9a,b). In all our runs for $\delta_0 < \frac{\pi}{2}$ the evolution finds a numerical steady state, i.e. a discrete approximation to the known minimizer from Deckelnick and Grunau (2007, Theorem 2). The run for $\delta_0 = 45°$ in Figure 8 is such an example. For contact angles $\delta_0 > \frac{\pi}{2}$ we can observe unlimited growth, as can be seen on the right of Figure 8.

For contact angles $\delta_0 > 120°$, our algorithm could not integrate the evolution starting from a semicircle. However, when starting with the solution $\vec{X}(1)$ obtained from above for $\delta_0 = 115°$, then also evolutions for larger contact angles can be computed. In Figure 9 we show the evolutions for $\delta_0 = -\delta_1 = 135°,\ 180°,\ 225°,\ 250°$. For the latter two evolutions

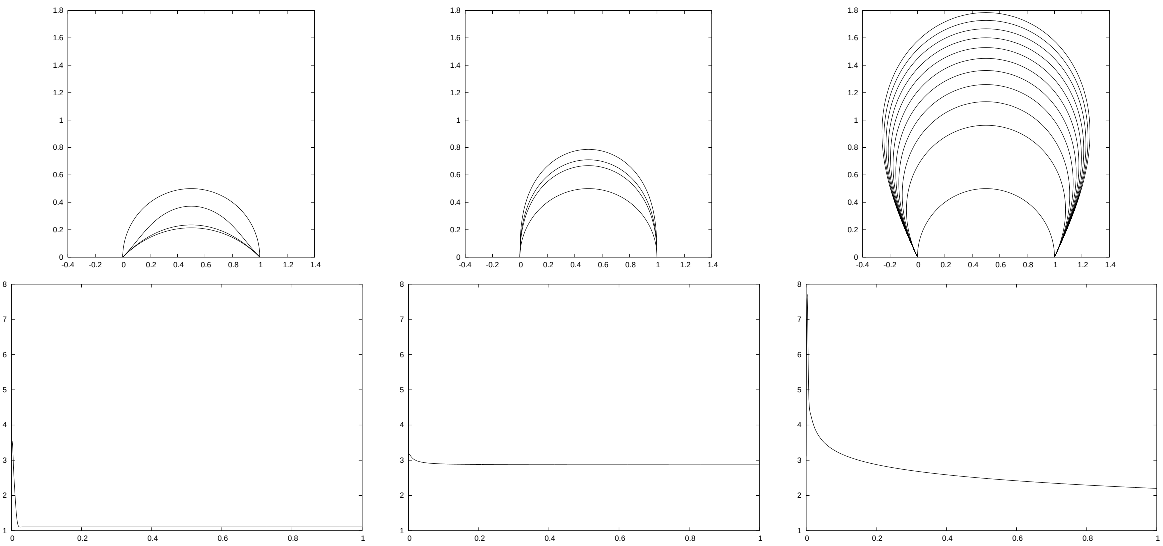

Figure 8: ($d = 2$) Evolutions for (4.9a,b) with (6.2) and $\delta_0 = -\delta_1 = 45°$, $90°, 115°$. Plots show $\vec{X}(t)$ at times $t = 0, 0.01, 0.02, 1$, at times $t = 0, 0.1, 0.2, 1$ and at times $t = 0, 0.1, \ldots, 1$, respectively. Plots of the discrete energy $E_0^h(\Gamma^m, \vec{Q}_\star^m\, \vec{Y}^{m+1})$ below.

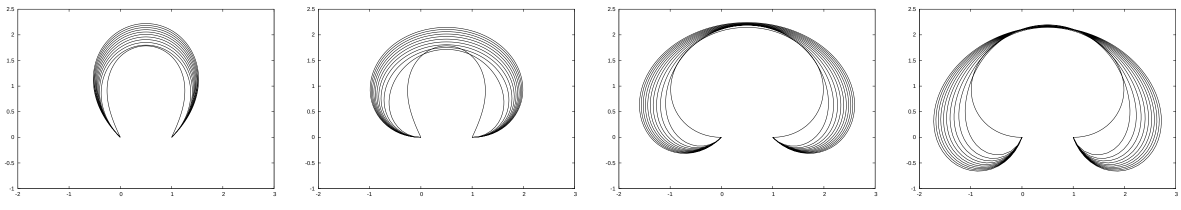

Figure 9: ($d = 2$) Evolutions for (4.9a,b) with (6.2) and $\delta_0 = -\delta_1 = 135°$, $180°$, $225°$, $250°$. Plots show $\vec{X}(t)$ at times $t = 0, 0.1, \ldots, 1$.

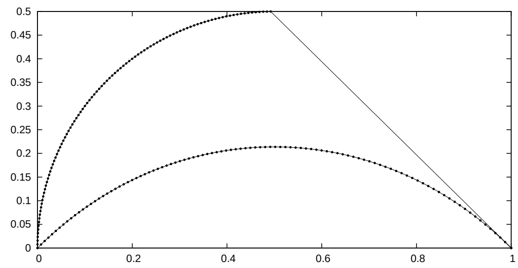

Figure 10: ($d = 2$) Evolution for (4.9a,b) with (6.2) and $\delta_0 = -\delta_1 = 45°$. Plots of $\vec{X}(0)$ and $\vec{X}(10)$.

we used the final curve $\vec{X}(1)$ from the run with $\delta_0 = 180°$ as initial data. The continued growth in the curves in Figure 8 and 9 is easily explained with the scaling behaviour of the elastic energy, and it also complements the theoretical results from Deckelnick and Grunau (2007, Theorem 2).

Next we present an experiment that demonstrates the equidistribution property proven for the semidiscrete schemes in e.g. Theorem 3.7. To this end, we repeated the first computation in Figure 8, but now use a very nonuniform polygonal approximation of the initial semi-circle. In particular, Γ^0 consists of a sequence of vertices on the left half of the semi-circle together with the vertex at $\vec{\alpha}_1$. The numerical results for our approximation (4.9a,b) can be seen in Figure 10, where we observe that at time $T = 10$ the polygonal curve Γ^M is close to being equidistributed.

In addition, we also present some evolutions for the even choice (6.2) with $\delta_0 = \delta_1$. Starting from a straight line, we present computations for $\delta_0 = \delta_1 = 15°$, $25°$, $45°$, $65°$, $80°$, $90°$, where when necessary we used the final solution of the previous run as initial data for the next larger δ_0. Here we recall from Deckelnick and Grunau (2007, Theorem 4), that for $\delta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the existence of a steady state solution is guaranteed. Interestingly, when we increase δ_0 to a value larger than $90°$, then unlimited growth of the curve can in general be observed again. See Figure 12 for an example with $\delta_0 = \delta_1 = 100°$. Here we take as initial data the final solution from Figure 11 for $\delta_0 = 90°$. In Figure 13 we present a numerical steady state for the flow with a $\delta_0 = 90°$ contact angle prescribed on the left and a $\delta_1 = 0°$ contact angle at the right.

In Figure 14 we show evolutions for some nonzero values of λ. We set $\delta_0 = -\delta_1 = 75°$ in (6.2) and let $\lambda = -2$, 0 or 20. As expected, the length of the curve is increasing/decreasing for negative/positive λ compared to the minimizer of the elastic energy, i.e. the steady state for the flow with $\lambda = 0$.

For the first experiment in three dimensions we set $\vec{\zeta}_0 = \frac{1}{2}(1, 1, 2^{\frac{3}{2}})^T$ and $\vec{\zeta}_1 = \frac{1}{2}(1, -1, 2^{\frac{3}{2}})^T$ in (4.9a,b). The numerical results for the same initial curve as in Figure 8 can be seen in Figure 15, where we show results for $\lambda = 0$ and $\lambda = 1$. For the

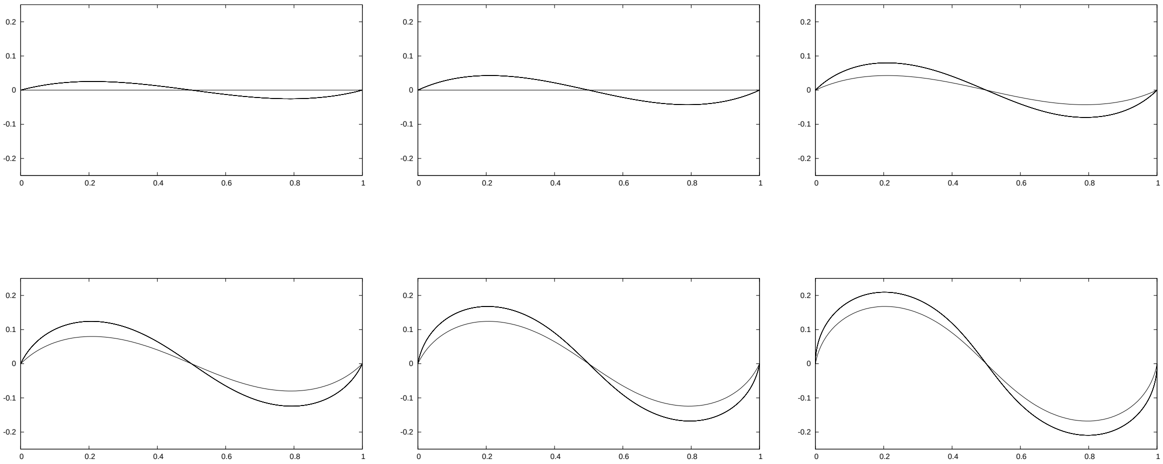

Figure 11: ($d = 2$) Evolutions for (4.9a,b) with (6.2) and $\delta_0 = \delta_1 = 15°, 25°, 45°, 65°, 80°, 90°$. Plots show $\vec{X}(0)$ and $\vec{X}(1)$.

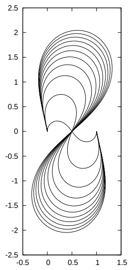

Figure 12: ($d = 2$) Evolution for (4.9a,b) with (6.2) and $\delta_0 = \delta_1 = 100°$. The plot shows $\vec{X}(t)$ at times $t = 0, 0.1, \ldots, 1$.

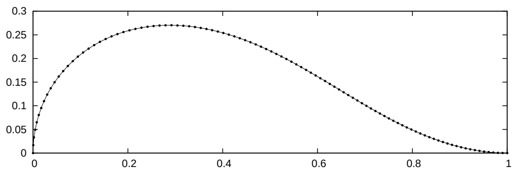

Figure 13: ($d = 2$) Numerical steady state for (6.2) with $\delta_0 = 90°$ and $\delta_1 = 0°$.

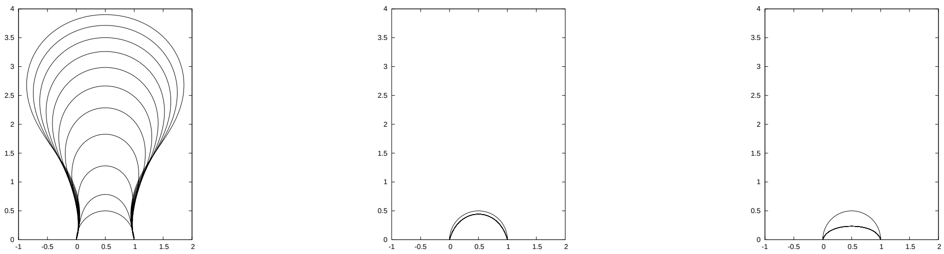

Figure 14: ($d = 2$) Evolutions for (4.9a,b) with (6.2) and $\delta_0 = -\delta_1 = 75°$ for $\lambda = -2,\ 0,\ 20$. Plots show $\vec{X}(t)$ at times $t = 0, 0.1, \ldots, 1$.

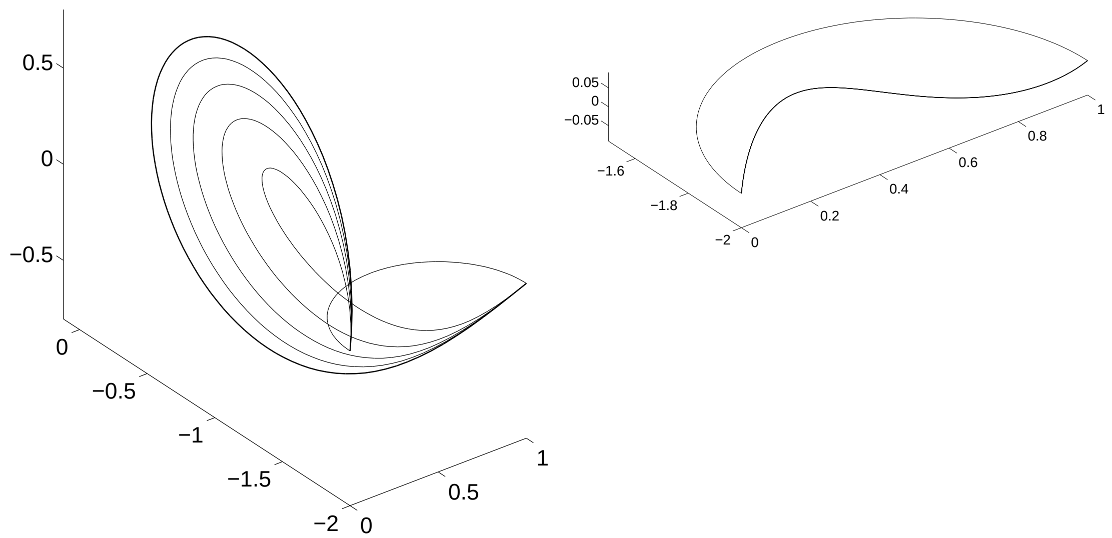

Figure 15: ($d = 3$) Evolutions for (4.9a,b) with $\vec{\zeta}_0 = \frac{1}{2}\,(1, 1, 2^{\frac{3}{2}})^T$, $\vec{\zeta}_1 = \frac{1}{2}\,(1, -1, 2^{\frac{3}{2}})^T$ and $\lambda = 0$ (left) or $\lambda = 1$ (right). Plots show $\vec{X}(t)$ at times $t = 0, 0.2, \ldots, 1$.

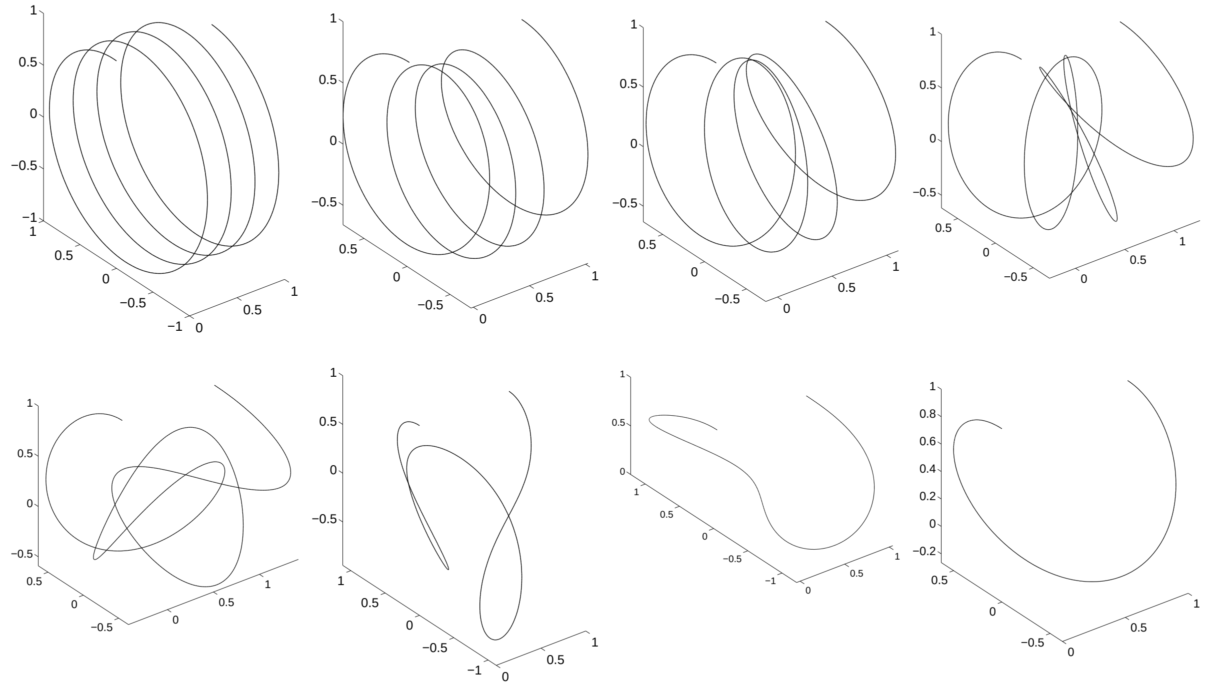

Figure 16: ($d = 3$) Evolution for (4.9a,b) with $\vec{\zeta}_0 = \vec{\zeta}_1 = \vec{e}_2$ and $\lambda = 1$. Plots show $\vec{X}(t)$ at times $t = 0, 2, 3, \ldots, 7, 10$.

helix experiment in Figure 16 we chose $\vec{\zeta}_0 = \vec{\zeta}_1 = \vec{e}_2$ and set $\lambda = 1$. The helix itself is parameterized by

$$\vec{x}_0(\rho) = (\rho, \sin(8\,\pi\rho), \cos(8\,\pi\,\rho))^T\,, \quad \rho \in [0,1]\,.$$

Under our approximation of the elastic flow (2.42) with clamped boundary conditions (2.43) the helix unravels and attains a numerical steady state which we conjecture is an approximation of the global minimizer for (1.2) among all curves satisfying (2.43). For this experiment, we used the fine spatial discretization parameter $J = 512$.

Next we perform some experiments for the length preserving variant of (4.9a,b), recall Remark 4.5. First we repeat the experiments from Figure 8, but now for the finer time step $\tau = 10^{-4}$. In Figure 17 we show the obtained numerical steady states for certain boundary contact angles. In each case, the presented polygonal curve is an approximation to the elasticae of length $|\Gamma(0)| = \frac{\pi}{2}$ for the given boundary conditions. The length preserving variant of the helix experiment in Figure 16 can be seen in Figure 18. Here we used the discretization parameters $J = 512$ and $\tau = 10^{-2}$. Finally, we also show a numerical approximations for the flow (2.42) with the homogeneous boundary conditions (2.44), i.e. $\vec{\beta} = \vec{0}$, where $\lambda(t)$ in (2.42) is chosen such that length is preserved. For this computation we use the length preserving variant of the scheme discussed at the end of §4.3.1. See Figure 19 for the results of this numerical computation, where we used the fine spatial discretization parameter $J = 512$. We note that the curve displayed in the final plot in Figure 19 lies in a two-dimensional hyperplane. We conjecture that stable steady states of

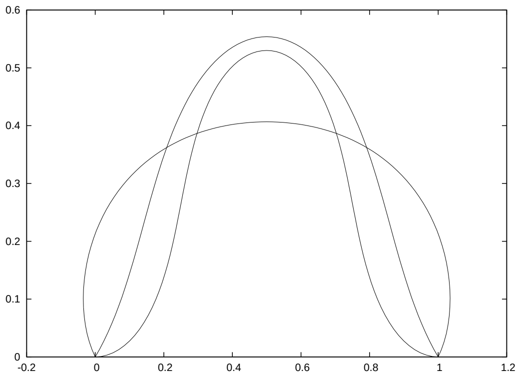

Figure 17: ($d = 2$) Numerical steady states for the length preserving variant of (4.9a,b) with (6.2) and $\delta_0 = -\delta_1 = 0°$, $45°$, $130°$. Here $|\Gamma^M| \approx \frac{\pi}{2}$.

the length preserving elastic flow with homogeneous Navier boundary conditions always lie in a two-dimensional hyperplane.

6.3.2 Navier conditions

As an initial value for $\vec{Y}^0 \in \underline{W}^h$ we choose the solution of

$$\langle \vec{Y}^0, \vec{\eta} \rangle_{\Gamma^0}^h + \langle \vec{\Phi}(\vec{X}_s^0)\, \vec{X}_s^0, \vec{\eta}_s \rangle_{\Gamma^0} = -\beta \, \langle \vec{\omega}^0, \vec{\eta} \rangle_{\Gamma^0}^h \qquad \forall\, \vec{\eta} \in \underline{W}^h\,.$$

It was shown in Deckelnick and Grunau (2007, Theorem 1) that a stable stationary solution in the shape of a smooth graph exists for the flow (2.42) with (2.46) if and only if $|\beta| < \beta_{\max} := 1.343799725...$. We now investigate this result numerically. To this end, we start the flow with the initial curve $\Gamma(0) = [0, 1]$ and observe the flow for increasing values of $|\beta|$. The results for our fully discrete finite element approximation (4.11a,b) are shown in Figure 20, where as expected we observe continued growth in the curve only for $|\beta| > \beta_{\max}$.

We should note that when we start with an initial curve far "above" the steady state profile, then, for $|\beta|$ sufficiently large, the flow does not settle on this stationary solution. Instead, it will continuously decrease the energy by expanding the length of the curve continuously. We show this behaviour in Figure 21 for an example with $\beta = -1 > -\beta_{\max}$, where we use a unit semicircle as initial data. For this experiment, we use the fine discretization parameters $J = 512$ and $\tau = 10^{-4}$.

Conclusions

On utilizing the ideas in Deckelnick and Dziuk (2009) and Barrett, Garcke, and Nürnberg (2010b), we introduced fully practical approximations of isotropic elastic flow of closed

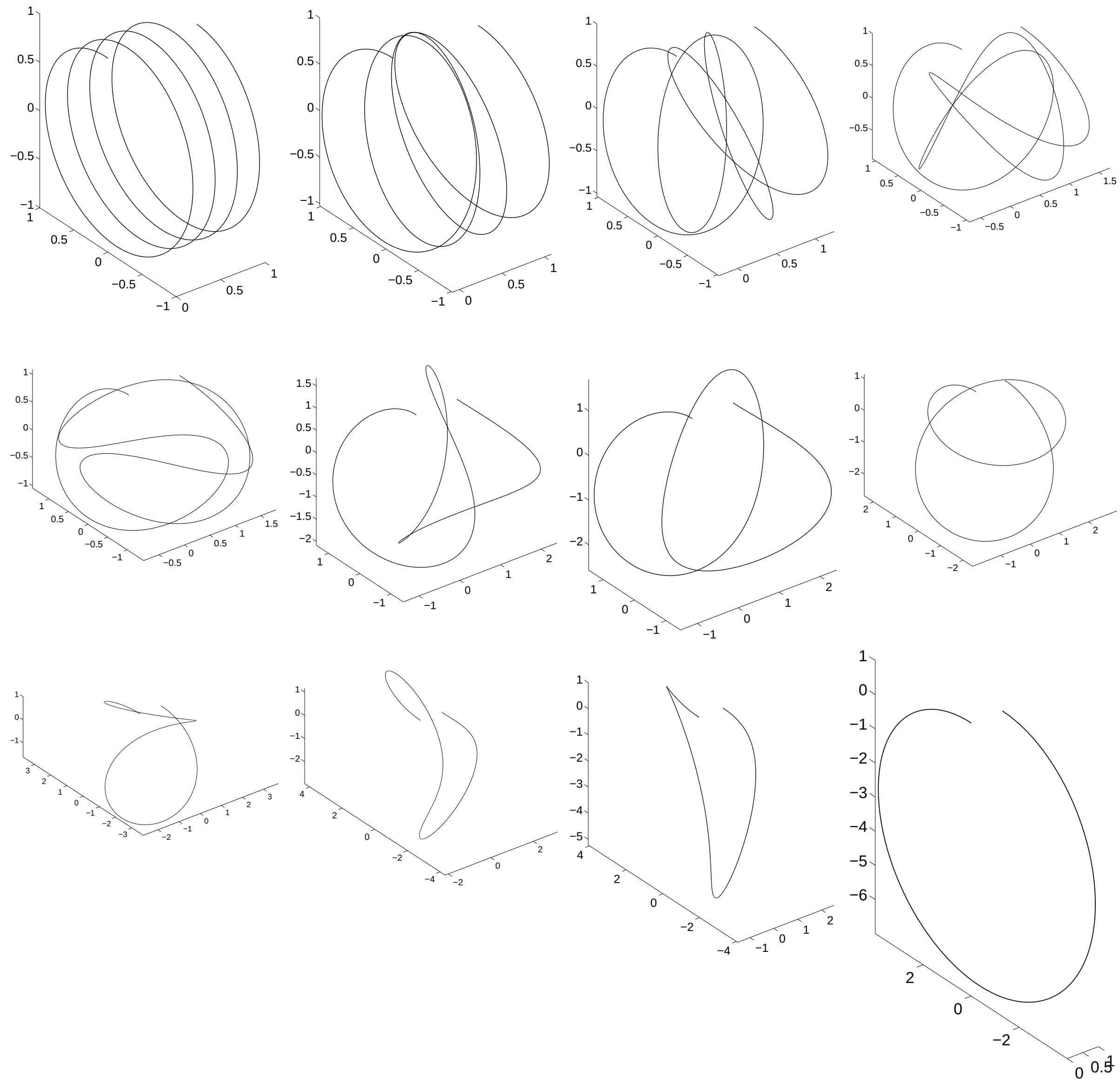

Figure 18: $(d = 3)$ Evolution for the length preserving variant of (4.9a,b) with $\vec{\zeta}_0 = \vec{\zeta}_1 = \vec{e}_2$. Plots show $\vec{X}(t)$ at times $t = 0, 15, 20, 25, 30, 40, 50, 100, 200, 300, 400, 1500$.

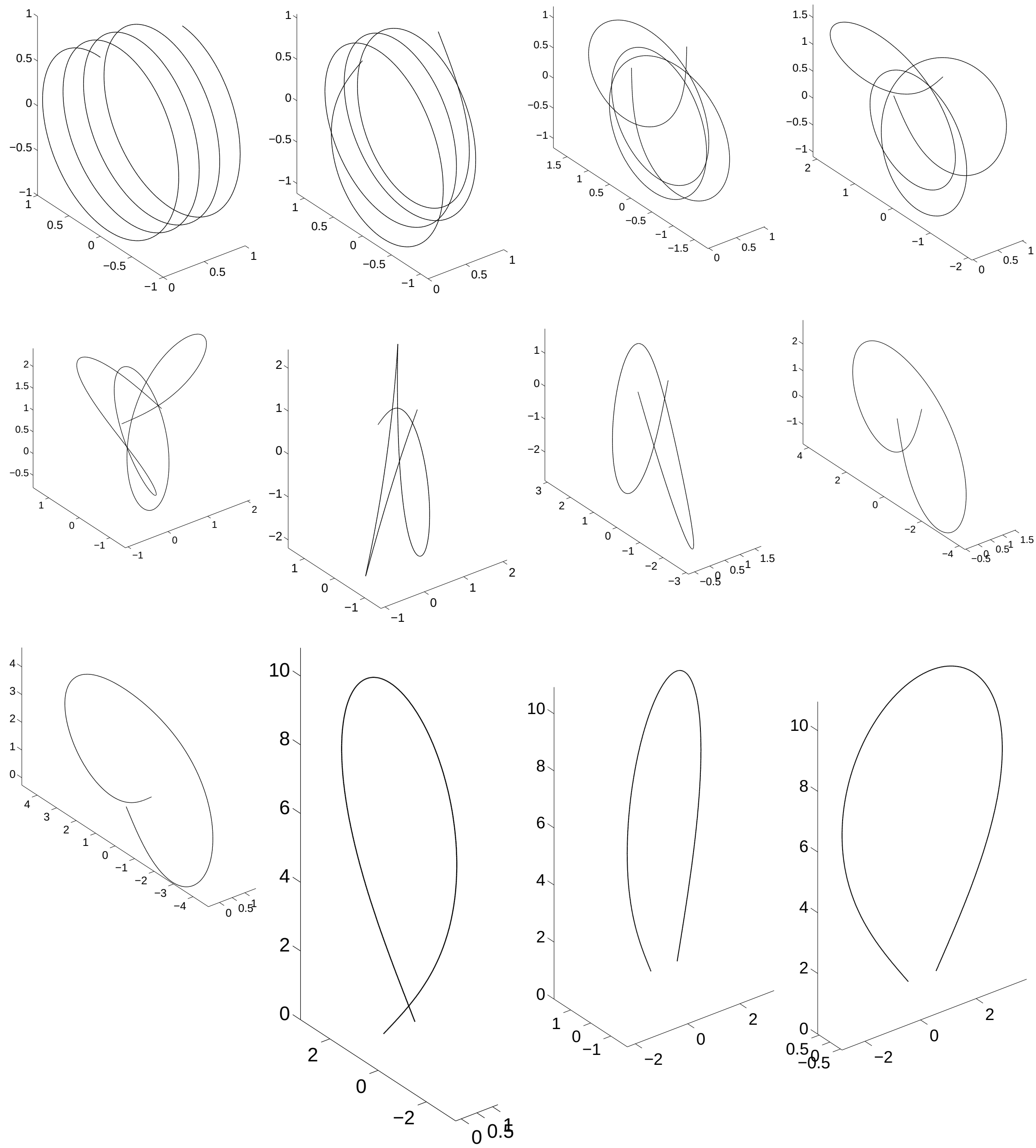

Figure 19: $(d = 3)$ Approximation of the length preserving variant of (2.42) with (2.44) with $\vec\beta = \vec 0$. Plots show $\vec X(t)$ at times $t = 0, 1, 5, 10, 20, 50, 100, 200, 300, 1000, 5000, 10000$.

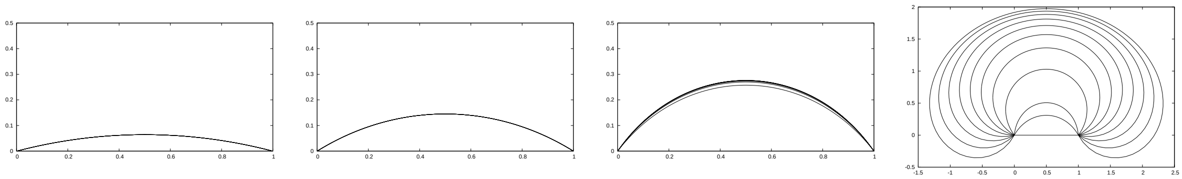

Figure 20: ($d = 2$) Evolutions of (4.11a,b) for $\beta = -0.5, -1.0, -1.34, -1.4$. Plots show $\vec{X}(t)$ at times $t = 0, 0.2, \ldots, 2$.

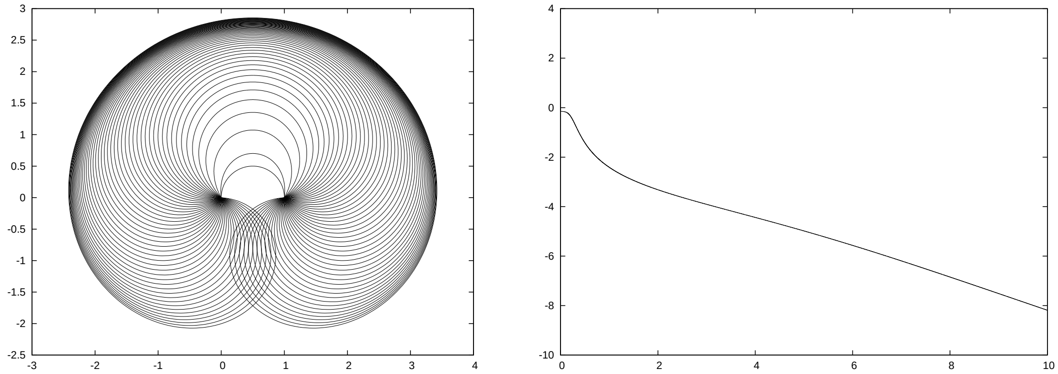

Figure 21: ($d = 2$) Evolution of (4.11a,b) and the energy $\widehat{E}^h_{\beta,0}(\Gamma^m, \vec{Y}^m \,.\, \vec{\omega}^m + \beta)$ for $\beta = -1$. Plot shows $\vec{X}(t)$ at times $t = 0, 0.2, \ldots, 10$.

curves in $\mathbb{R}^d$, $d \geq 2$, for which semidiscrete variants can be proven to be stable and to have an equidistribution property. These ideas extend to anisotropic flows, where the mesh equidistribution property is replaced with a nontrivial criterion that depends on the given anisotropy, yielding the first stable semidiscrete schemes for anisotropic elastic flow in the literature. For a single open curve, we considered the isotropic and anisotropic elastic flow under clamped and Navier boundary conditions, respectively. As before, the semidiscrete variants of these schemes are stable and satisfy an equidistribution property. In addition, as far as we are aware, these are the first approximations of these initial boundary value problems in the literature. The presented ideas can be extended to introduce similarly stable finite element approximations of the geodesic elastic flow, where the evolving curve is constrained to lie on a fixed two-dimensional manifold. This work is currently in progress; see Barrett, Garcke, and Nürnberg (2011a). Moreover, on utilizing ideas in Dziuk (2008), the techniques presented here for curves can be generalized to consider stable approximations for the Willmore flow of two-dimensional hypersurfaces in $\mathbb{R}^3$, which maintain the good mesh properties of the schemes in Barrett, Garcke, and Nürnberg (2008b). This work is also currently in progress; see Barrett, Garcke, and Nürnberg (2011b). Finally, extending the ideas presented here for a single open curve to the elastic flow of curve networks, where several open curves meet at junction points is another avenue for future research. It is hoped that the latter can be extended to the Willmore flow of surface clusters and surfaces with line energy, which play an important role in the modelling of biomembranes with locally varying physical properties.

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