# BANACH SPACE PROJECTIONS AND PETROV-GALERKIN ESTIMATES 

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#### Abstract

We sharpen the classic a priori error estimate of Babuška for Petrov-Galerkin methods on a Banach space. In particular, we do so by (i) introducing a new constant, called the Banach-Mazur constant, to describe the geometry of a normed vector space; (ii) showing that, for a nontrivial projection $P$, it is possible to use the Banach-Mazur constant to improve upon the naïve estimate $\|I-P\| \leq 1+\|P\|$; and (iii) applying that improved estimate to the Petrov-Galerkin projection operator. This generalizes and extends a 2003 result of Xu and Zikatanov for the special case of Hilbert spaces.


## 1. Introduction

In a landmark 1971 paper, Babuška [1] developed a framework for the analysis of finite element methods. This analysis encompassed not only Galerkin methods for coercive bilinear forms (as in the pioneering work of Céa [4), but also Galerkin methods for non-coercive bilinear forms (such as mixed finite element methods, cf. Brezzi and Fortin (3) and Petrov-Galerkin methods more generally. A key innovation in this work was the replacement of the coercivity assumption by the so-called inf-sup condition. (See also the essential contribution by Brezzi [2].) One of the main results of Babuška's paper is an a priori error estimate for Petrov-Galerkin methods satisfying this inf-sup condition.

Remarkably, more than three decades passed before a 2003 paper, by Xu and Zikatanov [12], pointed out that the constant in Babuška's estimate can be improved (by 1) when the space of trial functions is a Hilbert space. To develop this improved estimate, Xu and Zikatanov [12] used an identity concerning the operator norm of a projection on a Hilbert space. However, this identity is completely idiosyncratic to Hilbert spaces, and for arbitrary Banach spaces, Babuška's original estimate has yet to be improved.

The present paper aims to fill this gap, sharpening the constant in Babuška's estimate for Petrov-Galerkin methods on a Banach space. The degree of improvement depends on how "close" the trial space is to being Hilbert, in a sense related to Banach-Mazur distance. In particular, for the most pathological Banach spaces, such as non-reflexive spaces, no improvement is obtained over Babuška's estimate, while in the case of Hilbert spaces,

[^0]we recover Xu and Zikatanov's improved estimate. The paper is organized as follows:

- Section 2 briefly reviews the results of Céa [4, Babuška [1, and Xu and Zikatanov [12], as summarized above. In addition to providing the necessary background, this also serves to fix the notation and terminology used later in the paper.
- Section 3 introduces the Banach-Mazur constant of a normed vector space, which quantifies how "close" this space is to being an inner product space. We also show how this relates to the well-studied von Neumann-Jordan constant, which serves a similar purpose.
- Section 4 contains the main technical result: an estimate for projection operators on a normed vector space, generalizing the Hilbert space projection identity used by Xu and Zikatanov. This estimate depends fundamentally on the Banach-Mazur constant introduced in the previous section.
- Section 5 illustrates the preceding theory by applying it to an important class of Banach spaces: $L_{p}$ and Sobolev spaces. We compute the Banach-Mazur constants of these spaces and discuss the related properties of projection operators, showing that the main estimate of Section 4 is sharp.
- Finally, Section 6 contains the main theorem: a sharpened a priori error estimate for Petrov-Galerkin methods on a Banach space. This is proved by applying the estimate from Section 4 to the PetrovGalerkin projection operator.

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## 2. Background: analysis of Petrov-Galerkin methods

Let $X$ be a Banach space, $Y$ be a reflexive Banach space, and $a \in$ $\mathcal{L}(X \times Y, \mathbb{R})$ be a continuous bilinear form, so that $|a(x, y)| \leq M\|x\|_{X}\|y\|_{Y}$ for some $M>0$. Given $f \in Y^{*}$, we consider the linear problem:
(1) Find $u \in X$ such that $a(u, v)=\langle f, v\rangle$ for all $v \in Y$.

If $X_{h} \subset X$ and $Y_{h} \subset Y$ are closed (e.g., finite-dimensional) subspaces, then we also consider the related problem:
(2) Find $u_{h} \in X_{h}$ such that $a\left(u_{h}, v_{h}\right)=\left\langle f, v_{h}\right\rangle$ for all $v_{h} \in Y_{h}$.

The approximation of (1) by (2) is called the Petrov-Galerkin method, or simply the Galerkin method in the special case $X_{h}=Y_{h} \subset X=Y$.

The most elementary a priori error estimate for the Galerkin method is due to Céa [4, who proved that if the bilinear form satisfies the coercivity condition $a(x, x) \geq m\|x\|_{X}^{2}$ for some $m>0$, then

$$
\left\|u-u_{h}\right\|_{X} \leq \frac{M}{m} \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|_{X}
$$

Note that coercivity is sufficient (although not necessary) for problems (1) and (2) to be well-posed. Céa's theorem does not apply to the general form of the Petrov-Galerkin method (since coercivity is meaningless when $X \neq Y$ ), nor even to the Galerkin method with non-coercive bilinear forms, which arise in mixed finite element methods.

A more general condition for (1) to be well-posed-which is both necessary and sufficient-is given by the inf-sup condition

$$
\inf _{0 \neq x \in X} \sup _{0 \neq y \in Y} \frac{a(x, y)}{\|x\|_{X}\|y\|_{Y}}=m>0, \quad \inf _{0 \neq y \in Y} \sup _{0 \neq x \in X} \frac{a(x, y)}{\|x\|_{X}\|y\|_{Y}}=m^{*}>0
$$

This is proved by applying Banach's closed range and open mapping theorems to the operator $A: X \rightarrow Y^{*}, x \mapsto a(x, \cdot)$, and to its adjoint $A^{*}: Y \rightarrow$ $X^{*}, y \mapsto a(\cdot, y)$. In fact, when the inf-sup condition is satisfied, the constants $m$ and $m^{*}$ are equal, since

$$
m^{-1}=\left\|A^{-1}\right\|_{\mathcal{L}\left(Y^{*}, X\right)}=\left\|\left(A^{*}\right)^{-1}\right\|_{\mathcal{L}\left(X^{*}, Y\right)}=\left(m^{*}\right)^{-1}
$$

Likewise, the problem (2) is well-posed if and only if

$$
\inf _{0 \neq x_{h} \in X_{h}} \sup _{0 \neq y_{h} \in Y_{h}} \frac{a\left(x_{h}, y_{h}\right)}{\left\|x_{h}\right\|_{X}\left\|y_{h}\right\|_{Y}}=\inf _{0 \neq y_{h} \in Y_{h}} \sup _{0 \neq x_{h} \in X_{h}} \frac{a\left(x_{h}, y_{h}\right)}{\left\|x_{h}\right\|_{X}\left\|y_{h}\right\|_{Y}}=m_{h}>0
$$

which is called the discrete inf-sup condition.
Babuška [1] showed that, if the inf-sup conditions are satisfied, then the solutions to (1) and (2) satisfy the error estimate

$$
\left\|u-u_{h}\right\|_{X} \leq\left(1+\frac{M}{m_{h}}\right) \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|_{X}
$$

The proof relies on the Petrov-Galerkin projection operator on $X$, denoted by $P_{h}$, which maps each $u \in X$ to its Petrov-Galerkin approximation $u_{h} \in X_{h}$. Since $P_{h} x_{h}=x_{h}$ for all $x_{h} \in X_{h}$, we have
$\left\|u-u_{h}\right\|_{X}=\left\|\left(I-P_{h}\right) u\right\|_{X} \leq\left\|\left(I-P_{h}\right)\left(u-x_{h}\right)\right\|_{X} \leq\left\|I-P_{h}\right\|_{\mathcal{L}(X, X)}\left\|u-x_{h}\right\|_{X}$. Hence, the estimate follows by observing that

$$
\left\|I-P_{h}\right\|_{\mathcal{L}(X, X)} \leq 1+\left\|P_{h}\right\|_{\mathcal{L}(X, X)} \leq 1+\frac{M}{m_{h}}
$$

and by taking the infimum over all $x_{h} \in X_{h}$.
The Babuška estimate superficially resembles that of Céa, with the glaring exception of 1 being added to the constant. However, Xu and Zikatanov [12] observed that, in the case where $X$ is a Hilbert space, this additional term is unneccessary, and one obtains the sharpened estimate

$$
\left\|u-u_{h}\right\|_{X} \leq \frac{M}{m_{h}} \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|_{X}
$$

The key insight is that, in a Hilbert space, nontrivial projection operators $P$ satisfy $\|I-P\|_{\mathcal{L}(X, X)}=\|P\|_{\mathcal{L}(X, X)}$, so applying this identity to the PetrovGalerkin projection yields $\left\|I-P_{h}\right\|_{\mathcal{L}(X, X)} \leq \frac{M}{m_{h}}$. (See Szyld [10] for a discussion of this undeservedly obscure and frequently rediscovered identity.)

## 3. The Banach-Mazur constant of a normed vector space

In this section, we introduce the Banach-Mazur constant of a normed vector space $X$, which quantifies the degree to which $X$ fails to be an inner-product space. This constant will play a crucial role in the projection estimates of Section 4 and Section 6. First, we recall the definition of (multiplicative) Banach-Mazur distance between finite-dimensional normed vector spaces of equal dimension.

Definition 3.1. If $V$ and $W$ are finite-dimensional normed vector spaces with $\operatorname{dim} V=\operatorname{dim} W$, then the Banach-Mazur distance between $V$ and $W$ is

$$
d_{B M}(V, W)=\inf \left\{\|T\|\left\|T^{-1}\right\|: T \text { is a linear isomorphism } V \rightarrow W\right\} .
$$

Definition 3.2. If $X$ is a normed vector space with $\operatorname{dim} X \geq 2$, then we define the Banach-Mazur constant of $X$ to be

$$
C_{B M}(X)=\sup \left\{\left(d_{B M}\left(V, \ell_{2}^{2}\right)\right)^{2}: V \subset X, \operatorname{dim} V=2\right\},
$$

where $\ell_{2}^{2}$ denotes the two-dimensional $\ell_{2}$ space (i.e., $\mathbb{R}^{2}$ equipped with the Euclidean norm $\|\cdot\|_{2}$ ).

Notation. For notational brevity, we will omit subscripts from operator norms and from $\|\cdot\|_{X}$, denoting each of these simply by $\|\cdot\|$, where the norm is clear from context. The Euclidean norm will always be denoted by $\|\cdot\|_{2}$.

There are various other such "geometric constants" for normed vector spaces; see Kato and Takahashi [9] for a survey of recent results on several of these constants. Generally, these constants lie between 1 and 2 , equaling 1 in the case of an inner product space, and equaling 2 for the most pathological spaces, such as non-reflexive spaces. One of the oldest and best-known is the von Neumann-Jordan constant, dating to the 1935 paper of Jordan and von Neumann [7] (see also Clarkson [5), which measures the degree to which the norm satisfies (or fails to satisfy) the parallelogram law.

Definition 3.3. The von Neumann-Jordan constant of a normed vector space $X$ is

$$
C_{N J}(X)=\sup \left\{\frac{\|x+y\|^{2}+\|x-y\|^{2}}{2\left(\|x\|^{2}+\|y\|^{2}\right)}: x, y \in X \text { not both zero }\right\} .
$$

The following result establishes the relationship between the BanachMazur and von Neumann-Jordan constants.

Theorem 3.4. $1 \leq C_{N J}(X) \leq C_{B M}(X) \leq 2$.
Proof. The inequality $1 \leq C_{N J}(X)$ appears in Jordan and von Neumann [7]. John's theorem on maximal ellipsoids [6] implies that $d_{B M}\left(V, \ell_{2}^{2}\right) \leq \sqrt{2}$ for all $V \subset X$ with $\operatorname{dim} V=2$, and thus $C_{B M}(X) \leq 2$. To prove the remaining inequality, $C_{N J}(X) \leq C_{B M}(X)$, it suffices to show that

$$
\|x+y\|^{2}+\|x-y\|^{2} \leq 2 C_{B M}(X)\left(\|x\|^{2}+\|y\|^{2}\right),
$$

for all $x, y \in X$. This is obvious if $x$ and $y$ are linearly dependent, since in that case, they satisfy the parallelogram law exactly. Otherwise, take the two-dimensional subspace $V=\operatorname{span}\{x, y\}$. For any isomorphism $T: V \rightarrow \ell_{2}^{2}$,

$$
\begin{aligned}
\|x+y\|^{2}+\|x-y\|^{2} & \leq\left\|T^{-1}\right\|^{2}\left(\|T x+T y\|_{2}^{2}+\|T x-T y\|_{2}^{2}\right) \\
& =2\left\|T^{-1}\right\|^{2}\left(\|T x\|_{2}^{2}+\|T y\|_{2}^{2}\right) \\
& \leq 2\|T\|^{2}\left\|T^{-1}\right\|^{2}\left(\|x\|^{2}+\|y\|^{2}\right)
\end{aligned}
$$

where the parallelogram law for $\ell_{2}^{2}$ is applied in the second line. Finally, taking the infimum over all $T$ yields

$$
\begin{aligned}
\|x+y\|^{2}+\|x-y\|^{2} & \leq 2\left(d_{B M}\left(V, \ell_{2}^{2}\right)\right)^{2}\left(\|x\|^{2}+\|y\|^{2}\right) \\
& \leq 2 C_{B M}(X)\left(\|x\|^{2}+\|y\|^{2}\right)
\end{aligned}
$$

which completes the proof.
Theorem 3.5. $C_{B M}(X)=1$ if and only if $X$ is an inner product space.
Proof. If $X$ is an inner product space, then any two-dimensional subspace is unitarily isomorphic to $\ell_{2}^{2}$, so $C_{B M}(X)=1$. Conversely, if $C_{B M}(X)=1$, then Theorem 3.4 implies $C_{N J}(X)=1$, so by the Jordan-von Neumann theorem [7, $X$ is an inner product space.

Remark 3.6. The constants $C_{B M}(X)$ and $C_{N J}(X)$ agree in the most extreme cases. For Hilbert spaces, we have seen that $C_{B M}(X)=C_{N J}(X)=1$. At the opposite extreme, the most pathological spaces-including non-reflexive spaces, such as $L_{1}$ and $L_{\infty}$-have $C_{N J}(X)=2$, and hence $C_{B M}(X)=2$ by Theorem 3.4 (see also Clarkson [5]). More specifically, a theorem of Kato and Takahashi [8] states that, if $C_{N J}(X)<2$, then $X$ is super-reflexive. Consequently, if $X$ fails to be super-reflexive (in particular, if it is nonreflexive), then $C_{N J}(X)=C_{B M}(X)=2$.

## 4. A projection estimate for normed vector spaces

Having introduced the Banach-Mazur constant, we are now equipped to prove the main technical result: an estimate for projection operators on normed vector spaces. This generalizes the Hilbert space projection identity used by Xu and Zikatanov [12].

Theorem 4.1. Let $P$ be a nontrivial projection operator (i.e., $0 \neq P=$ $P^{2} \neq I$ ) on a normed vector space $X$. Then $\|I-P\| \leq C\|P\|$, where $C=\min \left\{1+\|P\|^{-1}, C_{B M}(X)\right\}$.
Proof. The inequality $\|I-P\| \leq\left(1+\|P\|^{-1}\right)\|P\|=1+\|P\|$ is elementary, so it suffices to show $\|(I-P) x\| \leq C_{B M}(X)\|P\|\|x\|$ for all $x \in X$.

If $(I-P) x=0$, then this inequality is trivial. On the other hand, if $P x=0$, then $(I-P) x=x$. Moreover, $\|P\| \geq 1$ since $P$ is a nontrivial projection, while $C_{B M}(X) \geq 1$ by Theorem 3.4. Hence, in this case we have $\|(I-P) x\|=\|x\| \leq C_{B M}(X)\|P\|\|x\|$.

We may now assume that we are in the remaining case, where neither $P x$ nor $(I-P) x$ vanishes, so $V=\operatorname{span}\{P x,(I-P) x\}$ is a two-dimensional subspace of $X$. If $T: V \rightarrow \ell_{2}^{2}$ is a linear isomorphism, then there exist unit vectors $u, v \in \ell_{2}^{2}$ and scalars $a, b \in \mathbb{R}$ such that $P x=a T^{-1} u$ and $(I-P) x=b T^{-1} v$. Thus, we may write $x=P x+(I-P) x=a T^{-1} u+b T^{-1} v$.

Now, take $y=b T^{-1} u+a T^{-1} v$, so that $P y=b T^{-1} u$ and $(I-P) y=a T^{-1} v$. It follows that

$$
\begin{aligned}
\|(I-P) x\| & =\left\|b T^{-1} v\right\| \\
& \leq|b|\left\|T^{-1}\right\| \\
& \leq\|T\|\left\|T^{-1}\right\|\left\|b T^{-1} u\right\| \\
& =\|T\|\left\|T^{-1}\right\|\|P y\| \\
& \leq\|T\|\left\|T^{-1}\right\|\|P\|\|y\|
\end{aligned}
$$

Next, since $\ell_{2}^{2}$ is an inner product space, we have

$$
\|a u+b v\|_{2}=\left(a^{2}+2 a b u \cdot v+b^{2}\right)^{1 / 2}=\|b u+a v\|_{2} .
$$

Therefore,

$$
\begin{aligned}
\|y\| & =\left\|b T^{-1} u+a T^{-1} v\right\| \\
& \leq\left\|T^{-1}\right\|\|b u+a v\|_{2} \\
& =\left\|T^{-1}\right\|\|a u+b v\|_{2} \\
& \leq\|T\|\left\|T^{-1}\right\|\left\|a T^{-1} u+b T^{-1} v\right\| \\
& =\|T\|\left\|T^{-1}\right\|\|x\|
\end{aligned}
$$

Altogether, we have now shown that

$$
\|(I-P) x\| \leq\|T\|\left\|T^{-1}\right\|\|P\|\left(\|T\|\left\|T^{-1}\right\|\|x\|\right)=\left(\|T\|\left\|T^{-1}\right\|\right)^{2}\|P\|\|x\|
$$

Finally, taking the infimum over all isomorphisms $T$ yields

$$
\|(I-P) x\| \leq\left(d_{B M}\left(V, \ell_{2}^{2}\right)\right)^{2}\|P\|\|x\| \leq C_{B M}(X)\|P\|\|x\|
$$

which completes the proof.
Corollary 4.2. If $X$ is an inner product space, then $\|I-P\|=\|P\|$.
Proof. Since $X$ is an inner product space, Theorem 3.5 implies $C_{B M}(X)=1$, so Theorem 4.1 gives $\|I-P\| \leq\|P\|$. The reverse inequality follows by symmetry of the projections $P$ and $I-P$.

Remark 4.3. Theorem 4.1 is strictly sharper than the obvious estimate $\|I-P\| \leq 1+\|P\|$ whenever $C_{B M}(X)<1+\|P\|^{-1}$. In particular, since $1<1+\|P\|^{-1} \leq 2$, this result is always sharper when $C_{B M}(X)=1$ (i.e., for Hilbert spaces) and never sharper for the opposite extreme, $C_{B M}(X)=2$ (e.g., for non-reflexive spaces). Intermediate cases $1<C_{B M}(X)<2$ depend on the particular projection operator $P$.

## 5. Application to $L_{p}$ and Sobolev spaces

In this section, we apply the foregoing theory to $L_{p}$ and Sobolev spaces, which are the most important and commonly-encountered Banach spaces in finite element analysis.

The simplest possible example is $X=\ell_{p}^{2}$, the two-dimensional $\ell_{p}$ space (i.e., $\mathbb{R}^{2}$ equipped with the $p$-norm), where $1 \leq p \leq \infty$. In this case, it is known that $d_{B M}\left(\ell_{p}^{2}, \ell_{2}^{2}\right)=2^{|1 / p-1 / 2|}$ (cf. Wojtaszczyk [11, Proposition II.E.8]), so the Banach-Mazur constant is $C_{B M}(X)=\left(d_{B M}\left(\ell_{p}^{2}, \ell_{2}^{2}\right)\right)^{2}=2^{|2 / p-1|}$. If $1 \leq p \leq 2$, consider the pair of projections

$$
P\left(x_{0}, x_{1}\right)=\left(x_{0}+x_{1}, 0\right), \quad(I-P)\left(x_{0}, x_{1}\right)=\left(-x_{1}, x_{1}\right) .
$$

It can be seen that the operator norms are attained at

$$
\|P\|=\frac{\|P(1,1)\|_{p}}{\|(1,1)\|_{p}}=\frac{\|(2,0)\|_{p}}{\|(1,1)\|_{p}}=\frac{2}{2^{1 / p}}=2^{1-1 / p}
$$

and

$$
\|I-P\|=\frac{\|(I-P)(0,1)\|_{p}}{\|(0,1)\|_{p}}=\frac{\|(-1,1)\|_{p}}{\|(0,1)\|_{p}}=\frac{2^{1 / p}}{1}=2^{1 / p}
$$

Hence, $\|I-P\|=2^{2 / p-1}\|P\|=C_{B M}(X)\|P\|$; the same can be shown for $2 \leq p \leq \infty$, simply by switching $P$ and $I-P$. Therefore, Theorem 4.1 is sharp for $X=\ell_{p}^{2}$.

More generally, consider $X=L_{p}(\mu)$ for some measure $\mu$. In this case, it is known that $d_{B M}\left(V, \ell_{2}^{2}\right) \leq 2^{|1 / p-1 / 2|}$ for any two-dimensional subspace $V$ (cf. Wojtaszczyk [11, Corollary III.E.9]). Hence, taking $V$ isometrically isomorphic to $\ell_{p}^{2}$-for instance, the span of two unit-norm functions with disjoint support-implies $C_{B M}(X)=2^{|2 / p-1|}$, as above. In particular, we obtain the "best" case, $C_{B M}(X)=1$, only for $p=2$; the "worst" case, $C_{B M}(X)=2$, only for $p=1, \infty$; and the strict inequality $1<C_{B M}(X)<2$ for $1<p<\infty$.
Remark 5.1. In fact, here we have $C_{B M}(X)=C_{N J}(X)$, since Clarkson 5 proved that $C_{N J}(X)=2^{|2 / p-1|}$ for $L_{p}$ spaces.

Finally, consider the Sobolev space $X=W_{p}^{1}(U)$ for $U \subset \mathbb{R}^{n}$. If $U^{\sqcup(n+1)}$ denotes the disjoint union of $n+1$ copies of $U$, then we can isometrically embed $X \hookrightarrow L_{p}\left(U^{\sqcup(n+1)}\right)$ by taking $u \mapsto u \oplus \partial_{1} u \oplus \cdots \oplus \partial_{n} u$. Thus, any twodimensional subspace of $X$ is isometrically isomorphic to a two-dimensional subspace of $L_{p}\left(U^{\sqcup(n+1)}\right)$, and we can again realize $\ell_{p}^{2} \subset X$ by taking the span of two unit-norm functions with disjoint support. Hence, it follows from the previous discussion that, once again, $C_{B M}(X)=2^{|2 / p-1|}$. More generally, this argument holds for $X=W_{p}^{k}(U), k \in \mathbb{N}$, since the map $u \mapsto \bigoplus_{|\alpha| \leq k} \partial_{\alpha} u$, where $\alpha$ denotes a multi-index, embeds $X$ isometrically into the space of $L_{p}$ functions on sufficiently many disjoint copies of $U$.

## 6. The sharpened Petrov-Galerkin estimate

We now finally apply Theorem 4.1 to the Petrov-Galerkin projection $P_{h}$, using the formalism reviewed in Section 2.

Theorem 6.1. Let $u \in X$ and $u_{h} \in X_{h}$ be the solutions to (1) and (2), respectively. As before, let $M, m_{h}>0$ denote the continuity and discrete inf-sup constants for the bilinear form $a(\cdot, \cdot)$. Then we have the error estimate

$$
\left\|u-u_{h}\right\| \leq C \frac{M}{m_{h}} \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|
$$

where $C=\min \left\{1+\frac{m_{h}}{M}, C_{B M}(X)\right\}$.
Proof. As in Babuška's argument (summarized in Section 2), we have

$$
\left\|u-u_{h}\right\| \leq\left\|I-P_{h}\right\| \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|,
$$

where $P_{h}$ is the Petrov-Galerkin projection on $X$. Applying Theorem 4.1 yields $\left\|I-P_{h}\right\| \leq C\left\|P_{h}\right\| \leq C \frac{M}{m_{h}}$, which completes the proof.

Corollary 6.2 (Xu and Zikatanov [12]). If $X$ is a Hilbert space, then

$$
\left\|u-u_{h}\right\| \leq \frac{M}{m_{h}} \inf _{x_{h} \in X_{h}}\left\|u-x_{h}\right\|
$$

Proof. By Theorem 3.5, we have $C=C_{B M}(X)=1$, so the result follows immediately from Theorem 6.1.

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