Strongly elliptic pseudodifferential equations on the sphere with radial basis functions

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Abstract

Spherical radial basis functions are used to define approximate solutions to strongly elliptic pseudodifferential equations on the unit sphere. These equations arise from geodesy. The approximate solutions are found by the Galerkin and collocation methods. A salient feature of the paper is a *unified theory* for error analysis of both approximation methods.

Keywords: pseudodifferential equation, sphere, radial basis function, Galerkin method, collocation method

1 Introduction

Pseudodifferential operators have long been used [8, 11] as a modern and powerful tool to tackle linear boundary-value problems. Svensson [25] introduces this approach to geodesists who study [6, 7] these problems on the sphere which is taken as a model of the earth. Efficient solutions to pseudodifferential equations on the sphere become more demanding when given data are collected by satellites. In this paper, we study the use of spherical radial basis functions to find approximate solutions to these equations.

The use of spherical radial basis functions results in meshless methods which, over the past decades, become more and more popular [2, 13, 29, 30]. These methods are alternatives to finite-element methods. Solving pseudodifferential equations on the sphere by using spherical radial basis functions with the collocation method has been studied by Morton and Neamtu [16]. Error bounds have later been improved by Morton [15]; see also Morton's PhD dissertation [14]. From the point of view of application, the collocation method is easier to implement, in particular when the given data are scattered. However, it is well-known that collocation methods in

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general elicit a complicated error analysis. The crux of the analysis in [15, 16] is the transformation of the collocation problem to a Lagrange interpolation problem.

In this paper, first we solve strongly elliptic pseudodifferential equations on the sphere by the Galerkin method. (A precise definition of these equations is delayed until Section 2.) Error analysis is performed with well-known knowledge on the Galerkin method. Next, we solve the equations by the collocation method. A salient feature of the paper is that error estimates for collocation methods (as considered in References [14, 15, 16]) are obtained as a by-product of the analysis for the Galerkin method. This *unified error analysis* is thanks to an observation that the collocation equation can be viewed as a Galerkin equation, due to the reproducing kernel property of the space in use. Efforts to perform error analysis for the collocation method based on that for the Galerkin method have been made by several authors to solve quasilinear parabolic equations [5], pseudodifferential equations on closed curves [1], and boundary integral equations [4]. These approaches use either a special set of collocation points or the duality inner product.

In an earlier paper [22], we analyse a collocation approximation to negative order strongly elliptic pseudodifferential equations. The results in the present paper are more general for operators of any order, negative or positive. Results for elliptic operators will be presented in another paper.

Our error estimates, as compared to those by Morton and Neamtu [15, 16], cover a wider range of Sobolev norms. Indeed, these authors only provide error estimates in the Sobolev norm $\|\cdot\|_{2\alpha}$, where 2α is the order of the operator.

A study of preconditioning techniques for the Galerkin method applied to these equations is carried out in [27]. In the present paper, we only discuss error estimates.

The paper is organised as follows. In Section 2, we provide necessary ingredients, and define the operators and the problem in consideration. Section 3 is devoted to the introduction of spherical radial basis functions and the approximation spaces to be employed. Numerical methods are introduced in Section 4. Analysis for the Galerkin and collocation methods is carried out in Sections 5 and 6. Section 7 is devoted to numerical experiments.

Throughout this paper, C, C_1 and C_2 denote generic constants which may take different values at different occurrences.

2 Preliminaries

2.1 Sobolev spaces

Throughout this paper, for $n \geq 3$ we denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, i.e., $\mathbb{S}^{n-1} := \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| = 1\}$ where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. A spherical harmonic of order ℓ on $\mathbb{S}^{n-1}$ is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial of degree ℓ in $\mathbb{R}^n$. The space of all spherical harmonics of order ℓ is the eigenspace of the Laplace–Beltrami operator $\Delta_{\mathbb{S}}$ corresponding to the eigenvalue

$\lambda_\ell = -\ell(\ell+n-2)$. The dimension of this space being

$$N(n,0) = 1 \quad \text{and} \quad N(n,\ell) = \frac{2\ell+n-2}{\ell}\binom{\ell+n-3}{\ell-1}, \quad \ell \neq 0,$$

see e.g. [17, page 4], one may choose for it an $L_2(\mathbb{S}^{n-1})$-orthonormal basis $\{Y_{\ell,m}\}_{m=1}^{N(n,\ell)}$. Note that $N(n,\ell) = O(\ell^{n-2})$. The collection of all the spherical harmonics $Y_{\ell,m}$, $m = 1,\ldots,N(n,\ell)$ and $\ell = 0,1,\ldots$, forms an orthonormal basis for $L_2(\mathbb{S}^{n-1})$.

For $s \in \mathbb{R}$, the Sobolev space H^s is defined as usual by

$$H^s := \Big\{ v \in \mathcal{D}'(\mathbb{S}^{n-1}) : \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s}|\widehat{v}_{\ell,m}|^2 < \infty \Big\},$$

where $\mathcal{D}'(\mathbb{S}^{n-1})$ is the space of distributions on $\mathbb{S}^{n-1}$ and $\widehat{v}_{\ell,m}$ are the Fourier coefficients of v,

$$\widehat{v}_{\ell,m} = \int_{\mathbb{S}^{n-1}} v(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{x}) \mathrm{d}\sigma_{\boldsymbol{x}}.$$

Here $\mathrm{d}\sigma_{\boldsymbol{x}}$ is the element of surface area. The space H^s is equipped with the following norm and inner product:

$$\|v\|_s := \left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s}|\widehat{v}_{\ell,m}|^2 \right)^{1/2} \tag{2.1}$$

and

$$\langle v, w \rangle_s := \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s}\widehat{v}_{\ell,m}\widehat{w}_{\ell,m}.$$

We note that the series on the right hand side also converges when $v \in H^{s+\sigma}$ and $w \in H^{s-\sigma}$ for any $\sigma > 0$. Therefore, in the following we use the same notation $\langle \cdot, \cdot \rangle_s$ for the duality product between $H^{s+\sigma}$ and $H^{s-\sigma}$.

When $s = 0$ we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_0$; this is in fact the L_2-inner product. In the sequel, we will frequently use the Cauchy–Schwarz inequality

$$|\langle v, w \rangle_s| \le \|v\|_s \|w\|_s \quad \text{for all } v, w \in H^s, \text{ for all } s \in \mathbb{R}, \tag{2.2}$$

and the following identity which can be easily proved

$$\|v\|_{s_1} = \sup_{\substack{w \in H^{s_2} \\ w \neq 0}} \frac{\langle v, w \rangle_{\frac{s_1+s_2}{2}}}{\|w\|_{s_2}} \quad \text{for all } v \in H^{s_1}, \text{ for all } s_1, s_2 \in \mathbb{R}. \tag{2.3}$$

Identity (2.3) will be used frequently in the proof of Proposition 3.3 with different values of s_1 and s_2.

2.2 Pseudodifferential operators

Let $\{\widehat{L}(\ell)\}_{\ell\geq 0}$ be a sequence of real numbers. A pseudodifferential operator L is a linear operator that assigns to any $v \in \mathcal{D}'(\mathbb{S}^{n-1})$ a distribution

$$Lv := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\widehat{v}_{\ell,m} Y_{\ell,m}.$$

The sequence $\{\widehat{L}(\ell)\}_{\ell\geq 0}$ is referred to as the *spherical symbol* of L. Let $\mathcal{K}(L) := \{\ell : \widehat{L}(\ell) = 0\}$. Then

$$\ker L = \operatorname{span}\{Y_{\ell,m} : \ell \in \mathcal{K}(L),\ m = 1, \dots, N(n,\ell)\}.$$

Denoting $M := \dim \ker L$, we assume that $0 \leq M < \infty$.

Definition 2.1. A pseudodifferential operator L is said to be *strongly elliptic of order* 2α if

$$C_1(\ell+1)^{2\alpha} \leq \widehat{L}(\ell) \leq C_2(\ell+1)^{2\alpha} \quad \text{for all } \ell \notin \mathcal{K}(L), \tag{2.4}$$

for some positive constants C_1 and C_2.

More general pseudodifferential operators can be defined via Fourier transforms by using local charts; see e.g., [9, 21]. It can be easily seen that if L is a pseudodifferential operator of order 2α then $L : H^{s+\alpha} \to H^{s-\alpha}$ is bounded for all $s \in \mathbb{R}$. Examples of strongly elliptic operators of various orders can be found in [25]; see also [27].

The problem we are solving in this paper is posed as follows.

Problem A: *Let L be a strongly elliptic pseudodifferential operator of order 2α. Given, for some $\sigma \geq 0$,*

$$g \in H^{\sigma-\alpha} \quad \textit{satisfying} \quad \widehat{g}_{\ell,m} = 0 \quad \textit{for all } \ell \in \mathcal{K}(L),\ m = 1, \dots, N(n,\ell), \tag{2.5}$$

find $u \in H^{\sigma+\alpha}$ satisfying

$$\begin{aligned} Lu &= g, \\ \langle \mu_i, u \rangle &= \gamma_i, \quad i = 1, \dots, M, \end{aligned} \tag{2.6}$$

where $\gamma_i \in \mathbb{R}$ and $\mu_i \in H^{-\sigma-\alpha}$ are given. Here $\langle\cdot,\cdot\rangle$ denotes the duality product between $H^{-\sigma-\alpha}$ and $H^{\sigma+\alpha}$, which coincides with the H^0-inner product when μ_i and u belong to H^0.

An explanation for the inclusion of σ in (2.5) is in order. For the Galerkin approximation, the energy space is H^α. Thus it suffices to assume (2.5) with $\sigma = 0$. However, for the collocation approximation, it is required that g be at least continuous. Moreover, we will reformulate the collocation equation into a Galerkin equation which requires $g \in H^\tau$ for some $\tau > 0$ to be specified in Section 6. Therefore, we include the constant σ in (2.5).

Problem A is uniquely solvable under the following assumption.

Assumption B: *The functionals* $\mu_1, \dots, \mu_N$ *are assumed to be* unisolvent *with respect to* $\ker L$*, i.e., for any* $v \in \ker L$ *if* $\langle \mu_i, v \rangle = 0$ *for all* $i = 1, \dots, M$*, then* $v = 0$*.*

The following result is proved in [16]. We include the proof here for completeness.

Proposition 2.2. *Under Assumption B, Problem A has a unique solution.*

Proof. Since $\ker L$ is a finite-dimensional subspace of $H^{\sigma+\alpha}$, we can represent $H^{\sigma+\alpha}$ as

$$H^{\sigma+\alpha} = \ker L \oplus (\ker L)^{\perp}_{H^{\sigma+\alpha}},$$

where $(\ker L)^{\perp}_{H^{\sigma+\alpha}}$ is the orthogonal complement of $\ker L$ with respect to the $H^{\sigma+\alpha}$-inner product. Writing the solution u in the form

$$u = u_0 + u_1 \quad \text{where} \quad u_0 \in \ker L \quad \text{and} \quad u_1 \in (\ker L)^{\perp}_{H^{\sigma+\alpha}}, \tag{2.7}$$

and noting that $L|_{(\ker L)^{\perp}_{H^{\sigma+\alpha}}}$ is injective, we can define u_1 by $u_1 = L^{-1}g$ and find $u_0 \in \ker L$ by solving

$$\langle \mu_i, u_0 \rangle = \gamma_i - \langle \mu_i, u_1 \rangle, \quad i = 1, \dots, M. \tag{2.8}$$

Since $u_0 \in \ker L$, it can represented as

$$u_0 = \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} c_{\ell,m} Y_{\ell,m}.$$

Substituting this into (2.8) yields

$$\sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} c_{\ell,m} \langle \mu_i, Y_{\ell,m} \rangle = \gamma_i - \langle \mu_i, u_1 \rangle, \quad i = 1, \dots, M. \tag{2.9}$$

Recalling that $M = \dim \ker L$, we note that there are M unknowns $c_{\ell,m}$. The unisolvency assumption B assures us that equation (2.8) with zero right-hand side has a unique solution $u_0 = 0$. Therefore, the matrix arising from (2.9) is invertible, which in turn implies unique existence of $c_{\ell,m}$, $m = 1, \dots, N(n,\ell)$ and $\ell \in \mathcal{K}(L)$. The proposition is proved. □

We define a bilinear form $a(\cdot,\cdot) : H^{\alpha+s} \times H^{\alpha-s} \to \mathbb{R}$, for any $s \in \mathbb{R}$, by

$$a(w, v) := \langle Lw, v \rangle \quad \text{for all } w \in H^{\alpha+s},\ v \in H^{\alpha-s}. \tag{2.10}$$

In particular, when $s = \sigma$ we have by noting (2.6)

$$a(u_1, v) = \langle g, v \rangle \quad \text{for all } v \in H^{\alpha-\sigma}. \tag{2.11}$$

In the sequel, for any $x, y \in \mathbb{R}$, $x \simeq y$ means that there exist positive constants C_1 and C_2 satisfying $C_1 x \le y \le C_2 x$. The following simple results are often used in the next sections.

Lemma 2.3. *Let s be any real number.*

1. *The bilinear form $a(\cdot,\cdot) : H^{\alpha+s} \times H^{\alpha-s} \to \mathbb{R}$ is bounded, i.e.,*

$$|a(w,v)| \le C\|w\|_{\alpha+s}\|v\|_{\alpha-s} \quad \text{for all } w \in H^{\alpha+s},\ v \in H^{\alpha-s}. \tag{2.12}$$

2. *If $w, v \in H^s$, then*

$$|\left\langle Lw, v\right\rangle_{s-\alpha}| \le C\|w\|_s\|v\|_s. \tag{2.13}$$

3. *Assume that L is strongly elliptic. If $v \in (\ker L)^{\perp}_{H^s}$, then*

$$\left\langle Lv, v\right\rangle_{s-\alpha} \simeq \|v\|_s^2. \tag{2.14}$$

   In particular, setting $s = \alpha$ in (2.14), there holds $a(v,v) \simeq \|v\|_\alpha^2$ for all $v \in (\ker L)^{\perp}_{H^\alpha}$.

Here C is a constant independent of v and w.

Proof. Let $w \in H^{\alpha+s}$ and $v \in H^{\alpha-s}$. Noting (2.4) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|a(w,v)| &\le \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} |\widehat{L}(\ell)||\widehat{w}_{\ell,m}||\widehat{v}_{\ell,m}| \le C\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\alpha}|\widehat{w}_{\ell,m}||\widehat{v}_{\ell,m}| \\
&= C\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{\alpha+s}|\widehat{w}_{\ell,m}|(\ell+1)^{\alpha-s}|\widehat{v}_{\ell,m}| \\
&\le C\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(\alpha+s)}|\widehat{w}_{\ell,m}|^2\right)^{1/2}\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(\alpha-s)}|\widehat{v}_{\ell,m}|^2\right)^{1/2} \\
&= C\|w\|_{\alpha+s}\|v\|_{\alpha-s},
\end{aligned}$$

proving (2.12). The proof for (2.13) and (2.14) can be done similarly, noting the definition (2.4) of strongly elliptic operators, and noting that $v \in (\ker L)^{\perp}_{H^s}$ if and only if $v \in H^s$ and $\widehat{v}_{\ell,m} = 0$ for all $\ell \in \mathcal{K}(L)$ and $m = 1,\dots,N(n,\ell)$. $\square$

In the next section, we shall define finite-dimensional subspaces in which approximate solutions are sought for.

3 Approximation subspaces

The finite-dimensional subspaces to be used in the approximation will be defined from spherical radial basis functions, which in turn are defined from kernels.

3.1 Positive-definite kernels

A continuous function $\Theta : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ is called a *positive-definite kernel* on $\mathbb{S}^{n-1}$ if it satisfies

(i) $\Theta(\boldsymbol{x}, \boldsymbol{y}) = \Theta(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^{n-1}$,

(ii) for any positive integer N and any set of distinct points $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_N\}$ on $\mathbb{S}^{n-1}$, the $N \times N$ matrix $\boldsymbol{B}$ with entries $\boldsymbol{B}_{i,j} = \Theta(\boldsymbol{y}_i, \boldsymbol{y}_j)$ is positive-semidefinite.

If the matrix $\boldsymbol{B}$ is positive-definite then Θ is called a *strictly positive-definite* kernel; see [24, 31].

We characterise the kernel Θ by a *shape function* θ as follows. Let $\theta : [-1, 1] \to \mathbb{R}$ be a univariate function having a series expansion in terms of Legendre polynomials,

$$\theta(t) = \sum_{\ell=0}^{\infty} \omega_n^{-1} N(n, \ell) \widehat{\theta}(\ell) P_\ell(n; t), \tag{3.1}$$

where ω_n is the surface area of the sphere $\mathbb{S}^{n-1}$, and $\widehat{\theta}(\ell)$ is the Fourier–Legendre coefficient,

$$\widehat{\theta}(\ell) = \omega_{n-1} \int_{-1}^{1} \theta(t) P_\ell(n; t)(1 - t^2)^{(n-3)/2} \, dt.$$

Here, $P_\ell(n; t)$ denotes the degree ℓ normalised Legendre polynomial in n variables so that $P_\ell(n; 1) = 1$, as described in [17]. Using this shape function θ, we define

$$\Theta(\boldsymbol{x}, \boldsymbol{y}) := \theta(\boldsymbol{x} \cdot \boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^{n-1}, \tag{3.2}$$

where $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the dot product between $\boldsymbol{x}$ and $\boldsymbol{y}$. We note that $\boldsymbol{x} \cdot \boldsymbol{y}$ is the cosine of the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, which is called the geodesic distance between the two points. Thus the kernel Θ is a zonal kernel. By using the well-known addition formula for spherical harmonics [17],

$$\sum_{m=1}^{N(n,\ell)} Y_{\ell,m}(\boldsymbol{x}) \, Y_{\ell,m}(\boldsymbol{y}) = \omega_n^{-1} N(n, \ell) \, P_\ell(n; \boldsymbol{x} \cdot \boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^{n-1}, \tag{3.3}$$

we can write

$$\Theta(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{\theta}(\ell) Y_{\ell,m}(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{y}). \tag{3.4}$$

Remark 3.1. In [3], a complete characterisation of strictly positive-definite kernels is established: the kernel Θ is strictly positive-definite if and only if $\widehat{\theta}(\ell) \geq 0$ for all $\ell \geq 0$, and $\widehat{\theta}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ; see also [24] and [31].

In the remainder of this section, we shall define a specific shape function ϕ and a specific kernel Φ which will be used to define the approximation subspace. The notations θ and Θ are reserved for future general reference.

3.2 Spherical radial basis functions

We choose a shape function ϕ such that there exists $\tau \in \mathbb{R}$ satisfying

$$\widehat{\phi}(\ell) \simeq (\ell+1)^{-2\tau} \quad \text{for all } \ell \geq 0. \tag{3.5}$$

The corresponding kernel Φ defined by (3.2), i.e., $\Phi(\boldsymbol{x},\boldsymbol{y}) = \phi(\boldsymbol{x}\cdot\boldsymbol{y})$, is then strictly positive-definite; see Remark 3.1. The native space associated with ϕ is defined by

$$\mathcal{N}_\phi := \Big\{ v \in \mathcal{D}'(\mathbb{S}^{n-1}) : \|v\|_\phi^2 = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} < \infty \Big\}.$$

This space is equipped with an inner product and a norm defined by

$$\langle v, w \rangle_\phi = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{v}_{\ell,m}\widehat{w}_{\ell,m}}{\widehat{\phi}(\ell)} \quad \text{and} \quad \|v\|_\phi = \left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} \right)^{1/2}.$$

Since $\widehat{\phi}(\ell)$ satisfies (3.5), the native space $\mathcal{N}_\phi$ can be identified with the Sobolev space H^τ, and the corresponding norms are equivalent.

Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ be a set of data points on the sphere. Two important parameters characterising the set X are the *mesh norm* h_X and *separation radius* q_X, defined by

$$h_X := \sup_{\boldsymbol{y}\in\mathbb{S}^{n-1}} \min_{1\leq j\leq N} \cos^{-1}(\boldsymbol{x}_j\cdot\boldsymbol{y}) \quad \text{and} \quad q_X := \frac{1}{2} \min_{\substack{i\neq j\\ 1\leq i,j\leq N}} \cos^{-1}(\boldsymbol{x}_i\cdot\boldsymbol{x}_j). \tag{3.6}$$

The *spherical radial basis functions* Φ_j, $j = 1, \ldots, N$, associated with X and the kernel Φ are defined by (see (3.4))

$$\Phi_j(\boldsymbol{x}) := \Phi(\boldsymbol{x}, \boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_j) Y_{\ell,m}(\boldsymbol{x}). \tag{3.7}$$

We note that

$$\widehat{(\Phi_j)}_{\ell,m} = \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_j), \quad j = 1, \ldots, N. \tag{3.8}$$

It follows from (3.5) that, for any $s \in \mathbb{R}$,

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} \left| \widehat{(\Phi_j)}_{\ell,m} \right|^2 \simeq \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(s-2\tau)} |Y_{\ell,m}(\boldsymbol{x}_j)|^2.$$

By using (3.3) and noting $P_\ell(n; \boldsymbol{x}_j\cdot\boldsymbol{x}_j) = P_\ell(n;1) = 1$ we obtain, recalling that $N(n,\ell) = O(\ell^{n-2})$,

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} \left| \widehat{(\Phi_j)}_{\ell,m} \right|^2 \simeq \sum_{\ell=0}^{\infty} (\ell+1)^{2(s-2\tau)+n-2}.$$

The latter series converges if and only if $s < 2\tau + (1-n)/2$. Hence,

$$\Phi_j \in H^s \iff s < 2\tau + (1-n)/2. \tag{3.9}$$

The finite-dimensional subspace to be used in our approximation is defined by $\mathcal{V}_X^\phi := \operatorname{span}\{\Phi_1, \ldots, \Phi_N\}$. This space is used by Kansa [10] for collocation approximation. For brevity of notation we write $\mathcal{V}^\phi$ for $\mathcal{V}_X^\phi$ since there is no confusion. Due to (3.9), we have

$$\mathcal{V}^\phi \subset H^s \quad \text{for all } s < 2\tau + \frac{1-n}{2}. \tag{3.10}$$

We note that if $\tau > (n-1)/2$, then $\mathcal{V}^\phi \subset \mathcal{N}_\phi \simeq H^\tau \subset C(\mathbb{S}^{n-1})$, which is essentially the Sobolev embedding theorem.

It is noted that if $\tau > (n-1)/2$ then any function $v \in \mathcal{N}_\phi$ satisfies

$$v(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{v}_{\ell,m}\widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_j)}{\widehat{\phi}(\ell)} = \langle v, \Phi_j \rangle_\phi, \quad j = 1, \ldots, N. \tag{3.11}$$

This property is crucial in our analysis for the collocation method in Section 6.

We finish this subsection by proving the approximation property of $\mathcal{V}^\phi$ as a subspace of Sobolev spaces. This property is obtained by using the interpolation error which is derived in [18, Theorem 5.5]. This theorem states that if $v \in H^{s^*}$ for some s^* satisfying $(n-1)/2 < s^* \le \tau$ then for $0 \le t^* \le s^*$ there holds

$$\|v - I_X v\|_{t^*} \le C \rho_X^{\tau - s^*} h_X^{s^* - t^*} \|v\|_s. \tag{3.12}$$

Here, $\rho_X = h_X / q_X$, and $I_X v \in \mathcal{V}^\phi$ is the interpolant of v at $\boldsymbol{x}_j$, $j = 1, \ldots, N$, given by

$$I_X v(\boldsymbol{x}_j) = v(\boldsymbol{x}_j), \quad j = 1, \ldots, N.$$

(In fact, it is required that $v \in \mathcal{N}_\phi$ so that $I_X v$ is well-defined.) When solving pseudodifferential equations of order 2α by the Galerkin method, it is natural to carry out error analysis in the energy space H^α. Since the order 2α may be negative (as in the case of the weakly-singular integral equation discussed after Definition 2.1) it is necessary to show an approximation property of the form (3.12) for a wider range of t and s, including negative real values.

Before stating and proving the above mentioned approximation property (Proposition 3.3), we recall the following property of interpolation spaces which will be frequently used in the proof of that proposition.

Lemma 3.2. *[12, Theorem B.2] Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ be such that $s_1 \le s_2$ and $t_1 \le t_2$. Assume that $T : H^{s_i} \to H^{t_i}$, $i = 1, 2$, are bounded linear operators satisfying*

$$\|Tv\|_{t_i} \le M_i \|v\|_{s_i} \quad \forall v \in H^{s_i},$$

for some $M_i \ge 0$, $i = 1, 2$. Then for any $\theta \in [0, 1]$, $T : H^{\theta s_1 + (1-\theta)s_2} \to H^{\theta t_1 + (1-\theta)t_2}$ is bounded and there holds

$$\|Tv\|_{\theta t_1 + (1-\theta)t_2} \le M_1^\theta M_2^{1-\theta} \|v\|_{\theta s_1 + (1-\theta)s_2} \quad \forall v \in H^{\theta s_1 + (1-\theta)s_2}.$$

Proposition 3.3. *Assume that* (3.5) *holds for some* $\tau > (n-1)/2$. *For any* $s^*, t^* \in \mathbb{R}$ *satisfying* $t^* \le s^* \le 2\tau$ *and* $t^* \le \tau$, *if* $v \in H^{s^*}$ *then there exists* $\eta \in \mathcal{V}^\phi$ *such that*

$$\|v - \eta\|_{t^*} \le C h_X^{s^*-t^*} \|v\|_{s^*} \tag{3.13}$$

for $h_X \le h_0$, *where* C *and* h_0 *are independent of* v *and* h_X.

Proof. For $k = 0, 1, 2, \ldots$, we denote $\mathcal{I}_k = [-k\tau, -(k-1)\tau]$ and prove by induction on k that (3.13) holds for $t^* \in \mathcal{I}_k$ for all k.

• We first prove that (3.13) is true when $t^* \in \mathcal{I}_0$. Indeed, let $t^* \in \mathcal{I}_0$. In this step, we consider two cases when s^* belongs to $[\tau, 2\tau]$ and $[t^*, \tau]$, respectively.

Case 1.1. $\tau \le s^* \le 2\tau$.

Let t and s be real numbers satisfying $0 \le t \le \tau \le s \le 2\tau$. Let $I_X v \in \mathcal{V}^\phi$ be the interpolant of v at $\boldsymbol{x}_i$, $i = 1, \ldots, N$. Then, by using (3.11), we deduce

$$\langle v - I_X v, w \rangle_\phi = 0 \quad \text{for all } w \in \mathcal{V}^\phi.$$

Hence, by using (3.5) and the Cauchy–Schwarz inequality, we obtain for $v \in H^{2\tau}$

$$\begin{aligned}
\|v - I_X v\|_\tau^2 &\simeq \|v - I_X v\|_\phi^2 = \langle v - I_X v, v - I_X v \rangle_\phi = \langle v - I_X v, v \rangle_\phi \\
&\le \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m} - \widehat{(I_X v)}_{\ell,m}| |\widehat{v}_{\ell,m}|}{\widehat{\phi}(\ell)} \\
&\simeq \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\tau} |\widehat{v}_{\ell,m} - \widehat{(I_X v)}_{\ell,m}| |\widehat{v}_{\ell,m}| \le \|v - I_X v\|_0 \|v\|_{2\tau}
\end{aligned} \tag{3.14}$$

Proposition 3.5 in [26] gives

$$\|v - I_X v\|_0 \le C h_X^{2\tau} \|v\|_{2\tau}, \tag{3.15}$$

which, together with (3.14), implies

$$\|v - I_X v\|_\tau \le C h_X^{\tau} \|v\|_{2\tau}. \tag{3.16}$$

Noting the inequalities (3.15), (3.16), and applying Lemma 3.2 with $T = I - I_X$, $s_1 = s_2 = 2\tau$, $t_1 = 0$, $t_2 = \tau$, and $\theta = (\tau - t)/\tau$, we obtain

$$\|v - I_X v\|_t \le C h_X^{2\tau - t} \|v\|_{2\tau}, \quad 0 \le t \le \tau. \tag{3.17}$$

On the other hand, by using (3.12) with t^* and s^* replaced by t and τ, respectively, we obtain

$$\|v - I_X v\|_t \le C h_X^{\tau - t} \|v\|_\tau, \quad 0 \le t \le \tau. \tag{3.18}$$

Using Lemma 3.2 again with $T = I - I_X$, $t_1 = t_2 = t$, $s_1 = \tau$, $s_2 = 2\tau$, and $\theta = 2 - s/\tau$, we deduce

$$\|v - I_X v\|_t \le C h_X^{s-t} \|v\|_s, \quad 0 \le t \le \tau.$$

Hence, we have proved

$$\begin{cases} 0 \le t^* \le \tau \le s^* \le 2\tau, \\ \forall v \in H^{s^*},\ \exists \eta_v = I_X v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \le C h_X^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{3.19}$$

Case 1.2. $t^* \le s^* < \tau$.

Let s and t be real numbers such that $0 \le s < \tau$ and $2s - 2\tau \le t \le s$. Let $P_s : H^s \to \mathcal{V}^\phi$ be defined by

$$\langle P_s v, w \rangle_s = \langle v, w \rangle_s \quad \forall w \in \mathcal{V}^\phi. \tag{3.20}$$

It is easily seen that

$$\|v - P_s v\|_s \le \|v\|_s. \tag{3.21}$$

If $2s - 2\tau \le t \le 2s - \tau$ so that $\tau \le 2s - t \le 2\tau$ then we apply (3.19) with t^* and s^* replaced by s and $2s - t$, respectively, to deduce that for any $w \in H^{2s-t}$, there exists $\eta_w \in \mathcal{V}^\phi$ such that

$$\|w - \eta_w\|_s \le C h_X^{s-t} \|w\|_{2s-t}. \tag{3.22}$$

Since $\langle v - P_s v, \eta_w \rangle_s = 0$, it follows from (2.3), (2.2), (3.21) and (3.22) that

$$\begin{aligned} \|v - P_s v\|_t &= \sup_{\substack{w \in H^{2s-t} \\ w \ne 0}} \frac{\langle v - P_s v, w \rangle_s}{\|w\|_{2s-t}} = \sup_{\substack{w \in H^{2s-t} \\ w \ne 0}} \frac{\langle v - P_s v, w - \eta_w \rangle_s}{\|w\|_{2s-t}} \\ &\le \|v - P_s v\|_s \sup_{\substack{w \in H^{2s-t} \\ w \ne 0}} \frac{\|w - \eta_w\|_s}{\|w\|_{2s-t}} \le C h_X^{s-t} \|v\|_s. \end{aligned}$$

In particular, for $t = 2s - \tau$ we have

$$\|v - P_s v\|_{2s-\tau} \le C h_X^{-s+\tau} \|v\|_s. \tag{3.23}$$

If $2s - \tau < t \le s$ then by noting (3.21) and (3.23), and applying Lemma 3.2 with $T = I - P_s$, $s_1 = s_2 = s$, $t_1 = 2s - \tau$, $t_2 = s$, and $\theta = (t - s)/(s - \tau)$ we obtain $\|v - P_s v\|_t \le C h_X^{s-t} \|v\|_s$.

Combining both cases 1.1 and 1.2, we have proved that

$$\begin{cases} t^* \in \mathcal{I}_0,\ t^* \le s^* \le 2\tau, \\ \forall v \in H^{s^*}, \exists \eta_v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \le C h_X^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{3.24}$$

• Assume that for some $k_0 \ge 0$, (3.13) is true when $t^* \in \mathcal{I}_k$, for all $k = 0, 1, \ldots, k_0$, i.e., the following statement holds,

$$\begin{cases} t^* \in \bigcup_{k=0}^{k_0} \mathcal{I}_k,\ t^* \le s^* \le 2\tau, \\ \forall v \in H^{s^*},\ \exists \eta_v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \le C h_X^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{3.25}$$

• We now prove that (3.13) is also true when $t^* \in \mathcal{I}_{k_0+1}$. Analogously to the case when $t^* \in \mathcal{I}_0$, we consider two cases when s^* belongs to $[-k_0\tau, 2\tau]$ and $[t^*, -k_0\tau)$, respectively.

Case 2.1. $-k_0\tau \le s^* \le 2\tau$.

Let t and s be real numbers satisfying $t \in \mathcal{I}_{k_0+1}$ and $s \in [-k_0\tau, 2\tau]$. Let $P_{-k_0\tau} : H^{-k_0\tau} \to \mathcal{V}^\phi$ be the projection defined by

$$P_{-k_0\tau}v \in \mathcal{V}^\phi : \quad \langle P_{-k_0\tau}v, w\rangle_{-k_0\tau} = \langle v, w\rangle_{-k_0\tau} \quad \forall w \in \mathcal{V}^\phi.$$

Then $P_{-k_0\tau}v$ is the best approximation of v from $\mathcal{V}^\phi$ in the $H^{-k_0\tau}$-norm. It follows from (3.25) with $-k_0\tau$ and s in place of t^* and s^*, respectively, that

$$\|v - P_{-k_0\tau}v\|_{-k_0\tau} \le Ch_X^{s+k_0\tau}\|v\|_s \quad \forall v \in H^s. \tag{3.26}$$

Since $t \in \mathcal{I}_{k_0+1}$ so that $-k_0\tau \le -t - 2k_0\tau \le 2\tau$, statement (3.25) with t^* and s^* replaced by $-k_0\tau$ and $-t - 2k_0\tau$, respectively, assures that for any $w \in H^{-t-2k_0\tau}$, there exists $\eta_w \in \mathcal{V}^\phi$ such that

$$\|w - \eta_w\|_{-k_0\tau} \le Ch_X^{-t-k_0\tau}\|w\|_{-t-2k_0\tau}. \tag{3.27}$$

Since $\langle v - P_{-k_0\tau}v, \eta_w\rangle_{-k_0\tau} = 0$, it follows from (2.3) and (2.2) that

$$\begin{aligned}\|v - P_{-k_0\tau}v\|_t &= \sup_{\substack{w\in H^{-t-2k_0\tau}\\ w\ne 0}} \frac{\langle v - P_0v, w\rangle_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}} = \sup_{\substack{w\in H^{-t-2k_0\tau}\\ w\ne 0}} \frac{\langle v - P_0v, w - \eta_w\rangle_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}} \\ &\le \|v - P_{-k_0\tau}v\|_{-k_0\tau} \sup_{\substack{w\in H^{-t-2k_0\tau}\\ w\ne 0}} \frac{\|w - \eta_w\|_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}}.\end{aligned}$$

Inequalities (3.26) and (3.27) imply $\|v - P_{-k_0\tau}v\|_t \le Ch_X^{s-t}\|v\|_s$.

Hence, we have proved that

$$\begin{cases} -(k_0+1)\tau \le t^* \le -k_0\tau, \ -k_0\tau \le s^* \le 2\tau, \\ \forall v \in H^{s^*}, \exists \eta_v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \le Ch_X^{s^*-t^*}\|v\|_{s^*}. \end{cases} \tag{3.28}$$

Case 2.2. $t^* \le s^* < -k_0\tau$.

Let s and t be real numbers such that $-(k_0+1)\tau \le s < -k_0\tau$ and $2s - 2\tau \le t \le s$. Let $P_s : H^s \to \mathcal{V}^\phi$ be defined by (3.20) with this new value of s.

If $2s - 2\tau \le t \le 2s + k_0\tau$ so that $-k_0\tau \le 2s - t \le 2\tau$ then we can use the same argument as in Case 1.2 with (3.19) replaced by (3.28) to obtain $\|v - P_sv\|_t \le Ch_X^{s-t}\|v\|_s$.

If $2s + k_0\tau < t \le s$ then we use Lemma 3.2 in the same manner as in Case 1.2 to obtain the same estimate.

Combining both cases 2.1 and 2.2 we obtain the result for $k = k_0 + 1$, completing the proof. □

4 Approximate solutions

4.1 Approach

Noting (2.7), we shall seek an approximate solution $\widetilde{u} \in H^{\sigma+\alpha}$ in the form

$$\widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1 \quad \text{where} \quad \widetilde{u}_0 \in \ker L \quad \text{and} \quad \widetilde{u}_1 \in \mathcal{V}^\phi.$$

The solution $\widetilde{u}_1$ will be found by the Galerkin or collocation method. Having found $\widetilde{u}_1$, we will find $\widetilde{u}_0 \in \ker L$ by solving the equations (cf. (2.6))

$$\langle \mu_i, \widetilde{u}_0 \rangle = \gamma_i - \langle \mu_i, \widetilde{u}_1 \rangle, \quad i = 1, \ldots, M,$$

so that

$$\langle \mu_i, \widetilde{u} \rangle = \langle \mu_i, u \rangle, \quad i = 1, \ldots, M. \tag{4.1}$$

The unique existence of $\widetilde{u}_0$ follows from Assumption B in exactly the same way as that of u_0; see Proposition 2.2.

We postpone until Sections 5 and 6 the issue of finding $\widetilde{u}_1$. It is noted that in general $\mathcal{V}^\phi \not\subseteq (\ker L)^\perp_{H^{\sigma+\alpha}}$. However, $\widetilde{u}$ can be rewritten in a form similar to (2.7) as follows. Let

$$u_0^* := \widetilde{u}_0 + \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m} \tag{4.2}$$

and

$$u_1^* = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m}. \tag{4.3}$$

Then

$$\widetilde{u} = u_0^* + u_1^* \quad \text{with} \quad u_0^* \in \ker L \quad \text{and} \quad u_1^* \in (\ker L)^\perp_{H^{\sigma+\alpha}}. \tag{4.4}$$

It should be noted that, in general, u_1^* does not belong to $\mathcal{V}^\phi$, and that this function is introduced purely for analysis purposes. We do not explicitly compute u_1^*, nor u_0^*.

4.2 Preliminary error analysis

Assume that the exact solution u and the approximate solution $\widetilde{u}$ of Problem A belong to H^t for some $t \in \mathbb{R}$, and assume that $\mu_i \in H^{-t}$ for $i = 1, \ldots, M$. Comparing (2.7) and (4.4) suggests that $\|u - \widetilde{u}\|_t$ can be estimated by estimating $\|u_0 - u_0^*\|_t$ and $\|u_1 - u_1^*\|_t$. It turns out that an estimate for the latter is sufficient, as shown in the following two lemmas.

Lemma 4.1. *Let u_0, u_1, u_0^* and u_1^* be defined by* (2.7), (4.2) *and* (4.3). *For $i = 1, \ldots, M$, if $\mu_i \in H^{-t}$ for some $t \in \mathbb{R}$, then*

$$\|u_0 - u_0^*\|_t \le C \|u_1 - u_1^*\|_t,$$

where C is independent of u.

Proof. For $i = 1, \ldots, M$, it follows from (4.1) that

$$\langle \mu_i, u_0 \rangle + \langle \mu_i, u_1 \rangle = \langle \mu_i, u_0^* \rangle + \langle \mu_i, u_1^* \rangle ,$$

implying $\langle \mu_i, u_0 - u_0^* \rangle = \langle \mu_i, u_1^* - u_1 \rangle$. Inequality (2.3) with $s_1 = t$ and $s_2 = -t$ yields

$$|\langle \mu_i, u_0 - u_0^* \rangle| = |\langle \mu_i, u_1 - u_1^* \rangle| \le \|\mu_i\|_{-t} \|u_1 - u_1^*\|_t.$$

This result holds for all $i = 1, \ldots, M$, implying

$$\|u_0 - u_0^*\|_\mu \ \le \ \mathcal{M} \, \|u_1 - u_1^*\|_t,$$

where $\mathcal{M} := \max_{i=1,\ldots,M} \|\mu_i\|_{-t}$, and $\|v\|_\mu := \max_{i=1,\ldots,M} |\langle \mu_i, v \rangle|$ for all $v \in \ker L$. (The unisolvency assumption assures us that the above norm is well-defined.) The subspace $\ker L$ being finite-dimensional, we deduce

$$\|u_0 - u_0^*\|_t \le C \, \|u_1 - u_1^*\|_t,$$

proving the lemma. $\square$

Lemma 4.2. *Under the assumptions of Lemma 4.1, there holds*

$$\|u - \widetilde{u}\|_t \le C \|u_1 - u_1^*\|_t.$$

Proof. Noting (2.7) and (4.4), the norm $\|u - \widetilde{u}\|_t$ can be rewritten as

$$\begin{aligned}
\|u - \widetilde{u}\|_t^2 &= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2t} \, |\widehat{u}_{\ell,m} - \widehat{(\widetilde{u})}_{\ell,m}|^2 \\
&\quad + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2t} \, |\widehat{u}_{\ell,m} - \widehat{(\widetilde{u})}_{\ell,m}|^2 \\
&= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2t} \, |\widehat{(u_0)}_{\ell,m} - \widehat{(u_0^*)}_{\ell,m}|^2 \\
&\quad + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2t} \, |\widehat{(u_1)}_{\ell,m} - \widehat{(u_1^*)}_{\ell,m}|^2 \\
&= \|u_0 - u_0^*\|_t^2 + \|u_1 - u_1^*\|_t^2.
\end{aligned}$$

The required result now follows from Lemma 4.1. $\square$

In the following sections, we describe methods to construct $\widetilde{u}_1$, and estimate $\|u_1 - u_1^*\|_t$ accordingly.

5 Galerkin approximation

Recalling (3.10), we choose the shape functions ϕ in this subsection such that

$$\tau > \frac{1}{2}\left(\alpha + \frac{n-1}{2}\right), \tag{5.1}$$

so that $\mathcal{V}^\phi \subset H^\alpha$. We find $\widetilde{u}_1 \in \mathcal{V}^\phi$ by solving the Galerkin equation

$$a(\widetilde{u}_1, v) = \langle g, v\rangle \quad \text{for all } \ v \in \mathcal{V}^\phi. \tag{5.2}$$

By writing $\widetilde{u}_1 = \sum_{i=1}^N c_i\Phi_i$ we derive from (5.2) the matrix equation $\boldsymbol{A}^{(G)}\boldsymbol{c} = \boldsymbol{g}$, where

$$\boldsymbol{A}_{ij}^{(G)} = a(\Phi_i, \Phi_j) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\, [\widehat{\phi}(\ell)]^2\, Y_{\ell,m}(\boldsymbol{x}_i)\, Y_{\ell,m}(\boldsymbol{x}_j), \tag{5.3}$$

$\boldsymbol{c} = (c_1, \ldots, c_N)$, and $\boldsymbol{g} = (\langle g, \Phi_1\rangle, \ldots, \langle g, \Phi_N\rangle)$.

Lemma 5.1. *The matrix $\boldsymbol{A}^{(G)}$ is symmetric positive-definite.*

Proof. Let θ be a shape function whose Fourier–Legendre coefficients are given by

$$\widehat{\theta}(\ell) = \begin{cases} \widehat{L}(\ell)[\widehat{\phi}(\ell)]^2 & \text{if } \ell \notin \mathcal{K}(L) \\ 0 & \text{if } \ell \in \mathcal{K}(L). \end{cases}$$

Then $\boldsymbol{A}_{ij}^{(G)} = \Theta(\boldsymbol{x}_i, \boldsymbol{x}_j)$ where Θ is the kernel defined from θ. Since $\widehat{\theta}(\ell) \geq 0$ for all $\ell \geq 0$, and $\widehat{\theta}(\ell) = 0$ only for a finite number of ℓ, it follows from Remark 3.1 that $\boldsymbol{A}^{(G)}$ is symmetric positive-definite. $\square$

As a consequence of this lemma, there exists a unique solution $\widetilde{u}_1$ to (5.2). With $\widetilde{u}_1$ given by (5.2), u_1^* defined by (4.3) satisfies $u_1^* \in H^\alpha$ and

$$a(u_1^*, v) = \langle g, v\rangle \quad \text{for all } v \in \mathcal{V}^\phi. \tag{5.4}$$

Even though in general u_1^* does not belong to $\mathcal{V}^\phi$, the following result is essentially Céa's Lemma.

Lemma 5.2. *If u_1 and u_1^* are defined by* (2.11) *and* (4.3) *with $\widetilde{u}_1$ given by* (5.2)*, then*

$$\|u_1 - u_1^*\|_\alpha \leq C\|u_1 - v\|_\alpha \quad \textit{for all } v \in \mathcal{V}^\phi.$$

Proof. It follows from the definition (4.3) of u_1^* that

$$a(w, u_1^*) = a(w, \widetilde{u}_1) \quad \text{for all } w \in H^\alpha. \tag{5.5}$$

Moreover, since $\mathcal{V}^\phi \subset H^\alpha \subset H^{\alpha-\sigma}$ (noting $\sigma \geq 0$) we infer from (2.11) and (5.4)

$$a(u_1 - u_1^*, v) = 0 \quad \text{for all } v \in \mathcal{V}^\phi. \tag{5.6}$$

Since $u_1 - u_1^* \in (\ker L)^\perp_{H^\alpha}$, Lemma 2.3 yields

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1 - u_1^*) = a(u_1 - u_1^*, u_1) - a(u_1 - u_1^*, u_1^*).$$

It follows from (5.5) and (5.6), noting $u_1 - u_1^* \in H^\alpha$ and $\widetilde{u}_1 \in \mathcal{V}^\phi$, that

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1) - a(u_1 - u_1^*, \widetilde{u}_1) = a(u_1 - u_1^*, u_1).$$

Hence, using again (5.6), we obtain for any $v \in \mathcal{V}^\phi$

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1 - v) \leq C\|u_1 - u_1^*\|_\alpha \, \|u_1 - v\|_\alpha,$$

where in the last step we used Lemma 2.3. By cancelling similar terms we obtain the required result. □

The above lemma and Proposition 3.3 will be used to estimate the error $u_1 - u_1^*$.

Lemma 5.3. *Assume that the shape function ϕ is chosen to satisfy* (3.5), (5.1) *and $\tau \geq \alpha$, $\tau > (n-1)/2$. Let u_1 and u_1^* be defined as in Lemma 5.2. Assume that $u_1 \in H^s$ for some s satisfying $\alpha \leq s \leq 2\tau$. Let $t \in \mathbb{R}$ satisfy $2(\alpha - \tau) \leq t \leq \alpha$. Then for h_X sufficiently small there holds*

$$\|u_1 - u_1^*\|_t \;\leq\; C h_X^{s-t} \, \|u_1\|_s. \tag{5.7}$$

The constant C is independent of u and h_X.

Proof. The result for the case when $t = \alpha$ is a direct consequence of Lemma 5.2 and Proposition 3.3 (with $t^* = \alpha$ and $s^* = s$).

The proof for the case $t < \alpha$ is standard, using Aubin–Nitsche's trick, and is included here for completeness. It follows from (2.3) and (2.4) that

$$\|u_1 - u_1^*\|_t \leq \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{\langle u_1 - u_1^*, v\rangle_\alpha}{\|v\|_{2\alpha-t}} \leq C \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{a(u_1 - u_1^*, v)}{\|v\|_{2\alpha-t}}.$$

By using successively (5.6), Lemma 2.3, (5.7) with t replaced by α, and (5.9), we deduce for any $\eta \in \mathcal{V}^\phi$

$$\begin{aligned}\|u_1 - u_1^*\|_t &\leq C \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{a(u_1 - u_1^*, v - \eta)}{\|v\|_{2\alpha-t}} \leq C\|u_1 - u_1^*\|_\alpha \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha-t}} \\ &\leq C h_X^{s-\alpha} \|u_1\|_s \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha-t}}. \end{aligned} \tag{5.8}$$

Since $2(\alpha-\tau) \le t < \alpha$, there holds $\alpha < 2\alpha - t \le 2\tau$. By invoking Proposition 3.3 again with t^* and s^* replaced by α and $2\alpha - t$, respectively, we can choose $\eta \in \mathcal{V}^\phi$ satisfying

$$\|v-\eta\|_\alpha \le C h_X^{\alpha-t} \|v\|_{2\alpha-t}. \tag{5.9}$$

This together with (5.8) yields the required estimate, proving the lemma. □

We are now ready to state and prove the main result of this section.

Theorem 5.4. *Assume that the shape function ϕ is chosen to satisfy* (3.5), (5.1) *and $\tau \ge \alpha$, $\tau > (n-1)/2$. Assume further that $u \in H^s$ for some s satisfying $\alpha \le s \le 2\tau$. If $\mu_i \in H^{-t}$ for $i = 1, \ldots, M$ with $t \in \mathbb{R}$ satisfying $2(\alpha-\tau) \le t \le \alpha$, then for h_X sufficiently small there holds*

$$\|u - \widetilde{u}\|_t \;\le\; C h_X^{s-t} \|u\|_s.$$

The constant C is independent of u and h_X.

Proof. Since $\mu_i \in H^{-t}$ for $i = 1, \ldots, M$, Lemma 4.2 gives

$$\|u - \widetilde{u}\|_t \le C \|u_1 - u_1^*\|_t.$$

The required result is a consequence of Lemma 5.3, noting that $\|u_1\|_s \le \|u\|_s$. □

6 Collocation approximation

Recall that for this method it is assumed that $g \in H^{\sigma-\alpha}$ for some positive σ so that $u \in H^{\sigma+\alpha}$; see Problem A. We will assume that

$$\max\{2\alpha, \alpha\} + \frac{n-1}{2} < \tau \le \min\{\sigma - \alpha, \sigma\}. \tag{6.1}$$

Recall that (3.5) implies $\mathcal{N}_\phi \simeq H^\tau$. Thus, the condition $\sigma - \alpha \ge \tau$ assures us that $g \in \mathcal{N}_\phi$. The condition $2\alpha + (n-1)/2 < \tau$ is to assure that $L\widetilde{u}_1 \in \mathcal{N}_\phi$. Indeed, this condition implies $\widetilde{u}_1 \in \mathcal{V}^\phi \subset H^{\tau+2\alpha}$ which is equivalent to $L\widetilde{u}_1 \in H^\tau \simeq \mathcal{N}_\phi$.

The functions $L\widetilde{u}_1$ and g are required to be in the native space $\mathcal{N}_\phi$ so that property (3.11) can be used. The conditions $\alpha + (n-1)/2 \le \tau$ and $\tau \le \sigma$ are purely technical requirements of our proof.

In this method we find $\widetilde{u}_1 \in \mathcal{V}^\phi$ by solving the collocation equation

$$L\widetilde{u}_1(\boldsymbol{x}_j) = g(\boldsymbol{x}_j), \quad j = 1, \ldots, N. \tag{6.2}$$

By writting $\widetilde{u}_1 = \sum_{j=1}^N c_j \Phi_j$, we derive from (6.2) the matrix equation $\boldsymbol{A}^{(C)}\boldsymbol{c} = \boldsymbol{g}$ where

$$\boldsymbol{A}^{(C)}_{ij} = L\Phi_i(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\, \widehat{\phi}(\ell)\, Y_{\ell,m}(\boldsymbol{x}_i)\, Y_{\ell,m}(\boldsymbol{x}_j),$$

$\boldsymbol{c} = (c_1, \ldots, c_N)$ and $\boldsymbol{g} = (g(\boldsymbol{x}_1, \ldots, g(\boldsymbol{x}_N))$. The symmetry and positive definiteness of the matrix $\boldsymbol{A}^{(C)}$ can be proved in the same manner as Lemma 5.1.

Since the function Φ defined as in (3.4) is a reproducing kernel for the Hilbert space $\mathcal{N}_\phi$, see (3.11), the collocation equation (6.2) can be rewritten as a Galerkin equation. This allows us to carry out error analysis in the same manner as in Section 5.

Recalling (3.11) and noting that $L\widetilde{u}_1, g \in \mathcal{N}_\phi$, we rewrite (6.2) as

$$\langle L\widetilde{u}_1, \Phi_j \rangle_\phi = \langle g, \Phi_j \rangle_\phi, \quad j = 1, \ldots, N. \tag{6.3}$$

In order to see that the above equation is a Galerkin equation, we introduce a new finite-dimensional subspace $\mathcal{V}^{\widetilde{\phi}}$:

$$\mathcal{V}^{\widetilde{\phi}} := \operatorname{span}\{\widetilde{\Phi}_1, \ldots, \widetilde{\Phi}_N\},$$

where the spherical radial basis functions $\widetilde{\Phi}_j$ are defined by

$$\widetilde{\Phi}_j(\boldsymbol{x}) := \widetilde{\phi}(\boldsymbol{x} \cdot \boldsymbol{x}_j), \quad j = 1, \ldots, N.$$

Here, $\widetilde{\phi}$ is a shape function given by

$$\widetilde{\phi}(t) := \sum_{\ell=0}^{\infty} \omega_n^{-1} N(n,\ell) \left[\widehat{\phi}(\ell)\right]^{1/2} P_\ell(n;t),$$

It is easily seen that (cf. (3.8))

$$\widehat{(\widetilde{\Phi}_j)}_{\ell,m} = [\widehat{\phi}(\ell)]^{1/2}\, Y_{\ell,m}(\boldsymbol{x}_j), \quad j = 1, \ldots, N. \tag{6.4}$$

It should be noted that this space $\mathcal{V}^{\widetilde{\phi}}$ is introduced purely for analysis purposes; it is not to be used in the implementation. Since (cf. (3.5))

$$c_1(\ell+1)^{-\tau} \le \widehat{(\widetilde{\phi})}(\ell) \le c_2(\ell+1)^{-\tau},$$

we have (cf. (3.10))

$$\mathcal{V}^{\widetilde{\phi}} \subset H^s \quad \text{for all } s < \tau + \frac{1-n}{2}. \tag{6.5}$$

In particular, $\mathcal{V}^{\widetilde{\phi}} \subset H^\alpha$ due to $\alpha + (n-1)/2 < \tau$ (see (6.1)).

The following lemma defines a weak equation equivalent to equation (2.11).

Lemma 6.1. *Let*

$$U_1 := \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \frac{(\widehat{u}_1)_{\ell,m}}{\left[\widehat{\phi}(\ell)\right]^{1/2}} Y_{\ell,m}, \tag{6.6}$$

where u_1 is the solution to (2.11). *Then U_1 belongs to $H^{\sigma+\alpha-\tau}$ and satisfies*

$$a(U_1, V) = \langle G, V \rangle \quad \textit{for all } V \in H^{\alpha-\sigma+\tau}, \tag{6.7}$$

where

$$G := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{g}_{\ell,m}}{\left[\widehat{\phi}(\ell)\right]^{1/2}} Y_{\ell,m}. \tag{6.8}$$

Proof. Since $u_1 \in H^{\sigma+\alpha}$, it is easily seen that $U_1 \in H^{\sigma+\alpha-\tau}$. For any $V \in H^{\alpha-\sigma+\tau}$ there holds

$$a(U_1, V) = a(u_1, v),$$

where

$$v := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{V}_{\ell,m}}{\left[\widehat{\phi}(\ell)\right]^{1/2}} Y_{\ell,m}.$$

Noting $v \in H^{\alpha-\sigma}$ we deduce from (2.11) that

$$a(U_1, V) = \langle g, v \rangle = \langle G, V \rangle,$$

finishing the proof of the lemma. $\square$

Analogously, the next lemma defines an equivalent to (6.3). It will be seen later that this equivalent is the Galerkin approximation to (6.7).

Lemma 6.2. *Let*

$$\widetilde{U}_1 := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{(\widetilde{u}_1)}_{\ell,m}}{\left[\widehat{\phi}(\ell)\right]^{1/2}} Y_{\ell,m} \tag{6.9}$$

where $\widetilde{u}_1$ *is given by* (6.2)*. Then* $\widetilde{U}_1$ *belongs to* $\mathcal{V}^{\widetilde{\phi}}$ *and satisfies*

$$a(\widetilde{U}_1, \widetilde{\Phi}_j) = \left\langle G, \widetilde{\Phi}_j \right\rangle, \quad j = 1, \dots, N. \tag{6.10}$$

Proof. Since $\widetilde{u}_1 \in \mathcal{V}^{\phi}$ we have $\widetilde{u}_1 = \sum_{j=1}^{N} c_j \Phi_j$ for some $c_j \in \mathbb{R}$, which together with (3.8) implies

$$\widehat{(\widetilde{u}_1)}_{\ell,m} = \widehat{\phi}(\ell) \sum_{j=1}^{N} c_j Y_{\ell,m}(\boldsymbol{x}_j).$$

This in turn gives

$$\widehat{(\widetilde{U}_1)}_{\ell,m} = [\widehat{\phi}(\ell)]^{1/2} \sum_{j=1}^{N} c_j Y_{\ell,m}(\boldsymbol{x}_j),$$

so that (see (6.4))

$$\widetilde{U}_1 = \sum_{j=1}^{N} c_j \widetilde{\Phi}_j,$$

i.e., $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}}$. By using successively (2.10), (6.4), (6.9), (6.3), (3.8) and (6.8), we deduce

$$a(\widetilde{U}_1, \widetilde{\Phi}_j) = \left\langle L\widetilde{U}_1, \widetilde{\Phi}_j \right\rangle = \langle L\widetilde{u}_1, \Phi_j \rangle_\phi = \langle g, \Phi_j \rangle_\phi = \left\langle G, \widetilde{\Phi}_j \right\rangle, \quad j = 1, \ldots, N,$$

completing the proof of the lemma. $\square$

Using the two above lemmas we can now estimate the error in the collocation approximation in the same manner as for the Galerkin approximation.

Theorem 6.3. *Let* (6.1) *hold. We choose the shape function ϕ such that* (3.5) *holds with $\tau > n - 1$. Assume further that $u \in H^s$ for some s satisfying $\tau + \alpha \le s \le 2\tau$. If $\mu_i \in H^{-t}$, $i = 1, \ldots, M$ for some t satisfying $2\alpha \le t \le \tau + \alpha$, then for h_X sufficiently small there holds*

$$\|u - \widetilde{u}\|_t \le C h_X^{s-t} \|u\|_s.$$

The constant C is independent of u and h_X.

Proof. Recall that $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}} \subset H^\alpha$ and $U_1 \in H^{\sigma+\alpha-\tau} \subset H^\alpha$ since $\tau \le \sigma$; see (6.1). Moreover, (6.7) and (6.10) imply

$$a(U_1 - \widetilde{U}_1, \widetilde{\Phi}_j) = 0, \quad j = 1, \ldots, N.$$

Hence, $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}}$ is the Galerkin approximation to U_1.

Analogously to (4.3) we define

$$U_1^* = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{U}_1)} Y_{\ell,m}. \tag{6.11}$$

Lemma 5.3 with $\mathcal{V}^\phi$ replaced by $\mathcal{V}^{\widetilde{\phi}}$ (and therefore, τ replaced by $\widetilde{\tau} := \tau/2$) and u_1, u_1^* replaced by U_1, U_1^*, gives

$$\|U_1 - U_1^*\|_{\widetilde{t}} \le C h_X^{\widetilde{s}-\widetilde{t}} \|U_1\|_{\widetilde{s}}, \quad \alpha \le \widetilde{s} \le 2\widetilde{\tau},\ 2(\alpha - \widetilde{\tau}) \le \widetilde{t} \le \alpha. \tag{6.12}$$

By the definition of $U_1, \widetilde{U}_1$ and U_1^*, see (6.6), (6.9) and (6.11), we have

$$\|u_1 - u_1^*\|_t \simeq \|U_1 - U_1^*\|_{t-\tau} \text{ and } \|u_1\|_s \simeq \|U_1\|_{s-\tau}. \tag{6.13}$$

Since t and s satisfy $2\alpha \le t \le \tau + \alpha$ and $\tau + \alpha \le s \le 2\tau$ so that $t - \tau$ and $s - \tau$ satisfy

$$2(\alpha - \widetilde{\tau}) \le t - \tau \le \alpha \quad \text{and} \quad \alpha \le s - \tau \le 2\widetilde{\tau},$$

the inequality (6.12) with $\widetilde{t} = t - \tau$ and $\widetilde{s} = s - \tau$ gives

$$\|U_1 - U_1^*\|_{t-\tau} \le C h_X^{s-t} \|U_1\|_{s-\tau}.$$

This together with (6.13) implies

$$\|u_1 - u_1^*\|_t \le Ch_X^{s-t} \|u_1\|_s$$

Since $\mu_i \in H^{-t}$, for $i = 1, \ldots, M$, by using Lemma 4.2 and noting that $\|u_1\|_s \le \|u\|_s$, we deduce

$$\|u - \widetilde{u}\|_t \le C\|u_1 - u_1^*\|_t \le Ch_X^{s-t} \|u_1\|_s \le Ch_X^{s-t} \|u\|_s,$$

completing the proof of the theorem.

□

Remark 6.4. In comparison with the results obtained by Morton and Neamtu, our error estimates for the collocation approximation cover a wider range of Sobolev norms. In fact, these two authors only proved [14]

$$\|u - \widetilde{u}\|_{2\alpha} \le ch_X^{\lfloor 2(\tau-\alpha) \rfloor} \|u\|_{2\tau}.$$

This is a special case of the results in Theorems 6.3.

7 Numerical experiments

In this section, we solved the Dirichlet problem

$$\begin{aligned} \Delta U &= 0 \text{ in } \mathbb{B}_e, \\ U &= U_D \text{ on } \mathbb{S}^{n-1}, \\ U(\boldsymbol{x}) &= O(1/|\boldsymbol{x}|) \text{ as } |\boldsymbol{x}| \to \infty, \end{aligned} \tag{7.1}$$

where $\mathbb{B}_e := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| > 1\}$. It is well-known, see e.g. [23], that the problem (7.1) is equivalent to

$$Su = g \text{ on } \mathbb{S}^{n-1}, \tag{7.2}$$

where

$$g = -\frac{1}{2}U_D + DU_D, \tag{7.3}$$

and

$$Dv(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{S}^{n-1}} v(\boldsymbol{y}) \frac{\partial}{\partial \nu_{\boldsymbol{y}}} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\sigma_{\boldsymbol{y}}.$$

Here, S is the weakly singular integral operator defined by

$$Sv(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \frac{v(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\sigma_{\boldsymbol{y}},$$

which is a pseudodifferential operator of order -1 and $\widehat{S}(\ell) = 1/(2\ell + 1)$; see the examples following Definition 2.1.

We solved the problem (7.1) with the boundary data

$$U_D(\boldsymbol{x}) := U_D(x_1, x_2, x_3) = \frac{1}{(1.0625 - 0.5x_3)^{1/2}}$$

so that the exact solution to the Dirichlet problem (7.1) is given by

$$U(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x} - \boldsymbol{q}|} \quad \text{with} \quad \boldsymbol{q} = (0, 0, 0.25),$$

and hence, the exact solution to the weakly singular integral equation (7.2) is $u(\boldsymbol{x}) = \partial_\nu U(\boldsymbol{x})$; see e.g. [23], i.e.,

$$u(\boldsymbol{x}) = \frac{-1 + \boldsymbol{x} \cdot \boldsymbol{q}}{|\boldsymbol{x} - \boldsymbol{q}|^3} = \frac{0.25x_3 - 1}{(1.0625 - 0.5x_3)^{3/2}}.$$

For the approximation of (7.2), we use spherical radial basis functions suggested by Wendland [28, page 128]. The sets $X := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ of points are chosen purely to observe the order of convergence. Experiments with real data can be found in [23].

The shape function $\phi : [-1, 1] \to \mathbb{R}$ which is used to define the kernel Φ is given by

$$\phi(t) = \rho(\sqrt{2 - 2t}), \tag{7.4}$$

where ρ is Wendland's functions [30, page 128] defined by

$$\rho(r) = (1 - r)_+^2.$$

Narcowich and Ward [19, Proposition 4.6] prove that $\widehat{\phi}(\ell) \sim (1 + \ell)^{-2\tau}$ for all $\ell \geq 0$, where $\tau = 3/2$. The spherical radial basis functions Φ_i, $i = 1, \ldots, N$, are computed by

$$\Phi_i(\boldsymbol{x}) = \rho(\sqrt{2 - 2\boldsymbol{x} \cdot \boldsymbol{x}_i}), \quad \boldsymbol{x} \in \mathbb{S}^{n-1}. \tag{7.5}$$

We first found an approximate solution $u_X^G \in \mathcal{V}_X^\phi := \operatorname{span}\{\Phi_1, \Phi_2, \ldots, \Phi_N\}$ satisfying the Galerkin equation

$$a_S(u_X^G, v) := \left\langle S u_X^G, v \right\rangle = \langle g, v \rangle \quad \forall v \in \mathcal{V}_X^\phi. \tag{7.6}$$

The stiffness matrix arising from (7.6) has entries given by

$$a_S(\Phi_i, \Phi_j) = \sum_{\ell=0}^{\infty} \frac{|\widehat{\phi}(\ell)|^2}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}_j) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} |\widehat{\phi}(\ell)|^2 P_\ell(\boldsymbol{x}_i \cdot \boldsymbol{x}_j).$$

The right-hand side of (7.6) is computed by using (7.3), noting $\widehat{D}(\ell) = -1/(4\ell + 2)$ (see [20, page 122]),

$$\begin{aligned}
\langle g, \Phi_i \rangle &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(-\frac{1}{2} - \frac{1}{2(2\ell + 1)} \Big) \widehat{(U_D)}_{\ell,m} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i) \\
&= -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell + 1)}{2\ell + 1} \widehat{(U_D)}_{\ell,m} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i).
\end{aligned}$$

The errors are computed by

$$\|u - u_X^G\|_{-1/2} = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\left| \widehat{u}_{\ell,m} - \widehat{(u_X^G)}_{\ell,m} \right|^2}{\ell+1} \right)^{1/2}. \tag{7.7}$$

Our theoretical result (Theorem 5.4) predicts an order of convergence of $2\tau + 1/2$ in the $H^{-1/2}$-norm. We carried out the experiment and observed some agreement between the experimented orders of convergence (EOC) and our theoretical results; see Tables 1.

Table 1: Galerkin method: Errors in $H^{-1/2}$-norm, $\tau = 1.5$. Expected order of convergence : 3.5

N	h_X	$H^{-1/2}$-norm	EOC
20	0.65140	0.120349381	
30	0.51210	0.054895875	3.262
40	0.44180	0.025612135	5.163
51	0.37500	0.015883257	2.915
101	0.26720	0.006082010	2.832
200	0.19420	0.001977985	3.520
500	0.12370	0.000492078	3.084

The collocation solution $u_X^C \in \mathcal{V}_X^\phi$ is found by solving

$$Su_X^C(\boldsymbol{x}_i) = g(\boldsymbol{x}_i), \quad i = 1, \ldots, N. \tag{7.8}$$

By writing $u_X^C = \sum_{i=1}^N c_i \Phi_i$, we derive from (7.8) the matrix equation

$$\boldsymbol{S}^C \boldsymbol{c} = \boldsymbol{g},$$

where $\boldsymbol{c} = (c_i)_{i=1,\ldots,N}$, $\boldsymbol{g} = (g(\boldsymbol{x}_i))_{i=1,\ldots,N}$ and

$$\boldsymbol{S}_{ij}^C = S\Phi_i(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{\phi}(\ell)}{2\ell+1} Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}_j), \quad i, j = 1, \ldots, N.$$

By using the addition formula (3.3), we obtain

$$\boldsymbol{S}_{ij}^C = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \widehat{\phi}(\ell) P_\ell(\boldsymbol{x}_i \cdot \boldsymbol{x}_j). \tag{7.9}$$

The errors are then computed similarly as in (7.7). There is agreement between the experimented order of convergence (EOC) and our theoretical result (which is $2\tau + 1/2$); see Tables 2.

Table 2: Collocation method: Errors in $H^{-1/2}$-norm, $\tau = 1.5$. Expected order of convergence : 3.5

N	h_X	$H^{-1/2}$-norm	EOC
20	0.65140	0.139479793	
30	0.51210	0.047806025	4.450
40	0.44180	0.020666895	5.679
51	0.37500	0.011785692	3.426
101	0.26720	0.003674365	3.439
400	0.12370	0.000277996	3.352

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